

1 Asymptotic shape and the speed of propagation of
2 continuous-time continuous-space birth processes

3 Viktor Bezborodov ^{*1}, Luca Di Persio ^{†1}, Tyll Krueger ^{‡2}, Mykola Lebid ^{§3}, and
4 Tomasz Ożański ^{¶2}

5 ¹ The University of Verona

6 ²Wroclaw University of Technology

7 ³(D-BSSE) ETH Zürich

8 July 17, 2017

9 **Abstract**

10 We formulate and prove a shape theorem for a continuous-time continuous-space stochas-
11 tic growth model under certain general conditions. Similarly to the classical lattice growth
12 models the proof makes use of the subadditive ergodic theorem. A precise expression for
13 the speed of propagation is given in the case of a truncated free branching birth rate.

14 *Mathematics subject classification:* 60K35, 60J80.

15 **1 Introduction**

16
17 Shape theorems have a long history. Richardson [Ric73] proved the shape theorem for the
18 Eden model. Since then, shape theorems have been proven in various settings, most notably for
19 first passage percolation and permanent and non-permanent growth models. Garet and Marc-
20 hand [GM12] not only prove a shape theorem for the contact process in random environment,
21 but also have a nice overview of existing results.

22 Most of literature is devoted to discrete-space models. A continuous-space first passage
23 percolation model was analyzed by Howard and Newman [HN97], see also references therein.
24 A shape theorem for a continuous-space growth model was proven by Deijfen [Dei03], see also

*Email: viktor.bezborodov@univr.it

†Email: luca.dipersio@univr.it

‡Email: tyll.krueger@pwr.wroc.pl

§Email: mykola.lebid@bsse.ethz.ch

¶Email: tomasz.ozanski@pwr.edu.pl

Keywords: *shape theorem, spatial birth process, growth model*

25 Gouéré and Marchand [GM08]. Our model is naturally connected to that model, see the end of
 26 Section 2.

27 Questions addressed in this article are motivated not only by probability theory but also
 28 by studies in natural sciences. In particular, one can mention a demand to incorporate spatial
 29 information in the description and analysis of 1) ecology 2) bacteria populations 3) tumor growth
 30 4) epidemiology 5) phylogenetics among others, see e.g. [WBP+15], [TSH+13], [VDPP15], and
 31 [TM15]. Authors often emphasize that it is preferable to use the continuous-space spaces R^2
 32 and R^3 as the basic, or ‘geographic’ space, see e.g. [VDPP15]. More on connections between
 33 theoretical studies and applications can be found in [MW03].

34 The paper is organized as follows. In Section 2 we describe the model and formulate our
 35 results, which are proven in Sections 3 and 4. Technical results, in particular on the construction
 36 of the process, are collected in the Section 5.

37 2 The model, assumptions and results

38 We consider a growth model represented by a continuous-time continuous-space Markov birth
 39 process. Let Γ_0 be the collection of finite subsets of \mathbb{R}^d ,

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

40 where $|\eta|$ is the number of elements in η . Γ_0 is also called the *configuration space*, or the *space*
 41 *of finite configurations*.

42 The evolution of the spatial birth process on \mathbb{R}^d admits the following description. Let $\mathcal{B}(X)$
 43 be the Borel σ -algebra on the Polish space X . If the system is in state $\eta \in \Gamma_0$ at time t , then
 44 the probability that a new particle appears (a “birth”) in a bounded set $B \in \mathcal{B}(\mathbb{R}^d)$ over time
 45 interval $[t; t + \Delta t]$ is

$$\Delta t \int_B b(x, \eta) dx + o(\Delta t),$$

46 and with probability 1 no two births happen simultaneously. Here $b : \mathbb{R}^d \times \Gamma_0 \rightarrow \mathbb{R}_+$ is some
 47 function which is called the *birth rate*. Using a slightly different terminology, we can say that
 48 the rate at which a birth occurs in B is $\int_B b(x, \eta) dx$. We note that it is conventional to call the
 49 function b the ‘birth rate’, even though it is not a rate in the usual sense (as in for example ‘the
 50 Poisson process (N_t) has unit jumps at rate 1 meaning that $\frac{P\{N_{t+\Delta t} - N_t = 1\}}{\Delta t} = 1$ as $\Delta t \rightarrow 0$)
 51 but rather a version of the Radon–Nikodym derivative of the rate with respect to the Lebesgue
 52 measure.

53 **Remark 2.1.** We characterize the birth mechanism by the birth rate $b(x, \eta)$ at each spatial
 54 position. Oftentimes the birth mechanism is given in terms of contributions of individual parti-
 55 cles: a particle at y , $y \in \eta$, gives a birth at x at rate $c(x, y, \eta)$ (often $c(x, y, \eta) = \gamma(y, \eta)k(y, x)$,
 56 where $\gamma(y, \eta)$ is the proliferation rate of the particle at y , whereas the dispersion kernel $k(y, x)$
 57 describes the distribution of the offspring), see e.g. Fournier and Méléard [FM04]. As long
 58 as we are not interested in the induced genealogical structure, the two ways of describing the

59 process are equivalent under our assumptions. Indeed, given c , we may set

$$b(x, \eta) = \sum_{y \in \eta} c(x, y, \eta), \quad (1)$$

60 or, conversely, given b , we may set

$$c(y, x, \eta) = \frac{g(x - y)}{\sum_{y \in \eta} g(x - y)} b(x, \eta), \quad (2)$$

61 where $g : \mathbb{R}^d \rightarrow (0, \infty)$ is a continuous function. Note that b is uniquely determined by c , but
62 not vice versa.

63 We equip Γ_0 with the σ -algebra $\mathcal{B}(\Gamma_0)$ induced by the sets

$$\mathbf{Ball}(\eta, r) = \{\zeta \in \Gamma_0 \mid |\eta| = |\zeta|, \text{dist}(\eta, \zeta) < r\}, \quad \eta \in \Gamma_0, r > 0, \quad (3)$$

64 where $\text{dist}(\eta, \zeta) = \min \left\{ \sum_{i=1}^{|\eta|} |x_i - y_i| \mid \eta = \{x_1, \dots, x_{|\eta|}\}, \zeta = \{y_1, \dots, y_{|\eta|}\} \right\}$. For more detail on
65 configuration spaces see e.g. Röckner and Schied [RS99] or Kondratiev and Kutovyi [KK06].
66 In particular, the dist above coincides with the restriction to the space of finite configurations
67 of the metric ρ used in [RS99], and the σ -algebra $\mathcal{B}(\Gamma_0)$ introduced above coincides with the
68 σ -algebra from [KK06].

69 We say that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ has an exponential moment if there exists $\theta > 0$ such
70 that

$$\int_{\mathbb{R}^d} e^{\theta|x|} f(x) dx < \infty.$$

71 Of course, if f has an exponential moment, then automatically $f \in L^1(\mathbb{R}^d)$.

72 *Assumptions on b .* We will need several assumptions on the birth rate b .

73 **Condition 2.2** (Sublinear growth). The birth rate b is measurable and there exists a function
74 $a : \mathbb{R}^d \rightarrow \mathbb{R}_+$ with an exponential moment such that

$$b(x, \eta) \leq \sum_{y \in \eta} a(x - y). \quad (4)$$

75 **Condition 2.3** (Monotonicity). For all $\eta \subset \zeta$,

$$b(x, \eta) \leq b(x, \zeta), \quad x \in \mathbb{R}^d. \quad (5)$$

76 The previous condition ensures attractiveness, see below.

Condition 2.4 (Rotation and translation invariance). The birth rate b is translation and
rotation invariant: for every $x, y \in \mathbb{R}^d$, $\eta \in \Gamma_0$ and $M \in \text{SO}(d)$,

$$\begin{aligned} b(x + y, \eta + y) &= b(x, \eta), \\ b(Mx, M\eta) &= b(x, \eta). \end{aligned}$$

Here $\text{SO}(d)$ is the orthogonal group of linear isometries on \mathbb{R}^d , and for a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$
and $y \in \mathbb{R}^d$,

$$\begin{aligned} B + y &= \{z \mid z = x + y, x \in B\} \\ MB &= \{z \mid z = Mx, x \in B\}. \end{aligned}$$

77 **Condition 2.5** (Non-degeneracy). Let there exist $c_0, r > 0$ such that

$$b(x, \eta) \geq c_0 \quad \text{whenever} \quad \min_{y \in \eta} |x - y| \leq r. \quad (6)$$

78 **Remark 2.6.** Condition 2.5 is used to ensure that the system grows at least linearly. The
79 condition could be weakened for example as follows:

80 *For some $r_2 > r_1 \geq 0$ and all $x, y \in \mathbb{R}^d$,*

$$b(y, \{x\}) \geq c_0 \mathbf{1}\{r_1 \leq |x - y| \leq r_2\}.$$

81 Respectively, the proof would become more intricate.

82 **Remark 2.7.** If b is like in (7) and f has polynomial tails, then the result of Durrett [Dur83]
83 suggests that we should expect a superlinear growth. This is in contrast with Deijfen's model, for
84 which Gouéré and Marchand [GM12] give a sharp condition on the distribution of the outbursts
85 for linear or superlinear growth.

86 Examples of a birth rate are

$$b(x, \eta) = \lambda \sum_{y \in \eta} f(|x - y|), \quad (7)$$

87 and

$$b(x, \eta) = k \wedge \left(\lambda \sum_{y \in \eta} f(|x - y|) \right), \quad (8)$$

88 where λ, k are positive constants and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous, non-negative, non-increasing
89 function with compact support.

90 We denote the underlying probability space by (Ω, \mathcal{F}, P) . Let \mathcal{A} be a sub- σ -algebra of \mathcal{F} .
91 A random element A in Γ_0 is \mathcal{A} -measurable if

$$\Omega \ni \omega \rightarrow A = A(\omega) \in \Gamma_0 \quad (9)$$

92 is a measurable map from the measure space (Ω, \mathcal{A}) to $(\Gamma_0, \mathcal{B}(\Gamma_0))$. Such an A will also be
93 called an \mathcal{A} -measurable finite random set.

94 The birth process will be obtained as a unique solution to a certain stochastic equation. The
95 construction and the proofs of key properties, such as the rotation invariance and the strong
96 Markov property, are given in Section 5. We place the construction toward the end because it is
97 rather technical and the methods used there do not shed much light on the ideas of the proofs of
98 our main results. Denote by $(\eta_t^{s,A})_{t \geq s} = (\eta_t^{s,A}, t \geq s)$ the process started at time $s \geq 0$ from an
99 \mathcal{S}_s -measurable finite random set A . Here $(\mathcal{S}_s)_{s \geq 0}$ is a filtration of σ -algebras to which $(\eta_t^{s,A})_{t \geq s}$
100 is adapted; it is introduced after (74). Furthermore, $(\eta_t^{s,A})_{t \geq s}$ is a strong Markov process with
101 respect to $(\mathcal{S}_s)_{s \geq 0}$ - see Proposition 5.8.

102 The construction method we use has the advantage that the stochastic equation approach
103 resembles graphical representation (see e.g. Durrett [Dur88] or Liggett [Lig99]) in the fact that
104 it preserves monotonicity: if $s \geq 0$ and a.s. $A \subset B$, A and B being \mathcal{S}_s -measurable finite random
105 sets, then a.s.

$$\eta_t^{s,A} \subset \eta_t^{s,B}, \quad t \geq s. \quad (10)$$

106 This property is proven in Lemma 5.10 and is often referred to as *attractiveness*.

107 The process started from a single particle at $\mathbf{0}$ at time zero will be denoted by $(\eta_t)_{t \geq 0}$; thus,
 108 $\eta_t = \eta_t^{0, \{\mathbf{0}\}}$. Let

$$\xi_t := \bigcup_{x \in \eta_t} B(x, r) \tag{11}$$

109 and similarly

$$\xi_t^{s,A} := \bigcup_{x \in \eta_t^{s,A}} B(x, r),$$

110 where $B(x, r)$ is the closed ball of radius r centered at x (recall that r appears in (6)).

111 The following theorem represents the main result of the paper.

112 **Theorem 2.8.** *There exists $\mu > 0$ such that for all $\varepsilon \in (0, 1)$ a.s.*

$$(1 - \varepsilon)B(\mathbf{0}, \mu^{-1}) \subset \frac{\xi_t}{t} \subset (1 + \varepsilon)B(\mathbf{0}, \mu^{-1}) \tag{12}$$

113 for sufficiently large t .

114 **Remark 2.9.** Let us note that the statement of Theorem 2.8 does not depend on our choice
 115 for the radius in (11) to be r ; we could just as well take any positive constant, for example

$$\bigcup_{x \in \eta_t} B(x, 1)$$

116 In particular, μ in (12) does not depend on r .

117 The proof of Theorem 2.8 and the outline of the proof are given in Section 3. It is common
 118 to write the ball radius as the reciprocal μ^{-1} , probably because μ comes up in the proof as the
 119 limiting value of a certain sequence of random variables after applying the subadditive ergodic
 120 theorem; see e.g. Durrett [Dur88] or Deijfen [Dei03]. We decided to keep the tradition not only
 121 for historic reasons, but also because μ comes up as a certain limit in our proof too, even though
 122 we do not obtain μ directly from the subadditive ergodic theorem. The value μ^{-1} is called the
 123 *speed of propagation*. The subadditive ergodic theorem is a cornerstone in the majority of shape
 124 theorem proofs, and our proof relies on it.

125 *Formal connection to Deijfen's model.* The model introduced in [Dei03] with deterministic
 126 outburst radius, that is, when in the notation of [Dei03] the distribution of outbursts F is the
 127 Dirac measure: $F = \delta_R$ for some $R \geq 0$, can be identified with

$$\zeta_t^R = \bigcup_{x \in \eta_t} B(x, R)$$

128 for the birth process (η_t) with birth rate

$$b(x, \eta) = \mathbf{1}\{\exists y \in \eta : |x - y| \leq R\}.$$

129 *Explicit growth speed for a particular model.* The precise evaluation of speed appears to
 130 be a difficult problem. For a general one dimensional branching random walk the speed of
 131 propagation is given by Biggins [Big95]. An overview of related results for different classes of
 132 models can be found in Auffinger, Damron, and Hanson [ADH15].

133 Here we give the speed for a model with interaction.

134 **Theorem 2.10.** *Let $d = 1$ and*

$$b(x, \eta) = 2 \wedge \left(\sum_{y \in \eta} \mathbb{1}\{|x - y| \leq 1\} \right). \quad (13)$$

135 *Then the speed of propagation is given by*

$$\mu^{-1} = \frac{144 \ln(3) - 144 \ln(2) - 40}{25} \approx 0.73548... \quad (14)$$

136 Section 4 contains the proof of Theorem 2.10.

137 3 Proof of Theorem 2.8

138 *Outline of the proof.* The proof can roughly be divided into three parts. In the first part we
 139 show that the system grows not faster than linearly, which is the content of Proposition 3.1.
 140 The proof of Proposition 3.1 relies on Lemma 5.10, which allows a comparison of birth processes
 141 with different rates, and on the results on the spread of the supercritical branching random walk
 142 by Biggins [Big95].

143 In the second part we show that the system grows at least linearly. Strictly speaking, in this
 144 part we only give exponential estimates on the probability of certain linearly growing balls not
 145 to be filled with the particles of our system (Lemma 3.5) as opposed to an a.s. statement about
 146 the entire trajectory as in Proposition 3.1. This is however sufficient for our purposes. The
 147 main ingredients here are exponential estimates for the Eden model (or first passage percolation
 148 model), comparison of the Eden model with our process, and once again Lemma 5.10. The Eden
 149 model is described on page 7.

150 In the third part, the most technical in our opinion, we actually prove the theorem using the
 151 previous two parts. We define a specially designed collection of stopping times $\{T_\lambda(x), x \in \mathbb{R}^d\}$
 152 and $\{T_\lambda(x, y), x, y \in \mathbb{R}^d\}$ depending on an additional parameter $\lambda > 0$ (see (24) and (25)). The
 153 strong Markov property of (η_t) (Proposition 5.8 and Corollary 5.9) allows us to apply Liggett's
 154 subadditive ergodic theorem to show that for any $x \in \mathbb{R}^d$, $(T_\lambda(tx))_{t \geq 0}$ grows linearly with t
 155 ((32) and Lemma 3.8). We then move on to prove that the limit $\lim_{t \rightarrow \infty} \frac{T_\lambda(tx)}{t}$ does not depend on
 156 x (Lemma 3.9) and is strictly positive (Lemma 3.10). The bulk of the final part of the proof of
 157 Theorem 2.8 is contained in Lemmas 3.12 and 3.13, where we show the necessary a.s. inclusions
 158 dropping λ along the way.

159 **Proposition 3.1.** *There exists $C_{upb} > 0$ such that a.s. for large t ,*

$$\eta_t \subset B(\mathbf{0}, C_{upb}t) \quad (15)$$

160 **Remark.** The index ‘upb’ hints on ‘upper bound’.

161 **Proof.** It is sufficient to show that for $\mathbf{e} = (1, 0, \dots, 0) \in \mathbb{R}^d$ there exists $C > 0$ such that
 162 a.s. for large t

$$\max\{\langle x, \mathbf{e} \rangle : x \in \eta_t\} \subset Ct. \quad (16)$$

163 Indeed, if (16) holds, then by Proposition 5.7 it is true if we replace \mathbf{e} with any other unit vector
 164 along any of the 2d directions in \mathbb{R}^d , and hence (15) holds too.

165 For $z \in \mathbb{R}$, $y = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$ we define $z \circ y$ to be the concatenation $(z, y_1, \dots, y_{d-1}) \in$
 166 \mathbb{R}^d . In this proof we denote by $(\bar{\eta}_t)$ the birth process with $\bar{\eta}_0 = \eta_0$ and the birth rate given by
 167 the right hand side of (4), namely

$$\bar{b}(x, \eta) = \sum_{y \in \eta} a(x - y). \quad (17)$$

168 Since $b(x, \eta) \leq \bar{b}(x, \eta)$, $x \in \mathbb{R}^d$, $\eta \in \Gamma_0$, we have by Lemma 5.10 a.s. $\eta_t \subset \bar{\eta}_t$ for all $t \geq 0$.
 169 Thus, it is sufficient to prove the proposition for $(\bar{\eta}_t)$. The process $(\bar{\eta}_t)$ with rate (17) is in fact a
 170 continuous-time continuous-space branching random walk (for an overview of branching random
 171 walks and related topics, see e.g. Shi [Shi15]). Denote by $\bar{\eta}_t^{\mathbf{e}}$ the element-wise projection of $\bar{\eta}_t$
 172 onto the line determined by \mathbf{e} ; that is $\bar{\eta}_t^{\mathbf{e}} = \{x \in \mathbb{R}^1 \mid x = \langle y, \mathbf{e} \rangle \text{ for some } y \in \eta_t\}$. The process
 173 $(\bar{\eta}_t^{\mathbf{e}})$ is itself a branching random walk, and by Corollary 2 in Biggins [Big95], the position of
 174 the rightmost particle $X_t^{\mathbf{e}}$ of $(\bar{\eta}_t^{\mathbf{e}})$ at time t satisfies

$$\lim_{t \rightarrow \infty} \frac{X_t^{\mathbf{e}}}{t} \rightarrow \gamma \quad (18)$$

175 for a certain $\gamma \in (0, \infty)$. The conditions from the Corollary 2 from [Big95] are satisfied because
 176 of Condition 2.2. Indeed, $(\bar{\eta}_t^{\mathbf{e}})$ is the branching random walk with the birth kernel

$$\bar{a}^{\mathbf{e}}(z) = \int_{y \in \mathbb{R}^{d-1}} a(z \circ y) dy,$$

177 that is, $(\bar{\eta}_t^{\mathbf{e}})$ is the a birth process on \mathbb{R}^1 with the birth rate

$$\bar{b}(x, \eta) = \sum_{y \in \eta} \bar{a}^{\mathbf{e}}(x - y), \quad x \in \mathbb{R}, \eta \in \Gamma_0(\mathbb{R}).$$

178 Note that $a^{\mathbf{e}}(z) = a(z)$ if $d = 1$. Hence, in the notation of [Big95] for $\theta < 0$

$$m(\theta, \phi) = \int_{\mathbb{R} \times \mathbb{R}_+} e^{-\theta z} e^{-\phi \tau} \bar{a}^{\mathbf{e}}(z) dz d\tau = \frac{1}{\phi} \int_{\mathbb{R}} e^{-\theta|z|} \bar{a}^{\mathbf{e}}(z) dz = \frac{1}{\phi} \int_{\mathbb{R}} e^{-\theta|z|} dz \int_{y \in \mathbb{R}^{d-1}} a(z \circ y) dy$$

179

$$= \frac{1}{\phi} \int_{\mathbb{R}^d} e^{-\theta|\langle x, \mathbf{e} \rangle|} a(x) dx \leq \frac{1}{\phi} \int_{\mathbb{R}^d} e^{-\theta|x|} a(x) dx,$$

180 and thus $\alpha(\theta) < \infty$ for a negative θ satisfying $\int_{\mathbb{R}^d} e^{-\theta|x|} a(x) dx < \infty$ (the functions $m(\theta, \phi)$ and

181 $\alpha(\theta)$ are defined in [Big95] at the beginning of Section 3).

182 Since (16) follows from (18), the proof of the proposition is now complete. \square

183 Next, using a comparison with the Eden model (see Eden [Ede61]), we will show that the
 184 system grows not slower than linearly (in the sense of Lemma 3.5 below). The Eden model is
 185 a model of tumor growth on the lattice \mathbb{Z}^d . The evolution starts from a single particle at the
 186 origin. A site once occupied stays occupied forever. A vacant site becomes occupied at rate
 187 $\lambda > 0$ if at least one of its neighbors is occupied. Let us mention that this model is closely

188 related to the first passage percolation model, see e.g. Kesten [Kes87] and Auffinger, Damron,
 189 and Hanson [ADH15]. In fact, the two models coincide if the passage times ([Kes87]) have
 190 exponential distribution.

191 For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, let $|z|_1 = \sum_{i=1}^d |z_i|$.

192 **Lemma 3.2.** *Consider the Eden model starting from a single particle at the origin. Then there*
 193 *exists a constant $\tilde{C} > 0$ such that for every $z \in \mathbb{Z}^d$ and time $t \geq \frac{4e^2}{\lambda^2(e-1)^2} \vee \tilde{C}|z|_1$,*

$$P\{z \text{ is vacant at } t\} \leq e^{-\sqrt{t}}. \quad (19)$$

194 **Proof.** Let σ_z be the time when z becomes occupied. Let v be a path on the integer lattice
 195 of length $m = \text{length}(v)$ starting from $\mathbf{0}$ and ending in z , so that $v_0 = \mathbf{0}$, $v_m = z$, $v_i \in \mathbb{Z}^d$
 196 and $|v_i - v_{i-1}| = 1$, $i = 1, \dots, m$. Define $\sigma(v)$ as the time it takes for the Eden model to move
 197 along the path v ; that is, if v_0, \dots, v_j are occupied, then a birth can only occur at v_{j+1} . By
 198 construction $\sigma(v)$ is distributed as the sum of $\text{length}(v)$ independent unit exponentials (the so
 199 called passage times; see e.g. [Kes87] or [ADH15]). We have

$$\sigma_z = \inf\{\sigma(v) : v \text{ is a path from } \mathbf{0} \text{ to } z\}.$$

200 Hence σ_z is dominated by the sum of $|z|_1$ independent unit exponentials, say $\sigma_z \leq Z_1 + \dots + Z_{|z|_1}$.

201 We have the equality of the events

$$\{z \text{ is vacant at } t\} = \{\sigma_z > t\}.$$

202 Note that $Ee^{\lambda(1-\frac{1}{e})Z_1} = e$. Using Chebyshev's inequality $P\{Z > t\} \leq Ee^{\lambda(1-\frac{1}{e})(Z-t)}$, we get

$$\begin{aligned} P\{\sigma_z > t\} &\leq P\{Z_1 + \dots + Z_{|z|_1} > t\} \leq E \exp\{\lambda(1 - \frac{1}{e})(Z_1 + \dots + Z_{|z|_1} - t)\} \\ &= \left[Ee^{\lambda(1-\frac{1}{e})Z_1} \right]^{|z|_1} e^{-\lambda(1-\frac{1}{e})t} = e^{|z|_1} e^{-\lambda(1-\frac{1}{e})t}. \end{aligned}$$

204 Since

$$\frac{1}{2}\lambda(1 - \frac{1}{e})t \geq \sqrt{t},$$

205 for $t \geq \frac{4e^2}{\lambda^2(e-1)^2}$, we may take $\tilde{C} = \frac{2e}{\lambda(e-1)}$. □

206 We now continue to work with the Eden model.

207 **Lemma 3.3.** *For the Eden model starting from a single particle at the origin, there are constants*
 208 *$c_1, t_0 > 0$ such that*

$$P\{\text{there is a vacant site in } B(0, c_1 t) \cap \mathbb{Z}^d \text{ at } t\} \leq e^{-\sqrt[4]{t}}, t \geq t_0 \quad (20)$$

209 **Proof.** By the previous lemma for $c_1 < \frac{1}{\tilde{C}}$,

$$\begin{aligned} &P\{\text{there is a vacant site in } B(0, c_1 t) \cap \mathbb{Z}^d \text{ at } t\} \\ &\leq \sum_{z \in B(0, c_1 t) \cap \mathbb{Z}^d} P\{z \text{ is vacant at } t\} \end{aligned}$$

211

$$\leq |B(0, c_1 t)| e^{-\sqrt{t}},$$

212 where $|B(0, c_1 t)|$ is the number of integer points (that is, points whose coordinates are integers)
213 inside $B(0, c_1 t)$. It remains to note that $|B(0, c_1 t)|$ grows only polynomially fast in t . \square

214 **Definition 3.4.** Let the growth process $(\alpha_t)_{t \geq 0}$ be a $\mathbb{Z}_+^{\mathbb{Z}^d}$ -valued process with

$$\alpha(z) \rightarrow \alpha(z) + 1 \quad \text{at rate } \lambda \mathbf{1} \left\{ \sum_{\substack{y \in \mathbb{Z}^d: \\ |z-y| \leq 1}} \alpha(y) > 0 \right\}, \quad z \in \mathbb{Z}^d, \alpha \in \mathbb{Z}_+^{\mathbb{Z}^d}, \sum_{y \in \mathbb{Z}^d} \alpha(y) < \infty, \quad (21)$$

215 where $\lambda > 0$.

216 Clearly, Lemma 3.3 also applies to $(\alpha_t)_{t \geq 0}$, since it dominates the Eden process. Recall that
217 r appears in (6), and (ξ_t) is defined in (11).

218 **Lemma 3.5.** *There are $c, s_0 > 0$ such that*

$$P\{B(\mathbf{0}, cs) \not\subset \xi_s\} \leq e^{-\sqrt[4]{s}}, \quad s \geq s_0. \quad (22)$$

219 **Proof.** For $x \in \mathbb{R}^d$ let $z_x \in \frac{r}{2d}\mathbb{Z}^d$ be uniquely determined by $x \in z_x + (-\frac{r}{4d}, \frac{r}{4d}]^d$. Recall
220 that c_0 appears in Condition 2.5. Define

$$\bar{b}(x, \eta) = c_0 \mathbf{1}\{z_x \sim z_y \text{ for some } y \in \eta\}, \quad (23)$$

221 where $z_x \sim z_y$ means that z_x and z_y are neighbors on $\frac{r}{2d}\mathbb{Z}^d$. Let $(\bar{\eta}_t)_{t \geq 0}$ be the birth process
222 with birth rate \bar{b} . Note that by (6) for every $\eta \in \Gamma_0$,

$$\bar{b}(x, \eta) \leq b(x, \eta), \quad x \in \mathbb{R}^d,$$

223 hence a.s. $\bar{\eta}_t \subset \eta_t$ by Lemma 5.10, $t \geq 0$. Then the ‘projection’ process defined by

$$\bar{\bar{\eta}}_t(z) = \sum_{x \in \bar{\eta}_t} \mathbf{1}\{x \in z + (-\frac{r}{4d}, \frac{r}{4d}]^d\}, \quad z \in \frac{r}{2d}\mathbb{Z}^d,$$

224 is the process $(\alpha_t)_{t \geq 0}$ from Definition 3.4 with $\lambda = c_0 \left(\frac{r}{2d}\right)^d$ and the ‘geographic’ space $\frac{r}{2d}\mathbb{Z}^d$
225 instead of \mathbb{Z}^d , that is, taking values in $\mathbb{Z}_+^{\frac{r}{2d}\mathbb{Z}^d}$ instead of $\mathbb{Z}_+^{\mathbb{Z}^d}$. Since $\bar{\bar{\eta}}_t(z_x) > 0$ implies that
226 $x \in \xi_t$, the desired result follows from Lemma 3.3 and the fact that Lemma 3.3 also applies to
227 $(\alpha_t)_{t \geq 0}$. \square

228 *Notation and conventions.* In what follows for $x, y \in \mathbb{R}^d$ we define

$$[x, y] = \{z \in \mathbb{R}^d \mid z = tx + (1-t)y, t \in [0, 1]\}.$$

229 We call $[x, y]$ an interval. Similarly, open or half-open intervals are defined, for example

$$(x, y] = \{z \in \mathbb{R}^d \mid z = tx + (1-t)y, t \in (0, 1]\}.$$

230 We also adopt the convention $B(x, 0) = \{x\}$.

231 For $x \in \mathbb{R}^d$ and $\lambda \in (0, 1)$ we define a stopping time $T_\lambda(x)$ (here and below, all stopping
 232 times are considered with respect to the filtration (\mathcal{S}_t) introduced after (74)) by

$$T_\lambda(x) = \inf\{t > 0 : |\eta_t \cap B(x, \lambda|x|)| > 0\}, \quad (24)$$

233 and for $x, y \in \mathbb{R}^d$, we define

$$T_\lambda(x, y) = \inf\left\{t > T_\lambda(x) : |\eta_t^{T_\lambda(x), \{z_\lambda(x)\}} \cap B(y + z_\lambda(x) - x, \lambda|y - x|)| > 0\right\} - T_\lambda(x), \quad (25)$$

234 where $z_\lambda(x)$ is uniquely defined by $\{z_\lambda(x)\} = \eta_{T_\lambda(x)} \cap B(x, \lambda|x|)$. Note that $\{z_\lambda(x)\}$ is a $\mathcal{S}_{T_\lambda(x)}$ -
 235 measurable finite random set. Also, $T_\lambda(\mathbf{0}) = 0$ and $T_\lambda(x, x) = 0$ for $x \in \mathbb{R}^d$. To reduce the
 236 number of double subscripts, we will sometimes write $z(x)$ instead of $z_\lambda(x)$.

Since for $q \geq 1$

$$\{x_1 + x_2 : x_1 \in B(x, \lambda|x|), x_2 \in B((q-1)x, \lambda(q-1)|x|)\} = B(qx, \lambda q|x|),$$

237 we have by attractiveness (recall (10))

$$T_\lambda(qx) \leq T_\lambda(x) + \left(\inf\{t > 0 : |\eta_t^{T_\lambda(x), \eta_{T_\lambda(x)}} \cap B(qx, \lambda q|x|)| > 0\} - T_\lambda(x)\right)$$

238

$$\leq T_\lambda(x) + \left(\inf\{t > 0 : |\eta_t^{T_\lambda(x), \{z_\lambda(x)\}} \cap B(z_\lambda(x) + (q-1)x, \lambda(q-1)|x|)| > 0\} - T_\lambda(x)\right),$$

239 that is,

$$T_\lambda(qx) \leq T_\lambda(x) + T_\lambda(x, qx), \quad x \in \mathbb{R}^d \setminus \{\mathbf{0}\}. \quad (26)$$

240 Note that by the strong Markov property (Proposition 5.8 and Corollary 5.9),

$$T_\lambda(x, qx) \stackrel{(d)}{=} T_\lambda((q-1)x). \quad (27)$$

241 The following elementary lemma is used in the proof of Lemma 3.7.

242 **Lemma 3.6.** *Let $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ be two d-dimensional balls.*

243 (i) *There exists a constant $c_{ball}(d) > 0$ depending on d only such that if B_1 and B_2 are two*
 244 *balls in \mathbb{R}^d and $x_1 \in B_2$ then*

$$Vol(B_1 \cap B_2) \geq c_{ball}(d)(Vol(B_1) \wedge Vol(B_2)), \quad (28)$$

245 *where $Vol(B)$ is the d-dimensional volume of B . (ii) The intersection $B_1 \cap B_2$ contains a ball*
 246 *of radius r_3 provided that*

$$2r_3 \leq (r_1 + r_2 - |x_1 - x_2|) \wedge r_1 \wedge r_2.$$

247 **Proof.** (i) Without loss of generality we can assume that $r_1 \leq r_2$. Indeed, if $r_1 > r_2$,
 248 then $x_2 \in B_1$, so we can swap B_1 and B_2 . Let $B'_1 = B(x'_1, r_1)$ be the shifted ball B_1 with
 249 $x'_1 = x_1 + r_1 \frac{x_2 - x_1}{|x_2 - x_1|}$ (see Figure 1). The intersection $B'_1 \cap B_1$ is a subset of B_2 and is a union
 250 of two identical d-dimensional hyperspherical caps with height $\frac{r_1}{2}$. Using the standard formula
 251 for the volume of a hyperspherical cap, we see that we can take

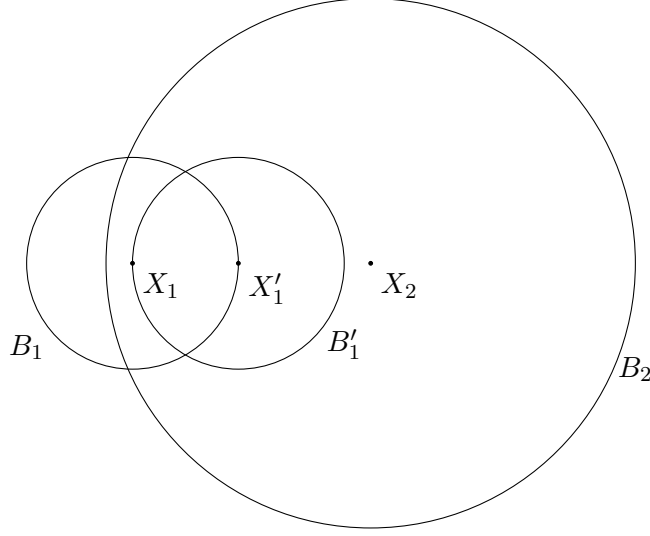


Figure 1: for Lemma 3.6 (i)

$$c_{\text{ball}}(d) = \frac{V(B'_1 \cap B_1)}{V(B_1)} = 2 \frac{\Gamma(\frac{d}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{d+1}{2})} \int_0^{\frac{\pi}{3}} \sin^d(s) ds.$$

252 (ii) We have $B_3 \subset B_1 \cap B_2$, where $B_3 = B(x_3, r_3)$ and x_3 is the middle point of the interval
 253 $[x_1, x_2] \cap B_1 \cap B_2$.

254 **Lemma 3.7.** For every $x \in \mathbb{R}^d$ and $\lambda > 0$ there exist $A_{x,\lambda}, q_{x,\lambda} > 0$ such that

$$P\{T_\lambda(x) > s\} \leq A_{x,\lambda} e^{-q_{x,\lambda} \sqrt[4]{s}}, \quad s \geq 0. \quad (29)$$

255 **Proof.** Let

$$\tau_x = \inf\{s > 0 : x \in \xi_s\}$$

256 (recall that (ξ_t) is defined in (11)), that is τ_x is the moment when the first point in the ball
 257 $B(x, r)$ appears. By Lemma 3.5 for $s \geq s_0 \vee \frac{|x|}{c}$

$$P\{\tau_x > s\} \leq P\{x \notin \xi_s\} \leq P\{B(\mathbf{0}, |x|) \not\subseteq \xi_s\} \leq P\{B(\mathbf{0}, cs) \not\subseteq \xi_s\} \leq e^{-\sqrt[4]{s}}. \quad (30)$$

258 In the case $r \leq \lambda|x|$ we have a.s. $T_\lambda(x) \leq \tau_x$, and the statement of the lemma follows from
 259 (30) since for $s \geq s_0 \vee \frac{|x|}{c}$

$$P\{T_\lambda(x) > s\} \leq P\{\tau_x > s\} \leq e^{-\sqrt[4]{s}}.$$

260 Let us now consider the case $r > \lambda|x|$. Denote by $\bar{x} \in B(x, r)$ the place where the particle
 261 is born at τ_x . For $t \geq 0$ on $\{t > \tau_x\}$ we have

$$\int_{y \in B(x, \lambda|x|)} b(y, \eta_t) dy \geq \int_{y \in B(x, \lambda|x|)} b(y, \{\bar{x}\}) dy \geq \int_{y \in B(x, \lambda|x|)} c_0 \mathbf{1}\{y \in B(\bar{x}, r)\} dy,$$

262 so that by Lemma 3.6 on $\{t > \tau_x\}$

$$\int_{y \in B(x, \lambda|x|)} b(y, \eta_t) dy \geq \int_{y \in B(x, \lambda|x|)} c_0 \mathbf{1}\{y \in B(\bar{x}, r)\} dy$$

263

$$= c_0 \text{Vol}(B(x, \lambda|x|) \cap B(\bar{x}, r)) \geq c_0 c_{\text{ball}}(d) \text{Vol}(B(x, \lambda|x|)) = c_0 c_{\text{ball}}(d) V_d \lambda^d |x|^d,$$

where $V_d = \text{Vol}(B(\mathbf{0}, 1))$, hence

$$P\{T_\lambda(x) - \tau_x > s'\} \leq P\{\inf\{t > 0 : \eta_t^{\tau_x, \{\bar{x}\}} \cap B(x, r) \neq \emptyset\} - \tau_x > s'\} \leq e^{-c_0 c_{\text{ball}}(d) V_d \lambda^d |x|^d s'}.$$

264

Combining this with (30) yields the desired result. \square

265

Let us fix an $x \in \mathbb{R}^d$, $x \neq \mathbf{0}$, and define for $k, n \in \mathbb{N}$, $k < n$,

$$s_{k,n} = T_\lambda(kx, nx). \quad (31)$$

266

Note that the random variables $s_{k,n}$ are integrable by Lemma 3.7. The conditions of Liggett's subadditive ergodic theorem, see [Lig85], are satisfied here. Indeed, condition (1.7) in [Lig85] is ensured by (26), while conditions (1.8) and (1.9) in [Lig85] follow from (27) and the strong Markov property of (η_t) (Proposition 5.8 and Corollary 5.9). Thus, there exists $\mu_\lambda(x) \in [0, \infty)$ such that a.s. and in L^1 ,

270

$$\frac{s_{0,n}}{n} \rightarrow \mu_\lambda(x). \quad (32)$$

271

Lemma 3.8. *Let $\lambda > 0$. For every $x \neq \mathbf{0}$,*

$$\lim_{t \rightarrow \infty} \frac{T_\lambda(tx)}{t} = \mu_\lambda(x). \quad (33)$$

272

Proof. We know that for every $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$

$$\lim_{n \rightarrow \infty} \frac{T_\lambda(nx)}{n} = \mu_\lambda(x). \quad (34)$$

273

Denote $\sigma_n = \inf_{y \in [nx, (n+1)x]} T_\lambda(y)$. Since there are only a finite number of particles born in a

274

bounded time interval, this infimum is achieved. So, let \tilde{z}_n be such that $\eta_{\sigma_n} \setminus \eta_{\sigma_n -} = \{\tilde{z}_n\}$. By

275

definition of σ_n , the set

$$\{y \in [nx, (n+1)x] \mid \tilde{z}_n \in B(y, \lambda|y|)\}$$

276

is not empty. $\{\tilde{z}_n\}$ is an \mathcal{S}_{σ_n} -measurable finite random set, so we can apply Corollary 5.9 here.

277

Define now another stopping time

$$\tilde{\sigma}_n = \inf\{t > 0 : \xi_t^{\sigma_n, \{\tilde{z}_n\}} \supset B(\tilde{z}_n, \lambda|x| + |x| + 2r)\}.$$

278

Let us show that

$$\sup_{y \in [nx, (n+1)x]} T_\lambda(y) \leq \tilde{\sigma}_n. \quad (35)$$

279

For any $y \in [nx, (n+1)x]$,

$$|y - \tilde{z}_n| \leq |\tilde{z}_n - nx| \vee |\tilde{z}_n - (n+1)x| \leq \lambda(n+1)|x| + |x|.$$

280 Therefore the intersection of the balls $B(\tilde{z}_n, \lambda|x| + |x| + 2r)$ and $B(y, \lambda|y|)$ contains a ball \tilde{B} of
 281 radius r by Lemma 3.6, (ii), since

$$\lambda|x| + |x| + 2r + \lambda|y| - \lambda(n+1)|x| - |x| \geq \lambda|x| + 2r + \lambda n|x| - \lambda(n+1)|x| = 2r.$$

282 Since the radius of \tilde{B} is r and $\xi_{\tilde{\sigma}_n}^{\sigma_n, \{\tilde{z}_n\}} \supset B(\tilde{z}_n, \lambda|x| + |x| + 2r) \supset \tilde{B}$,

$$\eta_{\tilde{\sigma}_n}^{\sigma_n, \{\tilde{z}_n\}} \cap \tilde{B} \neq \emptyset,$$

283 and hence

$$\eta_{\tilde{\sigma}_n} \cap \tilde{B} \neq \emptyset. \quad (36)$$

284 Since $\tilde{B} \subset B(y, \lambda|y|)$ for all $y \in [n|x|, (n+1)|x|]$, (36) implies (35).

285 For $q \geq (\lambda|x| + |x| + 2r) \vee cs_0$, by Lemma 3.5

$$\begin{aligned} P\{\tilde{\sigma}_n - \sigma_n \geq \frac{q}{c}\} &= P\{B(\tilde{z}_n, \lambda|x| + |x| + 2r) \not\subseteq \xi_{\frac{q}{c} + \sigma_n}^{\sigma_n, \{\tilde{z}_n\}}\} \\ &\leq P\{B(\tilde{z}_n, q) \not\subseteq \xi_{\frac{q}{c} + \sigma_n}^{\sigma_n, \{\tilde{z}_n\}}\} \leq e^{-\sqrt[4]{\frac{q}{c}}}, \end{aligned}$$

287 hence

$$P\{\tilde{\sigma}_n - \sigma_n \geq q'\} \leq e^{-\sqrt[4]{q'}}, \quad q' \geq \left(\frac{\lambda|x| + |x| + 2r}{c}\right) \vee s_0. \quad (37)$$

288 By the Borel–Cantelli lemma

$$P\{\tilde{\sigma}_n - \sigma_n > \sqrt{n} \text{ for infinitely many } n\} = 0,$$

289 and since $\sigma_n \leq T_\lambda(nx) \leq \tilde{\sigma}_n$, a.s. for large n

$$\tilde{\sigma}_n < T_\lambda(nx) + \sqrt{n}$$

290 and

$$\sigma_n \geq T_\lambda(nx) - \sqrt{n}.$$

291 By (35)

$$\limsup_{n \rightarrow \infty} \frac{\sup_{y \in [nx, (n+1)x]} T_\lambda(y)}{n} \leq \limsup_{n \rightarrow \infty} \frac{\tilde{\sigma}_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{T_\lambda(nx) + \sqrt{n}}{n} \leq \mu_\lambda(x),$$

292 and

$$\liminf_{n \rightarrow \infty} \frac{\inf_{y \in [nx, (n+1)x]} T_\lambda(y)}{n} = \liminf_{n \rightarrow \infty} \frac{\sigma_n}{n} \geq \limsup_{n \rightarrow \infty} \frac{T_\lambda(nx) - \sqrt{n}}{n} \geq \mu_\lambda(x).$$

293 □

294 **Lemma 3.9.** *The ratio $\frac{\mu_\lambda(x)}{|x|}$ in (32) does not depend on x , $x \neq \mathbf{0}$.*

295 **Proof.** First let us note that for every $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and every $q > 0$,

$$\mu_\lambda(x) = \frac{\mu_\lambda(qx)}{q} \quad (38)$$

296 by Lemma 3.8.

297 On the other hand, if $|x| = |y|$ then by Proposition 5.7

$$\mu_\lambda(x) = \mu_\lambda(y), \quad (39)$$

298 since the distribution of (η_t) is invariant under rotations and we can consider $\mu_\lambda(x)$ as a func-
299 tional acting on the trajectory $(\eta_t)_{t \geq 0}$. The statement of the lemma follows from (38) and
300 (39). \square

301 Set

$$\mu_\lambda := \frac{\mu_\lambda(x)}{|x|}, \quad x \neq \mathbf{0}.$$

302 As λ decreases, $T_\lambda(x)$ increases and therefore μ_λ increases too. Denote

$$\mu = \lim_{\lambda \rightarrow 0^+} \mu_\lambda. \quad (40)$$

303 **Lemma 3.10.** *The constants μ_λ and μ are strictly positive: $\mu_\lambda > 0$, $\mu > 0$.*

304 **Proof.** By Proposition 3.1 for x with large $|x|$,

$$\eta_{\frac{(1-\lambda)|x|}{C_{\text{upb}}}} \subset B(\mathbf{0}, (1-\lambda)|x|),$$

305 hence for every $\lambda \in (0, 1)$ for x with large $|x|$

$$T_\lambda(x) \geq \frac{(1-\lambda)|x|}{C_{\text{upb}}}.$$

306 Thus,

$$\mu_\lambda \geq \frac{(1-\lambda)}{C_{\text{upb}}}$$

307 and

$$\mu = \lim_{\lambda \rightarrow 0^+} \mu_\lambda \geq \frac{1}{C_{\text{upb}}}.$$

308 \square

309 **Lemma 3.11.** *Let $q, R > 0$. Suppose that for all $\varepsilon \in (0, 1)$ a.s. for sufficiently large $n \in \mathbb{N}$*

$$\frac{\eta_{qn}}{qn} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_{qn}}{qn} \right). \quad (41)$$

Then for all $\varepsilon \in (0, 1)$ a.s. for sufficiently large $t \geq 0$

$$\frac{\eta_t}{t} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_t}{t} \text{ respectively} \right).$$

Proof. We consider the first case only – the proof of the other one is similar. Since $\varepsilon \in (0, 1)$ is arbitrary, (41) implies that for all $\tilde{\varepsilon} \in (0, 1)$ a.s. for large $n \in \mathbb{N}$,

$$\frac{\eta_{q(n+2)}}{qn} \subset (1+\tilde{\varepsilon})B(\mathbf{0}, R).$$

Since a.s. $(\eta_t)_{t \geq 0}$ is monotonically growing, it is sufficient to note that

$$\frac{\eta_t}{t} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \text{if} \quad \frac{\eta_{\lceil \frac{t}{q} \rceil q + q}}{\lceil \frac{t}{q} \rceil q} \subset (1+\varepsilon)B(\mathbf{0}, R).$$

310 \square

311 Recall that c is a constant from Lemma 3.5.

312 **Lemma 3.12.** *Let $\varepsilon \in (0, 1)$. Then a.s.*

$$(1 - \varepsilon)B(\mathbf{0}, \mu^{-1}) \subset \frac{\xi_m}{m} \quad (42)$$

313 *for large m of the form $m = (1 + \frac{\lambda\mu_\lambda^{-1}}{c})n$, $n \in \mathbb{N}$.*

314 **Proof.** Let $\lambda = \lambda_\varepsilon > 0$ be chosen so small that

$$(1 - \varepsilon)\mu^{-1} \leq \frac{1 - \frac{\varepsilon}{2}}{1 + \frac{\lambda\mu_\lambda^{-1}}{c}}\mu_\lambda^{-1}. \quad (43)$$

315 Such a λ exists since

$$\lim_{\lambda \rightarrow 0^+} \frac{\mu_\lambda^{-1}}{1 + \frac{\lambda\mu_\lambda^{-1}}{c}} = \mu^{-1}.$$

316 Choose a finite sequence of points $\{x_j, j = 1, \dots, N\}$ such that $x_j \in (1 - \frac{\varepsilon}{2})B(\mathbf{0}, \mu_\lambda^{-1})$ and

$$\bigcup_j B(x_j, \frac{\varepsilon}{4}c) \supset (1 - \frac{\varepsilon}{2})B(\mathbf{0}, \mu_\lambda^{-1}).$$

317 Let $\delta > 0$ be so small that $(1 + \delta)(1 - \frac{\varepsilon}{2}) \leq (1 - \frac{\varepsilon}{4})$. Since a.s.

$$\frac{T_\lambda(nx_j)}{n|x_j|} \rightarrow \mu_\lambda,$$

318 for large n for every $j \in \{1, \dots, N\}$

$$T_\lambda(nx_j) \leq n|x_j|(1 + \delta)\mu_\lambda \leq n(1 - \frac{\varepsilon}{2})(1 + \delta) \leq n(1 - \frac{\varepsilon}{4}), \quad (44)$$

319 so that the system reaches the ball $B(nx_j, \lambda n|x_j|)$ before the time $n(1 - \frac{\varepsilon}{4})$. Let Q_n be the
320 random event

$$\{T_\lambda(nx_j) \leq n(1 - \frac{\varepsilon}{4}) \text{ for } j = 1, \dots, N\} = \{\eta_{n(1 - \frac{\varepsilon}{4})} \cap B(nx_j, \lambda n|x_j|) \neq \emptyset, \text{ for } j = 1, \dots, N\}.$$

321 Note that $P(Q_n) \rightarrow 1$ by (44), and even

$$P\{\bigcup_{m \in \mathbb{N}} \bigcap_{i=m}^{\infty} Q_i\} = 1. \quad (45)$$

322 In other words, a.s. for large i all Q_i occur.

323 Let $\bar{z}(nx_j)$ be defined as $z(nx_j)$ on Q_n and as nx_j on the complement $\Omega \setminus Q_n$ (recall
324 that $z(x) = z_\lambda(x)$, $x \in \mathbb{R}^d$, was defined after (25)). The set $\{\bar{z}(nx_j)\}$ is a finite random
325 $\mathcal{S}_{n(1 - \frac{\varepsilon}{4})}$ -measurable set.

326 Using Lemma 3.5, we will show that after an additional time interval of length $(\frac{\varepsilon}{4} + \frac{\lambda\mu_\lambda^{-1}}{c})n$
327 the entire ball $(1 - \frac{\varepsilon}{2})nB(\mathbf{0}, \mu_\lambda^{-1})$ is covered by (ξ_t) , that is, a.s. for large n

$$(1 - \frac{\varepsilon}{2})nB(\mathbf{0}, \mu_\lambda^{-1}) \subset \xi_{n(1 - \frac{\varepsilon}{4}) + (\frac{\varepsilon}{4} + \frac{\lambda\mu_\lambda^{-1}}{c})n} = \xi_{n + \frac{\lambda n \mu_\lambda^{-1}}{c}}. \quad (46)$$

328 Indeed, since

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda|x_j|n) \subset B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n),$$

329 the series

$$\begin{aligned} & \sum_{n \in \mathbb{N}} P\{B(nx_j, c\frac{\varepsilon}{4}n) \not\subset \xi_{n + \frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}), \{\bar{z}(nx_j)\})} \text{ for some } j\} \\ 330 & \leq \sum_{n \in \mathbb{N}} P\{B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n) \not\subset \xi_{n + \frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}), \{\bar{z}(nx_j)\})} \text{ for some } j\} \end{aligned}$$

331 converges by Lemma 3.5, thus a.s. for large n ,

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset \xi_{n + \frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}), \{\bar{z}(nx_j)\})}, \quad j = 1, \dots, N. \quad (47)$$

332 By (45) a.s. for large n

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset \xi_{n + \frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}), \{z(nx_j)\})}, \quad j = 1, \dots, N. \quad (48)$$

333 Hence the choice of $\{x_j, j = 1, \dots, N\}$ and (48) yield (46). Because of our choice of λ ,

$$(1 - \varepsilon)nB(\mathbf{0}, \mu^{-1}) \subset \frac{(1 - \frac{\varepsilon}{2})}{(1 + \frac{\lambda\mu_\lambda^{-1}}{c})}nB(\mathbf{0}, \mu_\lambda^{-1}),$$

334 which in conjunction with (46) implies that (42) holds a.s. for large m of the form $(1 + \frac{\lambda\mu_\lambda^{-1}}{c})n$,
335 where $n \in \mathbb{N}$. □

336 **Lemma 3.13.** *Let $\varepsilon \in (0, 1)$. Then a.s. for large $n \in \mathbb{N}$*

$$\frac{\eta_n}{n} \subset (1 + \varepsilon)B(\mathbf{0}, \mu^{-1}). \quad (49)$$

337 **Proof.** Let $\lambda = \lambda_\varepsilon > 0$ be so small that

$$(1 + \frac{\varepsilon}{2})B(\mathbf{0}, \mu_\lambda^{-1}) \subset (1 + \varepsilon)B(\mathbf{0}, \mu^{-1}) \quad (50)$$

338 Let $q \in (\varepsilon, \infty)$ and A be the annulus

$$A := (1 + q)B(\mathbf{0}, \mu_\lambda^{-1}) \setminus (1 + \frac{1}{2}\varepsilon)B(\mathbf{0}, \mu_\lambda^{-1}), \quad (51)$$

339 and $\{x_j, j = 1, \dots, N\}$ be a finite sequence such that $x_j \in A$ and

$$\bigcup_j B(x_j, \lambda|x_j|) \supset A.$$

340 Define $F := \{\eta_n \cap nA \neq \emptyset \text{ infinitely often}\}$. On F there exists a (random) $i \in \{1, \dots, N\}$ such
341 that the intersection

$$\eta_n \cap nB(x_i, \lambda|x_i|) \quad (52)$$

342 is non-empty infinitely often. Define also

$$F_i := \{\eta_n \cap nB(x_i, \lambda|x_i|) \neq \emptyset \text{ infinitely often}\} \quad (53)$$

343 Note that $F \subset \bigcup_{i=1}^N F_i$.
 On F_i we have

$$T_\lambda(nx_i) \leq n$$

344 infinitely often, hence our choice of A implies

$$\liminf_{n \rightarrow \infty} \frac{T_\lambda(nx_i)}{n|x_i|} \leq \liminf_{n \rightarrow \infty} \frac{n}{(1 + \frac{1}{2}\varepsilon)\mu_\lambda^{-1}n} = \mu_\lambda \frac{1}{(1 + \frac{1}{2}\varepsilon)}.$$

345 The last inequality and Lemma 3.8 imply that $P(F_i) = 0$ for every $i \in \{1, \dots, N\}$. Hence
 346 $P(F) = 0$ too. Setting $q = 2\mu_\lambda C_{upb} + 1$, so that the radius of the ball on the left-hand side of
 347 (50)

$$q\mu_\lambda^{-1} > 2C_{upb},$$

348 by Proposition 3.1 and the definition of F we get a.s. for large n ,

$$\frac{\eta_n}{n} \subset (1 + \frac{1}{2}\varepsilon)B(\mathbf{0}, \mu_\lambda^{-1}) \quad (54)$$

349 and the statement of the lemma follows from (50) and (54). \square

350 **Proof of Theorem 2.8.** The theorem follows from Lemmas 3.11, 3.12, and 3.13. Note that

$$\frac{\xi_n}{n} \subset (1 + \varepsilon)B(\mathbf{0}, \mu^{-1}). \quad (55)$$

352 is obtained from Lemma 3.13 by replacing ε in (49) with $\frac{\varepsilon}{2}$.

353 \square

354 4 Proof of Theorem 2.10

355 We precede the proof of Theorem 2.10 with an auxiliary lemma about Markovian functionals
 356 of a general Markov chain.

Let $(S, \mathcal{B}(S))$ be a Polish (state) space. Consider a (time-homogeneous) Markov chain on $(S, \mathcal{B}(S))$ as a family of probability measures on S^∞ . Namely, on the measurable space $(\bar{\Omega}, \mathcal{F}) = (S^\infty, \mathcal{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s \in S}$ such that for the coordinate mappings

$$\begin{aligned} X_n &: \bar{\Omega} \rightarrow S, \\ X_n(s_1, s_2, \dots) &= s_n, \end{aligned}$$

the process $X := \{X_n\}_{n \in \mathbb{Z}_+}$ is a Markov chain such that for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$

$$P_s\{X_{n+m_j} \in A_j, j = 1, \dots, l \mid \mathcal{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, \dots, l\}.$$

Here $A_j \in \mathcal{B}(S)$, $m_j \in \mathbb{N}$, $l \in \mathbb{N}$, $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathcal{B}(S) \rightarrow [0, 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S, A \in \mathcal{B}(S).$$

357 Consider a transformation of the chain X , $Y_n = f(X_n)$, where $f : S \rightarrow \mathbb{R}$ is a Borel-
 358 measurable function. Here we will give sufficient conditions for $Y = \{Y_n\}_{n \in \mathbb{Z}_+}$ to be a Markov
 359 chain. A very similar question was discussed by Burke and Rosenblatt [BR58] for discrete space
 360 Markov chains.

361 **Lemma 4.1.** *Assume that for any bounded Borel function $h : S \rightarrow S$*

$$E_s h(X_1) = E_q h(X_1) \text{ whenever } f(s) = f(q), \quad (56)$$

362 *Then Y is a Markov chain.*

363 **Remark.** Condition (56) is the equality of distributions of X_1 under two different measures,
 364 P_s and P_q .

365 **Proof.** For the natural filtrations of the processes X and Y we have an inclusion

$$\mathcal{F}_n^X \supset \mathcal{F}_n^Y, \quad n \in \mathbb{N}, \quad (57)$$

366 since Y is a function of X . For $k \in \mathbb{N}$ and bounded Borel functions $h_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, 2, \dots, k$,

$$E_s \left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathcal{F}_n^X \right] = E_{X_n} \prod_{j=1}^k h_j(f(X_j)) = \int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) \dots \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) \Big|_{x_0=X_n} \quad (58)$$

367 To transform the last integral, we introduce a new kernel: for $y \in f(S)$ chose $x \in S$ with
 368 $f(x) = y$, and then for $B \in \mathcal{B}(\mathbb{R})$ define

$$\bar{Q}(y, B) = Q(x, f^{-1}(B)). \quad (59)$$

369 The expression on the right-hand side does not depend on the choice of x because of (56). To
 370 make the kernel \bar{Q} defined on $\mathbb{R} \times \mathcal{B}(\mathbb{R})$, we set

$$\bar{Q}(y, B) = \mathbf{1}_{\{0 \in B\}}, \quad y \notin f(S).$$

Then, setting $z_n = f(x_n)$, we obtain from the change of variables formula for the Lebesgue
 integral that

$$\int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) = \int_{\mathbb{R}} \bar{Q}(f(x_{n-1}), dz_n) h_n(z_n).$$

371 Likewise, setting $z_{n-1} = f(x_{n-1})$, we get

$$\begin{aligned} \int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) = \\ \int_S Q(x_{n-2}, dx_{n-1}) h_n(f(x_{n-1})) \int_{\mathbb{R}} \bar{Q}(f(x_{n-1}), dz_n) h_n(z_n) = \\ \int_{\mathbb{R}} \bar{Q}(f(x_{n-2}), dz_{n-1}) h_n(z_{n-1}) \int_{\mathbb{R}} \bar{Q}(z_{n-1}, dz_n) h_n(z_n). \end{aligned}$$

373 Proceeding further, we obtain

$$\int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) \dots \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_{\mathbb{R}} \bar{Q}(z_0, dz_1) h_1(z_1) \int_{\mathbb{R}} \bar{Q}(z_1, dz_2) h_2(z_2) \dots \int_{\mathbb{R}} \bar{Q}(z_{n-1}, dz_n) h_n(z_n),$$

375 where $z_0 = f(x_0)$.

376 Thus,

$$E_s \left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathcal{F}_n^X \right] =$$

$$\int_{\mathbb{R}} \bar{Q}(f(X_0), dz_1) h_1(z_1) \int_{\mathbb{R}} \bar{Q}(z_1, dz_2) h_2(z_2) \dots \int_{\mathbb{R}} \bar{Q}(z_{n-1}, dz_n) h_n(z_n).$$

378 This equality and (57) imply that Y is a Markov chain. \square

379 **Remark 4.2.** From the proof it follows that \bar{Q} is the transition probability kernel for the chain
 380 $\{f(X_n)\}_{n \in \mathbb{Z}_+}$.

381 **Remark 4.3.** Clearly, this result holds for a Markov chain which is not necessarily defined on
 382 a canonical state space because the property of a process to be a Markov chain depends on its
 383 distribution only.

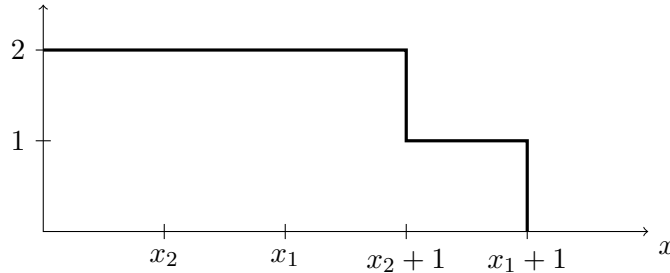


Figure 2: The plot of $b(\cdot, \eta_t)$.

384 **Proof of Theorem 2.10.** Without any loss of generality, we will consider the speed of
 385 propagation in one direction only, say toward $+\infty$. Let $x_1(t)$ and $x_2(t)$ denote the positions
 386 of the rightmost particle and the second rightmost particle, respectively ($x_2(t) = 0$ until first
 387 two births occurs inside $(0, +\infty)$). Let us observe that $b(x, \eta_t) \equiv 2$ on $(0, x_2(t) + 1]$, and
 388 $X = (x_1(t), x_2(t))$ is a continuous-time pure jump Markov process on $\{(x_1, x_2) \mid x_1 \geq x_2 \geq$
 389 $0, x_1 - x_2 \leq 1\}$ with transition densities

$$\begin{aligned} (x_1, x_2) &\rightarrow (v, x_1) && \text{at rate 1, } v \in (x_2 + 1, x_1 + 1]; \\ (x_1, x_2) &\rightarrow (v, x_1) && \text{at rate 2, } v \in (x_1, x_2 + 1]; \\ (x_1, x_2) &\rightarrow (x_1, v) && \text{at rate 2, } v \in (x_2, x_1]. \end{aligned} \tag{60}$$

390 (to be precise, the above is true from the moment the first birth inside \mathbb{R}_+ occurs).

391 Furthermore, $z(t) := x_1(t) - x_2(t)$ satisfies

$$E\{f(z(t+\delta)) \mid x_1(t) = x_1, x_2(t) = x_2\} = E\{f(z(t+\delta)) \mid x_1(t) = x_1 + h, x_2(t) = x_2 + h\}$$

392 for every $h > 0$ and every Borel bounded function f . In other words, transition rates of $(z(t))_{t \geq 0}$
 393 are entirely determined by the current state of $(z(t))_{t \geq 0}$. Therefore, by Lemma 4.1, $(z(t))_{t \geq 0}$
 394 is itself a pure jump Markov process on $[0, 1]$ (Lemma 4.1 ensures that the embedded Markov
 395 chain of $(z(t))_{t \geq 0}$ is indeed a discrete-time Markov process). The transition densities of $(z(t))_{t \geq 0}$
 396 are

$$\begin{aligned} q(x, y) &= 4\mathbb{1}\{y \leq x\} + 2\mathbb{1}\{x \leq y \leq 1-x\} + \mathbb{1}\{y \geq 1-x\}, & x \leq \frac{1}{2}, y \in [0, 1], \\ q(x, y) &= 4\mathbb{1}\{y \leq 1-x\} + 3\mathbb{1}\{1-x \leq y \leq x\} + \mathbb{1}\{y \geq x\}, & x \geq \frac{1}{2}, y \in [0, 1]. \end{aligned} \quad (61)$$

397 Note that the total jump rate out of x is $q(x) := \int_0^1 q(x, y) dy = 2+x$. The process $(z(t))_{t \geq 0}$ is
 398 a regular Harris recurrent Feller process with the Lebesgue measure on $[0, 1]$ being a supporting
 399 measure (see e.g. [Kal02, Chapter 20]). Hence a unique invariant measure exists and has a
 400 density g with respect to the Lebesgue measure. The equation for g is

$$\int_0^1 q(x, y)g(x)dx = q(y)g(y). \quad (62)$$

401 Set

$$f(x) = g(x)q(x) \left(\int_0^1 g(y)q(y)dy \right)^{-1}, \quad x \in [0, 1].$$

402 It is clear that f is again a density (as an aside we point out that f is the density of invariant
 403 distribution of the embedded Markov chain of $(z(t))_{t \geq 0}$). Equation (62) becomes

$$f(y) = \int_0^1 \frac{q(x, y)}{q(x)} f(x) ds,$$

which after some calculations transforms into

$$f(y) = 2 \int_0^{\frac{1}{2}} \frac{f(x)dx}{2+x} + 2 \int_y^{\frac{1}{2}} \frac{f(x)dx}{2+x} + 3 \int_{\frac{1}{2}}^1 \frac{f(x)dx}{2+x} + \int_{\frac{1}{2}}^{1-y} \frac{f(x)dx}{2+x}, \quad y \leq \frac{1}{2}, \quad (63)$$

$$f(y) = \int_0^{\frac{1}{2}} \frac{f(x)dx}{2+x} + \int_0^{1-y} \frac{f(x)dx}{2+x} + \int_{\frac{1}{2}}^1 \frac{f(x)dx}{2+x} + 2 \int_y^1 \frac{f(x)dx}{2+x}, \quad y \geq \frac{1}{2}. \quad (64)$$

404 Differentiating (63), (64) with respect to y , we find that f solves the equation

$$\frac{df}{dx}(x) = -2 \frac{f(x)}{2+x} - \frac{f(1-x)}{3-x}, \quad x \in [0, 1]. \quad (65)$$

Let

$$\varphi(x) := [(2+x)^2(3-x)^2]f(x), \quad x \in [0, 1].$$

405 Then (65) becomes

$$(3-x)\frac{d\varphi}{dx}(x) + 2\varphi(x) + \varphi(1-x) = 0, \quad x \in [0, 1]. \quad (66)$$

406 Looking for solutions to (66) among polynoms, we find that $\varphi(x) = c(4-3x)$ is a solution. By
407 direct substitution we can check that

$$f(x) = \frac{c(4-3x)}{(2+x)^2(3-x)^2} \quad x \in [0, 1] \quad (67)$$

408 solves (63)-(64). The constant $c > 0$ can be computed, but is irrelevant for our purposes. Hence,
409 after some more computation,

$$g(x) = \frac{36(4-3x)}{(2+x)^3(3-x)^2}, \quad x \in [0, 1]. \quad (68)$$

410 Note that we do not prove analytically that equation (63), (64) has a unique solution.
411 However, uniqueness for non-negative integrable solutions follows from the uniqueness of the
412 invariant distribution for $(z(t))_{t \geq 0}$. Let l be the Lebesgue measure on \mathbb{R} . By an ergodic theorem
413 for Markov processes, see e.g. [Kal02, Theorem 20.21 (i)], for any $0 \leq p < p' \leq 1$,

$$\lim_{t \rightarrow \infty} \frac{l\{s : z(s) \in [p, p'], 0 \leq s \leq t\}}{t} \rightarrow \int_p^{p'} g(x)dx. \quad (69)$$

414 Conditioned on $z(t) = z$, the transition densities of $x_1(t)$ are

$$\begin{aligned} x_1 &\rightarrow x_1 + v && \text{at rate } 2, && v \in (0, 1-z]; \\ x_1 &\rightarrow x_1 + v && \text{at rate } 1, && v \in (1-z, 1]. \end{aligned} \quad (70)$$

415 Hence by (68) the speed of propagation is

$$\int_0^1 g(z)dz \left[\int_0^{1-z} 2ydy + \int_{1-z}^1 ydy \right] = \int_0^1 g(z)(1-z + \frac{1}{2}z^2)dz = \frac{144 \ln(3) - 144 \ln(2) - 40}{25}. \quad \square$$

416 **Remark 4.4.** We see from the proof that the speed can be computed in a similar way for the
417 birth rates of the form

$$b_k(x, \eta) = k \wedge \left(\sum_{y \in \eta} \mathbf{1}\{|x-y| \leq 1\} \right), \quad (71)$$

418 where $k \in (1, 2)$. However, the computations quickly become unwieldy.

419 5 The construction and properties of the process

420 Here we proceed to construct the process as a unique solution to a stochastic integral equation.
421 First such a scheme was carried out by Massoulié [Mas98]. This method can be deemed an
422 analog of the construction from graphical representation. We follow here [Bez15].

423 **Remark 5.1.** Of course, the process starting from a fixed initial condition we consider here
424 can be constructed as the minimal jump process (pure jump type Markov processes in the
425 terminology of [Kal02]) as is done for example in [EW03]. Note however that we use coupling
426 of infinitely many processes starting at different time points from different initial conditions, so
427 we here employ another method.

428 Recall that

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

429 and the σ -algebra on Γ_0 was introduced in (3). To construct the family of processes $(\eta_t^{q,A})_{t \geq q}$,
430 we consider the stochastic equation with Poisson noise

$$|\eta_t \cap B| = \int_{(q,t] \times B \times [0,\infty)} \mathbf{1}_{[0,b(x,\eta_{s-})]}(u) N(ds, dx, du) + |\eta_q \cap B|, \quad t \geq q, \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (72)$$

431 where $(\eta_t)_{t \geq q}$ is a cadlag Γ_0 -valued solution process, N is a Poisson point process on $\mathbb{R}_+ \times$
432 $\mathbb{R}^d \times \mathbb{R}_+$, the mean measure of N is $ds \times dx \times du$. We require the processes N and η_0 to be
433 independent of each other. Equation (72) is understood in the sense that the equality holds a.s.
434 for every bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq q$. In the integral on the right-hand side of (72), x is
435 the location and s is the time of birth of a new particle. Thus, the integral over B from q to t
436 represents the number of births inside B which occurred before t .

437 Let us assume for convenience that $q = 0$. We will make the following assumption on the
438 initial condition:

$$E|\eta_0| < \infty. \quad (73)$$

439 We say that the process N is *compatible* with an increasing, right-continuous and complete
440 filtration of σ -algebras $(\mathcal{F}_t, t \geq 0)$ if N is adapted, that is, all random variables of the type
441 $N(\bar{T}_1, U)$, $\bar{T}_1 \in \mathcal{B}([0; t])$, $U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are \mathcal{F}_t -measurable, and all random variables of the
442 type $N(t+h, U) - N(t, U)$, $h \geq 0, U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are independent of \mathcal{F}_t , $N(t, U) =$
443 $N([0; t], U)$.

444 **Definition 5.2.** A (*weak*) *solution* of equation (72) is a triple $((\eta_t)_{t \geq 0}, N), (\Omega, \mathcal{F}, P), (\{\mathcal{F}_t\}_{t \geq 0})$,
445 where

- 446 (i) (Ω, \mathcal{F}, P) is a probability space, and $\{\mathcal{F}_t\}_{t \geq 0}$ is an increasing, right-continuous and
447 complete filtration of sub- σ -algebras of \mathcal{F} ,
- 448 (ii) N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity $ds \times dx \times du$,
- 449 (iii) η_0 is a random \mathcal{F}_0 -measurable element in Γ_0 satisfying (73),
- 450 (iv) the processes N and η_0 are independent, N is compatible with $\{\mathcal{F}_t\}_{t \geq 0}$,
- 451 (v) $(\eta_t)_{t \geq 0}$ is a cadlag Γ_0 -valued process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, $\eta_t|_{t=0} = \eta_0$,
- 452 (vi) all integrals in (72) are well-defined,

$$E \int_0^t ds \int_{\mathbb{R}^d} b(x, \eta_{s-}) < \infty, \quad t > 0,$$

- 453 (vii) equality (72) holds a.s. for all $t \in [0, \infty]$ and all Borel sets B .

Let

$$\begin{aligned} \mathcal{S}_t^0 &= \sigma\{\eta_0, N([0, q] \times B \times C), \\ & q \in [0, t], B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}(\mathbb{R}_+)\}, \end{aligned} \quad (74)$$

and let \mathcal{S}_t be the completion of \mathcal{S}_t^0 under P . Note that $\{\mathcal{S}_t\}_{t \geq 0}$ is a right-continuous filtration (see Remark 6.2).

Definition 5.3. A solution of (72) is called *strong* if $(\eta_t)_{t \geq 0}$ is adapted to $(\mathcal{S}_t, t \geq 0)$.

Remark 5.4. In the definition above we considered solutions as processes indexed by $t \in [0, \infty)$. The reformulations for the case $t \in [0, T]$, $0 < T < \infty$, are straightforward. This remark also applies to many of the results below.

Definition 5.5. We say that *joint uniqueness in law* holds for equation (72) with an initial distribution ν if any two (weak) solutions $((\eta_t), N)$ and $((\eta'_t), N')$ of (72), $Law(\eta_0) = Law(\eta'_0) = \nu$, have the same joint distribution:

$$Law((\eta_t), N) = Law((\eta'_t), N').$$

Theorem 5.6. *Pathwise uniqueness, strong existence and joint uniqueness in law hold for equation (72). The unique solution is a Markov process.*

Proof. Without loss of generality assume that $P\{\eta_0 \neq \emptyset\} = 1$. Define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf\{t > 0 : \int_{(\sigma_n, \sigma_n+t] \times B \times [0, \infty)} \mathbb{1}_{[0, b(x, \zeta_{\sigma_n})]}(u) N(ds, dx, du) > 0\} + \sigma_n, \quad \sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d : N(\{\sigma_{n+1}\} \times \{x\} \times [0, b(x, \zeta_{\sigma_n})]) > 0\}$. The points z_n are uniquely determined a.s. Furthermore, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s. by (6). We define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n, \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is \mathcal{F}_{σ_n} -measurable. By direct substitution we see that $(\zeta_t)_{t \geq 0}$ is a strong solution to (72) on the time interval $t \in [0, \lim_{n \rightarrow \infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead we are going to show that

$$\lim_{n \rightarrow \infty} \sigma_n = \infty \quad \text{a.s.} \quad (75)$$

The process $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n)}$ has the Markov property, because the process N has the strong Markov property and independent increments. Indeed, conditioning on \mathcal{S}_{σ_n} ,

$$E[\mathbb{1}_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x \in B\}} \mid \mathcal{S}_{\sigma_n}] = \frac{\int b(x, \zeta_{\sigma_n}) dx}{\int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n}) dx},$$

477 thus the chain $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$, $\sigma_{n+1} - \sigma_n$ are distributed
 478 exponentially:

$$E\{\mathbb{1}_{\{\sigma_{n+1} - \sigma_n > a\}} \mid \{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}\} = \exp\{-a \int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n}) dx\}.$$

479 Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1}) \int_{\mathbb{R}^d} b(x, \zeta_{\sigma_n}) dx$ constitute an independent of
 480 $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$ sequence of independent unit exponentials. Theorem 12.18 in [Kal02] implies that
 481 $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n)}$ is a pure jump type Markov process.

482 The jump rate of $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n)}$ is given by

$$c(\alpha) = \int_{\mathbb{R}^d} b(x, \alpha) dx.$$

483 Condition 2.2 implies that $c(\alpha) \leq \|a\|_1 \cdot |\alpha|$, where $\|a\|_1 = \|a\|_{L^1(\mathbb{R}^d)}$. Consequently,

$$c(\zeta_{\sigma_n}) \leq \|a\|_1 \cdot |\zeta_{\sigma_n}| = \|a\|_1 \cdot |\eta_0| + n \|a\|_1.$$

484 We see that $\sum_n \frac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [Kal02] implies that $\sigma_n \rightarrow \infty$.

485 We have proved the existence of a strong solution. The uniqueness follows by induction on
 486 jumps of the process. Namely, let $(\tilde{\zeta}_t)_{t \geq 0}$ be a solution to (72). From (vii) of Definition 5.2 and
 487 the equality

$$\int_{(0, \sigma_1) \times \mathbb{R}^d \times [0, \infty]} \mathbb{1}_{[0, b(x, \eta_0)]}(u) N(ds, dx, du) = 0,$$

488 it follows that $P\{\tilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time, the equality

$$\int_{\{\sigma_1\} \times \mathbb{R}^d \times [0, \infty]} \mathbb{1}_{[0, b(x, \eta_0)]}(u) N(ds, dx, du) = 1,$$

489 which holds a.s., yields that $\tilde{\zeta}$ too has a birth at the moment σ_1 , and in the same point of space
 490 at that. Therefore, $\tilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide
 491 up to σ_n a.s., and, since $\sigma_n \rightarrow \infty$ a.s.,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t \geq 0\} = 1.$$

492 Thus, pathwise uniqueness holds. Joint uniqueness in law follows from the functional de-
 493 pendence between the solution to the equation and the ‘input’ η_0 and N . \square

494 **Proposition 5.7.** *If b is rotation invariant, then so is (η_t) .*

495 **Proof.** It is sufficient to note that $(M_d \eta_t)$, where $M_d \in \text{SO}(d)$, is the unique solution to
 496 (72) with N replaced by $M_d^{-1} N$ defined by

$$M_d^{-1} N([0, q] \times B \times C) = N([0, q] \times M_d^{-1} B \times C), \quad q \geq 0, B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}(\mathbb{R}_+).$$

497 $M_d^{-1} N$ is a Poisson point process with the same intensity, therefore by uniqueness in law
 498 $(M_d \eta_t) \stackrel{d}{=} (\eta_t)$.

499 **Proposition 5.8. (The strong Markov property)** Let τ be an $(\mathcal{S}_t, t \geq 0)$ -stopping time and
500 let $\tilde{\eta}_0 \stackrel{d}{=} \eta_\tau$. Then

$$(\eta_{\tau+t}, t \geq 0) \stackrel{d}{=} (\tilde{\eta}_t, t \geq 0). \quad (76)$$

Furthermore, for any $\mathcal{D} \in \mathcal{B}(D_{\Gamma_0}[0, \infty))$,

$$P\{(\eta_{\tau+t}, t \geq 0) \in \mathcal{D} \mid \mathcal{S}_\tau\} = P\{(\eta_{\tau+t}, t \geq 0) \in \mathcal{D} \mid \eta_\tau\};$$

501 that is, given η_τ , $(\eta_{\tau+t}, t \geq 0)$ is conditionally independent of $(\mathcal{S}_t, t \geq 0)$.

502 **Proof.** Note that

$$|\eta_{\tau+t} \cap B| = \int_{(\tau, \tau+t] \times B \times [0, \infty)} \mathbf{1}_{[0, b(x, \eta_{s-})]}(u) N(ds, dx, du) + |\eta_\tau \cap B|, \quad t \geq 0, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

503 Since the unique solution is adapted to the filtration generated by the noise and initial
504 condition, the conditional independence follows, and (76) follows from the uniqueness in law.
505 We rely here on the strong Markov property of the Poisson point process, see Proposition 6.1
506 below. \square

507 **Corollary 5.9.** Let τ be an $(\mathcal{S}_t, t \geq 0)$ -stopping time and $\{y\}$ be an \mathcal{S}_τ -measurable finite
508 random singleton. Then

$$(\eta_{\tau+t}^{\tau, \{y\}} - y)_{t \geq 0} \stackrel{(d)}{=} (\eta_t)_{t \geq 0}.$$

509 **Proof.** This is a consequence of Theorem 5.6 and Proposition 5.8. \square

510 Consider two growth processes $(\zeta^{(1)})_t$ and $(\zeta^{(2)})_t$ defined on the common probability space
511 ans satisfying equations of the form (72),

$$|\zeta_t^{(k)} \cap B| = \int_{(q, t] \times B \times [0, \infty)} \lambda \mathbf{1}_{[0, b_k(x, \zeta_{s-}^{(k)})]}(u) N(ds, dx, du) + |\zeta_q^{(k)} \cap B|, \quad k = 1, 2. \quad (77)$$

512 Assume that and the rates b_1 and b_2 satisfy the conditions of imposed on b in Section 2. Let
513 $(\zeta_t^{(k)})_{t \in [0, \infty)}$ be the unique strong solutions.

514 **Lemma 5.10.** Assume that a.s. $\zeta_0^{(1)} \subset \zeta_0^{(2)}$, and for any two finite configurations $\eta^1 \subset \eta^2$,

$$b_1(x, \eta^1) \leq b_2(x, \eta^2), \quad x \in \mathbb{R}^d. \quad (78)$$

515 Then a.s.

$$\zeta_t^{(1)} \subset \zeta_t^{(2)}, \quad t \in [0, \infty). \quad (79)$$

516 **Proof.** Let $(\sigma_n)_{n \in \mathbb{N}}$ be the ordered sequence of the moments of births for $(\zeta_t^{(1)})$, that is,
517 $t \in (\sigma_n)_{n \in \mathbb{N}}$ if and only if $|\zeta_t^{(1)} \setminus \zeta_{t-}^{(1)}| = 1$. It suffices to show that for each $n \in \mathbb{N}$, σ_n is a moment
518 of birth for $(\zeta_t^{(2)})_{t \in [0, \infty)}$ too, and the birth occurs at the same place. We use induction on n .

Here we deal only with the base case, the induction step is done in the same way. Assume that

$$\zeta_{\sigma_1}^{(1)} \setminus \zeta_{\sigma_1-}^{(1)} = \{x_1\}.$$

The process $(\zeta^{(1)})_{t \in [0, \infty)}$ satisfies (77), therefore $N(\{x\} \times [0, b_k(x_1, \zeta_{\sigma_1-}^{(1)})]) = 1$. Since

$$\zeta_{\sigma_1-}^{(1)} = \zeta_0^{(1)} \subset \zeta_0^{(2)} \subset \zeta_{\sigma_1-}^{(2)},$$

by (78)

$$N_1(\{x\} \times \{\sigma_1\} \times [0, b_k(x_1, \zeta_{\sigma_1-}^{(2)})]) = 1,$$

hence

$$\zeta_{\sigma_1}^{(2)} \setminus \zeta_{\sigma_1-}^{(2)} = \{x_1\}.$$

519

□

520 6 Appendix. The strong Markov property of a Poisson point 521 process

522 We need the strong Markov property of a Poisson point process. Denote $X := \mathbb{R}^d \times \mathbb{R}_+$ (compare
523 the proof of Proposition 5.8), and let l be the Lebesgue measure on X . Consider a a Poisson
524 point process N on $\mathbb{R}_+ \times X$ with intensity measure $dt \times l$. Let N be compatible with a right-
525 continuous complete filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and τ be a finite a.s. $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping time . Introduce
526 another Point process \bar{N} on $\mathbb{R}_+ \times X$,

$$\bar{N}([0; s] \times U) = N((\tau; \tau + s] \times U), \quad U \in \mathcal{B}(X).$$

527 **Proposition 6.1.** *The process \bar{N} is a Poisson point process on $\mathbb{R}_+ \times X$ with intensity $dt \times l$,*
528 *independent of \mathcal{F}_τ .*

529 **Proof.** To prove the proposition, it suffices to show that

530 (i) for any $b > a > 0$ and open bounded $U \subset X$, $\bar{N}((a; b), U)$ is a Poisson random variable
531 with mean $(b - a)l(U)$, and

532 (ii) for any $b_k > a_k > 0$, $k = 1, \dots, m$, and any open bounded $U_k \subset X$, such that $((a_i; b_i) \times$
533 $U_i) \cap ((a_j; b_j) \times U_j) = \emptyset$, $i \neq j$, the collection $\{\bar{N}((a_k; b_k) \times U_k)\}_{k=1, m}$ is a sequence of independent
534 random variables, independent of \mathcal{F}_τ .

535 Indeed, \bar{N} is determined completely by values on sets of type $(b - a)\beta(U)$, a, b, U as in (i),
536 therefore it must be an independent of \mathcal{F}_τ Poisson point process if (i) and (ii) hold.

Let τ_n be the sequence of $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping times, $\tau_n = \frac{k}{2^n}$ on $\{\tau \in (\frac{k-1}{2^n}, \frac{k}{2^n}]\}$, $k \in \mathbb{N}$. Then
 $\tau_n \downarrow \tau$ and $\tau_n - \tau \leq \frac{1}{2^n}$. Note that the stopping times τ_n take countably many values only. The
process N satisfies the strong Markov property for τ_n : the processes \bar{N}_n , defined by

$$\bar{N}_n([0; s] \times U) := N((\tau_n; \tau_n + s] \times U),$$

537 are Poisson point processes, independent of \mathcal{F}_{τ_n} . To prove this, take k with $P\{\tau_n = \frac{k}{2^n}\} > 0$
538 and note that on $\{\tau_n = \frac{k}{2^n}\}$, \bar{N}_n coincides with process the Poisson point process $\tilde{N}_{\frac{k}{2^n}}$ given by

$$\tilde{N}_{\frac{k}{2^n}}([0; s] \times U) := N\left(\left(\frac{k}{2^n}; \frac{k}{2^n} + s\right] \times U\right), \quad U \in \mathcal{B}(\mathbb{R}^d).$$

539 Conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is again a Poisson point process, with the same intensity.
 540 Furthermore, conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is independent of $\mathcal{F}_{\frac{k}{2^n}}$, hence it is independent
 541 of $\mathcal{F}_\tau \subset \mathcal{F}_{\frac{k}{2^n}}$.

542 To prove (i), note that $\bar{N}_n((a; b) \times U) \rightarrow \bar{N}((a; b) \times U)$ a.s. and all random variables
 543 $\bar{N}_n((a; b) \times U)$ have the same distribution, therefore $\bar{N}((a; b) \times U)$ is a Poisson random variable
 544 with mean $(b - a)\lambda(U)$. The random variables $\bar{N}_n((a; b) \times U)$ are independent of \mathcal{F}_τ , hence
 545 $\bar{N}((a; b) \times U)$ is independent of \mathcal{F}_τ , too. Similarly, (ii) follows. \square

546 **Remark 6.2.** We assumed in Proposition 6.1 that there exists an increasing, right-continuous
 547 and complete filtration $\{\mathcal{S}_t\}_{t \geq 0}$ compatible with N . Let us show that such filtrations exist.

548 Introduce the natural filtration of N ,

$$\mathcal{S}_t^0 = \sigma\{N_k(C, B), B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}([0; t])\},$$

549 and let $\bar{\mathcal{S}}_t$ be the completion of \mathcal{S}_t^0 under P . Then N is compatible with $\{\bar{\mathcal{S}}_t\}$. We claim
 550 that $\{\bar{\mathcal{S}}_t\}_{t \geq 0}$, defined in such a way, is right-continuous (this may be regarded as an analog
 551 of Blumenthal's 0-1 law). Indeed, as in the proof of Proposition 6.1, we can check that \tilde{N}_a is
 552 independent of $\bar{\mathcal{S}}_{a+}$. Since $\bar{\mathcal{S}}_\infty = \sigma(\tilde{N}_a) \vee \bar{\mathcal{S}}_a$, $\sigma(\tilde{N}_a)$ and $\bar{\mathcal{S}}_a$ are independent and $\bar{\mathcal{S}}_{a+} \subset \bar{\mathcal{S}}_\infty$,
 553 we see that $\bar{\mathcal{S}}_{a+} \subset \bar{\mathcal{S}}_a$. Thus, $\bar{\mathcal{S}}_{a+} = \bar{\mathcal{S}}_a$.

554 Acknowledgement

555 Luca Di Persio would like to acknowledge the Informatics Department at the University of
 556 Verona for having funded the project "Stochastic differential equations with jumps in Mathe-
 557 matical Finance: applications to pricing, hedging and dynamic risk measure's problems" and
 558 the Gruppo Nazionale per l'analisi matematica, la Probabilità e le loro applicazioni (GNAMPA).
 559 Viktor Bezborodov (VB) is supported by the Department of Computer Science at the Univer-
 560 sity of Verona. VB was also partially supported by the DFG through the SFB 701 "Spektrale
 561 Strukturen und Topologische Methoden in der Mathematik" and by the European Commission
 562 under the project STREVCOMS PIRSES-2013-612669. Mykola Lebid (ML) is supported by
 563 the Department of Biosystems Science and Engineering at the ETH Zürich. VB, ML, TK and
 564 TO are grateful for the support of the ZIF Cooperation group "Multiscale modelling of tumor
 565 evolution, progression and growth", and Wrocław University of Science and Technology, Faculty
 566 of Electronics. TK is also supported by the National Science Center in Poland (NCN) through
 567 grant 2013/11/B/HS4/01061: "Agent based modeling of innovation diffusion".

568 References

- 569 [ADH15] A. Auffinger, M. Damron, and J. Hanson. 50 years of first passage percolation.
 570 2015. <https://arxiv.org/abs/1511.03262>.
- 571 [Bez15] V. Bezborodov. Spatial birth-and-death Markov dynamics of finite particle systems.
 572 2015. <http://arxiv.org/abs/1507.05804>.

- 573 [Big95] J. D. Biggins. The growth and spread of the general branching random walk. *Ann.*
574 *Appl. Probab.*, 5(4):1008–1024, 1995.
- 575 [BR58] C. J. Burke and M. Rosenblatt. A Markovian function of a Markov chain. *E Ann.*
576 *Math. Statist.*, 29:1112–1122, 1958.
- 577 [Dei03] M. Deijfen. Asymptotic shape in a continuum growth model. *Adv. in Appl. Probab.*,
578 35(2):303–318, 2003.
- 579 [Dur83] R. Durrett. Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete*,
580 62(2):165–170, 1983.
- 581 [Dur88] R. Durrett. *Lecture notes on particle systems and percolation*. The Wadsworth
582 & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced
583 Books & Software, Pacific Grove, CA, 1988.
- 584 [Ede61] M. Eden. A two-dimensional growth process. In *Proc. 4th Berkeley Sympos. Math.*
585 *Statist. and Prob.*, Vol. IV, pages 223–239. Univ. California Press, Berkeley, Calif.,
586 1961.
- 587 [EW03] A. Eibeck and W. Wagner. Stochastic interacting particle systems and nonlinear
588 kinetic equations. *Ann. Appl. Probab.*, 13(3):845–889, 2003.
- 589 [FM04] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regu-
590 lated population and macroscopic approximations. *Ann. Appl. Probab.*, 14(4):1880–
591 1919, 2004.
- 592 [GM08] J.-B. Gouéré and R. Marchand. Continuous first-passage percolation and continuous
593 greedy paths model: linear growth. *Ann. Appl. Probab.*, 18(6):2300–2319, 2008.
- 594 [GM12] O. Garet and R. Marchand. Asymptotic shape for the contact process in random
595 environment. *Ann. Appl. Probab.*, 22(4):1362–1410, 2012.
- 596 [HN97] C. D. Howard and C. M. Newman. Euclidean models of first-passage percolation.
597 *Probab. Theory Related Fields*, 108(2):153–170, 1997.
- 598 [Kal02] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications.
599 Springer-Verlag, second edition, 2002.
- 600 [Kes87] H. Kesten. Percolation theory and first-passage percolation. *Ann. Probab.*,
601 15(4):1231–1271, 1987.
- 602 [KK06] Y. G. Kondratiev and O. V. Kutoviy. On the metrical properties of the configuration
603 space. *Math. Nachr.*, 279(7):774–783, 2006.
- 604 [Lig85] T. M. Liggett. An improved subadditive ergodic theorem. *Ann. Probab.*, 13(4):1279–
605 1285, 1985.

- 606 [Lig99] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*,
607 volume 324. Springer-Verlag, Berlin, 1999.
- 608 [Mas98] L. Massoulié. Stability results for a general class of interacting point processes
609 dynamics, and applications. *Stochastic Process. Appl.*, 75(1):1–30, 1998.
- 610 [MW03] J. Møller and R. Waagepetersen. *Statistical inference and simulation for spatial
611 point processes*. CRC Press, 2003.
- 612 [Ric73] D. Richardson. Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*,
613 74:515–528, 1973.
- 614 [RS99] Michael Röckner and Alexander Schied. Rademacher’s theorem on configuration
615 spaces and applications. *J. Funct. Anal.*, 169(2):325–356, 1999.
- 616 [Shi15] Z. Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*.
617 Springer, Cham, 2015. Lecture notes from the 42nd Probability Summer School
618 held in Saint Flour, 2012, École d’Été de Probabilités de Saint-Flour. [Saint-Flour
619 Probability Summer School].
- 620 [TM15] D. Tartarini and E. Mele. Adult stem cell therapies for wound healing: Biomaterials
621 and computational models. *Frontiers in bioengineering and biotechnology*, 3, 2015.
- 622 [TSH⁺13] K. Treloar, M. Simpson, P. Haridas, K. Manton, D. Leavesley, S. McElwain, and
623 R. Baker. Multiple types of data are required to identify the mechanisms influencing
624 the spatial expansion of melanoma cell colonies. *BMC systems biology*, 7(1):1, 2013.
- 625 [VDPP15] B. Vo, C. Drovandi, A. Pettitt, and G. Pettet. Melanoma cell colony expansion
626 parameters revealed by approximate bayesian computation. *PLoS Comput Biol*,
627 11(12):e1004635, 2015.
- 628 [WBP⁺15] B. Waclaw, I. Bozic, M.E. Pittman, R.H. Hruban, B Vogelstein, and M.A. Nowak.
629 A spatial model predicts that dispersal and cell turnover limit intratumour hetero-
630 geneity. *Nature. Research letter*, 525:261–277, 2015.