Abstract—In the present work we consider a novel approach to model the dynamic of financial bubbles. In particular, we exploit a technique based on trinomial trees, which is mainly inspired by the typical Market Order Book (MOB) structure. We use a bottom-up approach, according to the typical MOB rules, to derive the relevant generator process for the financial quantities characterizing the market we are considering. Our proposal pays specific attention to consider the real world changes in probability levels characterizing the bid-ask preferences driven by market movements. We show that financial bubbles are, indeed, originated by these movements which, in turn, amplify their growth. Numerical experiments are provided to show the effectiveness of our results within real volatility wars scenarios. Namely, we study realistic economic frameworks characterized by volatility levels showing great fluctuations, in relatively small times.

Keywords—Bubbles, Constraints, Stochastic Dynamics, Trinomial Models.

I. INTRODUCTION

The rise of financial bubbles is a well-known phenomenon in real world markets. It substantially consists in episodes in which abnormal and unpredictable increasing of prices are followed by sudden collapses of the same.

Famous examples could be easily recognized in the past as well as during recent times. The general dynamic of such financial phenomena present rather similar characteristics, even if each bubble has its own history. In fact, the financial assets involved are part of very different economic scenarios, spanning from contracts based on natural products to exclusively speculative structured derivatives. So that, during history, we can recognize both old bubble bursts, as in the case of Dutch Tulipomania (1634/6-1637), see, e.g., [7], the Kipper- und Wipperzeit bubble, see, e.g., [27], which was triggered by the lack of an effective taxation and proper methods to identify the real value of currencies coined by different countries which aimed at raise revenue for the Thirty Years’ War, and more recent rises and sudden falls of strictly financial products, as happened during the 2007-2008 US crisis of sub-prime mortgages market, see, e.g., [10], whose effects have clearly shown the effectiveness of the so called globalisation phenomena, see, e.g., [23], namely the intimate interconnections between economic markets around the world, see also,e.g., [12] for more nested commodities related bubbles, and, e.g., [3, 13, 26], and references therein, with respect to (also potential) bubbles based on the new digital economy.

Therefore, researchers have been focused on these phenomena, rigorously studying them from the macroeconomic perspective as well as from the mathematical point of views, see, e.g., the recent papers [17, 18, 33], where the local martingale approach is used to study stock financial bubbles within both complete and incomplete markets, also providing a methodology for bubble detecting. In what follows, we focus our attention on the relation between bubbles formation and how stocks are exchanged through market order books. Inspired by the MOB structure and rules, we built up a model for stock prices. In particular we exploit a methodology based on trinomial tree. Thus, in this framework the market price is given endogenously. It is worth to mention that within the MOB scenario, characterized by a rather strict structure, a trader who wants to buy or sell a certain security can put orders in two different way: limit orders and market orders. The trader could place a limit order to buy or sell a security, or, broadly speaking, a general type of a financial instrument, at a specified price.

That kind of orders may not be executed if the price set by the trader cannot be met during the period of time in which the order is left open. Alternatively, the trader could place a market order and instantly transact the desired quantity at the best bid or at the best ask price. Additionally, a market order will cause a shock to the order book removing liquidity. At each time the MOB consists of a queue of buyers and a queue of sellers. Prices and orders arrival times, both determine the place where a new order is ordered in the MOB. The ranking procedure is of the First In First Out (FIFO) type, with respect to any specific price. When there is an overlap between the two queues, the intersection of all orders will be transacted, and a new price is formed. We will analyse the dynamic of possible bubbles in such a MOB scheme.

Another really important characteristic of stocks that we consider here is the birth of a bubble. We underline that the model we are considering are based on the macroeconomic theory that sees the great investors as triggering factors for the birth of the financial bubbles. In order to model this phenomenon we considered the financial movements, characterizing the bid-ask preferences, once some great investor put its order, shifting real world probabilities for subsequent orders.

The volatility clustering property of financial returns comes in a naturally way from the model’s dynamic. We give much more insights also on the relation between the boundedness of the MOB and the volatility clustering property. The paper is structured as follows: in Section II we provide the model setup, also by showing some examples illustrating the model’s dynamic; in Section III, we shall describe the way we model the birth and the death of financial bubbles within the endogenously determined stock price; while in Section IV, resp. in Section V, we will provide, using different probability distributions, numeri-
We consider a market consisting of a single stock which is traded through a Limit Order Book (LOB). In general LOBs have different bid-ask levels so that the resulting structure could easily become rather nested. In order to simplify this structure, we will consider just two LOB levels, namely the first bid, resp. the first ask, levels. The latter implies that traders can only exercise their orders using such two levels. We underline that this hypothesis is not unrealistic, since any more general order, can be subdivided as a sum of the aforementioned twos. Fig. 1 represents the typical LOB queue at time $n \geq 0$.

In particular, the first ask level has position 0 with quantity $q_{0,0}$ that is the quantity that can be traded at time $n$, while the first bid level has position 1, with quantity $q_{1,1}$. The bid-ask spread is fixed as $2\epsilon$, where $\epsilon > 0$ represents the distance between the actual price and the bid/ask price offers.

It follows that the LOB representation here provided is of static type, while the dynamic of the order book will be determined by orders arrival. In particular, traders put their orders at each trading day, composed by $N$ trading instants. As an example, $N$ could be the number of seconds or the number of milliseconds between the opening of the market and its closure. Then, in each trading day we have $0 \leq n \leq N$ trading moments, with orders arrivals that are randomly distributed over time. Moreover, their magnitude is random, too. Last but not least, each order could be of bid, or ask type. In what follows, we model orders arrivals supposing that at each time $0 \leq n \leq N$, an order can be placed or not with certain probabilities. Sources of randomness characterizing the LOB are

- Order’s arrival;
- Bid or ask level of the order;
- Quantity of the order.

It is worth to mention that the first two sources of randomness are binary. In fact an order can arrive or not and the order can be on bid level or ask level, While we model the last component by a probability distribution for the orders quantities. Furthermore traders can make limit orders or market orders. Market orders can only remove a specific quantity from the level chosen, because a market order is traded at the best bid or ask level. Instead, limit orders can also add quantity to the level chosen.

We will model this property of orders by assuming a symmetric probability distribution and considering the sign of the quantity as a discriminant factor. If the quantity is positive we will add quantity to the level chosen, otherwise we will subtract it.

Let us consider an example of the just introduced LOB dynamic:

- $n = 0$, $q_{n,0} = 50$, $q_{n,1} = 38$. We have an order of bid level (position 1), with quantity $-20$.
- $n = 1$, $q_{n,0} = 50$, $q_{n,1} = 18$. We have no order.
- $n = 2$, $q_{n,0} = 50$, $q_{n,1} = 18$. We have an order of ask level (position 0), with quantity $+50$.
- $n = 3$, $q_{n,0} = 100$, $q_{n,1} = 18$. We have an order of bid level (position 1), with quantity $-30$.
- $n = 4$, $q_{n,0} = 100$, $q_{n,1} = 12$. We have no order.

The last two steps of the above example produce a price change. Indeed, since the last order fills completely the bid quantity, then a new price is formed. The remaining quantity $12 = 50 - 18$, becomes the new quantity for the new bid level. Remind that a price is formed whenever the ask or the bid level is completely filled by an order, namely when there is an overlap between the bid queue and the ask queue.

In particular, at time $n + 1$, considering a price variation $\epsilon > 0$, the price $X_{n+1}$ can be $X_n + \epsilon$ or still $X_n$ if there is no overlap. For example if a trader buy $q_{n,0}$ price shares at $X_n + \epsilon$ then a new price $X_{n+1} = X_n + \epsilon$ is formed. Equivalently, if a trader sell $q_{n,1}$ price shares at $X_n - \epsilon$ then the new price $X_{n+1} = X_n - \epsilon$ is formed. In other cases a price is not formed and the resulting price will be still $X_{n+1} = X_n$. We will now make the whole structure more precise and formal. Let $(\Omega, F, \mathbb{P})$ be a probability space, and, discretizing time, let us define $o_n : \Omega \to \mathbb{R}$ as the binary random variable describing if an order is arrived at time $n$, being $0 \leq n \leq N$, for a given horizon $N \in \mathbb{N}^+$. We assume that $\mathbb{P}(o_n = 0) = \lambda = 1 - \mathbb{P}(o_n = 1)$, where $\lambda \in (0, 1)$. Moreover, we define $P_n : \Omega \to \mathbb{R}$, as the random variable describing the bid-ask level of the market order, once this is arrived. Since also $P_n$ is a Bernoulli random variable, taking $p \in (0, 1)$, we set $\mathbb{P}(P_n = 0) = p = 1 - \mathbb{P}(P_n = 1)$. Concerning the magnitude $q_n^{*}$ of the $n$-th order, we model it as follows: $q_n^{*} \sim N(\mu, \sigma^2)$. It is worth to mention that latter Gaussian-type assumption is coherent with our aim to have a benchmark, or starting point, to calibration purposes. Let us remind that once an order has arrived, and the trader has chosen its position, a negative, resp. positive, quantity corresponds to a subtraction, resp. an addition. Since the formation of a new price, when dealing with a bid order, depends on the fact that $q_n^{*} \leq q_{n,0}$, or $q_{n,1}$, then the price at time $n + 1$ is defined as follows:

$$X_{n+1} = (X_n + \epsilon) I \{ o_n = 1 \} I \{ q_n^{*} < -q_{n,0} \} I \{ P_n = 0 \} + (X_n - \epsilon) I \{ o_n = 1 \} I \{ q_n^{*} < -q_{n,1} \} I \{ P_n = 1 \} + X_n I \{ o_n = 1 \} I \{ q_n^{*} > q_{n,0} \} I \{ P_n = 0 \} + X_n I \{ o_n = 1 \} I \{ q_n^{*} > q_{n,1} \} I \{ P_n = 1 \} + \epsilon I \{ o_n = 0 \}.$$

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III. THE BIRTH OF A BUBBLE
In this section we will focus our attention on the financial bubbles
bursting mechanism, within the framework defined along the
previous section. A ever present feature of a bubble is that it is
characterized by changes that strongly deviate a market price from its intrinsic value, see, e.g., [33]. We will characterize the
birth of a bubble by the arrival of a great order in the order book.
It is worth to mention that such great orders could be triggered by
different micro as well as macro economic factors, as, e.g.,
market uncertainty, speculative politics acted by big financial
players, insider trading and mass media attacks, etc.
As a matter of fact, empirical evidences based on the analysis of a
huge quantity of time series, show that small to medium
investors are likely to suddenly change their trading intentions
when great magnitude orders hit the markets. Such type of dy-
namics for crowds, or collective aggregation, which are mostly
ethero directed have been widely analysed, also because of simil-
arities across etheroogeneous frameworks, and then applied in
population biology, particle physics, social sciences, etc., see, e.g., [2, 9, 11, 20, 22, 35], and references therein. In our setting,
the latter means that preferences on bid/ask level change as well as probabilities. In particular, we have a change of
the probability level $p$. We identify the arrival of huge orders with $q_n^*$ being under a certain threshold. Having assumed that
$q_n^* \sim N(\mu, \sigma^2)$, we choose $q_n^* \sim N(-1, 1)$, so that to have a threshold $q_n^* = -5.6$, if $N = 10000$ and the number of days of the
simulation $n^* = 252 \times 10$. The reason of the latter choice lies in
the fact that the intra-day price process simulation brings $N$ realization of the process $\{q_n^*\}_{0 \leq n < N}$ each day and for $n^* = 252 \times 10$
days, then taking $q_n^* = -5.6$ we have $N(q_n^*) < 2.11 \cdot 10^{-6}$. Is such an order arrive, then many traders start to replicate it, mem-
orizing its arrival position, in time, and then changing accordingly
the probabilities in the bid/ask level choice. We underline that
this approach has been inspired by to the one suggested in [5],
see Remark 3.3.
A further peculiarity of bubbles concerns their duration. Even
limiting our attention to the financial scenario, bubbles could last
days or months. Moreover, due to the increasing relevance of
high frequency trading, smaller bubbles have been studied with very short durations, see, e.g., [8, 19, 32]. We will focus our
study on bubbles with fixed duration $t_{vol}$, analysing their decay
dynamic. Indeed, whenever we are in a bubble regime, we can
exploit empirical data to predict the time region where the bubble
will end up. It should be underlined that, in most of the cases
the end of a financial bubble does not consist in a instantaneous
renormalization of prices, but rather it follows a renormalization
regime that may happens at slow speed. In what follows we will
consider the changes in probability until we arrive at $\frac{t_{vol}}{2}$
and after that we use reversed probabilities. As to make an
example, supposing $p = 0.5$, with a huge order arrival at time $n$,
namely $(o_n = 1, p_n = k, q_n^* < -5.6)$, $k \in \{0, 1\}$ being fixed, then for $n \in [n + 1, N]$, in the next $t_{vol}/2$ days, the probabilities of the
$P_n$ process change, so that

$$P_{n^{t_{vol}/2}}(\omega) = \left\{ \begin{array}{ll}
  k & p_1 = 0.6 \\
  1 - k & 1 - p_1 = 0.4 
\end{array} \right.$$ 

From $t_{vol}/2$ to $t_{vol}$ we invert probabilities to simulate the bubble’s
death, therefore

$$P_{n^{>t_{vol}/2}}(\omega) = \left\{ \begin{array}{ll}
  k & p_1 = 0.4 \\
  1 - k & 1 - p_1 = 0.6 
\end{array} \right.$$ 

and we will refer to this scheme as to the first scheme. Even if we
have considered instantaneous jumps in probability, to make the
example more clear, probabilities changes could happen in less
sudden ways, e.g. following some smooth dynamic of expen-
tional type. Indeed, in real world settings, bubbles are characterized
by three important phases: birth, death and a period, separating
the first twos, of more quiescent volatility levels. Such time
interval results is typically longer than the bubble’s burst and
death periods. Aiming at improving our approach, we consider the
bubble life as subdivided into four parts, as follows

$$P_{n^{<t_{vol}/4}}(\omega) = \left\{ \begin{array}{ll}
  k & p_1 = 0.6 \\
  1 - k & 1 - p_1 = 0.4 
\end{array} \right.$$ 

hence recovering a middle part where we restore the initial prob-
obabilities. Therefore, we are now exploiting a regime switching
approach

$$P_{n^{\frac{t_{vol}}{4}}}^{\frac{3t_{vol}}{4}}(\omega) = \left\{ \begin{array}{ll}
  k & p_1 = 0.5 \\
  1 - k & 1 - p_1 = 0.5 
\end{array} \right.$$
where, from $\frac{3t_{\text{vol}}}{4}$ to $t_{\text{vol}}$, we invert probabilities, to model the bubble’s death. In particular, we will have

$$P_{n}^{T_{vol}}(\omega) = \begin{cases} k & p_1 = 0.4 \\ 1-k & 1-p_1 = 0.6 \end{cases}$$

and we define such scheme as the "second scheme".

**IV. SIMULATION RESULTS**

i. Gaussian case

In this section we provide numerical results related to the methods introduced during previous sections. We consider $n^* = 252 \cdot 10$, ten financial years, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments.

We fix $\epsilon = 0.2$, and the initial price $X_0 = 100$. We suppose that the initial order book is normalized and characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n \geq 0}$, and that quantities of orders are distributed as $q_{n} \sim N(\mu = -1, \sigma^2 = 1)$, the choice of $\mu = -1$ being forced to obtain non trivial dynamics. Indeed, choosing $\mu = 0$, implies $50\%$ of orders adding/subtracting some quantity to the bid/ask level chosen.

To optimize our approach, we give a greater probability of having subtracting operation on bid/ask level, instead of addition ones. n. Regarding probabilities levels, we set $\lambda = \frac{1}{2}, p = \frac{1}{2}$.

Great orders arrive within the threshold $q_{n}^* < -5.6$ and we fixed $t_{\text{vol}} = 200$ and $p_1 = 0.515$. Figures from 2a to 3b, show the results obtained following the *first scheme*. Fig.2a shows the price process during days namely we report the last price of each simulation of the intra-day price process $\{X_n\}_{0 \leq n \leq N}$, plotting the new sequence $(X_{N_1}, X_{N_2}, X_{N_3}, \ldots, X_{N_n})$, $X_{N_i}$ which represents the last price of the $i$-th day. Furthermore, we made a simple analysis of our price as if they are given as exogenous quantities.

In figure 2b we consider the volatility process fitted with the Auto Regressive Conditional Heteroskedasticity (GARCH) method.

The logarithmic returns of prices are represented in figure 3a below

while in figure 3b we report their empirical distribution, again using the *first scheme* approach.
We would like to underline how the obtained results clearly show that the proposed model well represent realistic market dynamics. In particular, we outline how the computations highlight that logarithmic returns share the volatility clustering property. It is also worth to mention that the variables here considered are independent between each other. Hence, it seems that the volatility clustering property is intrinsic, being mainly realized by the structure of the order book, instead of being triggered by the structure of the involved random variables. Furthermore this may suggest that, in a realistic framework where order book have various bid/ask levels, as well as different characteristics due to social, economic and financial factors, it is reasonable that the volatility clustering property comes from the order book structure. It is worth to mention that the red lines which compare in figures 2a and 3a, represent the bubble’s regime. As we can see, initially the bubble burst and after $t_{vol} = 100$ days, the preferences in bid/ask level change. The graphs presented in figures from 4a to 5b, are related to simulation results obtained using the same constant aforementioned specified, but now the method used is the second scheme one.

Looking at figure 4a

we can see that, after the bubble burst, price fluctuate showing an average volatility behavior. Then, the bubble tends to its end. As before we can fit the volatility levels obtained by computations, using the GARCH method, in order to have the following graph

Fig. (3b): Distribution of logarithmic returns. First Scheme.

Fig. (4a): Price process simulated. Second Scheme.

Fig. (4b): Volatility fitted using GARCH methodologies. Second Scheme.

Fig. (5a): Logarithmic returns. Second Scheme.

While figures 5a and 5b respectively show the logarithmic returns behavior and the associated empirical distribution.
Remark 1. Aiming at deriving a more complete analysis of our approach, in what follows we consider different type of probability distributions, hence moving away from the Gaussian perspective. In particular, we will show how the flexibility of the proposed model allows to consider the dynamic of analyzed financial quantities by using different types of probability distributions. The latter permits to see how the involved priced behaviors change passing from a given random framework to another. By the very definition of our model, possible distribution choices have to be done with respect to random variable distributed over all the real line. This is motivated by the fact that quantities related to our trinomial model must have a sign. Therefore, we will discard all distribution defined just in the real positive line and we consider the Cauchy distribution and the t-Student distribution. Of course, previous options have to be handled with care, namely a great attention has to be put concerning the parameters selection. This is to have dynamics similar to the Gaussian case in regarding the presence of bubbles, their lifetime as well as concerning the probability assigned to the presence of orders of great magnitude.

ii. Cauchy case

As in the Gaussian case, we again consider $n^* = 252 \cdot 10$, namely ten financial years, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments. We fix $\varepsilon = 0.2$, and the initial price $X_0 = 100$. We suppose that the initial order book is normalized and characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n \geq 0}$, and that quantities of orders are distributed as $q_n^* \sim C(-q_{n-1}^*, 1)$, where we considered a Cauchy distribution defined as:

$$C(x_0, \gamma) = \frac{1}{\pi \gamma (1 + (\frac{x-x_0}{\gamma})^2)}, \quad (5)$$

so that we choose the location parameter to be the quantity of the previous financial moment and the scale interpreted by the gamma factor as 1. Regarding probabilities levels, we set $\lambda = \frac{1}{2}$, $p = \frac{1}{2}$. We also fix $t_{vol} = 200$ and $p_1 = 0.515$. To model great orders arrival, we then must choose a threshold in such a way that the probability of such an event results as a very rare case. In order to do this we consider $q_n^* < -150000$. Therefore, the resulting cumulative distribution function of the Cauchy distribution give us a very little probability to have a great order. Namely, we have $C(-q_n^*) = 2.12 \cdot 10^{-6}$. Figures from 6a to 7b show the computational results for the present Cauchy approach. In Fig. 6a we can see the simulation for the price dynamic, according with the second scheme while the following figure 6b is about the volatility levels.
The logarithmic returns as well as their empirical distribution, are shown in figures 7a and 7b below.

![Logarithmic returns in the Cauchy case. Second Scheme.](image)

**Fig.(7a):** Logarithmic returns in the Cauchy case. Second Scheme.

![Distribution of logarithmic returns in the Cauchy case. Second Scheme.](image)

**Fig.(7b):** Distribution of logarithmic returns in the Cauchy case. Second Scheme.

iii. t-Student case

In what follows we adopt the t-Student probability distribution to model the random quantities we are interested in. Therefore, as before, we set \( n^* = 252 \cdot 10 \), namely ten financial years, and we simulate the dynamic of our order book in \( N = 10000 \) financial intra-day moments. We also fix \( \varepsilon = 0.2 \), while the initial price is taken as \( X_0 = 100 \). We suppose that the initial order book is normalized, being also characterized by \( \{ q_{0,0} = 1, q_{0,1} = 1 \}_{n \geq 0} \). Moreover, the quantities of orders are distributed as follow \( q_n \sim t(-q_{n-1}, 1, 1) \), where we have considered the t-Student probability distribution function defined by:

\[
t(x, df, \gamma) = \frac{\gamma((df+1)/2)}{\sqrt{(\pi \cdot df) \cdot \gamma(df/2) \cdot (1 + x^2 \gamma)((df+1)/2)}} , \tag{6}
\]

therefore we choose the location parameter to be the quantity of the previous financial moment, the parameter \( df \) as 1, while is taken as \( \gamma = 1 \). Regarding probabilities levels, we set \( \lambda = \frac{1}{2} \), \( p = \frac{1}{2} \). We also fix \( t_{vol} = 200 \) and \( p_1 = 0.515 \). Great orders arrivals are modeled by considering a a threshold in such a way that the probability of the correspondent random event results as a very rare case.

The latter implies large negative values for \( q_n^* \). In particular, we consider \( q_n^* < -150000 \), so that the resulting cumulative distribution function results in a tiny probability of having a great order, namely we obtain \( t(-q_n^*) = 2.12 \cdot 10^{-6} \).

In Fig. 8a we show the price process in days. This is the last price of each simulation of the intra-day price process \( \{ X_n \}_{0 \leq n \leq N} \). In particular we plotted the new sequence \((X_{N_1}, X_{N_2}, X_{N_3}, ..., X_{N_{n^*}})\), \( X_{N_i} \) representing the last price of the \( i \)-th day.

![Price process simulated in the t-Student case. Second Scheme.](image)

**Fig.(8a):** Price process simulated in the t-Student case. Second Scheme.

while Fig.8b below represents the associated volatility levels fitted with GARCH method.

![Volatility in the t-Student case. Second Scheme.](image)

**Fig.(8b):** Volatility in the t-Student case. Second Scheme.
Moreover, in figures 9a, 9b we show the related logarithmic returns of the price, with their empirical distribution.

**Fig.(9a):** Logarithmic returns in the t-Student case. Second Scheme.

**Fig.(9b):** Distribution of logarithmic returns in the t-Student case. Second Scheme.

V. CONCLUSION

In this paper we provided a mathematical method, discrete in time, to model stocks’ bubbles dynamics, focusing the attention on the market micro-structure in terms of Market Order Book (MOB). In particular we described how micro-economic trading interactions determine a shift in probabilistic traders’ perspectives. We proposed a constructive methodology that determine endogenously price processes. We also provided a model that, starting from the changes in bid-ask preferences, determines the formation of a bubble.

Numerical simulations have been also reported to show the effectiveness of the proposed approach. Our setting is very flexible, indeed it can be adapted to a large class of financial scenario, also taking into account different types of market and socio-economics factors. The latter is also witnessed by the use of different probability distributions setting, namely spanning from the Gaussian to the t-Student one, we have considered. It is worth to mention that obtained results seem to be very stable under such kind of random changes, hence maintaining the possibility to sharp model bubbles’ dynamics. Future improvements are related to the continuous counterpart of the proposed schemes, as well as concerning its mathematical relationships with the general local martingale approach.

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