Subnets of Proof-nets in $\text{MLL}^-$

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Abstract

The paper studies the properties of the subnets of proof-nets. Very simple proofs are obtained of known results on proof-nets for $\text{MLL}^-$, Multiplicative Linear Logic without propositional constants.

1. Preface

The theory of proof-nets for $\text{MLL}^-$, multiplicative linear logic without the propositional constants 1 and $\perp$, has been extensively studied since Girard's fundamental paper [9]. The improved presentation of the subject given by Danos and Regnier [3] for propositional $\text{MLL}^-$ and by Girard [7] for the first-order case has become canonical: the notions are defined of an arbitrary proof-structure and of a 'context-forgetting' map $\Phi$ from sequent derivations to proof-structures which preserves cut-elimination; correctness conditions are given that characterize proof-nets, the proof-structures $R$ such that $R = (\mathcal{D})^*$, for some sequent calculus derivation $\mathcal{D}$. Although Girard's original correctness condition is of an exponential computational complexity over the size of the proof-structure, other correctness conditions are known of quadratic computational complexity.

A further simplification of the canonical theory of proof-nets has been obtained by a more general classification of the subnet of a proof-net. Given a proof-net $\mathcal{R}$ and a formula $A$ in $\mathcal{R}$, consider the set of subnets that have $A$ among their conclusions, in particular the largest and the smallest subnet in this set, called the empire and the kingdom of $A$, respectively. One must give a construction proving that such a set is not empty: in Girard's fundamental paper a construction of the empires is given which is linear in the size of the

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proof-net. When the notion of kingdom is introduced, the essential properties of proof-nets – including the existence of a sequent derivation D such that \( R = (D)^- \) (Theorem 1, sequentialization theorem) – can be easily proved using simple properties of the kingdoms and empires, in particular the fact that the relation X is in the kingdom of Y is a strict ordering.

Moreover the map \((\cdot)^-\) identifies equivalence classes of sequent derivations, where \( D_i \) and \( D_j \) are equivalent if they differ only for permutations of inferences. Now consider the set of derivations \( B \) which have \( A \) as a conclusion, and that are subderivations of some derivation \( D_i \) in an equivalence class. The kingdom and the empire of a formula \( A \) in the proof-net \((D_i)^-\) yield the notions of the minimum and the maximum, respectively, in such a set of subderivations (Theorem 2). This fact gives evidence that the notions is question do not depend on accidental features of the representation; therefore satisfactory generalizations of our results to larger fragments or to other logics should include Theorem 2.

Such a generalization is impossible in any logic with any form of Weakening, e.g., in the fragment MLL of multiplicative linear logic with the rule for the constant \( \bot \). Indeed a minimal subderivation in which a formula \( A \) may be introduced by Weakening is an axiom; but the process of permuting Weakening upwards in a derivation is non-deterministic and does not always identify a unique axiom as the minimum in our set of subderivations; hence in such a logic we cannot have a meaningful notion of kingdom.

2 Proof Nets for Propositional MLL\(^-\)

We give a simple presentation of the well-known basic theory of proof nets for Multiplicative Linear Logic without propositional constants (MLL\(^-\)). The main novelty is the use of the structural properties of subnets of a proof-net, in particular the tight relations between kingdoms and empires. A pay-off is a simple and elegant proof of the following theorems:

Theorem 1. There exists a “context-forgetting” map \((\cdot)^-\) from sequent derivations in MLL\(^-\) to proof nets for MLL\(^-\) with the following properties:
(a) Let \( D \) be a derivation of \( \Gamma \) in the sequent calculus for MLL\(^-\); then \((D)^-\) is a proof net with conclusions \( \Gamma \).

(b) (Sequentialization) If \( R \) is a proof net with conclusions \( \Gamma \) for MLL\(^-\) then there is a sequent calculus derivation \( D \) of \( \Gamma \) such that \( R = (D)^- \).

(c) If \( D \) reduces to \( D' \), then \( D^- \) reduces to \( (D')^- \).

(d) If \( D^- \) reduces to \( R' \) then there is a \( D' \) such that \( D \) reduces to \( D' \) and \( R' = (D')^- \).

Theorem 2. (Permutability of Inferences)
(i) Let \( D \) and \( D' \) be a pair of derivation of the same sequent \( \Gamma \) in propositional MLL\(^-\). Then \((D)^- = (D')^-\) if and only if there exists a sequence of derivations \( D = D_1, D_2, \ldots, D_n = D' \) such that \( D_i \) and \( D_{i+1} \) differ only for a permutation of two consecutive inferences.

(ii) Let \( R \) be a proof-net and let \( A \) be a formula occurrence in \( R \). Then there exists a derivation \( D \) with \((D)^- = R \) and a subderivation \( B \) of \( D \) such that \((B)^- = eA \). A similar statement holds for \( eA \).

2.1 Propositional Proof Structures and Proof Nets

A link is an \( m+n \)-ary relation between formula occurrences, for some \( m, n \geq 0, m + n \neq 0 \). Suppose \( X_1, \ldots, X_m \) are in a link: if \( m > 0 \), then \( X_1, \ldots, X_m \) are called the premises of the link; if \( n > 0 \), then \( X_{m+1}, \ldots, X_{m+n} \) are called the conclusions of the link. If \( m = 0 \), the link is called an axiom link. Links are graphically represented as

\[
\begin{array}{c}
X_1, \ldots, X_m \\
X_{m+1}, \ldots, X_{m+n}
\end{array}
\]

We consider links of the following forms:

**Identity Links:**

\[
\text{axiom links: } \begin{array}{c}
A \\
A^\bot
\end{array}
\]

**Cut Links:**

\[
\text{cut links: } \begin{array}{c}
A \\
A^\bot
\end{array}
\]

**Multiplicative Links:**

\[
\text{times links: } \begin{array}{c}
A \\
A \otimes B
\end{array}
\]

**Par Links:**

\[
\text{par links: } \begin{array}{c}
A \\
A \otimes B
\end{array}
\]

Convention. We assume that the logical axioms and cut links are symmetric relations. Other links are not regarded as symmetric. The word “cut” in a cut link is not a formula, but a place-holder; following common practice, we may sometimes omit it.
Definitions 1. (i) A proof structure $S$ for propositional MLL$^-$ consists of
1. a nonempty set of formula-occurrences together with (ii) a set of identity
occurrences, multiplicative links satisfying the properties:
2. Every formula-occurrence in $S$ is the conclusion of one and only one link;
3. Every formula-occurrence in $S$ is the premise of at most one link.

We write $X \leftarrow Y$ if $X$ is a hereditary premise of $Y$; in this case we also say
that ‘$X$ is above $Y$’. We shall draw proof structures in the familiar way as
non-empty, not necessarily planar, graphs.

(ii) We define the following reductions on propositional MLL$^-$ proof structures:

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
Y & X & X^* & Y^* \\
X \otimes Y & X^* \otimes Y^* & X^1 \otimes Y^1 & X \otimes Y^1 & X^1 \otimes Y \\
\end{array}
\]

reduce to

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
Y & X & X^* & Y^* \\
X \otimes Y & X^* \otimes Y^* & X^1 \otimes Y^1 & X \otimes Y^1 & X^1 \otimes Y \\
\end{array}
\]

reduce to

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
Y & X & X^* & Y^* \\
X \otimes Y & X^* \otimes Y^* & X^1 \otimes Y^1 & X \otimes Y^1 & X^1 \otimes Y \\
\end{array}
\]

reduce to

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
Y & X & X^* & Y^* \\
X \otimes Y & X^* \otimes Y^* & X^1 \otimes Y^1 & X \otimes Y^1 & X^1 \otimes Y \\
\end{array}
\]

reduce to

Symmetric Reductions

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\
Y & X & X^* & Y^* \\
X \otimes Y & X^* \otimes Y^* & X^1 \otimes Y^1 & X \otimes Y^1 & X^1 \otimes Y \\
\end{array}
\]

reduce to

Definitions 2. Let $R$ be a propositional proof structure for MLL$^-$. 

(i) A Danos-Regnier switching $s$ for $R$ consists in the choice for each $par$ link
$\varepsilon$ in $R$ of one of the premises of $\varepsilon$.

(ii) Given a switching $s$ for $R$, we define the undirected Danos-Regnier graph
$s(R)$ as follows:

- the vertices of $s(R)$ are the formulas of $R$;
- there is an edge between vertices $X$ and $Y$ exactly when:

1. $X$ and $Y$ are the conclusions of a logical axioms or the premises of a cut
link; or
2. $X$ is a premise and $Y$ the conclusion of a times link; or else

3. $Y$ is the conclusion of a $par$ and $X$ is the occurrence selected by the
switching $s$.

Definition 3. Let $R$ be a multiplicative proof-structure. $R$ is a proof-net for
propositional MLL$^-$ if for every switching $s$ of $R$, the graph $s(R)$ is acyclic
and connected (i.e., an undirected tree).

2.2 Subnets

Definitions 4. Let $m : S \rightarrow R$ be any injective map of MLL$^-$ proof structures
(regarded as sets of formula occurrences) such that $X$ and $m(X)$ are
occurrences of the same formula.

(i) We say that $m$ preserves the links if for every $\varepsilon$ in $S$ there is a link $\varepsilon'$ in
$R$ of the same kind such that

\[
\varepsilon : \frac{X_1, \ldots, X_k}{X_{k+1}, \ldots, X_{k+n}} \quad \Rightarrow \quad \varepsilon' : \frac{mX_1, \ldots, mX_k}{mX_{k+1}, \ldots, mX_{k+n}}
\]

(ii) A proof-structure $S$ is a substructure of a proof-structure $R$ if there is an
injective map $i : S \rightarrow R$ preserving links. If $S$ is a substructure of $R$, then the
lowermost formula occurrences of $S$ are also called the doors of $S$.

(iii) We write $st(\Sigma)$ for the smallest substructure of $R$ containing $\Sigma$.

(iv) A subnet is a substructure which satisfies the condition of proof-nets.

Remark. In definition 4.(ii) let $i$ be the identity map. A subnet $S$ of $R$ (with
the links of $R$ holding among the occurrences in $S$) is a substructure if and only if

1. $S$ is closed under hereditary premises and
2. if $X_i \leftarrow X_j$ is an axiom and $X_i \in S$ then $X_{i+1} \in S$.

In particular, the set of formula occurrences in $st(\Sigma)$ consists of $\Sigma$, of all the
hereditary premises of $\Sigma$ and of the axioms above them:

\[
\text{st}(\Sigma) = \bigcup_{Z \in \Sigma} \{ X \mid X \leq Z \} \cup \bigcup_{Z \in \Sigma} \{ X \in \overline{X} \mid Y \leq Z \}
\]

Lemma 1. Let $R_1$ and $R_2$ be subnets of the proof net $R$. Then

(i) $S = R_1 \cup R_2$ is a subnet if and only if $R_1 \cap R_2 = \emptyset$.

(ii) If $R_1 \cap R_2 = \emptyset$, then $R_0 = R_1 \cap R_2$ is a subnet.

Proof. Let $R$ be a proof net and $R'$ any substructure. Given a switching $s'$ for
$R'$, extend $s'$ to a switching $s$ for $R$; then $s' \cup s$ is a subgraph of $sR$, hence $s' \cup s$
is acyclic, since $s\mathcal{R}$ is. Therefore we need only to consider the connectedness of $s\mathcal{S}$ and $s\mathcal{R}_0$.

To prove (i), assume $\mathcal{R}_1$ and $\mathcal{R}_2$ are subnetts with nonempty intersection and fix a switching $s$ for $S = \mathcal{R}_1 \cup \mathcal{R}_2$. For $i = 1, 2$ let $s\mathcal{R}_i$ be the restriction of $s\mathcal{R}$ to $\mathcal{R}_i$; then $s\mathcal{R}_i$ is connected since $\mathcal{R}_i$ is a subnet. Let $A$ be in $\mathcal{R}_1$ and $B$ in $\mathcal{R}_2$; if $C \in \mathcal{R}_1 \cap \mathcal{R}_2$, then $A$ is connected with $C$ since $s\mathcal{R}_1$ is connected and $B$ is connected with $C$ since $s\mathcal{R}_2$ is connected, hence $A$ is connected with $B$ as required. The converse is immediate, namely, if $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$, then any Danos-Regnier graph on $\mathcal{R}_1 \cup \mathcal{R}_2$ is disconnected.

To prove (ii), let $s_0$ be a switching for $\mathcal{R}_0 = \mathcal{R}_1 \cap \mathcal{R}_2$; let $s_1, s_2$ be extensions of $s_0$ to $\mathcal{R}_1$, $\mathcal{R}_2$, respectively; then $s = s_1 \cup s_2$ is a switching of $\mathcal{R}_1 \cup \mathcal{R}_2$. If $A$ and $B$ occur in $\mathcal{R}_0$, then they are connected by a path $\pi_1$ in $s_1\mathcal{R}_1$ and by a path $\pi_2$ in $s_2\mathcal{R}_2$; if $\pi_1 \neq \pi_2$, then there is a cycle in $s\mathcal{S}$, which is impossible. But $\pi_1 = \pi_2$ means that $A$ and $B$ are connected in $s_0\mathcal{R}_0$.

**Proposition 1:** (i) Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be proof nets and let

$$S = \text{Times}(\mathcal{R}_1, \mathcal{R}_2) = \frac{A B}{A \otimes B} \quad \text{or} \quad S = \text{Cut}(\mathcal{R}_1, \mathcal{R}_2) = \frac{A A^\perp}{\text{cut}}$$

Then $S$ is a proof net if and only if $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$.

(ii) Let $\mathcal{R}_0$ be a substructure of the proof net $\mathcal{R}$ and let

$$S = \text{Par}(\mathcal{R}_0) = \frac{A_1 A_2}{A_0 B}$$

Then $S$ is a subnet if and only if $\mathcal{R}_0$ is a subnet.

**Proof.** (i) Let $s$ be a switching of $S = \text{Times}(\mathcal{R}_1, \mathcal{R}_2)$; since $\mathcal{R}_1$ and $\mathcal{R}_2$ are proof nets, each of the graphs $s\mathcal{R}_1$ and $s\mathcal{R}_2$ are acyclic and connected; in addition to $s\mathcal{R}_1 \cup s\mathcal{R}_2$, $s\mathcal{S}$ has the vertex $A \otimes B$ and two edges $(A, A \otimes B)$ and $(B, A \otimes B)$, which establish a connection between $s\mathcal{R}_1$ and $s\mathcal{R}_2$; this is the only connection since $\mathcal{R}_1$ and $\mathcal{R}_1$ are disjoint.

Conversely, if $\mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset$, then by Lemma 1(i) $\mathcal{R}_1 \cup \mathcal{R}_2$ is a subnet. Therefore given any switching $s$ of $S$, the nodes $A$ and $B$ in are connected already in $s(\mathcal{R}_1 \cup \mathcal{R}_2)$; also the edges along link $A \otimes B$ yield another connection between the vertices $A$ and $B$, hence there is a cycle in $s\mathcal{S}$.

Part (ii) is immediate: for any switching $s$ of $\mathcal{R}$, $s\mathcal{S}$ comes from $s\mathcal{R}_0$ by introducing an additional edge $(A_1, A_1 \perp A_2)$ to a leaf $A_1$, where $A \perp B$ is a new leaf.

By induction on the definition of a sequent derivation in MLL we define the map $(\cdot )^\perp$ from sequent derivations to proof structures ("forgetting the context").

**Theorem 1.** (a) Let $D$ be a derivation in the sequent calculus for MLL; then $(D)^\perp$ is a proof net.

**Proof.** Axioms are proof nets, and the property of being a net is preserved under the times, cut and par rules by Proposition 1.

**Definitions 5.** Let $\Sigma$ be a set of formula-occurrences in a proof-net $\mathcal{R}$.

(i) The territory $\Sigma_\mathcal{R}$ of is the smallest subnet of $\mathcal{R}$ including $\Sigma$ (not necessarily as doors).

(ii) The kingdom $kA$ [the empire $eA$] of a formula-occurrence $A$ in a proof-net $\mathcal{R}$ is the smallest [the largest] subnet of $\mathcal{R}$ having $A$ as a door.

(iii) Let $X \ll Y \iff X \in kY$.

**Remarks.** (i) Given a proof-net $\mathcal{R}$ and formula occurrences $\Sigma$ in $\mathcal{R}$, the subnet $\Sigma_\mathcal{R}$ always exists by Lemma 1.

(ii) Suppose for no $X, Y$ in $\Sigma$ we have that $X$ is a hereditary premise of $Y$ ($X \ll Y$). Then $\Sigma_\mathcal{R}$, the smallest substructure containing $\Sigma$, has all the occurrences in $\Sigma$ among its doors. On the other hand, there may not be a subnet having all of $\Sigma$ among its doors.

(iii) The existence of $kA$ and $eA$ is immediate by Lemma 1 once we prove there exists a subnet having $A$ as a door. This can be done by giving an explicit construction of $eA$ as in [5, 7] and in the following section.

### 2.3 Empires and Kingdoms: Existence and Properties

Among the results in this section, for the proof of the Sequentialization theorem we need only the fact that for each formula occurrence $A$ in a proof-net $\mathcal{R}$ there exists a subnet having $A$ as a door.

**Definition 6.** Let $A$ be a formula occurrence in the proof net $\mathcal{R}$. For a given D-R-switching $s$, let $s(\mathcal{R}, A)$ be (the set of formula occurrences and of links occurring in) the connected component of the graph $s\mathcal{R}$ which is obtained as follows:

- if $A$ is a premise of a link in $\mathcal{R}$ with conclusion $Z$ and there is an edge $(A, Z)$ in the D-R-graph $s\mathcal{R}$, then remove $(A, Z)$ and let $s(\mathcal{R}, A)$ be the component containing the vertex $A$.
- otherwise, let $s(\mathcal{R}, A) = s\mathcal{R}$.

We write $s(\mathcal{R}, A)$ for the connected component not containing $A$ after the
removal of the edge \((A, Z)\) from \(s \mathcal{R}\), if such an edge exists; \(s(\mathcal{R}, A)\) is empty otherwise.

**Definition 7.** Let \(\mathcal{R}\) be a proof-net and let \(\Sigma\) be a set of formula-occurrences in \(\mathcal{R}\). We write \(\text{path}_s(\Sigma)\) for the smallest subgraph of \(s \mathcal{R}\) connecting all formula-occurrences in \(\Sigma\). Clearly \(\text{path}_s(A, B)\) is a path of \(s \mathcal{R}\), for every \(A, B \in \mathcal{R}\) and every switching \(s\) for \(\mathcal{R}\).

**Proposition 2.** (Characterisation of empires; cf. [3, 5, 7]) Let \(\mathcal{R}\) be a proof net. Then \(e(\mathcal{A})\) (the largest subnet of \(\mathcal{R}\) containing \(A\) as a conclusion) exists and is characterized by the following equivalent conditions:

(a) \(\bigcap s(\mathcal{R}, A)\), where \(s\) varies over all possible switchings;

(b) the smallest set of formula occurrences in \(\mathcal{R}\) closed under the following conditions:

(i) \(A \in e(\mathcal{A})\);

(ii) if \(X_1 \rightarrow X_2\) is a link in \(\Sigma\) and \(Y \in e(\mathcal{A})\), then \(X_1, X_2 \in e(\mathcal{A})\) (\(\rightarrow\) step);

(iii) if \(X_1 \rightarrow X_2\) is an axiom in \(\Sigma\) and \(X_1 \in e(\mathcal{A})\), then \(X_1 \rightarrow \in e(\mathcal{A})\) (\(\rightarrow\) step);

(iv) if \(X_1 \rightarrow X_2\) is a link in \(\Sigma\), and for \(i = 1\) or \(2\), \(X_i \neq A\) and \(X_i \in e(\mathcal{A})\), then \(X_1 \otimes X_2 \in e(\mathcal{A})\) (\(\otimes\) step);

(v) if \(X_1 \rightarrow X_2\) is a link in \(\Sigma\), \(X_1 \neq A \neq X_2\) and \(\{X_1, X_2\} \subset e(\mathcal{A})\), then \(X_1 \otimes X_2 \in e(\mathcal{A})\) (\(\otimes\) step).

(According to our conventions, \(X_1 \neq A\) means that \(X_1\) and \(A\) are different formula occurrences.)

**Proof.** The following proof of \((a) \implies (b)\) follows the argument in [7]. To show that \((b) \implies (a)\), we show that the set \((a)\) is closed under definitions \((i)\) -- \((v)\) defining \((b)\). This is easy for clauses \((i)\), \((iii)\), \((iv)\), \((v)\) of \((b)\), and also for clause \((ii)\), if the link in question is a times link. Now suppose that for some \(\Sigma\) the conclusion \(X_1 \otimes X_2 \in e(\mathcal{A})\), but \(s\) does not, for the premise \(X_2\) we have \(X_2 \notin \bigcap s(\mathcal{R}, A)\). Then for some \(s\) we have that \(X_1 \otimes X_2 \in s(\mathcal{R}, A)\) and \(X_2\) does not. Therefore \(A\) is a premise of a link with conclusion \(Z\) and \(X_2\) belongs to the same connected component as \(Z\), i.e., to \(s(\mathcal{R}, A)\); let \(\Sigma = \text{path}_s(X_2, Z)\), the path connecting \(X_2\) and \(Z\) in \(\mathcal{A}\). Since the switching \(s\) is left and the edge \((X_1, X_1 \otimes X_2)\) belongs to \(s(\mathcal{R}, A)\), it plays no role in the connections \(\pi\) between \(X_2\) and \(Z\). Therefore if \(s\) is like \(x\), except that the switch \(\Sigma\) is changed from \(L\) to \(R\), then we still have a connection \(\pi\) between \(X_2\) and \(Z\), since \(X_1 \otimes X_2 \in \bigcap s(\mathcal{R}, A)\), \(\pi\) can be extended to a connection \(\text{path}_s(A, Z)\), between \(A\) and \(Z\) in \(\mathcal{A}\); but then in \(s\) we have a cycle, and this is a contradiction. Therefore \(\{X_1, X_2\} \subset e(\mathcal{A})\).

To show that \((a) \subset (b)\) we consider a principal switching \(s\) for \(A\); this is a switching such that for every par link \(\Sigma\), if a premise \(X_1\) of \(\Sigma\) is \((b)\), but the conclusion \(X_1 \otimes X_1\) is not, \(s\) chooses \(X_1\). We claim that if \(s\) is a principal switching, then \(s(\mathcal{R}, A)\) is precisely \((b)\).

Notice that any set \(S\) closed under rules \((i)\) -- \((v)\) has the property that if \(S\) contains \(X\), then it contains also every formula occurrence such that \(X\) and \(Z\) are in a link \(\Sigma\); in all cases except perhaps the following:

(1) \(X\) is a premise of \(\Sigma\), while \(Z\) is the conclusion of \(\Sigma\);

(2) \(\Sigma\) is a par link, \(X\) is a premise and \(Z\) the conclusion of \(\Sigma\), and the other premise \(Y\) is not in \(S\).

It follows that the set \((b)\) is a substructure of \(\mathcal{R}\) whose doors can only be conclusions of \(\mathcal{R}\), or cuts, or occurrences \(X\) as in \((1)\) or \((2)\).

Now suppose a formula-occurrence \(W\) is in \(a\) but not in \(b\); choose a switching \(s\) principal for \(A\). Since \(s(\mathcal{R}, A)\) is connected and \(b\) is a substructure, the path \(\pi\) connecting \(\Sigma\) with \(\mathcal{A}\) must exit \(b\) from a door \(X\) as in cases \((1)\) or \((2)\). But this is impossible by the definition of principal switching and of \(s(\mathcal{R}, A)\). Hence \((a) \subset (b)\) as claimed.

We must show that \(A\) is a door of the substructure equivalently defined by \((a)\) and \((b)\). Let \(Z \in e(\mathcal{A})\) and suppose \(A \not< Z\). Choose a switching \(s\) such that if \(X_1 \rightarrow X_1\) is a link such that \(A \leq X_1 < Z\), then \(s\) chooses \(X_1\).

We claim that there must be a times link \(B \otimes C\) in \(s(\mathcal{R}, A)\) such that, say, \(A \leq C < Z\); otherwise, \(Z \notin s(\mathcal{A})\), by the choice of \(s\) and the definition of \(s(\mathcal{R}, A)\). Thus let \(B\) be the uppermost such link: then the path \(\pi\) connecting \(A\) and \(B\) in \(s(\mathcal{R}, A)\) does not pass through \(C\); but then in \(s\) we have two distinct paths connecting \(A\) and \(B\), which contradicts the acyclicity of \(s\).

Since \(b\) is a substructure satisfying the condition \((a)\), for each \(s\) the restriction of \(s(\mathcal{R}, A)\) to \(b\) is acyclic and connected, hence \((b)\) is a subnet. We have proved that given a proof-net \(\mathcal{R}\) and a formula-occurrence \(A\) in \(\mathcal{R}\), a subnet with conclusion \(A\) always exists.

But \((a)\) is also the largest among such subnets: let \(S\) be a substructure of \(\mathcal{R}\) with \(A\) as a door and suppose \(Z \in S\)'s \((a)\); then for some \(s\), we have \(Z \notin s(\mathcal{R}, A)\), from which it follows that no path connects \(A\) and \(Z\) in \(sS\); hence \(S\) is not a subnet. We conclude that \(e(\mathcal{A}) = (a) = (b)\).

The construction of a principal switching was given first in Girard's Trip Theorem (cf. [5], 2.9.5.); using Girard's notion of a trip the principal switching
constructed ‘dynamically’, by making the following choices during a trip.

Starting from A, the trip proceed upwards in R, and at a branching point, i.e., at times link, we choose arbitrarily;

- if the trip reaches a par link for the first time from below, then we fix s arbitrarily and the trip continues to the chosen premise;
- if the trip reach a par link for the first time from a premise, then we let s choose the other premise.

The Trip Theorem shows that eA is exactly the set of occurrences visited between the first and the second visit to A. The algorithm is transferred to our setting using the correspondence between trips and D-R-graphs established by Danos and Regnier [9]. One advantage of such a formulation is that the following corollary becomes completely obvious.

Corollary. The complexity of the computation of eA is linear on the size of the proof-net.

Proposition 3.(I) (properties of territories). Let R be a proof-net and let Σ be a set of occurrences in R. Then the territory tΣ satisfies

\[ tΣ = t(\text{path}_s(Σ)) = \bigcup_{X \in \text{path}_s(Σ)} tX \]

for any switching s.

Proposition 3.(II) (characterizations of kingdoms). Let R be a proof-net. Then the kingdom kA of A in R (the smallest subnet of R having A as a conclusion), exists and is characterized by the following equivalent conditions:

(a) tA;

(b) the smallest set satisfying the following conditions (Danos et al.):

(i) A ∈ kA.

(ii) Let \( \overline{X \quad X^1} \) occur in R. Then

\[ kX = k(X, X^1) = kX^1. \]

(iii) Let \( \frac{A \quad B}{A \otimes B} \) be a link in R. Then

\[ kX \otimes Y = kX \cup kY \cup \{X \otimes Y\}. \]

(c) the smallest set of formula occurrences closed under the following conditions:

(i) A ∈ k(A);

(ii) if \( \frac{X \quad Y}{X \otimes Y} \) is a link in S and Y ∈ k(A), then X₁, X₂ ∈ k(A) [similarly, if \( \frac{X \quad Y}{X \otimes Y} \) is a link in S and \( \exists y. X \in k(A) \), then \( X \otimes [y] \in k(A) \) (1-step);

(iii) if \( \overline{X_0 \quad X_1} \) is an axiom in S and X₁ ∈ k(A), then X₁₋₁ ∈ k(A) (→-step);

(iv) if \( \overline{X \quad Y} \) is a link in S X ≠ A ≠ Y, X ∈ k(A), then Y ∈ kA if and only if A /∈ eX (1-step).

· The proof is left to the reader; for case (c)(iv), see the following Lemma 2.

2.4 Sequentialization Theorem

Lemma 2. (Empire-Kingdom Nesting) Let \( \frac{A \quad C}{A \otimes C} \) and \( \frac{B \quad D}{B \otimes D} \) be distinct links in a proof net R for MLL⁻. Suppose C ∈ eA; then D /∈ eA if and only if C /∈ kD.

Proof. Clearly B ∈ eA ∩ kD, hence R₀ = eA ∩ kD and S = eA ∪ kD are subnets of R. If C /∈ kD and D /∈ eA, then S is a subnet with conclusion A, which is larger than eA, since it contains D; this contradicts the definition of the empire of A. If C ∈ kD and D ∈ eA, then R₀ is a subnet with conclusion D, which is smaller than kD since it does not contain C; this contradicts the definition of the kingdom of D.

Lemma 3. (Kingdom Ordering) (i) Let R be a proof net and let X, Y occur in R. If X ⊏ Y and Y ⊏ X then either X and Y are the same occurrence or they occur in an axiom \( \overline{X \quad Y} \) of R. (ii) Hence ⊏ is an ordering of the conclusions of non-axiom links.

Proof. For an axiom A = \( \overline{X \quad X^1} \) we have kX = kA = kX^1. Otherwise, let X ∈ kY, with X and Y distinct; if also Y ∈ kX, then kY ∩ kX is a subnet, and necessarily kX = kX ∩ kY = kY.

If X is X₁X₂ in a link \( \frac{X_1 \quad X_2}{X_1 \otimes X_2} \) then the result of removing X and L from kY is still a subnet, and this contradicts the definition of kY.
If \( X \) is \( X_1 \otimes X_2 \) in a link \( \frac{X_1 \cdot X_2}{X_1 \otimes X_2} \), then clearly \( kX = k(X_1) \cup k(X_2) \cup \{ X \} \), hence for \( i = 1 \) or \( 2, Y \in k(X_i) \); but by Lemma 2, \( Y \) is not even in \( e(X_i) \).

**Theorem 1** (b) (Sequentialization) If \( R \) is a proof net with conclusions \( \Gamma \), then there is a sequent calculus derivation \( D \) of \( \Gamma \) such that \( R = (D^-) \).

**Proof.** By induction on the size of \( R \). If \( R \) is an axiom, then \( D \) is an axiom sequent. If \( R \) is a lowermost link, it is a \( \text{par} \) or \( \text{for all} \) link, then we remove such a link, we apply the induction hypothesis to the resulting subnet and we conclude by applying a suitable \( \text{par} \) inference. Now suppose that all the conclusions of \( R \) are conclusions either of an axiom or of a times link: we choose a terminal times link \( L \) whose conclusion \( X = A \otimes B \) is maximal w.r.t. \( \sqsubseteq \). In this case \( eA \) and \( eB \) split \( R \setminus \{ A \otimes B \} \). Suppose not; then there is a link \( L' = \frac{A \otimes B}{C} \) such that, say, \( D \in eB \) and \( C \notin eB \). But \( C \) occurs at or above another conclusion \( Y = A_j \otimes B_j \). By the lemma 2 \( X = A_i \otimes B_i \in kC \); also \( C \in kY \) hence \( kC \subset kY \); thus we obtain \( X \in kY \), contradicting the choice of \( X \).

**Remark.** The computational complexity of Girard's no-short-trip condition and of Danos-Regnier's requirement that all DRT-graphs be acyclic and connected is clearly exponential on the size of the given proof-structure. It is known (see, e.g., \([3, 4, 1]\)) that there are procedures to decide whether or not a proof-structure \( R \) for MLL~ is a proof-net in time quadratic over the cardinality of \( R \).

### 2.5 Permutability of Inferences in the Sequent Calculus

Given a derivation \( D \) and two formula-occurrences \( X_1 \) and \( X_2 \) in some sequents of \( D \), if \( X_1 \) is an ancestor of \( X_2 \) then certainly the inference introducing \( X_1 \) must occur above the inference introducing \( X_2 \). We are concerned with occurrences \( X_1 \) and \( X_2 \) in \( D \) such that neither one is an ancestor of the other. Suppose \( X_1 \) is introduced above \( X_2 \) in \( D \), we ask whether there is a derivation \( D' \) which is obtained from \( D \) by successive permutation of the inferences and such that \( X_1 \) is introduced below \( X_2 \) in \( D' \).

**Counterexample.** The following is a derivation in MLL~ in which the applications of the \( \otimes \)-rule and of the \( \beta \)-rule cannot be permuted.

\[
\begin{align*}
\Gamma & \vdash p_1, p_2 & \Gamma & \vdash q_1, q_2 \\
\vdash p_1, p \otimes q_1, q_2 & \otimes & \vdash q_1, p_1, p \otimes q & \text{exchange} \\
\vdash q_1, p_1, p \otimes q & \beta & \vdash q_1, p_1, p \otimes q
\end{align*}
\]

**Remark.** In the sequent calculus for propositional MLL~ \( \otimes/p, \text{cut}/p \) and \( \exists/y \) are the only exceptions to the permutability of inferences where neither one of the principal formulas is an ancestor of the other.

A full characterization of permutability of inference in MLL~ is obtained using the 'context-forgetting' map \( .^- \) of derivations into proof-nets and the notions of universe and kingdom. Such a map uniquely associates each inference \( \Gamma \in D \) other than Exchange with a link \( L \in (D^-) \) and the principal formula(s) of \( \Gamma \) with the conclusion(s) of \( L \).

**Theorem 2.** (i) Let \( D \) and \( D' \) be a pair of derivation of the same sequent \( \vdash \Gamma \) in propositional MLL~. Then \( (D^-) = (D')^- \) if and only if there exists a sequence of derivations \( D = D_1, D_2, \ldots, D_n = D' \) such that \( D_i \) and \( D_{i+1} \) differ only for a permutation of two consecutive inferences.

(ii) Let \( R \) be a proof-net and let \( A \) be a formula occurrence in \( R \). Then there exists a derivation \( D \) with \( (D^-) = R \) and a subderivation \( B \) of \( D \) such that \( (B^-) = eA \). A similar statement holds for \( kA \).

**Proof.** (i) The "if" part is clear. To prove the "only if" part, let \( (D^-) = R = (D')^- \); consider a branch of \( D \) and let \( \Delta_0 \) the last inference from bottom up where \( D \) agrees with \( D' \). If \( \Delta_0 \) is an axiom, then \( D \) and \( D' \) are clearly in the order of inferences in this branch. Otherwise, let \( \Delta_4 \) be the inference immediately above \( \Delta_0 \) in the branch of \( D \) under consideration, and let \( \Delta_3 \) be the inference of \( D' \) such that the principal formulas of \( \Delta_4 \) and \( \Delta_3 \) are mapped to the same formula occurrence \( A \) of \( R \); such an \( \Delta_3 \) exists, since \( (D^-) = (D')^- \).

Moreover, let \( \Delta_1, \ldots, \Delta_k \) be the inferences which occur in \( D' \) between \( \Delta_3 \) and \( \Delta_0 \) (proceeding downwards). Notice that if the principal formula of any \( \Delta_i \) for \( i \leq k \) is mapped to a formula \( B \) of \( R \), then the inference \( \Delta_i \) of \( D \) whose principal formula is mapped to \( B \) also occurs above the inference \( \Delta_0 \), by our assumption that \( D \) and \( D' \) agree in the given branch up to \( \Delta_0 \). It follows that no descendant of \( A \) is active in \( \Delta_1, \ldots, \Delta_k \).

If the inference \( \Delta_3 \) is an instance of the \( \text{par} \) rule, then clearly it can be permuted below \( \Delta_1, \ldots, \Delta_{k-1} \). If \( \Delta_3 \) is a \( \text{times} \) rule, say, \( A = A_1 \otimes A_2 \), then we have

\[
\begin{align*}
& \vdash \Gamma_1, A_1, \Gamma_2, A_2 \\
& \vdash \Gamma_1, \Gamma_2, \Gamma_3, A_1 \otimes A_2 \\
& \vdash \Delta_1, A_1, \Delta_2, A_2 \\
\end{align*}
\]

If \( \Gamma_3 \) is another \( \text{times} \) rule, then clearly it can be permuted above \( \Gamma_3 \). If \( \Gamma_3 \) is a \( \text{par} \) rule, then consider the inference \( \Gamma_3 \in D \) such that the principal formulas of \( \Gamma_3 \) and \( \Gamma_0 \) are mapped to the same formula occurrence \( C = C_{xy} x_{y} \in R \). Now \( (B_j^-) \) is a subnet of \( R \) with \( \Delta_j \) as a conclusion, hence \( (B_j^-) \in e(\Delta_j) \);
3 Proof Nets for First Order MLL\(^-\)

This section is essentially based on Girard [7].

3.1 First-Order Proof-Structures

We work with a first-order language for MLL\(^-\) and consider multiplicative proof-structures with the addition of the following links.

First-order links:

\[
\text{for all: } \frac{A}{\forall x. A} \quad \text{exists: } \frac{A[t/x]}{\exists x. A}
\]

Definition 8. The variable \(x\) (possibly) occurring free in the premise of a for all link \(\frac{A}{\forall x. A}\) is called the eigenvariable associated with the link \(L\). Notice that the same variable \(x\) occurs free in the premise and bound in the conclusion of \(L\). We associate with each eigenvariable \(x\) a constant \(\bar{x}\). Obviously, a link of the form \(\frac{A[x/x]}{\forall x. A}\) is incorrect.

Definitions 9. (i) A proof structure for first order MLL\(^-\) is defined as before with the addition of the following conditions:

3. (a) Each occurrence of a quantifier link uses a distinct bound variable.

(b) If a variable occurs freely in some formula of the structure, then the variable is the eigenvariable of exactly one \(\forall\)-link.

(c) The conclusions of the proof structure are closed formulas.

4. We say that in a first-order proof-structure \(S\) eigenvariables are used strictly if no substitution of any set of occurrences of an eigenvariable \(x\) with the constant \(\bar{x}\) yields a correct proof structure with the same conclusions as \(S\). We require also that in first-order proof-structures eigenvariable is used strictly.\(^4\)

(ii) Let \(R\) be a proof structure for MLL\(^-\) and let \(x\) be an eigenvariable in \(R\). The free range of \(x\) in \(S\) is the set of all formula occurrences in which the eigenvariable \(x\) occurs freely. The existential border of \(x\) is the set of all the formula occurrences which are the conclusion of a link \(L : \frac{B[t/y]}{\exists y. B}\) where \(x\) occurs in the premise but not conclusion of \(L\). We say also that the link \(L\) is in the existential border of \(x\).

\(^4\)We modify the setting of Girard [7] only with the condition of a strict use of the eigenvariables; this is enough to give a smooth treatment of kingdom and empires.
(iii) We define the following additional reductions.

**Symmetric Reductions**

\[
\begin{array}{c|c}
\mathcal{R}(x) & A[t/x] \\
\hline
\exists x. A & A^1 \\
\end{array}
\quad \text{reduces to} \quad
\begin{array}{c|c}
\mathcal{R}[t/x] & A[t/x] \\
\hline
\forall x. A^1 & A^1[t/x] \\
\end{array}
\]

where \(\mathcal{R}(x)\) is the smallest substructure containing all occurrences of the eigen-variable \(x\) and \(\mathcal{R}[t/x]\) results from \(\mathcal{R}(x)\) by replacing \(t\) for \(x\) everywhere.

The definition of Danos-Regnier graph for first order proof structures is extended as follows.

**Definitions 10.** Let \(\mathcal{R}\) be a proof structure for first order MLL\(^-\).

(i) A Danos-Regnier switching \(s\) in a first order proof structure \(\mathcal{R}\) for MLL\(^-\) consists in a switch for each \(\text{par}\) and for all link of \(\mathcal{R}\), where

- a switch for a \(\text{par}\) link is the choice of one of the premises of the link and
- a switch for a \(\text{for all}\) link with associate eigenvariable \(x\) is a choice of either (1) the premise of the link or of a formula occurrence in (2) the free range or in (3) the existential border of \(x\) (case (1) is needed if \(x\) does not occur free in \(\mathcal{R}\)).

(ii) Given a switching \(s\) for \(\mathcal{R}\), we define the undirected Danos-Regnier graph \(s(\mathcal{R})\) as follows:

- the vertices of \(s(\mathcal{R})\) are the formulas of \(\mathcal{R}\);
- there is an edge between vertices \(X\) and \(Y\) exactly when:
  - (a) \(X\) and \(Y\) are the conclusions of a logical axioms or the premises of a cut link;
  - (b) \(X\) is a premise and \(Y\) the conclusion of a \(\text{times}\) or \(\exists\) link;
  - (c) \(Y\) is the conclusion of a \(\text{par}\) or \(\text{for all}\) link and \(X\) is the occurrence selected by the switching \(s\).

(iii) \(\mathcal{R}\) is a proof net for first order DL [MLL\(^-\)] if for every switching \(s\) of \(\mathcal{R}\), the graph \(s(\mathcal{R})\) is acyclic [and connected].

The requirement that eigenvariable should be used strictly guarantees that the following structure is incorrect:

\[
\begin{array}{c|c}
\mathcal{R}(x) & A[t/x] \\
\hline
\forall x. A & A^1 \\
\exists x. A^1 & A^1[t/x] \\
\end{array}
\]

and must be rewritten as

\[
\begin{array}{c|c}
\mathcal{R}(x) & A^1(t/x) \\
\hline
\forall x. A & A^1 \\
\exists x. A^1 & A^1[t/x] \\
\end{array}
\]

where \(c\) is a new constant.

The following is an equivalent way of characterizing the same property.

**Definition 11.** An \(\times\)-thread in a proof structure \(\mathcal{R}\) is a sequence \(C_1, \ldots, C_n\) of formula occurrences which contain the free variable \(x\) and such that for each \(i < n\) there is a link \(L\) such that either (1) \(C_i\) is the premise and \(C_{i+1}\) is the conclusion of \(L\) or (2) \(C_i\) and \(C_{i+1}\) are conclusions of \(L\) (an axiom link) or (3) \(C_i\) is the conclusion and \(C_{i+1}\) is the premise of \(L\).

**Fact 1.** In a proof structure eigenvariables are used strictly if and only if every occurrence of an eigenvariable \(x\) belongs to an \(\times\)-thread ending with the \(\forall\)-link associated with \(x\).

3.2 Subnets

The definition of a substructure \(S_0\) of a proof structure \(S\) must take into account the requirement that all conclusion of \(S_0\) should be closed formulas.

**Definitions 12.** (i) Let \(S\) be a proof structure for first order MLL. A set of formula occurrences and links \(S_0\) is a substructure of \(S\) if \(S_0\) is a proof structure and there is an injective map \(i: S_0 \rightarrow S\) preserving links such that \(X\) and \(i(X)\) are the same formula or \(X\) comes from \(i(X)\) by a substitution of a free variable \(x\) with \(x\). (We will usually omit to mention the map \(i\).)

As before, a subnet is a substructure which satisfies the condition of proof-nets.

**Fact 2.** If \(S\) is a substructure of a first order proof-structure \(\mathcal{R}\) and a link \(L: \forall x. A\) occurs in \(S\), then the free range of \(x\) and its existential border are contained in \(S\).
Proof. All eigenvariables are used strictly in \( S \) by definition. Suppose \( \mathcal{L} \) occurs in \( S \) but \( x \) occurs outside \( S \); then there is an \( x \)-thread ‘crossing the border of’ \( S \), say at a door \( C \). This means that any substitution of \( z \) for \( x \) in \( C \) spoils the link \( \mathcal{L} \), i.e., \( S \) cannot be a substructure, a contradiction. \( \blacksquare \)

**Lemma 1** (first order case) In first order \( \text{MLL}^- \), the intersection and the union of subnets are subnets if and only if the intersection is nonempty.

**Proof.** The argument for the propositional case applies here; we need only to make sure that if \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are subnets of a proof-net \( \mathcal{R} \) with \( \mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset \), then \( S = \mathcal{R}_1 \cup \mathcal{R}_2 \) and \( S = \mathcal{R}_1 \cap \mathcal{R}_2 \) are first-order substructures, and in particular, the eigenvariables are used strictly and their conclusions are closed.

If a \( \forall \)-link of \( \mathcal{R} \) does not occur in \( S \), then the associated eigenvariable \( x \) is replaced by \( x \) in the subnets \( \mathcal{R}_1 \) and in \( \mathcal{R}_2 \), hence in \( S \) too.

If a \( \forall \)-link with eigenvariable \( x \) occurs in \( \mathcal{R}_1 \), then (since eigenvariables are used strictly in \( \mathcal{R} \) \( x \) also occurs inside \( \mathcal{R}_1 \) but not in any door of \( \mathcal{R}_1 \), by the Fact 2.

Finally, if a \( \forall \) link with eigenvariable \( x \) occurs, say, in \( \mathcal{R}_1 \setminus \mathcal{R}_2 \), then any occurrence of \( x \) in the substructure \( \mathcal{R}_1 \) is replaced by \( x \). Moreover \( x \) does not occur in the doors of \( S \); indeed by the same corollary, \( x \) does not occur in the doors of \( \mathcal{R}_1 \), hence it does not occur in \( \mathcal{R}_1 \setminus \mathcal{R}_1 \) either. \( \square \)

**Proposition 1.** (first order cases) Let \( \mathcal{R}_0 \) be a substructure of the proof net \( \mathcal{R} \). Then:

(iii) \( S = \text{For All (} \mathcal{R}_0 \text{)} = \frac{\mathcal{R}_0[z/x]}{\forall z.A} \) is a subnet if only if \( \mathcal{R}_0 \) is a subnet and \( x \) does not occur in \( \Gamma \).

(iv) The substructure

\( S = \text{Exists (} \mathcal{R}_0 \text{)} = \frac{\mathcal{R}_0}{A[t/x]} \) is a subnet if \( \mathcal{R}_0 \) is one.

**Proof.** (iii) \( S \) is a substructure, since the substitution of \( x \) for \( z \) does not affect the conclusions of \( S \), which remain closed. Given a switching \( s \) for \( S \), \( sS \) differs from \( s\mathcal{R}_0 \) only for having a leaf \( \forall x.A \) connected by an edge to some vertex of \( \mathcal{R}_0 \); thus \( sS \) is acyclic and connected, since \( s\mathcal{R}_0 \) is. (iv) is similar but easier. \( \square \)

**Remark.** It is not true that if \( S = \text{Exists} \mathcal{R}_0 \) and \( S \) is a proof-net then \( \mathcal{R}_0 \) is a proof-net: for instance, in \( A[t/x] \) the term \( t \) may contain the eigenvariable of some for all link which occur in \( \mathcal{R}_0 \).

As before Theorem 1(a) follows as a corollary. (Notice that if \( \Gamma \) is the end sequent of \( \mathcal{D} \) and a free variable \( x \) occurs in \( \Gamma \), then \( (\mathcal{D}^\neg) = (\mathcal{D}[x/x]^\neg) \), a proof-structure with conclusions \( \Gamma[x/x] \).)

**Theorem 1.** (a) (first-order case) Let \( \mathcal{D} \) be a derivation in the sequent calculus for first order \( \text{MLL}^- \); then \( \mathcal{D}^\neg \) is a proof net. \( \square \)

### 3.3 Empires and Kingdoms: Existence and Properties

As in the propositional case, we need to prove that given a proof-net \( \mathcal{R} \) and a formula \( A \) in \( \mathcal{R} \), there always exists a subnet of \( \mathcal{R} \) having \( A \) among its conclusions.

**Proposition 2.** (Characterization of empires, first-order case; cf. [7]) Let \( \mathcal{R} \) be a proof net for first order \( \text{MLL}^- \) and let \( A \) occur in \( \mathcal{R} \). Then the empire \( eA \) of \( A \) in \( \mathcal{R} \) exists and is characterized by the following equivalent conditions:

(a) \( \exists s(\mathcal{R}, A) \), where \( s \) varies over all possible switchings;

(b) the smallest set of formula occurrences in \( \mathcal{R} \) closed under conditions (b)(i)-(v) of Proposition 2 for propositional multiplicative links and moreover

(bi) if \( \frac{X[t/y]}{\exists y.X} \) is a link in \( S \) and \( X[t/y] \neq A \), then \( \exists y.X \in e(A) \) if and only if \( X[t/y] \in s(A) \), (\( \uparrow \) - and \( \downarrow \)-steps);

(bii) if \( \frac{X}{\forall y.X} \) is a link in \( S \) and \( X \neq A \), then \( \forall y.X \in e(A) \) if and only if the free range of \( y \) and the occurrences in its existential border belong to \( eA \) (\( \uparrow \)- and \( \downarrow \)-steps).

**Proof.** We follow Girard [7]. (bii) Suppose \( \forall y.X \in e(A) \), but for some \( C \) in the free range of \( y \) we have \( C \notin e(A) \). Then \( A \) must be a premise of some link with conclusion \( Z \), and for some \( s \) we have \( \forall y.X \in s(R, A) \) and \( C \in s(R, A) \), where \( s(R, A) \) is the connected component not containing \( A \) after removal of the edge \( (A, Z) \) from \( sR \). Therefore in \( s(R, A) \) there is a path connecting \( A \) and \( y.X \) and moreover in \( s(R, A) \) there is a path connecting \( Z \) and \( C \) which obviously does not depend on the switch for \( \forall y.X \). Now if we change the switch for \( \forall y.X \) to choose \( C \) leaving all other choices unchanged, then we obtain a switch \( s' \) such that \( s'R \) is cyclic: indeed there still remains a connection between \( Z \) and \( C \) in \( s'(R, A) \) (which lies outside \( e(A) \) and there certainly is a distinct connection between \( A \) and \( \forall y.X \) in \( s'(R, A) \) (since \( \forall y.X \in e(A) \)). But then \( s'R \)}
contains a cycle, a contradiction.  

The example at the beginning of the present section shows that an eigen-
variable \( x \) can occur outside the domain of \( \forall x. A \), unless a strict use of eigen-
variables is required. We have the following characterisation of kingdoms in first
order MLL\(^-\) (which is not true in the setting of \([7]\)).

**Proposition 3.** (Inductive definition of kingdoms, first-order cases) Let \( R \)
be a proof net for first order MLL\(^-\). Then \( kA \), the kingdom of a in \( R \) exists
and is characterized as the smallest set of formula occurrences closed under
conditions (i)-(iv) of Proposition 3 for multiplicative propositional links and
moreover:

(i) if \( X[t/y] \) is a link in \( S \) and \( \exists y. X \in k(A) \), then \( X[t/y] \in k(A) \), (\( \forall \)-step);

(ii) if \( X \) is a link in \( S \) and \( \forall y.X \in k(A) \), then \( X \in k(A) \) \( (\exists \)-step).

\[ \forall y. X \]

3.4 Sequentialization

The proof of Lemma 2 extends to the first-order case without modifications.

**Lemma 2.** (Empire-Kingdom Nesting) Let \( L_1 : \frac{A \to B}{C} \) and \( L_2 : \frac{C \to B}{D} \) be distinct links in a proof net \( R \). Suppose \( B \in eA \); then \( D \notin eA \) if and only if \( C \in kD \).

**Lemma 3.** (Ordering of the kingdoms, first-order case) In proof-nets for first
order MLL\(^-\) the relation \( \ll \) is a strict ordering of formula-occurrences that are not conclusions of axioms links.

**Proof.** Suppose \( X \in kY \), where \( X \) and \( Y \) not the conclusions of axioms links.
Two cases are to be added to the propositional proof.

Let \( X \) be the conclusion of a link \( \frac{A[t/x]}{\exists x. A} \). It follows from the definition of kingdom and proposition 1 that \( kX = k(\exists x. A) = k(A[t/x]) \cup \{ \exists x. A \} \). If \( X \) and \( Y \) are distinct and also \( Y \in kX \), then \( Y \in k(A[t/x]) \) and this is absurd, since \( Y \notin e(A[t/x]) \) follows from \( \exists x. A \in kY \) by lemma 2.

Finally, let \( X \) be the conclusion of a link \( \frac{A}{\forall x. A} \). If follows from proposition 1 that \( kX \setminus \{ \forall x. A \} \subseteq eA \). If \( X \) and \( Y \) are distinct and also \( Y \in kX \), then \( Y \in eA \), and this contradicts lemma 2.

**Theorem 1.(b)** The Sequentialization Theorem holds in first order MLL\(^-\).

**Proof.** We consider first the lowermost \( \exists \) and for all \( \exists \), if such links exist. Otherwise, we choose a terminal link \( L \) whose conclusion is maximal w.r.t. \( \ll \).
If \( L \) is an \( \exists \)-link, then the result of removing it is still a proof-net. Suppose not; then \( L : \frac{A[t/x]}{\exists x. A} \) is in the existential border of \( y \), where \( y \) is associated with \( B \), \( \forall y.B \) then \( \exists x. A \in k(\forall y.B) \), by Fact 2, hence \( \exists x. A \) it cannot be maximal w.r.t. \( \ll \). The rest of the proof is as before.

3.5 Permutability of Inferences in the Sequent Calculus

**Counterexample.** Let \( x \) occur free in \( P \). The following is a derivation in MLL\(^-\) in which the applications of the \( \exists \)-rule and of the \( \forall \)-rule cannot be permuted.

\[ \vdash P \quad P, \exists \quad \exists \quad \forall \]

**Theorem 2.** (first order case) The Theorem on permutability of inferences holds in first order MLL\(^-\).

**Proof.** (i) Assuming the pure parameter property, the argument is similar to the propositional case, where for all rules behave like \( \exists \)-rules and \( \forall \)-rules like \( \exists \)-rules. The nontrivial case is the following: an inference \( I_A \) of \( \forall \) has the principal formula \( A = \exists x. A \) and must be permuted below \( \forall \) for all rule \( \forall \). As before we argue that in \( D \) we have an inference \( I_B \) such that \( I_1 \) and \( I_B \) are mapped to \( B = \forall y.B \) and that such an inference must occur above the inference \( I_D \) whose active formula is \( A[t/x] \); by the pure parameter property of \( D \), \( y \) does not occur in \( t \), and the permutation is permissible.

(ii) As before, the argument is by induction on \( eA \setminus \{ A \} \); to the propositional cases we add the following cases (the cases of existential links being unproblematic):

(\( \forall \)-step) for all link, clause (vii): By the pure parameter property the eigen-
variables occur only above the associated \( \forall \)-inference, which already occurs above \( I_A \) by induction hypothesis.

(\( \exists \)-step) for all links, clause (vii): Let \( I' \) be the inference introducing \( \forall y. X \) below \( I \), where \( \forall y. X \in eA \). By induction hypothesis the eigenvariable \( y \) occurs only in sequents above \( I_A \), except for one occurrence of a formula \( X(y) \) (an ancestor of \( \forall y. X \)) for each sequent between \( I_A \) and \( I' \). Hence we can always permute \( I' \) with the inference immediately above it.
NONCOMMUTATIVE PROOF NETS

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Introduction

The aim of this paper is to give a purely graph-theoretical definition of noncommutative proof nets, i.e. graphs coming from proofs in MNLL (multiplicative noncommutative linear logic, the (⊗, ⨬)-fragment of the one-sided sequent calculus for classical noncommutative linear logic, introduced in [Abr91]). Analogously, one of the aims of [Gir87] was to give a purely graph-theoretical definition of proof nets, i.e. graphs coming from the proofs in MLL (multiplicative linear logic, the (⊗, ⨬)-fragment of the one-sided sequent calculus for classical linear logic - better, for classical commutative linear logic). The relevance of the purely graph-theoretical definition of proof nets for the development of commutative linear logic is well-known; thus we hope the results of this paper will be useful for a similar development of noncommutative linear logic.

The language for MNLL is an extension of the language for MLL, obtained simply adding, as atomic formulas, propositional letters with an arbitrary finite number of negations written after the propositional letter (linear post-negation) or before the propositional letter (linear retro-negation). Every formula A of MNLL may be translated into a formula Tr(A) of MLL (simply by replacing each propositional letter with an even number of negations by the propositional letter without negations, and...
Advances in Linear Logic

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Preface

This volume is based to a large extent on the Linear Logic Workshop held June 14-18, 1993 at the MSI\(^1\) and partially supported by the US Army Research Office and the US Office of Naval Research. The workshop was attended by about 70 participants from the USA, Canada, Europe, and Japan. The workshop program committee was chaired by A. Scedrov (University of Pennsylvania) and included S. Abramsky (Imperial College, London), J.-Y. Girard (CNRS, Marseille), D. Miller (University of Pennsylvania), and J. Mitchell (Stanford). The principal speakers at the workshop were J.-M. Andreoli, A. Blass, V. Danos, J.-Y. Girard, A. Joyal, Y. Lafont, J. Lambeek, P. Lincoln, M. Moortgat, R. Pareschi, and V. Pratt. There were also a number of invited 30 minute talks and several software demonstration sessions.

Our intention was not only to publish a volume of proceedings. We also wanted to give an overview of a topic that started almost 10 years ago and that is of interest for mathematicians as well as for computer scientists. For these reasons, the book is divided into 5 parts:

1. Categories and Semantics
2. Complexity and Expressivity
3. Proof Theory
4. Proof Nets
5. Geometry of Interaction

The five parts are preceded by a general introduction to Linear Logic by J.-Y. Girard. Furthermore, parts 2 and 4 start with survey papers by P. Lincoln and Y. Lafont. We hope this book can be useful for those who work in this area as well as for those who want to learn about it. All papers have been refereed and the editors are grateful to A. Scedrov who took care of the refereeing process for the papers written by the the editors themselves.

Jean-Yves Girard
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January 1995

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\(^1\)Mathematical Sciences Institute, Cornell University, Ithaca, New York, USA. MSI is a US Army Center of Excellence.