SYMMETRIES IN AN OVERDETERMINED PROBLEM FOR THE GREEN'S FUNCTION

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Abstract. We consider in the plane the problem of reconstructing a domain from the normal derivative of its Green's function with pole at a fixed point in the domain. By means of the theory of conformal mappings, we obtain existence, uniqueness, (non-spherical) symmetry results, and a formula relating the curvature of the boundary of the domain to the normal derivative of its Green's function.

1. Introduction

Overdetermined boundary value problems in partial differential equations have connections to various fields in mathematics; they emerge in the study of isoperimetric inequalities, optimal design and ill-posed and free boundary problems, to name a few. In many such problems one's interest is focused on a specific feature: the shape of the domain considered; mainly, its (spherical) symmetry, as in Serrin's landmark paper [13] and its many offsprings (see [14], [1], [4], [8], [10], and the references therein).

With the present paper, we want to start a more detailed analysis of overdetermined problems in the plane, by exploiting the full power of the theory of analytic functions. As a case study, we shall analyse what appears to be the simplest situation: in a planar bounded simply connected domain Ω with boundary $\partial\Omega$ of class $C^{1,\alpha}$, we shall consider the problem

$$-\Delta U = \delta_{\zeta_c} \qquad \text{in } \Omega, \tag{1}$$

$$U = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial U}{\partial \nu} = \varphi \qquad \text{on } \partial\Omega. \tag{3}$$

where ν is the *interior* normal direction to $\partial\Omega$, δ_{ζ_c} is the Dirac delta centered at a given point $\zeta_c \in \Omega$ and $\varphi : \partial\Omega \to \mathbb{R}$ is a positive given function of arclength, measured counterclockwise from a reference point on $\partial\Omega$.

Problem (1)-(3) can be interpreted as a free-boundary problem: find a domain Ω whose Green's function U with pole at ζ_c has gradient with values on the boundary that fit those of the given function φ. This formulation serves as a basis to model, for example, the Hele-Shaw flow, as done in [6] and [12].

By means of the Riemann Mapping Theorem, the solution of (1)-(2) can be explicitly written in terms of a conformal mapping f from the unit disk D to Ω, which is uniquely determined if it satisfies some suitable normalizing conditions. Since it turns out that the normal derivative of U on $\partial\Omega$ is proportional to the modulus of

1991 *Mathematics Subject Classification.* Primary: 35N25; Secondary: 35J08, 35B06.

Key words and phrases. Overdetermined boundary values problems, Green's function, symmetries.

the inverse of f, then by (3) and classical results on holomorphic functions, we can derive an explicit formula for f in terms of φ (see section §2 for details). With the help of such a formula, we obtain the following results:

(i) existence and uniqueness theorems for a domain Ω satisfying (1)-(3) (Theorems 2.2 and 2.3);

(ii) symmetry results relating the invariance of φ under certain groups of transformations to that of Ω (Theorems 3.1 and 3.2); of course, when φ in constant, we obtain that Ω is a disk — a well-known result (see [10], [8] [1]);

(iii) a formula relating the interior normal derivative of the Green's function to the curvature of $\partial\Omega$.

2. Construction of a forward operator and its inverse

In what follows, D will always be the open unit disk in $\mathbb{C}$ centered at 0.

Let us recall some basic facts of harmonic and complex analysis. We refer the reader to [5] and [9] for more details. If $\Omega \subseteq \mathbb{C}$ is a simply connected domain bounded by a Jordan curve and $\zeta_c \in \Omega$, then, by the Riemann Mapping Theorem, Ω is the image of an analytic function $f : D \to \Omega$ which induces a homeomorphism between the closures $\overline{D}$ and $\overline{\Omega}$, has non-zero derivative f' in D and is such that $f(0) = \zeta_c$. Moreover, if Ω is of class $C^{1,\alpha}$, $0 < \alpha < 1$, that is its boundary $\partial\Omega$ is locally the graph of a function of class $C^{1,\alpha}$, then, by Kellogg's theorem, we can infer that $f \in C^{1,\alpha}(\overline{D})$ (see [5]).

In the following elementary lemma, which will be useful in the sequel, we relate f' to the so called *outer function* (see [2]).

Lemma 2.1. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ and $f : D \to \Omega$ be one-to-one and analytic with $f \in C^1(\overline{D})$. Then there exists $\gamma \in \mathbb{R}$ such that*

$$f'(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt\right\} \tag{4}$$

for every $z \in D$.

Proof. The function

$$f'(z) \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt\right\}, \qquad z \in D,$$

is analytic, never zero in D and has unitary modulus on ∂D; hence it equals the number $e^{i\gamma}$ for some $\gamma \in \mathbb{R}$. □

With these premises, given two distinct numbers ζ_c and $\zeta_b \in \mathbb{C}$, we consider

the set $\mathscr{O}$ of all $C^{1,\alpha}$, $0 < \alpha < 1$, simply connected
bounded domains such that $\zeta_c \in \Omega$ and $\zeta_b \in \partial\Omega$.

We can put $\mathscr{O}$ in one-to-one correspondence with

the class $\mathscr{F}$ of all one-to-one analytic mappings
$f \in C^{1,\alpha}(\overline{D})$ such that $f(0) = \zeta_c$ and $f(1) = \zeta_b$.

In fact, the arbitrary parameter γ in (4) can be determined by observing that

$$\zeta_b - \zeta_c = \int_0^1 f'(t)dt. \tag{5}$$

We now construct our forward operator $\mathcal{T}$ as the one that associates to each Ω in $\mathscr{O}$ the interior normal derivative $\frac{\partial U}{\partial \nu}$ — as function of the arclength, measured counterclockwise on $\partial\Omega$, starting from ζ_b — of the solution of (1)-(2). With our identification of $\mathscr{O}$ with $\mathscr{F}$ in mind, for $f \in \mathscr{F}$, $\mathcal{T}(f)$ is a function of arclength $s \in [0, |\partial\Omega|]$ and it is defined by the following remarks.

First, notice that, by Gauss-Green's formula, if U satisfies (1)-(2), then

$$v(\zeta_c) = \int_{\partial\Omega} v(\zeta) \frac{\partial U}{\partial \nu}(\zeta) ds(\zeta)$$

for every function $v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ which is harmonic in Ω.

Secondly, recall that any such function v satisfies the well-known Poisson integral formula

$$v(\zeta) = \frac{1}{2\pi} \int_{\partial\Omega} v(\zeta') \frac{1 - |f^{-1}(\zeta)|^2}{|f^{-1}(\zeta) - f^{-1}(\zeta')|} \frac{ds(\zeta')}{|f'(f^{-1}(\zeta'))|}, \qquad \zeta \in \Omega,$$

if $\partial\Omega$ is rectifiable (see [9]). By comparing the last two formulas (with $\zeta = \zeta_c = f(0)$), we obtain that

$$\frac{\partial U}{\partial \nu}(\zeta) = \frac{1}{2\pi |f'(f^{-1}(\zeta))|}, \qquad \zeta \in \partial\Omega.$$

Thirdly, since the arclength on $\partial\Omega$ is related to f by the formula

$$s(\theta) = \int_0^\theta |f'(e^{it})| dt, \qquad \theta \in [0, 2\pi], \tag{6}$$

the values $\mathcal{T}(f)(s)$, $s \in [0, |\partial\Omega|]$, can be defined parametrically by

$$s = \int_0^\theta |f'(e^{it})| dt, \quad \mathcal{T}(f) = \frac{1}{2\pi |f'(e^{i\theta})|}, \quad \theta \in [0, 2\pi]. \tag{7}$$

It is clear that $\mathcal{T}(f) \in C^{0,\alpha}([0, |\partial\Omega|])$ and also that

$$\int_0^{|\partial\Omega|} \mathcal{T}(f)(s) ds = 1, \qquad \mathcal{T}(f) > 0 \text{ on } [0, |\partial\Omega|],$$

for all $f \in \mathscr{F}$.

We shall now prove that $\mathcal{T}$ is injective by showing that each φ in the range of $\mathcal{T}$ determines only one $f \in \mathscr{F}$. In fact, for $\varphi \in C^{0,\alpha}([0, |\partial\Omega|])$ in the range of $\mathcal{T}$, by formulas (7) it turns out that

$$2\pi \varphi(s(\theta)) s'(\theta) = 1, \qquad \theta \in [0, 2\pi]. \tag{8}$$

This last formula, once integrated between 0 and θ, gives

$$s(\theta) = \Phi^{-1}(\theta), \qquad \theta \in [0, 2\pi], \tag{9}$$

where Φ^{-1} is the inverse of $\Phi : [0, |\partial\Omega|] \to [0, 2\pi]$ defined by

$$\Phi(s) = 2\pi \int_0^s \varphi(\sigma) d\sigma, \qquad s \in [0, |\partial\Omega|]. \tag{10}$$

By the same formulas (7), we then obtain that

$$|f'(e^{i\theta})| = \frac{1}{2\pi \varphi(\Phi^{-1}(\theta))}, \qquad \theta \in [0, 2\pi], \tag{11}$$

and hence (4) gives

$$f'(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{1}{2\pi \varphi(\Phi^{-1}(t))} dt \right\}, \qquad z \in D, \tag{12}$$

where γ is determined by (5). Therefore, for any φ in the range of $\mathcal{T}$, a unique $f \in \mathscr{F}$ such that $\mathcal{T}(f) = \varphi$ is determined by

$$f(z) = \zeta_c + \int_0^1 f'(tz) z dt, \qquad z \in D,$$

with f' given by (12).

We collect these remarks in the following theorem.

Theorem 2.2. *Given $\Omega \in \mathscr{O}$, let ζ_b be a reference point on $\partial\Omega$ from which the arclength on $\partial\Omega$ is measured counterclockwise.*

Let φ be in the range of $\mathcal{T}$, that is φ is the interior normal derivative of the Green's function on $\partial\Omega$ (as function of the arclength).

Then a function $f \in \mathscr{F}$ is uniquely determined such that $\mathcal{T}(f) = \varphi$ and its derivative is given by

$$f'(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{1}{2\pi\varphi(s(t))} dt \right\}, \quad z \in D, \tag{13}$$

where s and Φ are defined by (9) and (10), respectively.

Moreover, the constant γ is determined by

$$e^{i\gamma} \int_0^1 \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\tau} + t}{e^{i\tau} - t} \log \frac{1}{2\pi\varphi(s(\tau))} d\tau \right\} dt = \zeta_b - \zeta_c. \tag{14}$$

Theorem 2.2 tells us that the operator $\mathcal{T}$ is injective. A discussion about its surjectivity is beyond the aims of this paper. Far from being complete, we want here to suggest the following criterion.

Referring to [3], let us introduce the so called *boundary rotation* of a function f defined in D:

$$\rho = \lim_{r \to 1^-} \int_0^{2\pi} \left| \mathrm{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} \right| d\theta, \quad z = re^{i\theta} \in D.$$

We consider the class $\mathscr{V}$ of all normalized functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

which are analytic, locally univalent and with $\rho < +\infty$. The proof of the surjectivity of $\mathcal{T}$ relies on the problem of finding an analytic and univalent function f from the disk to $f(D) = \Omega$. The following theorem is based on a sufficient condition, due to Paatero, that says that any function in the class $\mathscr{V}$ with $\rho \leq 4\pi$ is univalent (see [3]).

Theorem 2.3. *Let $\varphi \in C^1(\mathbb{R})$ be L-periodic, strictly positive and satisfying the compatibility condition $\int_0^L \varphi(s) ds = 1$. If, moreover, φ satisfies the condition*

$$\max_{[0,L]} \left| \frac{\varphi'(s)}{\varphi^2(s)} \right| \leq 2\pi,$$

then there exists $\Omega \in \mathscr{O}$ with perimeter L and a solution of the overdetermined boundary value problem (1)-(3); thus, $\mathcal{T}$ is surjective.

Proof. By Theorem 2.2, we know that a function $f \in \mathscr{F}$ such that $\mathcal{T}(f) = \varphi$ must satisfy (13). Thus, we have to check Paatero's condition on (13). From that expression we deduce that

$$\frac{f''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{2e^{it}}{(e^{it}-z)^2}\log\frac{1}{2\pi\varphi(s(t))}dt,$$

being s defined as in (9) and (10). Now, by observing that

$$\frac{d}{dt}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{-2ize^{it}}{(e^{it}-z)^2},$$

we can integrate by parts and obtain that

$$\frac{-izf''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\frac{\varphi'(s(t))s'(t)}{\varphi(s(t))}dt.$$

By the maximum modulus principle, we can estimate, for $z \in D$,

$$\begin{aligned}\left|\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| &\le 1+\left|\frac{-izf''(z)}{f'(z)}\right| \\ &\le 1+\max_{[0,2\pi]}\left|\frac{\varphi'(s(t))s'(t)}{\varphi(s(t))}\right|,\end{aligned}$$

and, from (8), we have that $\varphi'(s)s'/\varphi(s) = \varphi'(s)/2\pi\varphi^2(s)$. Therefore, we can estimate the boundary rotation of f in the following way:

$$\rho \le \int_0^{2\pi}\left(1+\max_{[0,2\pi]}\left|\frac{\varphi'(s(t))}{2\pi\varphi^2(s(t))}\right|\right)d\theta = 2\pi\left(1+\max_{[0,L]}\left|\frac{\varphi'(s)}{2\pi\varphi^2(s)}\right|\right).$$

By our assumptions, it follows that $\rho \le 4\pi$ and hence, from Paatero's criterion for univalence, f is a homeomorphism from the disk onto $f(D)$. □

3. Symmetries

Remark 1. Theorem 2.2 allows us to rediscover a result already proved in [10] and also in [8] and [1]: if φ is constant, then Ω is a disk. More precisely, given $\Omega \in \mathscr{O}$ with perimeter L, let φ be constantly equal to $C > 0$. From (13), we obtain that

$$f'(z) = e^{i\gamma}\exp\left\{\frac{1}{2\pi}\log\frac{1}{2\pi C}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}dt\right\} = \frac{e^{i\gamma}}{2\pi C},$$

since $\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}dt = 2\pi$. Therefore, we get that

$$f(z) = \zeta_c + \frac{e^{i\gamma}}{2\pi C}z, \qquad z \in D,$$

that is Ω is the disk centered at ζ_c with radius $\frac{1}{2\pi C}$.

Now we want to show how some other symmetry properties of Ω can be derived from some invariance properties of φ and viceversa.

In what follows, for $\Omega \in \mathscr{O}$, let $L = |\partial\Omega|$ and let φ denote the values of the interior normal derivative on $\partial\Omega$ (as function of arclength) of the Green's function of Ω.

In the next theorem, we will identify φ with its L-periodic extension to $\mathbb{R}$ and $\mathcal{R}_{\zeta,\beta}$ will denote the clockwise rotation of an angle β around a point ζ.

Theorem 3.1. *Let $\Omega \in \mathscr{O}$ and $n \in \{2,3,4,...\}$. Then:*

$$\mathcal{R}_{\zeta_c,\frac{2\pi}{n}}(\Omega) = \Omega \text{ if and only if } \varphi \text{ is } \tfrac{L}{n}\text{-periodic.}$$

Proof. Let us fix n and suppose φ measured counterclockwise from $\zeta_b \in \partial\Omega$. Let $f \in \mathscr{F}$ be the unique analytic function from D to Ω such that $f(0) = \zeta_c$ and $f(1) = \zeta_b$.

(i) If Ω is invariant by rotations of angle $\frac{2\pi}{n}$ around ζ_c, then f satisfies

$$f(ze^{i\frac{2\pi}{n}}) = \zeta_c + [f(z) - \zeta_c]e^{i\frac{2\pi}{n}}, \qquad z \in D.$$

By differentiating this expression, we obtain $f'(ze^{i\frac{2\pi}{n}}) = f'(z)$, from which

$$s\left(\theta + \frac{2\pi}{n}\right) = \int_0^{\theta+\frac{2\pi}{n}} |f'(e^{it})|dt = s(\theta) + \int_\theta^{\theta+\frac{2\pi}{n}} |f'(e^{it})|dt,$$

and hence

$$s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + s\left(\frac{2\pi}{n}\right), \quad \theta \in \mathbb{R}. \tag{15}$$

Since

$$L = s(2\pi) = s\left(\frac{n-1}{n}2\pi\right) + s\left(\frac{2\pi}{n}\right) = ... = ns\left(\frac{2\pi}{n}\right),$$

we have that $s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + \frac{L}{n}$. Thus, (15) and (8)-(11) imply that

$$\varphi\left(s(\theta) + \frac{L}{n}\right) = \varphi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) = \frac{1}{2\pi|f'(e^{i(\theta+\frac{2\pi}{n})})|} = \frac{1}{2\pi|f'(e^{i\theta})|} = \varphi(s(\theta)),$$

and hence, for every $s \in \mathbb{R}$,

$$\varphi\left(s + \frac{L}{n}\right) = \varphi(s).$$

(ii) If now φ is $\frac{L}{n}$-periodic, from (10) we write

$$\Phi\left(s + \frac{L}{n}\right) = 2\pi \int_0^{s+\frac{L}{n}} \varphi(\sigma)d\sigma = \Phi(s) + \Phi\left(\frac{L}{n}\right). \tag{16}$$

Since (9) holds, it follows that

$$2\pi = \Phi(s(2\pi)) = \Phi(L) = \Phi\left(\frac{n-1}{n}L\right) + \Phi\left(\frac{L}{n}\right) = ... = n\Phi\left(\frac{L}{n}\right),$$

and hence

$$\Phi\left(\frac{L}{n}\right) = \frac{2\pi}{n} = \theta + \frac{2\pi}{n} - \theta = \Phi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) - \Phi(s(\theta)).$$

From this and (16), we infer that

$$\Phi\left(s\left(\theta + \frac{2\pi}{n}\right)\right) = \Phi(s(\theta)) + \Phi\left(\frac{L}{n}\right) = \Phi\left(s(\theta) + \frac{L}{n}\right),$$

and, thanks to the invertibility of Φ, we obtain

$$s\left(\theta + \frac{2\pi}{n}\right) = s(\theta) + \frac{L}{n}, \qquad \theta \in \mathbb{R}.$$

By this formula, (13) and the periodicity of φ, it follows that

$$\begin{aligned} f'(z) &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi\left(s\left(\frac{2\pi}{n}+t\right)-\frac{L}{n}\right)}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{2\pi\varphi\left(s\left(\frac{2\pi}{n}+t\right)\right)}dt\right\}. \end{aligned}$$

By a change of variables, we thus get

$$\begin{aligned} f'(z) &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_{\frac{2\pi}{n}}^{2\pi+\frac{2\pi}{n}}\frac{e^{i(t-\frac{2\pi}{n})}+z}{e^{i(t-\frac{2\pi}{n})}-z}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= e^{i\gamma}\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+ze^{i\frac{2\pi}{n}}}{e^{it}-ze^{i\frac{2\pi}{n}}}\log\frac{1}{2\pi\varphi(s(t))}dt\right\} \\ &= f'(ze^{i\frac{2\pi}{n}}). \end{aligned}$$

Finally we find

$$f(z)-\zeta_c=\int_0^1 f'(tz)zdt=\int_0^1 f'(tze^{i\frac{2\pi}{n}})zdt=[f(ze^{i\frac{2\pi}{n}})-\zeta_c]e^{-i\frac{2\pi}{n}},$$

and hence $\mathcal{R}_{\zeta_c,\frac{2\pi}{n}}\Omega=\Omega$. ☐

In what follows, $\mathcal{M}$ will denote mirror-reflection with respect to a given axis.

Theorem 3.2. *A domain $\Omega\in\mathscr{O}$ is symmetric with respect to a generic axis passing through ζ_c if and only if $\varphi(s)=\varphi(L-s)$ for all $s\in[0,L]$.*

Here φ is measured counterclockwise starting from an intersection point of the axis with $\partial\Omega$.

Proof. (i) Suppose Ω symmetric with respect to a given axis passing through ζ_c, that is $\mathcal{M}(\Omega)=\Omega$. Short of rotations and translations, we can assume the symmetry axis to coincide with the real axis, so that $\mathcal{M}z$ is the conjugate $\overline{z}$ of z.

Let $f\in\mathscr{F}$ be the unique mapping from D to Ω such that $f(0)=\zeta_c$ and $f(1)=\zeta_b$, where ζ_b is supposed to be one of the intersection point of $\partial\Omega$ with the symmetry axis. We keep in mind that arclength on $\partial\Omega$ is measured counterclockwise from ζ_b.

It is clear that $\zeta_c-\zeta_b\in\mathbb{R}$ and

$$\overline{f(z)}=f(\overline{z}); \tag{17}$$

thus,

$$\overline{f(e^{i\theta})}=f(e^{i(2\pi-\theta)}),\quad \theta\in[0,2\pi].$$

Differentiating the latter formula with respect to θ and taking the modulus, yields

$$|f'(e^{i\theta})|=|f'(e^{i(2\pi-\theta)})|,\quad \theta\in[0,2\pi]; \tag{18}$$

thus, from (6), we have that

$$s(2\pi-\theta)=L-s(\theta),\qquad \theta\in\mathbb{R}.$$

From this formula and (11), we obtain:

$$\varphi(L-s(\theta))=\varphi(s(2\pi-\theta))=\frac{1}{2\pi|f'(e^{i(2\pi-\theta)})|},\qquad \theta\in\mathbb{R}.$$

Finally, from (18), it follows that

$$\varphi(s) = \varphi(L-s), \qquad s \in [0, L].$$

(ii) Suppose now $\varphi(s) = \varphi(L-s)$ for all $s \in \mathbb{R}$. From (10) we write

$$\Phi(L-s) = 2\pi \int_0^{L-s} \varphi(\sigma)d\sigma = 2\pi \int_s^L \varphi(L-\sigma)d\sigma = 2\pi - \Phi(s), \quad s \in [0, L].$$

This property of Φ and (9) imply that

$$\Phi(s(2\pi-\theta)) = 2\pi - \theta = 2\pi - \Phi(s(\theta)) = \Phi(L - s(\theta)),$$

and hence

$$s(2\pi - \theta) = L - s(\theta), \qquad \theta \in \mathbb{R},$$

by the invertibility of Φ. Then, by differentiating, we have that

$$|f'(e^{i(2\pi-\theta)})| = s'(2\pi-\theta) = s'(\theta) = |f'(e^{i\theta})|$$

for every $\theta \in \mathbb{R}$. Thus, by a change of variable and by simple properties of the complex conjugate, we can write that, for $z \in D$,

$$\begin{aligned}
\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{it})|dt &= \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f'(e^{i(2\pi-t)})|dt \\
&= \int_0^{2\pi} \frac{e^{i(2\pi-t)}+z}{e^{i(2\pi-t)}-z} \log|f'(e^{it})|dt \\
&= \overline{\left(\int_0^{2\pi} \frac{e^{it}+\overline{z}}{e^{it}-\overline{z}} \log|f'(e^{it})|dt\right)}.
\end{aligned}$$

Therefore, modulo a rotation, we have obtained that

$$f'(z) = \overline{f'(\overline{z})}, \quad z \in D,$$

and hence

$$f(z) = \overline{f(\overline{z})}, \quad z \in D,$$

modulo a translation. Thus, $\mathcal{M}(\Omega) = \Omega$ for some reflection $\mathcal{M}$. □

4. A formula involving curvature

Recall that the curvature (with sign) κ of a planar curve can be defined by the formula

$$\kappa = \frac{d\psi}{ds}, \tag{19}$$

where ψ is the angle between the positive real axis and the tangent (unit) vector.

By using the conformal map $f : D \to \Omega$ already introduced and the Hilbert transform, we can express the curvature κ of $\partial\Omega$ in terms of the interior normal derivative φ of the Green's function of Ω.

Theorem 4.1. *Let $\Omega \in \mathscr{O}$ and φ be defined as usual. Then φ and the curvature κ of $\partial\Omega$ are related by the formula:*

$$\kappa(s) = 2\pi\varphi(s)\left[1 - \frac{1}{2\pi}\int_0^{|\partial\Omega|} \cot\left(\frac{\Phi(s)-\Phi(\sigma)}{2}\right)\frac{d}{d\sigma}(\log\varphi)(\sigma)d\sigma\right], \tag{20}$$

for $s \in [0, |\partial\Omega|]$, where Φ is defined as in (10).

Proof. Let $f : D \to \Omega$ be as usual. Now we compute κ in terms of f. Define

$$\omega(\theta) = \arg(f'(e^{i\theta}))$$

for $\theta \in [0, 2\pi]$; the angle ψ in (19) is given by

$$\psi(\theta) = \arg\left(\frac{d}{d\theta} f(e^{i\theta})\right) = \omega(\theta) + \frac{\pi}{2} + \theta.$$

From (19) and (8), we have that

$$\kappa(s) = \frac{d\psi}{d\theta}\frac{d\theta}{ds} = 2\pi\varphi(s)[1 + \omega'(\theta)], \qquad s \in [0, \partial\Omega]. \tag{21}$$

As is well-known (see [7] and [11]), since $\log|f'|$ and $\arg f'$ are the real and the imaginary part of the analytic function $\log f'$, we have that

$$\arg f'(e^{i\theta}) = \mathcal{H}(\log s')(\theta), \tag{22}$$

being $s'(\theta) = |f'(e^{i\theta})|$. Here, $\mathcal{H}$ is the Hilbert transformation on the unit circle, namely,

$$\mathcal{H}(\log s')(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta - t}{2}\right) \log(s'(t))dt.$$

In our notations, (22) can be rewritten as

$$\omega = \mathcal{H}(\log s');$$

thus,

$$\omega' = \mathcal{H}(s''/s'),$$

since $\mathcal{H}$ and $\frac{d}{d\theta}$ commute. From (21), we infer that

$$\kappa(s(\theta)) = 2\pi\varphi\left[1 + \mathcal{H}\left(\frac{s''}{s'}\right)(\theta)\right], \quad \theta \in [0, 2\pi],$$

and hence

$$\kappa(s(\theta)) = 2\pi\varphi(s(\theta))\left[1 - \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\theta - t}{2}\right)\frac{\varphi'(s(t))}{2\pi\varphi^2(s(t))}dt\right], \quad \theta \in [0, 2\pi],$$

from (8). Finally, we obtain (20) by operating the change of variable $\sigma = s(t)$ and by using (9). □

Remark 2. Let $\mathcal{D}2\mathcal{N}$ denote the Dirichlet-to-Neumann operator, that is $\mathcal{D}2\mathcal{N}$ maps the values on $\partial\Omega$ of any harmonic function in Ω to the values of its (interior) normal derivative on $\partial\Omega$. Then, formula (20) can be rewritten as

$$\kappa = 2\pi\varphi[1 + \mathcal{D}2\mathcal{N}(\log(\varphi))].$$

Acknowledgments

This research was partially supported by a PRIN grant of the italian MIUR and by INdAM-GNAMPA.

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Received September 2009; revised January 2010.

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