Introduction. We give a simple proof of Herbrand's Theorem for Gentzen's Calculi of Sequents in the general case, without restriction to sequents containing only prenex formulas; this proof holds, with little modifications, both for the classical calculus \( \mathcal{L} \) and the intuitionistic calculus \( \mathcal{I} \). Since we deal with the general case, we must use different techniques from Gentzen's verschärfter Hauptsatz; we follow instead Herbrand's original proof more closely.

Herbrand's Theorem is a fundamental topic in Predicate Calculus, closely connected with several other basic results, for instance Cut-Elimination Theorem, Completeness Theorem, Hilbert's definition of quantification in terms of his \( t \)-symbol and, finally, the proof procedures used in the Automatic Theorem Proving. Because of these connections, too many results are called Herbrand's Theorem today; first we give an informal account, with the attempt to make clear the connections and the differences between Herbrand's and Gentzen's results.

1. Given a formula \( A \) of Predicate Calculus, Herbrand constructs the sequence of domains \( D_1, D_2, D_3, \ldots \) whose union is called Herbrand Universe or lexicon (relatively to \( A \)) and then the expansion \( \mathcal{E}_p(A) \) over the domain \( D_p \). There are two equivalent definitions of expansion; following the most famous one, \( \mathcal{E}_p(A) \) is a disjunction of quantifier free formulas \( A_1, A_2, \ldots, A_k \) whose variables are elements of \( D_p \) or terms built up with the elements of \( D_p \).

I thank very much Prof. Dag Prawitz: he took care of me with a lot of patience. I thank also my friends in Padova, that gave me the ABC of logic: they try to study logic in the only possible way in Italy today, quite on the borders of the academic society.
Then Herbrand proves for classical logic in a Hilbert-
type system:
(a) If \( \vdash A \), then for some \( p \) \( \mathcal{E}_P(A) \) is a tautology;
(b) If for some \( p \), \( \mathcal{E}_P(A) \) is a tautology, then there is
a proof of \( A \) from \( \mathcal{E}_P(A) \) in which no use is made of Modus
Ponens.

Gentzen’s verschränker Hauptsatz gives, for classical
sequences \( S \) containing only pronex formulas:
If \( \vdash S \) then there is a cut-free proof of \( S \) of the shape

\[
\begin{array}{c}
\vdash S' \\
\vdash S
\end{array}
\]

where \( S' \) is a sequent containing no quantifiers and where
all propositional inferences are above \( S' \) and all quanti-
ficational inferences are below \( S' \).

The proof shows that it is always possible to permute
the inferences of a cut-free proof of \( S \) in order to get a
proof with this property.

Now we generalize the notion of expansion from formulas
to sequents; (by a suitable renomination of the variables in
the proof and) by adding, if necessary, some suitable quanti-
ifier free formulas to \( S' \) by Thinning, we obtain \( \mathcal{E}_P(S) \) as
midsequent; the new formulas disappear by Contraction after
the quantification of their variables. It is not fussiness to
note that, since the expansion \( \mathcal{E}_P(S) \) is generated mechan-
ically, it contains many formulas that are unnecessary in order
to get a proof of \( S \), while from Gentzen’s Theorem we get more
informations in order to single out the simplest midsequent
\( S' \); indeed Gentzen’s Hauptsatz contains an analysis of the
propositional inferences that lacks in Herbrand’s Theorem.

Herbrand’s Theorem holds for any formula \( A \) of the Predi-
cate Calculus, Gentzen’s verschränker Hauptsatz for sequents
\( S \) containing pronex formulas only; moreover Herbrand’s expan-
sion always separates the propositional and the quantificatio-
nal parts of a proof, but this last property depends on a
particularly of Herbrand’s system. In fact Herbrand assumes
among the primitive rules the so-called Rules of Passage,
allowing to move quantifiers inside and outside a formulas;
hence proofs in his system have the canonical form:

\[
\begin{array}{c}
\vdash \mathcal{E}_P(A) \\
\vdash \cdots \vdash \mathcal{E}_P(A) \\
\vdash \cdots \\
\vdash A
\end{array}
\]

where first, we quantify universally or existentially the
variables of \( \mathcal{E}_P(A) \) and second, we obtain \( A \) by applying the
Rules of Passage and then by eliminating redundant disjuncts
inside a formula (Generalized Rule of Simplification).

However the use of the Rules of Passage has a very high
price; firstly, a lot of complications arise in the proof of
the theorem because of these rules; (as Dreben and Denton
experimented when they amended an error of Herbrand [DREBEN
and DENTON 1969]); secondly, we cannot accept these rules if
we want to prove the theorem for the intuitionistic case.
Therefore we give up the Rules of Passage and consequently
the property of the missequent in the general case.

2. In the classical case, from Herbrand’s Theorem we get a
proof procedure for the Predicate Calculus; this procedure
is complete in the sense that either (i) there exists a \( p \)
such that \( \mathcal{E}_P(A) \) is a tautology, or (ii) for all \( p \), there
is an assignment of truth-values to the atomic formulas of
\( \mathcal{E}_P(A) \) such that \( \mathcal{E}_P(A) \) is false. It is well known [VAN
HEIJENBOERT 1967] that from Herbrand’s Theorem we get Comple-
teness Theorem just by showing that in the case (ii) it
holds that (iii) \( A \) is falsifiable in a denumerable model
(i.e., the set \( \bigcup_{p \in \mathbb{N}} D_p \) generated by \( A \), with a suitable inter-
pretation of the predicate letters, constants and functions
of \( A \)).

The proofs of Completeness Theorem in Gentzen’s type
calculi (see for instance [KLEENE 1967]) are very elegant and straightforward; if we consider the proof procedure sketched there, however, we have to generate mechanically the subformulas of a quantified formula, as in Herbrand. The computer scientists have tackled the problem of a practical use of these procedures by giving several 'search strategies': the aim is plainly to avoid to test all the expansion $\exists_x(A)$ for each $p$ and to consider only the part of it that is really relevant for its validity [NILSSON 1971].

3. It is clear that from Completeness Theorem, formulated in the Calculus of Sequents, we get the Cut-Elimination Theorem as a corollary. Besides we could try to derive the Hauptsatz directly from Herbrand's Theorem by the parts (a) and (b) together, if $A$ is provable with Modus Ponens, then $A$ is provable without Modus Ponens from a tautology $\exists_x(A)$ for some $p$. However no treatment is given in Herbrand's work of the Cut-Elimination for propositional logic. Obviously what we obtain in this way is only a reduction of the Cut-Elimination to the Propositional Calculus.

On the contrary, our proof of Herbrand's Theorem is highly simplified having assumed the Cut-Elimination Theorem for predicate logic also.

4. Gentzen's verschriftter Hauptsatz does not hold for LJ; as the counterexample $A(a) \forall(b) \rightarrow \exists x A(x)$ shows, this depends on the non permutability of the inferences $\exists$:right/$\forall$:left [KLEENE 1952].

Of course the theorem holds for sequents whose antecedent is empty; moreover as the succedent of an intuitionistic sequent consists of at most one formula, we know immediately that $\exists x A(x)$ has one only ancestor $A(t)$.

It is evident that (as pointed out by [BOWEN 1976]) since Herbrand's Theorem holds intuitionistically for a sequent $\rightarrow A$ with a prenex, the theorem fails in general because of the intuitionistic invalidity of the Rules of Passage. But if $A$ is prenex, a very special property of intuitionistic logic is involved, i.e. $A$ is decidable, and we do not suppose we shall prove so much when we try to prove Herbrand's Theorem for intuitionistic logic.

5. In order to do this, we use the alternative notion of expansion, defined by induction on the construction of the formulas. Then our method is the following: given a proof $(\alpha)$ in $\mathbb{L}K (\mathcal{L}J)$ of $S$ we construct, by induction on the length of $(\alpha)$, a proof $(\beta)$ of $\exists_x(S)$ for some $p$ in the propositional part of $\mathbb{L}K (\mathcal{L}J)$, and viceversa.

$$
\begin{array}{c}
(\alpha) \\
\forall \rightarrow \\
S \\
\exists_x(S) \\
\exists_x(S)
\end{array}
\iff
\begin{array}{c}
(\beta) \\
\forall \rightarrow \\
\exists_x(S) \\
\exists_x(S)
\end{array}
\iff
\begin{array}{c}
(\gamma) \\
\forall \rightarrow \\
\exists_x(S) \\
\exists_x(S)
\end{array}
$$

However, we cannot pass from any propositional proof $(\gamma)$ of $\exists_x(S)$ to a proof of $S$ in $\mathbb{L}K (\mathcal{L}J)$; in order to construct such a proof of $S$ we need to make the induction on a suitable proof $(\beta)$ where the inferences are in a certain order, so that the applications of $\forall$: right and $\exists$: left can be carried out accordingly with the restrictions on the eigenvariables. By the Permutability Theorem [KLEENE 1952], in the classical case from any proof $(\gamma)$ we can get a suitable proof $(\beta)$; in the intuitionistic case, only from proofs that satisfy a certain condition and that we call adequate. It is easy to see that a propositional proof that is constructed by our method from a quantificationals proof in $\mathcal{L}J$ is always adequate.

It would be very interesting to express the peculiarity of the intuitionistic case by a condition on the expansions themselves instead of a condition on their proofs, i.e. to establish which kind of expansions do not have an adequate proof. We were unable to do this.

6. A proof of Herbrand's Theorem for intuitionistic logic in a manuscript of Beth (1956) is mentioned by [KRESEL 1958]. We were not able to find this proof.

In the literature Herbrand's Theorem is considered a
a classical result that does not hold for intuitionistic
logic. The whole idea of Herbrand’s expansion is consid-
ered a finitistic version of model-theoretic concepts so that
Herbrand’s Theorem seems to be senseless without the clas-
sical notion of truth (see for instance the edition of [HER-
BRAND 1971] by Goldfarb).

On the contrary our proof shows that any reference to
the classical notion of truth is unnecessary for Herbrand’s
Theorem.

**Definitions.** Negation is defined ($\neg A$ is $A \downarrow$). We
denote always a sequent by $\Gamma \rightarrow \Delta$, where, for the intu-
itionistic case, $\Delta$ must contain at most one formula. We disregard
the structural rules Contraction and Exchange, but it is
intended that we are always able to find the ancestors and
the descendants of a formula in a proof (as it is required for
the proof of the Permutability Theorem). Therefore our
only structural rule is Thinning (left and right). We assume
that all the top sequents contain atomic formulas only.

Let us consider only sequents which contain no variable
occurring both free and bound, and which contain no two
occurrences of quantifiers with the same variable.

We define in a standard way the positive [negative]
ocurrences of a subformula in a sequent $\Gamma \rightarrow \Delta$. If $A$ belongs
to $\Delta$ then $A$ is positive; if $B$ belongs to $\Gamma$ then $B$ is nega-
tive. If $\land D$ or $\lor D$ or $\forall xD(x)$ or $\exists xD(x)$ are positive [negative] then
$C$ and $D$ or $C(t)$ are positive [negative]. If $\forall D$ is
positive [negative] then $D$ is positive [negative] and $C$ is
negative [positive].

Following Herbrand, we call a bound variable $x$ and its
quantifier $\exists x$ restricted if $\exists x$ is existential [universal] and
its scope $\exists xC(x)$ is positive [negative]; a variable $y$ and
its quantifier $\forall y$ (if any) are general either if $y$ is free
or if $\forall y$ is universal [existential] and its scope $\forall yD(y)$ is
positive [negative].
Here \( \bigwedge_{t \in D} \mathcal{E}_p(C(t)) \) \( \bigwedge_{t \in B} \mathcal{E}_p(C(t)) \) is the finite disjunction of all the formulas that result from \( \mathcal{E}_p(C(x)) \) by replacing a \( t \in D \) for \( x \).

Let \( y \) be a general variable and let \( QyD(y) \) be its scope in \( S \). Note that in \( \mathcal{E}_p(S) \) several subformulas \( \mathcal{E}_p(QyD(y)) \) can correspond to \( QyD(y) \), each of them having a different functional term in the place of \( y \). We shall call these functional terms the functional terms of \( y \).

Any sequence \( S_1, \ldots, S_k \) of consecutive sequents in a branch of the proof-tree will be called a fragment (of the proof-tree).

Let \( S \) be any sequent containing a subformula \( QyD(y) \), with \( y \) general; let \( \mathcal{E}_p(S) \) be the \( p \)-th expansion of \( S \) and let \( \mathcal{E}_p(D(t_i)) \) be an expansion of \( QyD(y) \) in \( \mathcal{E}_p(S) \).

Now let us consider any cut-free proof \( \langle y \rangle \) of \( \mathcal{E}_p(S) \). A fragment \( S_1, \ldots, S_k \) of \( \langle y \rangle \) is crucial for \( \langle y \rangle \) (the quantification of the variable \( t_i \)) if \( \mathcal{E}_p(D(t_i)) \) occurs just once in each sequent \( S_1, \ldots, S_k \) of the fragment, but only in \( S_i \) as the principal formula of a rule application and only in \( S_k \) as the side formula of a rule application \( \mathcal{A}_k \). Call \( \mathcal{A}_k \) crucial rule application for \( t_i \).

The end of this definition is clear: when we pass from a proof \( \langle a \rangle \) of \( S \) to a proof of its expansion \( \mathcal{E}_p(S) \), no inference corresponds in the new proof to any \( \forall \)-right or \( \exists \)-left application in \( \langle a \rangle \). Conversely, when we pass from a proof \( \langle y \rangle \) of \( \mathcal{E}_p(S) \) to a proof of \( S \) we do not find any instruction in \( \langle y \rangle \) for the \( \forall \)-right and \( \exists \)-left applications, but we know that such an inference with \( QyD(y) \) as principal formula can occur only in the part of the new proof corresponding to the crucial fragment for the variable \( t_i \).

It can happen that there are several crucial fragments for \( t_i \) but only because of a branching in the proof. Note that if different occurrences of the same formula \( \mathcal{E}_p(D(t_i)) \) are conlicated, the sequent \( S_i \) of the crucial fragment for \( t_i \) is the sequent that contains just one occurrence of \( \mathcal{E}_p(D(t_i)) \) as principal formula of the Contraction.

If in a proof \( \langle y \rangle \) of \( \mathcal{E}_p(S) \) some \( \bigwedge_{t \in D} \mathcal{E}_p(C(t)) \) or \( \bigwedge_{t \in B} \mathcal{E}_p(C(t)) \) comes from \( \mathcal{E}_p(C(t_i)) \) by repeated \( \forall \)-right or \( \exists \)-left applications then a critical fragment for \( t_i \) is defined to be the fragment of \( \langle y \rangle \) containing all the ancestors of \( \mathcal{E}_p(C(t_i)) \) in which \( t_i \) occurs.

Preliminaries. This is the basic condition for the "if" part of the theorem, both in the classical and intuitionistic cases:

(\( X \)) A crucial fragment for the quantification of \( t_i \) is not included in a critical fragment for \( t_i \).

It is easy to see that if the condition (\( X \)) holds for any \( t_i \), then there is always in the fragment of the new proof corresponding to the crucial fragment for \( t_i \) a sequent where \( t_i \) occurs just once; at this point we can make the required \( \forall \)-right or \( \exists \)-left application accordingly with the restrictions on the eigenvariable.

By the Formulatability Theorem, in the classical case we can always permute two propositional inferences: so from any proof \( \langle y \rangle \) of \( \mathcal{E}_p(S) \) we can obtain a proof \( \langle b \rangle \) having the property (\( X \)) for all the crucial fragments, just by shifting the crucial inference for any \( t_i \) below all critical fragments for \( t_i \).

But we have to show that there is a consistent procedure for making these permutations, i.e. a procedure that does not contain contradictory instructions.

In the classical case it can happen that a crucial fragment for \( t_i \) must be included in a critical fragment for \( t_j \) only when these conditions occur: \( t_j \) is a functional term of the variable \( y \), \( t_j \) takes the place of the variable \( x \) and in \( S \) the scope of the quantifier \( Qy \) is included in the scope of the quantifier \( Qx \).

By adapting an idea of Herbrand, we make this link explicit as follows.
Any array of functional terms preceded by a sign \( + \) or \( - \) and, possibly, connected with braces, will be called a schema. We construct the schema of a proof \( \gamma \) of \( \varphi(S) \) according to the following instructions:

i) if there is in \( \gamma \) a crucial fragment for \( t_1 \), write \( + t_1 \) in the schema;

ii) if there is in \( \gamma \) a critical fragment for \( t_2 \), write \( - t_2 \) in the schema;

iii) if the scope of the quantifier \( Qz \), \( z \) corresponding to \( \pm t_1 \) lies in the scope of the quantifier \( Qw \), \( w \) corresponding to \( \pm t_2 \), then write \( + t_1 \) on the right of \( - t_2 \);

iv) if two functional terms correspond to two disjoint quantifiers, then one term is below the other.

A brace can be introduced in order to make clear the dependence of several terms on one term.

For instance

\[
\{+y_1, -y_1, +y_2[1], -y_2[1], +y_2[y_2[1]], -y_2[y_2[1]]\}
\]

is the schema of the following proof of \( \varphi(S) \) with

\[
S: \forall y_1 \exists x_1 \forall y_2 \exists x_2 [p(x_1, y_1) \lor p(y_2, x_2)]
\]

We can easily establish a linear order between the functional terms of the array by ordering the lines of the schema as follows: let \( t_1, \ldots, t_k \) and \( t_1', \ldots, t_h' \) be all the negative terms of two lines \( L_1 \) and \( L_2 \) and let \( n_1, \ldots, n_k \) and \( m_1, \ldots, m_h \) be the numbers of order of these negative terms. Then \( L_1 \) precedes \( L_2 \) if

i) either \( \max(n_1, \ldots, n_k) < \max(m_1, \ldots, m_h) \)

ii) or, if \( \max(n_1, \ldots, n_k) = \max(m_1, \ldots, m_h) \), then \( n_1 < m_1 \) precedes \( m_1, \ldots, m_h \) in the lexicographical order.

In our example, as \( D_1 = \{y_1\} \), \( D_2 = D_1 \cup \{y_1, y_2[1]\} \), the linear order of the terms of the schema is given by the sequence:

\[+y_1, -y_1, +y_2[1], -y_2[1], +y_2[y_2[1]], -y_2[y_2[1]], -1\]

Now it is clear that we can permute the fragments in such a way that a fragment connected with the terms \( t \) is above all the fragment connected with the terms on the left of \( t \).

Moreover it is clear that the proof obtained by these permutations necessarily satisfies the condition \( \# \): indeed if in the sequence there are two occurrences of the same term with different signs, then the rightmost occurrence has the sign \( - \).

In the intuitionistic case there are the following exceptions to the permutability of propositional inferences: we cannot shift the following upper inferences \( \alpha_a \) below the lower one \( \alpha_b \):

\[\alpha_a \vdash \text{left} \quad \alpha_a \vdash \text{left or right} \quad \alpha_a \vdash \text{right} \quad \alpha_b \vdash \text{left}\]

In this case we cannot obtain from any proof \( \gamma \) a proof \( \beta \) satisfying the condition \( \# \). Let us suppose that in an intuitionistic proof \( \gamma \) a crucial fragment for the quantification of \( t_1 \) is included in a critical fragment for \( t_1' \).
Then we can shift the crucial inference $A_N$ below the critical fragment only if it is not the case that
i) $A_N$ is $v$-left and any application of $v$-right or of $v$-left occurs in the critical fragment below $A_N$.
ii) $A_N$ is $v$-right and any application of $v$-left occurs in the critical fragment below $A_N$.

Let us say that an intuitionistic proof $(\beta)$ of $E_p(s)$ is adequate if $(\beta)$ satisfies the condition $(\omega)$.

Then our procedure for the "if" part of the theorem in the intuitionistic case is the following. Given a sequent $S$ and its $p$-th expansion $E_p(s)$, for any $p$, first, by Gentzen's decision procedure for the propositional part of $L$, search for a proof of $E_p(s)$. If for some $p$ there is any proof $(\gamma)$ of $E_p(s)$, then consider if $(\gamma)$ is adequate, or if from $(\gamma)$ an adequate proof $(\beta)$ can be obtained by suitable permutations.

For an instructive example, consider the following classically but not intuitionistically provable sequent $S$: $\forall x(A(x) \lor B) \rightarrow \forall y(A(y) \lor B)$, where, for the sake of simplicity, $A(x)$ and $B$ are atomic. Look at the following proof $(\gamma)$ of $E_2(s)$:

\[
\begin{align*}
&v\text{-left} & &v\text{-right} & &B \rightarrow B \\
& \quad A(y) \rightarrow A(y) & & \quad A(y) \rightarrow A(y) \lor B & & B \rightarrow A(y) \lor B \\
&v\text{-left} & &v\text{-right} & &A(y) \lor B \rightarrow A(y) \lor B \\
& & & & & (A(y) \lor B) \rightarrow A(y) \lor B \\
& & & & & (A(y) \lor B) \rightarrow A(y) \lor B
\end{align*}
\]

Then we can shift the crucial inference $A_N$ below the critical fragment only if it is not the case that
i) $A_N$ is $v$-left and any application of $v$-right or of $v$-left occurs in the critical fragment below $A_N$.
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\[
\begin{align*}
&v\text{-left} & &v\text{-right} & &B \rightarrow B \\
& \quad A(y) \rightarrow A(y) & & \quad A(y) \rightarrow A(y) \lor B & & B \rightarrow A(y) \lor B \\
&v\text{-left} & &v\text{-right} & &A(y) \lor B \rightarrow A(y) \lor B \\
& & & & & (A(y) \lor B) \rightarrow A(y) \lor B \\
& & & & & (A(y) \lor B) \rightarrow A(y) \lor B
\end{align*}
\]

Here the crucial fragment for $y$ (i.e., the highest most sequent of the left branch) is included in a critical fragment for $x$. In the classical case we can shift the crucial inference $v$-right at the bottom of the proof, but in the intuitionistic case we cannot shift this inference below $v$-left. Therefore the proof is intuitionistic, but not adequate.

Herbrand's Theorem. For all classical sequents $S$

\[
\frac{}{L^K} S
\]

if and only if there exists a $p$ such that $E_p(s)$ is provable in the propositional part of $L^K$.

For all intuitionistic sequents $S$

\[
\frac{}{L^I} S \text{ if and only if there exists a } p \text{ such that } E_p(s) \text{ is provable with an adequate proof in the propositional part of } L^I.
\]

PROOF. (If). By the preliminary discussion we consider both for the classical and the intuitionistic cases only proofs $(\beta)$ that satisfy the condition $(\omega)$. The proof is by induction on the length of $(\beta)$.

Clearly we have only to take into account the inferences of $(\beta)$ in which a subformula of the shape $E_p(q \exists x(x))$ or $E_p(q \forall y(y'))$ (with $x$ restricted and $y$ general) is firstly introduced as (a part of) the principal formula, as in the other cases nothing as to be changed.

Case I. If the expansion of a quantified formula is (a part of) a formula $E_p(A)$ and
i) $E_p(A)$ is the principal formula of a Thinning, or
ii) $E_p(A) \lor E_p(B)$ is the principal formula of a $v$-right $v$-left whose side formula is $E_p(A)$, then

1) Introduce $B$ by Thinning.

2) Introduce $A \lor B$ from $A$ as side formula that is given by induction hypothesis.

Case II. $E_p(q \exists x(x))$ is introduced from $E_p(t)$ by repeated applications of $v$-right $v$-left; then introduce $q \exists x(x)$ by just an application of $\exists$-right $v$-left instead of these repeated propositional inferences.

Case III. The principal formula is $E_p(q \forall y(y'))$, i.e.,

$E_p(t_i)$, so that the crucial fragment for $t_i$ starts. Then we continue the construction accordingly with the precedent cases but we know that at a certain sequent $S_N$ of the new proof corresponding to a sequent $S_N$ of the crucial fragment we have to introduce $q \forall y(y')$ from $B(t_i)$ as side formula.

We know that $(\beta)$ satisfies the condition $(\omega)$. (A critical fragment for $t_i$ could begin inside the crucial fragment, because of an introduction of a formula containing $t_i$ by
Thinning, but this case is treated as the case I). So let $S_\downarrow$ be the first sequent of the crucial fragment such that all critical fragments for $t_1$ and above it. We show that the corresponding sequent $S_\downarrow$ satisfies the conditions on the eigenvariable $t_1$.

Note that we use in the new proof the same names for the free variables as in (3); but (3) is cut-free and because of the Subformula Property the variables that occur in the proof occur in $\xi_p(S)$ also.

Consider now any term $t_j$ occurring in $S_\downarrow$.

If in $\xi_p(S)$ $t_j$ is the functional term of a general variable $y^*$ different from $y$, then certainly $t_j \neq t_1$.

If in $\xi_p(S)$ $t_j$ takes the place of a restricted variable $x$ of $S$, then $S_\downarrow$ belongs to a critical fragment for $t_j$, so that necessarily $t_j \neq t_1$. (Indeed, let $\xi_p(C(x))$ be the scope of the restricted quantifier $\xi$; if $\xi_p(D(t_1))$ is included in $\xi_p(C(t_1))$ then $t_1$ is of the shape $y^*[t_1, \ldots, t_n, t_j]$; if $\xi_p(D(t_1))$ was included in $\xi_p(C(t_1))$, it would have already disappeared in the new proof; if $\xi_p(D(t_1))$ and $\xi_p(C(t_1))$ are disjoint, $t_j \neq t_1$ is true by the condition (5)).

(Only if). We need the following lemma:

**Lemma I.** If $\not\vdash \xi_p(S)$, then $\vdash \xi_p(S)$, where $\xi_p(S)$ is the $p$-th expansion of $S$ over $D_p$, and $\xi_p(S)$ is an expansion of $S$ over a $D_p$ such that $D_p \subseteq D_\downarrow$.

The proof is by induction on the cut-free proof ($\xi$) of $\xi_p(S)$.

The Theorem is proved by induction on the length of the cut-free proof ($\xi$) of $S$. For the induction step we define a strong analyzing function for a rule of inference (see [DREHER, DENTON and ANDERSON 1963]). The primitive recursive function $\tau$ is a strong analyzing function for the rule $\xi$, if the following condition is satisfied: if $S$ comes from $S_1$ and $S_2$ by the rule $\xi$, and if $\xi_p(S_1)$ and $\xi_p(S_2)$ is [are] the provable

expansion [$\xi$] of the upper sequent [$\xi$] $S_1$ and $S_2$, then

$\xi_p(\tau_p(S)) = \xi_p(\tau_p(S_1), \xi_p(S_2))$ is a provable expansion of the lower sequent $S$.

We shall show that there exist strong analyzing functions for all the rules of inference of $\mathcal{N}$ (without Cut). This proves the Theorem, as the basis of the induction is trivial.

It is easy to see that Identity is a strong analyzing function for all the rules with one premise, except $\xi$ right and $\forall$ left; by using Lemma I we see that $\max(p, q)$ is a strong analyzing function for all the rules with two premises (except Cut).

**Lemma II.** Successor is a strong analyzing function for $\forall$ left and $\exists$ right.

Let $S_1 \vdash \tau_\xi(A(t)) \rightarrow \Delta$ or $S_1 \vdash \tau_\xi(A(t)) \rightarrow \Delta$ (where $\Delta$ is empty in the case $S_1 \vdash \tau_\xi(A(x)) \rightarrow \Delta$ or $S_1 \vdash \tau_\xi(A(x)) \rightarrow \Delta$ intuitionistic case)

be an application of $\forall$ left or $\exists$ right; by induction hypothesis we have a proof ($\xi$) of $\tau_\xi(S_1)$.

We must distinguish the cases: if $t$ occurs or if $t$ does not occur in $S$. If not, replace everywhere in ($\xi$) the numeral 1 for $t$. Now $\tau_\xi$ occurs both in $D_1^{S_1}$ and in $D_2^{S_1}$ and the following argument holds again.

If some general quantifier lies in the scope $\forall_\xi A(x)$ of the restricted $\xi$ introduced by this rule application, then the set of the index functions of $S_1$ is different from that of $S$, the former having some function $y_1[x_1, \ldots, t_n, t_1]$ where the latter has $y_1[x_1, t_1, \ldots, t_n]$. (We consider this case only; otherwise the proof is trivial). So $D_1^{S_1} \neq D_2^{S_1}$ for $i \geq 2$.

But now substitute everywhere in ($\xi$) $y_1[x_1, \ldots, t_n, t_1]$ for $y_1[x_1, \ldots, t_n, t_1]$. We obtain a proof of the expansion $\xi_p(S_1)$ over a domain $D_\downarrow$ such that $D_\downarrow \subseteq D_\downarrow^{S_1}$. For instance, take the case of a variable $y$ whose index function is $y$ in $S_1$ and becomes $y[x]$ in $S$. So if the terms

$y_1, y_2, \ldots, y_n, y_1[\ldots, y_n, \ldots]$, $y_1[\ldots, y_1[\ldots, \ldots]]$, ...

that belong to $D_1^{S_1}, D_2^{S_1}, \ldots$, occur in $\xi_p(S_1)$, then
Now by the Lemma I and by repeated applications of \( \text{left} \) or viright we get a proof of \( \varepsilon_{p+1}(s) \).

It is immediate to see that the proof \( (\beta) \) of \( \varepsilon_p(s) \) obtained by this procedure satisfies the condition \( (\delta) \), so that in the intuitionistic case \( (\beta) \) is adequate.

REMARK. The "only if" part of the proof is very similar to the original exposition of Herbrand. Lemma II is similar also to the Corollary 7(ii) of the Normal Form Theorem for Intuitionistic Logic in [PRAWITZ 1965].

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