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The use of optical flow for the analysis of non-rigid motions

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Abstract.
This paper analyses the 2D motion field on the image plane produced by the 3D motion of a plane undergoing simple deformations. When the deformation can be represented by a planar linear vector field, the projected vector field, i.e. the 2D motion field of the deformation, is at most quadratic. This 2D motion field has one singular point, with eigenvalues identical to those of the singular point describing the deformation. As a consequence, the nature of the singular point of the deformation is a projective invariant. When the plane moves and experiences a linear deformation at the same time, the associated 2D motion field is at most quadratic with at most 3 singular points. In the case of a normal rototranslation, i.e. when the angular velocity is normal to the plane, and of a linear deformation, the 2D motion field has one singular point and substantial information on the rigid motion and on the deformation can be recovered from it. Experiments with image sequences of planes moving and undergoing linear deformations show that the proposed analysis can provide accurate results. In addition, experiments with deformable objects, such as water, oil, textiles and rubber show that the proposed approach can provide information on more general 3D deformations.

Introduction

The majority of available algorithms for the analysis and recovery of 3D motion of moving objects from image sequences, makes the assumption of opacity and rigidity (Fennema & Thompson, 1979; Hildreth, 1984; Longuet-Higgins, 1984; Nagel, 1983; Francois & Bouthemy, 1990). As these objects are seen by an imaging device, the 3D motion field of moving objects in the scene is transformed into a 2D motion field in the image plane (Gibson, 1950; Horn & Schunck, 1981). The assumption of opacity implies that at any location in the image plane, the 2D motion field is single valued, that is the 2D motion field is uniquely determined. In the case of transparent objects, the 2D motion field is not single valued and two different velocities can be assigned to the same location in the image plane. In the case of opaque objects, the 2D motion field is a planar vector field, which can be analysed with the tools of dynamical systems theory (Hirsch & Smale, 1974).

The assumption of rigidity is usually made in order to simplify the problem and, by using properties of singular points, useful information on the 3D motion can be recovered (Verri, Girosi & Torre, 1989). The problem of non-rigid motion has already been considered: Ullman (1984) introduced an incremental approach to recover the structure from motion, even in the case of non-rigid bodies, and Jasinschi & Yuille (1989) have recently used the same approach with more sophisticated mathematical tools. Bergholm and Carlsson (1991) used the flow of "visual directions" to recover information about complex motions. Penna (1992) considered the problem of shape from motion for objects undergoing non-rigid isometric motions. Yakamoto, Boulanger, Beraldin & Rioux (1993) introduced a deformable net model to find the constraints for a direct estimation of 3D motion. Other approaches to the non-rigid motion have been proposed by Koenderink & Van Doorn

(1986), Shulman and Aloimonos (1988), Subbarao (1989).

In this paper we relax the hypothesis of rigidity and we analyse a special case of moving and deformable objects, that is of deformations of a plane undergoing a normal rototranslation. Deformations on the plane are modelled by a linear vector field, allowing to introduce the three elementary deformations: expansions, rotations and shears (Helmholtz, 1858). The deformable plane is also supposed to move in the 3D space. The justification for studying this specific case is threefold. Firstly, given the 2D motion field, it is possible to provide almost a complete solution to the problem of the recovery of deformations and rigid motion . Secondly, the proposed approach is a good local approximation of a generic deformation occurring on the surface of an opaque object. Thirdly, as shown in the experimental section, several deformations of real objects can be treated in first approximation as linear planar deformations.

Planar deformations have already been studied by several authors. Wohn and Waxman (1990) analysed deformation components (up to the second order) of the 2D motion field on the image plane to study the 3D motion of rigid surfaces; Rao and Jain (1992) proposed the representation of flow patterns with linear differential equations; Shu and Jain (1993) and Ford, Strickland and Thomas (1994) proposed algorithms for the recovery of the linear deformation components of flow fields. This paper analyses the recovery of 3D linear planar deformations, given their projections on the image plane, presents some new analytical results and shows how to recover the 3D motion and linear deformations in a simple case. In addition, an extensive experimentation on synthetic and real images is presented.

The paper is organized as follows: Section 1 introduces linear planar deformations and proves that for this class of deformations there are useful perspective invariants. By using this property the combined case of deformable and moving plane can be treated almost completely. Section 2 shows that in the case of a normal rototranslation (i.e. when the angular velocity is perpendicular to the plane) a simple recovery of the 3D motion and of the linear deformation can be obtained. Additional properties of the singular points are presented in Section 3. Section 4 presents several experiments with synthetic and real images. Experiments with image sequences of real deformable objects (liquid, textiles, rubber...) show that the proposed procedure for recovering motion and deformation can also be used in the case of more general deformations and more complex objects.

1. Properties of linear deformations over a plane

In this section we introduce the problem under investigation and we prove the existence of some perspective invariants for planar linear deformations (see eqn. 17). A well known theorem of Helmoltz (1858) (see also Sommerfeld 1950) has shown that the most general motion of a sufficiently small element of a deformable (i.e. not rigid) body can be represented as the sum of a translation, a rotation and an extension (or contraction) in three mutually orthogonal directions. In the presence of opaque objects, the visible deformations are those occurring at the object surface. These deformations can be assumed to occur locally on the plane tangent to the surface. Following the approach of Sommerfeld (1950), let $\vec{X}^\pi$ be a point of an element of a plane π and X^π_α, X^π_β its coordinates in the reference system $(O^\pi, \hat{\alpha}, \hat{\beta})$ where O^π is the center of reference and $\hat{\alpha}, \hat{\beta}$ two orthogonal unit vectors. We want to study the motion field in the plane around the point $\vec{P^\pi} = (P^\pi_\alpha, P^\pi_\beta)$. In the case of a general non-rigid motion both points $\vec{X}^\pi$ and $\vec{P}^\pi$ will experience changes in position, which we denote $\vec{V}^\pi = (V^\pi_\alpha, V^\pi_\beta)$ and $\vec{V}^\pi_0 = (V^\pi_{0\alpha}, V^\pi_{0\beta})$ for $\vec{X}^\pi$ and $\vec{P}^\pi$ respectively. If the element is sufficiently small, it is possible to use a Taylor expansion up to the first order, so that:

$$\vec{V}^\pi(\vec{X}^\pi) = \vec{V}^\pi_0 + \mathbf{L}(\vec{X}^\pi - \vec{P}^\pi) \tag{1}$$

the term $\vec{V}^\pi_0$ is the rigid translation, while the second term, represented by the linear operator $\mathbf{L}$ is caused by the rotation and the non rigid component of the motion. In the system of reference previously introduced, $\mathbf{L}$ can be expressed as:

$$\begin{pmatrix} V^\pi_\alpha \\ V^\pi_\beta \end{pmatrix} = \begin{pmatrix} V^\pi_{0\alpha} \\ V^\pi_{0\beta} \end{pmatrix} + \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^\pi_\alpha - P^\pi_\alpha \\ X^\pi_\beta - P^\pi_\beta \end{pmatrix} \tag{2}$$

It is evident therefore that the matrix $\mathbf{L}$ can be used to characterize the rotation and non rigid part of the motion of a sufficiently small element of a plane or locally of a surface. Now we drop the translational term and we consider the plane π with a stationary point $\vec{P}^\pi$ and a 2D motion field determined by the linear deformation $\vec{V}_D^\pi$ on its surface, given by:

$$\vec{V}_D^\pi = \mathbf{L}(\vec{X}^\pi - \vec{P}^\pi) \tag{3}$$

where $\mathbf{L}$ is the matrix previously defined. It is well known that the matrix $\mathbf{L}$ describing the deformation can be decomposed as:

$$\mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \\ +\omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + S_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + S_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

where

$$E = \frac{L_{11} + L_{22}}{2} \quad \omega = \frac{L_{12} - L_{21}}{2} \\ S_1 = \frac{L_{11} - L_{22}}{2} \quad S_2 = \frac{L_{12} + L_{21}}{2} \tag{5}$$

are the elementary deformation components: expansion, rotation and shears respectively, reproduced in Fig. 2 A, C and E. From eqn. (5) it is evident that the component of expansion E is simply equal to Trace $\mathbf{L}/2$.

Now we suppose that the deforming plane is lying in the 3D space and is observed by an imaging device with the optical center centered in the origin O of a reference system (O, $\hat{e}_1, \hat{e}_2, \hat{e}_3$) as shown in Fig. 1.

[Fig. 1 near here]

$\vec{X} = (X_1, X_2, X_3)$ indicates a point in the 3D space and $\vec{V}$ its velocity. The optical axis of the imaging device is assumed to coincide with the $\hat{e}_3$ axis and the image plane to have equation $X_3 = f$ where f is the focal length of the imaging device. We want to analyse the relation between the 2D vector field $\vec{V}_D$ describing the deformation on the plane π and its perspective projection $\vec{v_D}$ on the image plane of the imaging device and we will show the existence of some useful perspective invariants (eqn.17). If $\hat{\gamma}$ is the unit vector perpendicular to the deforming plane and d is the distance between the plane and the optical center, the equation of the plane is:

$$\hat{\gamma} \cdot \vec{X} = d \tag{6}$$

Let us now write the motion field (3) of the deformation on π as a 3D vector field in the reference system $(O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$. If we choose the origin of the reference system on π so that the position of O^π in the reference system $(O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ is $\vec{O}$ such that $\vec{O}$ is perpendicular to the plane π, we have $\vec{O} \cdot \hat{\alpha} = 0$, $\vec{O} \cdot \hat{\beta} = 0$, $\vec{O} \cdot \hat{\gamma} = d$. If $\vec{X}$ is on the plane π, and $\vec{X}^\pi$ is the position of the same point in the reference system $(O^\pi, \hat{\alpha}, \hat{\beta})$, we have that $X_\alpha^\pi = \vec{X} \cdot \hat{\alpha}$ and $X_\beta^\pi = \vec{X} \cdot \hat{\beta}$. The motion field in the reference system $(O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ can be written as:

$$\vec{V_D} = (V_1, V_2, V_3) = \\ = \begin{cases} L_{11}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\hat{\alpha} + L_{22}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\hat{\beta} + \\ +L_{12}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\hat{\alpha} + L_{21}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\hat{\beta} \\ \qquad\qquad \text{if } \vec{X} \in \pi \\ 0 \qquad \text{otherwise} \end{cases} \tag{7}$$

where $\vec{P}$ is the 3D position of the stationary point of the deformation in the reference of the imaging device. If we rewrite the 3D motion field (7) in terms of its components we have:

$$V_i = L_{11}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\alpha_i + L_{22}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\beta_i + \\ +L_{12}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\alpha_i + L_{21}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\beta_i \\ (\vec{X} \in \pi) \tag{8}$$

where $\alpha_i = \hat{\alpha} \cdot \hat{e}_i$, $\beta_i = \hat{\beta} \cdot \hat{e}_i$ $(i = 1, 2, 3)$.

Now we want to find the perspective projection of $\vec{V}_D$ over the image plane (see Fig. 1). The well known perspective projection formulas are:

$$\vec{x} = \frac{f}{X_3}\vec{X} \tag{9}$$

$$\vec{v} = \frac{f}{X_3^2}[\hat{e_3} \times (\vec{V} \times \vec{X})] = \\ = \frac{f}{X_3^2}[(\hat{e_3} \cdot \vec{X})\vec{V} - (\vec{V} \cdot \hat{e_3})\vec{X}] \tag{10}$$

[Fig. 2 near here]

Because of eqn. (10) the vector field $\vec{V_D}\,(\vec{X} \in \pi)$ is transformed into the 2D motion field on the image plane $\vec{v}_D$ as:

$$\vec{v}_D\,(\vec{X}) = \tfrac{f}{X_3^2} X_3 \{L_{11}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\hat{\alpha} + \\ +L_{22}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\hat{\beta} + L_{12}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\hat{\alpha} + \\ +L_{21}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\hat{\beta}\} - \tfrac{f}{X_3^2}\{L_{11}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\alpha_3 + \\ +L_{22}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\beta_3 + L_{12}[\hat{\beta} \cdot (\vec{X} - \vec{P})]\alpha_3 + \\ +L_{21}[\hat{\alpha} \cdot (\vec{X} - \vec{P})]\beta_3\}\vec{X} \qquad (\vec{X} \in \pi) \tag{11}$$

Eq. (11) can be rewritten, using (9) as:

$$\vec{v}_D\,(\vec{x}) = (v_1, v_2) = L_{11}[\hat{\alpha} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\hat{\alpha} + \\ +L_{22}[\hat{\beta} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\hat{\beta} + L_{12}[\hat{\beta} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\hat{\alpha} + \\ +L_{21}[\hat{\alpha} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\hat{\beta} - \{L_{11}[\hat{\alpha} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\alpha_3 + \\ +L_{22}[\hat{\beta} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\beta_3 + L_{12}[\hat{\beta} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\alpha_3 + \\ +L_{21}[\hat{\alpha} \cdot (\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{P}}\vec{P})]\beta_3\}\tfrac{\vec{x}}{f} \tag{12}$$

and by introducing the elementary deformation components (5) and rearranging the different terms as shown in Appendix A, we obtain the simple expression:

$$v_1 = a_{13}x_1^2 + a_{23}x_1x_2 + (a_{33} - a_{11})fx_1 + \\ -a_{21}fx_2 - a_{31}f^2 \\ v_2 = a_{13}x_1x_2 + a_{23}x_2^2 + (a_{33} - a_{22})fx_2 + \\ -a_{12}fx_1 - a_{32}f^2 \tag{13}$$

with

$$a_{ij} \equiv E\frac{\gamma_i P_j}{fd} + \omega(\frac{\hat{\gamma} \cdot \hat{e_j} \times \hat{e_i}}{f} - \frac{\gamma_i(\vec{P} \times \hat{\gamma})_j}{fd}) + \\ +S_1\frac{\alpha_j(\vec{P} \times \hat{\beta})_i + \beta_j(\vec{P} \times \hat{\alpha})_i}{fd} + \\ +S_2\frac{-\alpha_j(\vec{P} \times \hat{\alpha})_i + \beta_j(\vec{P} \times \hat{\beta})_i}{fd} \tag{14}$$

The motion field (**??**) has the same mathematical structure of the one caused by a rigid motion of a plane (Verri, Girosi & Torre, 1989), and only the parameters a_{ij} are different. As a consequence, the motion field (13) and the motion field of an arbitrary rigid motion of a plane have some common properties. For instance both motion fields have at most three singular points and cannot have limit cycles (Aicardi & Verri, 1990).

The 2D motion field (**??**) has a singular point $\vec{p}$ (i.e. a point such that $\vec{v}_D\,(\vec{p}) = 0$) that is the perspective projection of the point $\vec{P}$ over the image plane: $\vec{p} = f\frac{\vec{P}}{P_3}$. In the case of a linear expansion, characterized by the coefficient E ($L_{11} = L_{22} = E, L_{12} = L_{21} = 0$) eqn. (**??**) becomes:

$$\vec{v}_E = E[(\hat{\alpha} \cdot \vec{x})\hat{\alpha} + (\hat{\beta} \cdot \vec{x})\hat{\beta}] - \frac{E(\hat{\gamma} \cdot \vec{x})}{d}[(\hat{\alpha} \cdot \vec{P})\hat{\alpha} + \\ +(\hat{\beta} \cdot \vec{P})\hat{\beta}] - \frac{E}{f}[(\hat{\alpha} \cdot \vec{x})\alpha_3 + (\hat{\beta} \cdot \vec{x})\beta_3]\vec{x} + \\ +\frac{E(\hat{\gamma} \cdot \vec{x})}{fd}[(\hat{\alpha} \cdot \vec{P})\alpha_3 + (\hat{\beta} \cdot \vec{P})\beta_3]\vec{x} \tag{15}$$

Remembering that $(\hat{\alpha}\cdot\vec{x})\hat{\alpha} + (\hat{\beta}\cdot\vec{x})\hat{\beta} + (\hat{\gamma}\cdot\vec{x})\hat{\gamma} = \vec{x}$ and $(\hat{\alpha} \cdot \vec{x})\alpha_3 + (\hat{\beta} \cdot \vec{x})\beta_3 + (\hat{\gamma} \cdot \vec{x})\gamma_3 = f$ we have:

$$\vec{v_E} = E\frac{\hat{\gamma} \cdot \vec{x}}{fd}(\vec{x}P_3 - f\vec{P}) \tag{16}$$

If we analyse, in the general case, the Jacobian matrix **M** of the 2D motion field on the image plane in its singular point $\vec{p}$, by simple, but long calculations (see Appendix A), we obtain the important result:

$$\begin{aligned} Trace\mathbf{L} &= Trace\mathbf{M} \\ Det\mathbf{L} &= Det\mathbf{M} \end{aligned} \tag{17}$$

where

$$\mathbf{L} = \begin{pmatrix} \left.\frac{\partial V_\alpha^\pi}{\partial X_\alpha^\pi}\right|_{\vec{X}^\pi=\vec{P}^\pi} & \left.\frac{\partial V_\beta^\pi}{\partial X_\alpha^\pi}\right|_{\vec{X}^\pi=\vec{P}^\pi} \\ \left.\frac{\partial V_\alpha^\pi}{\partial X_\beta^\pi}\right|_{\vec{X}^\pi=\vec{P}^\pi} & \left.\frac{\partial V_\beta^\pi}{\partial X_\beta^\pi}\right|_{\vec{X}^\pi=\vec{P}^\pi} \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} \left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} \\ \left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} \end{pmatrix} \tag{18}$$

Therefore, given a linear deformation, the center of deformation is projected into a singular point having eigenvalues equal to those of the center of deformation on the plane π, that is the eigenvalues of a linear deformation are perspective invariants. Also the elementary components E, ω and the sum of the squared components of shear (the components of shear depend on the choice of the unit vectors) $S_1^2 + S_2^2$ are perspective invariants. These invariant properties can also be seen in Fig. 2, which illustrates the perspective projection of an expansion (B), a rotation (D) and a shear (F). It is evident that the projected vector field is no more a linear vector field, but has the same kind of singular point.

2. The motion field of a moving plane with linear deformations

In this section we will consider properties of the 2D motion field produced by a translating plane undergoing a linear deformation of the kind discussed in the previous section.

In this section the plane π is assumed to move, so that its distance $d(t)$ from the optical center varies with time and its equation is:

$$\hat{\gamma} \cdot \vec{X} = d(t) \tag{19}$$

If π is rigid and moves by a pure translation $T = (T_1, T_2, T_3)$, the associated 2D motion field on the image plane is:

$$\vec{v}_T = \frac{\hat{\gamma} \cdot \vec{x}}{f d(t)} (\vec{T} f - \vec{x} T_3) \tag{20}$$

which has the same structure of the 2D motion field projected on the image plane by the linear expansion (16). The two 2D motion fields (16) and (20) become identical when:

$$E(t)\vec{P}(t) = -\vec{T} \tag{21}$$

As a consequence the 2D motion field of the plane $\hat{\gamma} \cdot \vec{X} = d(t)$ translating with speed $\vec{T}$ is instantaneously identical to the 2D motion field of a linear expansion on a fixed plane lying in the same position and centered in the point:

$$\vec{Q} = \frac{d(t)\vec{T}}{\hat{\gamma} \cdot \vec{T}} \tag{22}$$

which can be written in the reference system $(O^\pi, \hat{\alpha}, \hat{\beta})$ as:

$$\vec{Q}^\pi = (Q_\alpha^\pi, Q_\beta^\pi) = \frac{d(t)}{\hat{\gamma} \cdot \vec{T}} (\vec{T} \cdot \hat{\alpha}, \vec{T} \cdot \hat{\beta}) \tag{23}$$

The eigenvalue of the expansion is equal to:

$$E_T(t) = -\frac{\hat{\gamma} \cdot \vec{T}}{d(t)} \tag{24}$$

As a consequence, the 2D motion field of a plane undergoing the linear deformation $\vec{V}_D$ (sum of expansion, rotation and shear) with center $\vec{P}$ and the translation $\vec{T}$ is instantaneously equal to the perspective projection of the linear deformation $\vec{V}_D^\pi + \vec{V}_T^\pi$ occurring on the same plane with the center of deformation in an appropriate point $\vec{P}^* = (P_\alpha^*, P_\beta^*)$

$$\begin{aligned} \vec{V}_D^\pi + \vec{V}_T^\pi &= \begin{pmatrix} V_\alpha^\pi \\ V_\beta^\pi \end{pmatrix} = \\ &= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X_\alpha^\pi - P_\alpha^\pi \\ X_\beta^\pi - P_\beta^\pi \end{pmatrix} + \\ &+ E_T \begin{pmatrix} X_\alpha^\pi - Q_\alpha^\pi \\ X_\beta^\pi - Q_\beta^\pi \end{pmatrix} = \mathbf{L}' \begin{pmatrix} X_\alpha^\pi - P_\alpha^* \\ X_\beta^\pi - P_\beta^* \end{pmatrix} \end{aligned} \tag{25}$$

The Jacobian of this linear deformation is :

$$\mathbf{L}' = \mathbf{L} + E_T \mathbf{I} = \begin{pmatrix} L_{11} + E_T & L_{12} \\ L_{21} & L_{22} + E_T \end{pmatrix} \tag{26}$$

The 2D motion field on the image plane is then described by eqs. (13) and (14) replacing $\vec{P}$ with

$\vec{P}^*$ and E with $E + E_T$. The trace and determinant of the Jacobian $\mathbf{M}$ of the singular point of the 2D motion field are:

$$Det\mathbf{M} = Det\mathbf{L}' = L_{11}L_{22} - L_{12}L_{21} + E_T(L_{11} + L_{22}) + E_T^2 \quad (27)$$

$$Tr\mathbf{M} = Tr\mathbf{L}' = L_{11} + L_{22} + 2E_T \quad (28)$$

Let us see how these results can be used to recover some useful motion parameters. If we write the reciprocal of E_T:

$$\frac{1}{E_T} = -\frac{d(0) + (\hat{\gamma} \cdot \vec{T})t}{\hat{\gamma} \cdot \vec{T}} \quad (29)$$

and we set, for simplicity, the time origin so as to have $d(0) = 0$, we obtain $E_T = \frac{1}{t}$: the time variable t can now be interpreted as the time which has to elapse before collision between the moving plane and the optical center of the imaging device, quantity which we will call *time to collision*. It is evident that the value of t can be obtained by a simple derivation of $Tr\mathbf{M}/2$ (i.e. the component of expansion):

$$\frac{\partial(Tr\mathbf{M})}{\partial t} = \frac{\partial}{\partial t}\left[-\frac{\hat{\gamma} \cdot \vec{T}}{(\hat{\gamma} \cdot \vec{T})t}\right] = \frac{(\hat{\gamma} \cdot \vec{T})^2}{[(\hat{\gamma} \cdot \vec{T})t]^2} = \frac{1}{t^2} = E_T^2 \quad (30)$$

The sign of t can be recovered by a further derivation of $Tr\mathbf{M}/2$ and the eigenvalues of the real deformation can be obtained as :

$$Tr\mathbf{L} = L_{11} + L_{22} = Tr\mathbf{M} - 2E_T \quad (31)$$

$$Det\mathbf{L} = Det\mathbf{M} - E_T Tr\mathbf{M} + 2E_T^2 \quad (32)$$

In the same way we can recover the components of the real deformation:

$$E = Tr\mathbf{L}/2 = Tr\mathbf{M} - 2E_T \quad (33)$$

$$\omega = \frac{L_{12} - L_{21}}{2} = \frac{M_{12} - M_{21}}{2} \quad (34)$$

$$S_{tot}^2 = S_1^2 + S_2^2 = \frac{(L_{12} + L_{21})^2 + (L_{11} - L_{22})^2}{2} = \frac{(M_{12} + M_{21})^2 + (M_{11} - M_{22})^2}{2} \quad (35)$$

This result can be extended to the case of the normal rototranslation, i.e. when a rigid rotation ω' with axis perpendicular to the moving plane (so that $\vec{\omega} \parallel \hat{\gamma}$) is added. In this case the location of the singular point on the image plane is different but the motion field is still equivalent to a linear one, and the only change in the matrix $\mathbf{M}$ is that ω has to be replaced by $\omega + \omega'$.

3. Evolution of singular points of the 2D motion field of linear planar deformations

Singular points of the 2D motion field on the image plane, i.e. those points $\vec{p}$ such that $\vec{v}(\vec{p}) = 0$, have been shown to capture many features of the motion of rigid bodies (Verri et al., 1989). For instance, from the time evolution of $\vec{p}$ on the image plane and the analysis of its eigenvalues, it is possible to recover substantial information on the 3D rigid motion. Therefore it is interesting to see whether the analysis of the evolution of singular points of the 2D motion field of deformable objects may provide similar information. In this section we will analyse the evolution of the singular point of the 2D motion field originating from a plane undergoing a linear deformation and a translation

First of all it is useful to notice that the singular point of the 2D motion field projected by a linear planar deformation on the image plane evolves in such a way that the components of rotation and shear remain unchanged over time (see eqs. 34–35). This property allows to distinguish between the 2D motion field produced by a rigid plane undergoing an arbitrary rototranslation and the motion field of a deforming plane undergoing a normal rototranslation. In the former case the components of shear are expected to change with time, while in the latter case they are expected to remain unchanged.

The singular point $\vec{p}$ will change its location on the image plane with time and, as shown in Appendix B, its trajectory will be a conic in the general case (see eqn. (B22)). When the time to collision is very large (see eqn. (B15)) we have:

$$\lim_{t\to\infty}(p_1, p_2) = (\frac{T_1}{T_3}, \frac{T_2}{T_3}) \tag{36}$$

as a consequence the location of the singular point for large values of the time to collision is related to the direction of the 3D translation and this property can be used to recover the direction of translation.

[Fig. 3 near here]

When the plane is perpendicular to the optical axis and the translation is parallel to the optical axis, the trajectory of the singular point $\vec{p}$ has a very simple shape, as shown in Fig. 3A. As shown in Appendix B, the following cases can be found:

- in the case of a pure expansion the singular point moves on a segment with equation (B26) (solid line).
- in the case of a pure rotation the singular point moves on a circle with equation (B30) (broken line).
- in the case of a shear the singular point moves on a hyperbola with equation (B32) (dotted line).

Fig. 4 illustrates 2D motion fields obtained from eqs. (13) and (14) with $\hat{\gamma} = (0, 0, 1)$ and $\vec{T} = (0, 0, -T)$ in the case of a pure expansion (A, B), a pure rotation (D,E) and a pure shear (G,H).

[Fig. 4 near here]

In the first column the 2D motion fields are represented when the time to collision is very large and in the second column the 2D motion field are represented on the image plane just before the collision. The last column reproduces the trajectories of the singular point on the image plane for the three cases. The arrows point toward the collisions.

Because of the rigid motion the singular point $\vec{p}$ may change its qualitative nature during its evolution on the image plane, that is it may vary the value of the trace and of the determinant. The properties of the singular point can be described by analysing the trajectory of the point in the $(Tr\mathbf{M}, Det\mathbf{M})$ plane (Verri, Girosi & Torre 1989) as shown in Fig. 3B. This plane is divided into several regions, according to the sign of $Tr\mathbf{M}$ and of $Det\mathbf{M}$. When a point is above the parabola $Det\mathbf{M} = (Tr\mathbf{M}/2)^2$ (the dotted line in Fig. 3B), the singular point is a spiral; below this parabola the singular point is a node or a saddle point. Below the $Det\mathbf{M}$ axis, the singular point is a saddle point. For $t \to \infty$, from eqn. (??) we obtain that $E_T \to 0$ and the singular point on the image plane has the same eigenvalues as those of the pure deformation. As a consequence when t is very large, the direction of translation can be recovered from the location of the singular point on the image and the deformation can be characterized by the eigenvalues of the singular point. For $t \to 0$ (collision) we have

$$\begin{aligned} Det\mathbf{M} &\to \infty \\ Tr\mathbf{M} &\to \infty \end{aligned} \tag{37}$$

and the eigenvalues of $\mathbf{M}$ become very close to E_T giving immediate information about the time to collision.

It is easy to show that the singular point in the $(Tr\mathbf{M}, Det\mathbf{M})$ plane can never cross the above parabola and the only possible trajectories are shown as solid lines in Fig. 3B.

4. Experimental results

In order to verify the theoretical results presented in previous sections we have designed three sets of experiments. The first set was intended to test whether the invariant properties implied by (??) could be verified with image sequences of deforming planes. The second set aimed at verifying to which extent the recovery of the true 3D motion and of the deformations, outlined in Section 2, could be obtained in practice. In the third set of experiments we analysed image sequences of real deformable objects, such as liquids, textiles and rubber.

Deforming planes

[Fig. 5 near here]

Fig. 5 A and B illustrate synthetic images of a tiger undergoing a pure expansion, while Fig. 5 C and D show images of the same tiger undergoing a shear deformation. Under these conditions the properties of the original deformation were exactly known. A sequence of images of successive deformations was used and the optical flow was computed using the technique described in De Micheli et al. (1993). The optical flow was obtained by first convolving images with a symmetric Gaussian filter $\frac{1}{\sqrt{2\sigma}} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$ with $\sigma = 1.5$ pixels and by using the values of 0.4 and 0.3 for the control parameters d_H and c_H respectively, which allow the selection of reliable displacements. In this case a sparse optical flow is obtained. Fig. 6A and Fig. 6B reproduce the optical flows obtained from the sequence of the expanding and of the deforming tiger respectively. From these sparse optical flows, the best linear vector field through each optical flow can be estimated (see Fig. 6 C and D respectively). As a consequence, it is possible to estimate the eigenvalues of the matrix $\mathbf{M}$ and to recover the deformations of the tiger by using the results presented in Section 2. Table 1 illustrates a comparison between the recovered and the original deformations of Fig. 5.

[Fig. 6 near here]

[Table 1 near here]

The true expansion was $1.98 \cdot 10^{-2}$ frame^{-1} and the experimentally obtained value was correct with an error not greater than 3%. A similar precision was also observed for the image sequences reproducing the shear. Similar results were obtained in other image sequences in which a pattern was deformed under controlled conditions and the angle Θ between the image plane (i.e. the viewing camera) and the deforming plane was varied from 0 to 60 degrees. A comparison between recovered and original deformations, shown in Table 2, indicates a reasonable agreement between the recovered and original deformations when Θ is below 45 degrees. It is evident that the accuracy of the estimation deteriorates when Θ becomes larger than 45 degrees. In this case the second order term of the 2D motion field is predominant and the estimation of the linear term of the 2D motion field becomes rather sensitive to noise. These results suggest that the eigenvalues of linear planar deformations can be useful perspective invariants in a variety of experimental conditions.

[Table 2 near here]

Moving and deforming planes

The same deformations were used in another set of experiments, but in the presence of a known relative motion between the plane and the viewing camera. We have also used synthetic images of clouds deforming with a shear, a rotation and an expansion while the viewing camera was moving towards the deforming plane.

[Fig. 7 near here]

Fig. 7A illustrates a frame of an image sequence of simulated deforming clouds translating toward the camera. Fig. 7B and C illustrate a sparse optical flow obtained from the sequence and its best linear approximation . From this linear approximation it is possible to estimate the different parameters describing the motion and the deformation. The time to collision was estimated by using equation (??). The deformation components were computed from eqns. (33–35). A comparison between the estimated values and the real time to collision and deformation parameters is illustrated in Table 3. The three deformation components were recovered with a precision of about 90%. The estimation of the time to collision obtained with a differentiation (see eqn. (??)) can be rather noisy. In this case, the error can be efficiently reduced with an appropriate filtering (see figure caption). Fig. 7D shows a comparison between the filtered values (diamonds) and the true values (solid line).

[Table 3 near here]

Deforming objects

In the third series of experiments image sequences of liquids and deforming objects were analysed. In these experiments the real deformation could be measured in an indenpedent way and a qualitative and quantitative test of the proposed approach could be obtained.

[Fig. 8 near here]

Fig. 8A and B illustrate two frames of an image sequence taken while a drop of a black liquid was deforming in a jar filled with water. These images were acquired at video rate and the two images in A and B were the first and the 12th in the sequence. In this case the deformation is caused by convection and diffusion. Fig. 8 C and D illustrate optical flows computed with the techniques of De Micheli et al., 1993 and of Campani & Verri, 1992 respectively. The optical flow in C is not dense and is significant only at the edge of the black spot, while the optical flow in D is dense, but blurred. The average expansion around the edges of the black spot computed with the two procedures was 0.0142 and 0.0139 frame^{-1} respectively and corresponds fairly well to the real displacement of the edge of the black liquid, as illustrated in Fig 8E. The edge contours of the deforming drop in the two images in A and B are reproduced with the superimposed optical flow. This flow was magnified by 12 times corresponding to the number of frames separating image A and B. Fig 8 F illustrates the expansion coefficient computed from five consecutive optical flows obtained with the technique of De Micheli et al. (1993).

[Fig. 9 near here]

Fig 9 illustrates the case of a rotating magnet deforming a mixture of water and small dark particles. The angular velocity of the rotating magnet was 0.125 rad/frame. Two images of this deforming liquid are shown in Fig 9 A and B. The optical flow obtained with a correlation technique (described in the figure legend) is shown in Fig 9 C. Contrary to the case of a rigid object, the instantaneous velocity does not increase linearly with the distance from the singular point. However, the angular velocity computed very near the singular point was close to the true angular velocity of the stirring magnet. The decrease of the angular velocity (see Fig. 9 D) at more distant points is caused by the viscosity of the liquid.

Fig 10 illustrates the deformation of an eraser rubber deformed by the pressure exerted by two clamps. One clamp was fixed, while the other was moved by 1 mm. between each frame. The width of the eraser was 25 mm.

[Fig. 10 near here]

Fig 10 A and B illustrate the first and fifth frame of the sequence and the deformation of the squeezed eraser is clearly evident. C and D reproduce a sparse optical flow obtained from the entire image and a dense optical flow around the singular point respectively. The obtained optical flow around the singular point was decomposed in the four elementary components, providing the estimates: -0.014 frame^{-1}, -0.001 rad. frame^{-1}, 0.019 frame^{-1} and -0.001 frame^{-1} for E, ω, S_1 and S_2 respectively. Evidently, the major deformations were E and S_1. If we approximated the real deformation with a linear one, shrinking the rubber in the y direction by a factor 1.04 every frame (i.e. 25/24), the deformation had the two components: $E = -0.018$ frame^{-1} and $S_1 = 0.018$ frame^{-1}. The values of E and S_1 obtained by the proposed approach and those calculated assuming a linear deformation were in reasonable agreement.

The deformation of an elastic textile is illustrated in Fig 11. In this case two parallel sides of the textile were kept fixed, while the other two orthogonal sides were moved apart so as to have an expansion in one direction by a factor 1.02 at every frame, corresponding to an ideal linear deformation with $E = 0.01$ frame^{-1} and $S_1 = -0.01$ frame^{-1}.

[Fig. 11 near here]

Fig 11 A and B illustrate the first and the fifth frame of the sequence in which the textile is deformed. C and D reproduce a subsampled optical flow obtained from the entire image and a dense optical flow around the singular point respectively. The obtained optical flow around the singular point was decomposed in the four elementary components and had the values of $E = 0.010$ frame^{-1}, $\omega = 0.002$ rad. frame^{-1}, $S_1 = -0.014$ frame^{-1}, $S_2 = 0.002$ frame^{-1}.

These values agree with those corresponding to the real linear deformation.

5. Discussion

The aim of this paper is to begin an analysis of image sequences of deforming objects. The major contribution of the paper consists in the identification of some perspective invariants of planar linear deformations (see eqn. (17)). The use of these invariants allows the recovery of the 3D motion and of the parameters describing the deformations in the simple case of normal rototranslations (see Section 2). In addition, the paper shows that it is possible to obtain a meaningful optical flow also in the case of deformable objects, such as liquids and gases.

Optical flow and deformable objects

It is now well established that an almost correct optical flow can be computed from image sequences of moving rigid objects (Barron et al., 1994, Otte and Nagel, 1994) and it is therefore interesting to see whether a reasonable optical flow can be obtained also in more general cases. Our experimentation indicates that the optical flow of fluids and deforming objects can be estimated whenever the underlying dynamics is not too chaotic (Lichtenberg & Lieberman, 1992).

The analysis of deformations

This paper studies the simple case of a plane moving by a normal rototranslation (a motion during which the axis of rotation is perpendicular to the plane itself) and undergoing a linear deformation. The results obtained in sections 1–3 show that the 2D motion field of a linearly deforming and moving plane is at most quadratic. It is also shown that the translation and the deformations occurring on the plane can be recovered by looking at the properties of singular points, as proposed in Verri et al., 1989 for rigid moving objects. The proposed approach can be used to analyse a general deformation occurring at the surface of an opaque object provided that the Taylor expansion of eqn. (1) can be truncated up to the first order, as in the case of a sufficiently small surface element. As a consequence, it is adequate to analyse deformations of opaque objects locally.

Our approach makes use of the computation of optical flow and it is useful also to observe that its estimation and the recovery of its first order properties are usually reliable whenever the 2D motion field is essentially a linear or at most quadratic vector field. Therefore the proposed approach will fail when the approximation of eqn.(1) is valid over a too small surface element and when the normal vector to the surface becomes parallel to the image plane (see Table 2). In this case second order terms of the 2D optical flow will have to be considered, with a considerably heavier computation. The deformations reproduced in Fig. (8–11) could be satisfactorily analysed with the proposed technique, thus indicating that the underlying assumptions are verified in a variety of real cases.

Future work

The analysis of image sequences of deformable objects is an important field with a wide range of relevant applications. The analysis of the blood flow and cardiac motion (Amartur & Vesselle, 1993), and the study of calcium waves in living cells (Gallione et al., 1993) are two major applications to biology. The study of the convection of clouds and of other meteorological phenomena is another important field which may benefit from techniques developed in computer vision. However it is important to observe that these problems have specific constraints and peculiarities and the theoretical results obtained in this paper can certainly be extended to more realistic cases.

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Appendix A
First order properties of the singular point

Eqn. (12) can be rewritten as a function of the elementary deformation components. As $L_{11} = E+S_1$, $L_{22} = E-S_1$, $L_{12} = \omega+S_2$, $L_{21} = S_2-\omega$, we have for v_1:

$$
\begin{aligned}
v_1 = \frac{E}{fd}\{&[(\hat{\alpha}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\alpha}\cdot\vec{P})](\alpha_1 f - x_1\alpha_3) + [(\hat{\beta}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\beta}\cdot\vec{P})](\beta_1 f - x_1\beta_3)\}+\\
&+\frac{S_1}{fd}\{[(\hat{\alpha}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\alpha}\cdot\vec{P})](\alpha_1 f - x_1\alpha_3) - [(\hat{\beta}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\beta}\cdot\vec{P})](\beta_1 f - x_1\beta_3)\}+\\
&+\frac{\omega}{fd}\{[(\hat{\beta}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\beta}\cdot\vec{P})](\alpha_1 f - x_1\alpha_3) - [(\hat{\alpha}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\alpha}\cdot\vec{P})](\beta_1 f - x_1\beta_3)\}+\\
&+\frac{S_2}{fd}\{[(\hat{\beta}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\beta}\cdot\vec{P})](\beta_1 f - x_1\beta_3) - [(\hat{\alpha}\cdot\vec{x})(\hat{\gamma}\cdot\vec{P})+\\
&-(\hat{\gamma}\cdot\vec{x})(\hat{\alpha}\cdot\vec{P})](\alpha_1 f - x_1\alpha_3)\}
\end{aligned}
\tag{A1}
$$

As $\hat{\gamma}\cdot\vec{P} = d$ is the distance between the optical center and the deforming plane, using some

some well known properties of vector calculus (i.e. $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a}$, $(\hat{\alpha}\cdot\vec{x})\alpha_i+(\hat{\beta}\cdot\vec{x})\beta_i+(\hat{\gamma}\cdot\vec{x})\gamma_i = x_i$, $(a_ib_j - b_ia_j) = (\vec{a}\times\vec{b})\cdot\vec{e_k}$) we obtain:

$$
\begin{aligned}
v_1 = {} & \frac{E}{fd}(\hat{\gamma}\cdot\vec{x})(P_3x_1 - P_1f)+ \\
& +\frac{S_1}{fd}[(\vec{P}\times\hat{\beta})\cdot\vec{x}(\alpha_3x_1 - \alpha_1f)+ \\
& +(\vec{P}\times\hat{\alpha})\cdot\vec{x}(\beta_3x_1 - \beta_1f)]+ \\
& +\frac{\omega}{fd}[(\hat{e}_1\times\vec{x})\cdot\hat{\gamma}f - (\hat{e}_3\times\vec{x})\cdot\hat{\gamma}x_1+ \\
& -(\hat{\gamma}\cdot\vec{x})[(\hat{e}_1\times\vec{P})\cdot\hat{\gamma}f + (\hat{e}_3\times\vec{P})\cdot\hat{\gamma}x_1]+ \\
& +\frac{S_2}{fd}(\vec{P}\times\hat{\alpha})\cdot\vec{x})(\alpha_1f - \alpha_3x_1)+ \\
& -(\vec{P}\times\hat{\beta})\cdot\vec{x})(\beta_1f - \beta_3x_1)
\end{aligned}
\tag{A2}
$$

By rearranging eqn.(??) we obtain:

$$
\begin{aligned}
v_1 = {} & \tfrac{E}{fd}(\gamma_1P_3x_1^2 + \gamma_2P_3x_1x_2 + (\gamma_3P_3+ \\
& -\gamma_1P_1)fx_1 - \gamma_2P_1fx_2 - \gamma_3P_1f^2)+ \\
& +\tfrac{S_1}{fd}\{[(\vec{P}\times\hat{\beta})_1\alpha_3 + (\vec{P}\times\hat{\alpha})_1\beta_3]x_1^2+ \\
& +[(\vec{P}\times\hat{\beta})_2\alpha_3 + (\vec{P}\times\hat{\alpha})_2\beta_3]x_1x_2+ \\
& +[(\vec{P}\times\hat{\beta})_3\alpha_3 + (\vec{P}\times\hat{\alpha})_3\beta_3+ \\
& -(\vec{P}\times\hat{\beta})_1\alpha_1 + (\vec{P}\times\hat{\alpha})_1\beta_1]fx_1+ \\
& -[(\vec{P}\times\hat{\beta})_2\alpha_1 + (\vec{P}\times\hat{\alpha})_2\beta_1]fx_2+ \\
& -[(\vec{P}\times\hat{\beta})_3\alpha_1 + (\vec{P}\times\hat{\alpha})_3\beta_1]f^2\}+ \\
& +\tfrac{\omega}{fd}\{[\gamma_2fd + (\vec{P}\times\vec{\gamma})_3\gamma_1)]x_1^2+ \\
& +[-\gamma_1fd + (\vec{P}\times\vec{\gamma})_3\gamma_2]x_1x_2+ \\
& +[(\vec{P}\times\vec{\gamma})_3\gamma_3 - (\vec{P}\times\vec{\gamma})_1\gamma_1]fx_1+ \\
& -[\gamma_3fd + (\vec{P}\times\vec{\gamma})_1\gamma_2]fx_2+ \\
& -[\gamma_2f^2d + (\vec{P}\times\vec{\gamma})_1\gamma_3]f^2\}+ \\
& +\tfrac{S_2}{fd}\{[-(\vec{P}\times\hat{\alpha})_1\alpha_3 + (\vec{P}\times\hat{\beta})_1\beta_3]x_1^2+ \\
& +[-(\vec{P}\times\hat{\alpha})_2\alpha_3 + (\vec{P}\times\hat{\beta})_2\beta_3]x_1x_2+ \\
& +[-(\vec{P}\times\hat{\alpha})_3\alpha_3 + (\vec{P}\times\hat{\beta})_3\beta_3+ \\
& +(\vec{P}\times\hat{\alpha})_1\alpha_1 - (\vec{P}\times\hat{\beta})_1\beta_1]fx_1+ \\
& -[(\vec{P}\times\hat{\alpha})_2\alpha_1(\vec{P}\times\hat{\beta})_2\beta_1]fx_2+ \\
& -[-(\vec{P}\times\hat{\alpha})_3\alpha_1 + (\vec{P}\times\hat{\beta})_3\beta_1]f^2\}
\end{aligned}
\tag{A3}
$$

In the same way a similar expression for v_2 can be obtained and the two components can be written as:

$$
\begin{aligned}
v_1 &= a_{13}x_1^2 + a_{23}x_1x_2 + (a_{33} - a_{11})fx_1+ \\
&\qquad -a_{21}fx_2 - a_{31}f^2 \\
v_2 &= a_{13}x_1x_2 + a_{23}x_2^2 + (a_{33} - a_{22})fx_2+ \\
&\qquad -a_{12}fx_1 - a_{32}f^2
\end{aligned}
\tag{A4}
$$

where:

$$
\begin{aligned}
a_{ij} \equiv {} & E\frac{\gamma_iP_j}{fd} + \omega(\frac{\hat{\gamma}\cdot\hat{e_j}\times\hat{e_i}}{f} - \frac{\gamma_i(\vec{P}\times\hat{\gamma})_j}{fd}) \\
& +S_1\frac{\alpha_j(\vec{P}\times\hat{\beta})_i + \beta_j(\vec{P}\times\hat{\alpha})_i}{fd}+ \\
& S_2\frac{-\alpha_j(\vec{P}\times\hat{\alpha})_i + \beta_j(\vec{P}\times\hat{\beta})_i}{fd}
\end{aligned}
\tag{A5}
$$

Let us now compute the eigenvalues of the Jacobian matrix of the motion field $\mathbf{M}$ in the singular point $\vec{p} = f\vec{P}/P_3$. First we rewrite eqn.(12) as a function of $\vec{p}$:

$$
\begin{aligned}
\vec{v_D} = {} & L_{11}[\hat{\alpha}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\hat{\alpha} + L_{22}[\hat{\beta}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\hat{\beta}+ \\
& +L_{12}[\hat{\beta}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\hat{\alpha} + L_{21}[\hat{\alpha}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\hat{\beta}+ \\
& -\tfrac{1}{f}\{L_{11}[\hat{\alpha}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\alpha_3 + L_{22}[\hat{\beta}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\beta_3+ \\
& L_{12}[\hat{\beta}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\alpha_3 + L_{21}[\hat{\alpha}\cdot(\vec{x} - \tfrac{\hat{\gamma}\cdot\vec{x}}{\hat{\gamma}\cdot\vec{p}}\vec{p})]\beta_3\}\vec{x}
\end{aligned}
\tag{A6}
$$

We define, for simplicity:

$$\Theta = \hat{\alpha} \cdot (\vec{x} - \frac{\hat{\gamma} \cdot \vec{x}}{\hat{\gamma} \cdot \vec{p}} \vec{p}) \tag{A7}$$

$$\Phi = \hat{\beta} \cdot (\vec{x} - \frac{\hat{\gamma} \cdot \vec{x}}{\hat{\gamma} \cdot \vec{p}} \vec{p}) \tag{A8}$$

The motion field $\vec{v}_D$ can now be written as:

$$\begin{aligned} \vec{v}_D &= L_{11}\Theta\hat{\alpha} + L_{22}\Phi\hat{\beta} + L_{12}\Phi\hat{\alpha} + L_{21}\Theta\hat{\beta} + \\ &- \frac{\vec{x}}{f}(L_{11}\Theta\alpha_3 + L_{22}\Phi\beta_3 + L_{12}\Phi\alpha_3 + L_{21}\Theta\beta_3) \end{aligned} \tag{A9}$$

and the components are:

$$\begin{aligned} v_1 &= L_{11}\Theta\alpha_1 + L_{22}\Phi\beta_1 + L_{12}\Phi\alpha_1 + L_{21}\Theta\beta_1 + \\ &- \tfrac{x_1}{f}(L_{11}\Theta\alpha_3 + L_{22}\Phi\beta_3 + L_{12}\Phi\alpha_3 + L_{21}\Theta\beta_3) \\ v_2 &= L_{11}\Theta\alpha_2 + L_{22}\Phi\beta_2 + L_{12}\Phi\alpha_2 + L_{21}\Theta\beta_2 + \\ &- \tfrac{x_2}{f}(L_{11}\Theta\alpha_3 + L_{22}\Phi\beta_3 + L_{12}\Phi\alpha_3 + L_{21}\Theta\beta_3) \end{aligned} \tag{A10}$$

Let us now compute the partial derivative:

$$\begin{aligned} \frac{\partial v_1}{\partial x_1} &= L_{11}\alpha_1\frac{\partial\Theta}{\partial x_1} + L_{22}\beta_1\frac{\partial\Phi}{\partial x_1} + L_{12}\alpha_1\frac{\partial\Phi}{\partial x_1} + \\ & L_{21}\beta_1\frac{\partial\Theta}{\partial x_1} - \tfrac{1}{f}\{L_{11}\Theta\alpha_3 + L_{22}\Phi\beta_3 + L_{12}\Phi\alpha_3 + \\ & + L_{21}\Theta\beta_3\} - \frac{x_1}{f}\{L_{11}\alpha_3\frac{\partial\Theta}{\partial x_1} + L_{22}\beta_3\frac{\partial\Phi}{\partial x_1} + \\ & + L_{12}\alpha_3\frac{\partial\Phi}{\partial x_1} + L_{21}\beta_3\frac{\partial\Theta}{\partial x_1}\} \end{aligned} \tag{A11}$$

As $\Theta(\vec{p}) = \Phi(\vec{p}) = 0$, the above partial derivative, computed in $\vec{p}$ becomes:

$$\begin{aligned} \left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} &= L_{11}(\alpha_1 - \frac{\alpha_3 p_1}{f})\frac{\partial\Theta}{\partial x_1} + \\ & + L_{22}(\beta_1 - \frac{\beta_3 p_1}{f})\frac{\partial\Phi}{\partial x_1} + L_{12}(\alpha_1 - \frac{\alpha_3 p_1}{f})\frac{\partial\Phi}{\partial x_1} + \\ & + L_{21}(\beta_1 - \frac{\beta_3 p_1}{f})\frac{\partial\Theta}{\partial x_1} \end{aligned} \tag{A12}$$

Similarly we have:

$$\begin{aligned} \left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} &= L_{11}(\alpha_2 - \frac{\alpha_3 p_2}{f})\frac{\partial\Theta}{\partial x_2} + \\ & + L_{22}(\beta_2 - \frac{\beta_3 p_2}{f})\frac{\partial\Phi}{\partial x_2} + L_{12}(\alpha_2 - \frac{\alpha_3 p_2}{f})\frac{\partial\Phi}{\partial x_2} + \\ & + L_{21}(\beta_2 - \frac{\beta_3 p_2}{f})\frac{\partial\Theta}{\partial x_2} \end{aligned} \tag{A13}$$

$$\begin{aligned} \left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} &= L_{11}(\alpha_1 - \frac{\alpha_3 p_1}{f})\frac{\partial\Theta}{\partial x_2} + \\ & + L_{22}(\beta_1 - \frac{\beta_3 p_1}{f})\frac{\partial\Phi}{\partial x_2} + L_{12}(\alpha_1 - \frac{\alpha_3 p_1}{f})\frac{\partial\Phi}{\partial x_2} + \\ & + L_{21}(\beta_1 - \frac{\beta_3 p_1}{f})\frac{\partial\Theta}{\partial x_2} \end{aligned} \tag{A14}$$

$$\begin{aligned} \left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} &= L_{11}(\alpha_2 - \frac{\alpha_3 p_2}{f})\frac{\partial\Theta}{\partial x_1} + \\ & + L_{22}(\beta_2 - \frac{\beta_3 p_2}{f})\frac{\partial\Phi}{\partial x_1} + L_{12}(\alpha_2 - \frac{\alpha_3 p_2}{f})\frac{\partial\Phi}{\partial x_1} + \\ & + L_{21}(\beta_2 - \frac{\beta_3 p_2}{f})\frac{\partial\Theta}{\partial x_1} \end{aligned} \tag{A15}$$

$\frac{\partial\Theta}{\partial x_1}$, $\frac{\partial\Theta}{\partial x_2}$, $\frac{\partial\Phi}{\partial x_1}$, $\frac{\partial\Phi}{\partial x_2}$ do not depend on $\vec{x}$ and are equal to:

$$\frac{\partial\Theta}{\partial x_1} = \alpha_1 - \frac{\hat{\alpha} \cdot \vec{p}}{\hat{\gamma} \cdot \vec{p}}\gamma_1 \tag{A16}$$

$$\frac{\partial\Theta}{\partial x_2} = \alpha_2 - \frac{\hat{\alpha} \cdot \vec{p}}{\hat{\gamma} \cdot \vec{p}}\gamma_2 \tag{A17}$$

$$\frac{\partial\Phi}{\partial x_1} = \beta_1 - \frac{\hat{\beta} \cdot \vec{p}}{\hat{\gamma} \cdot \vec{p}}\gamma_1 \tag{A18}$$

$$\frac{\partial\Phi}{\partial x_2} = \beta_2 - \frac{\hat{\beta} \cdot \vec{p}}{\hat{\gamma} \cdot \vec{p}}\gamma_2 \tag{A19}$$

Substituting eqns. (A16–A19) into (A12–A15) we finally obtain:

$$\left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} = L_{11}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{22}(\beta_1 + -\tfrac{\beta_3 p_1}{f})(\beta_1 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\beta_1 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{21}(\beta_1 - \tfrac{\beta_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A20}$$

$$\left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} = L_{11}(\alpha_2 - \tfrac{\alpha_3 p_2}{f})(\alpha_2 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + L_{22}(\beta_2 + -\tfrac{\beta_3 p_2}{f})(\beta_2 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + L_{12}(\alpha_2 - \tfrac{\alpha_3 p_2}{f})(\beta_2 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + +L_{21}(\beta_2 - \tfrac{\beta_3 p_2}{f})(\alpha_2 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) \tag{A21}$$

$$\left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} = L_{11}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_2 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + L_{22}(\beta_1 + -\tfrac{\beta_3 p_1}{f})(\beta_2 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\beta_2 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) + +L_{21}(\beta_1 - \tfrac{\beta_3 p_1}{f})(\alpha_2 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_2) \tag{A22}$$

$$\left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} = L_{11}(\alpha_2 - \tfrac{\alpha_3 p_2}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{22}(\beta_2 + -\tfrac{\beta_3 p_2}{f})(\beta_1 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{12}(\alpha_2 - \tfrac{\alpha_3 p_2}{f})(\beta_1 - \tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{21}(\beta_2 - \tfrac{\beta_3 p_2}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A23}$$

Now we can compute the trace and the determinant of the Jacobian matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} \left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} \\ \left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} \end{pmatrix} \tag{A24}$$

We can choose the reference system $(O^\pi, \hat{\alpha}, \hat{\beta})$ such that $\hat{e_2} \equiv \hat{\beta}$, and therefore: $\beta_1 = \beta_3 = 0$, $\beta_2 = 1$, $\alpha_2 = \gamma_2 = 0$, $\alpha_1 = \gamma_3$, $\alpha_3 = -\gamma_1$. As a consequence, we have:

$$\left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} = L_{11}(\alpha_1 - \frac{\alpha_3 p_1}{f})(\alpha_1 - \frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{12}(\alpha_1 - \frac{\alpha_3 p_1}{f})(-\frac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A25}$$

$$\left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} = L_{22} + L_{12}(-\frac{\alpha_3 p_2}{f}) \tag{A26}$$

$$\left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} = L_{12}(\alpha_1 - \frac{\alpha_3 p_1}{f}) \tag{A27}$$

$$\left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} = L_{11}(-\frac{\alpha_3 p_2}{f})(\alpha_1 - \frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{22}(-\frac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{12}(-\frac{\alpha_3 p_2}{f})(-\frac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{21}(\alpha_1 - \frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A28}$$

and the trace of $\mathbf{M}$ becomes:

$$Tr\mathbf{M} = \left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} + \left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} = = L_{11}(\alpha_1 - \frac{\alpha_3 p_1}{f})(\alpha_1 - \frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + +L_{12}(\alpha_1 - \frac{\alpha_3 p_1}{f})(-\frac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{22} + L_{12}(-\frac{\alpha_3 p_2}{f}) \tag{A29}$$

Remembering that $\alpha_1 = \gamma_3$ and $\gamma_1 = -\alpha_3$, we have:

$$Tr\mathbf{M} = \frac{L_{11}}{f}[\alpha_1(\gamma_3 f + \gamma_1 p_1) + \alpha_3\frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}(\gamma_1 p_1 + +\gamma_3 f)] + L_{22} + \frac{L_{12}}{f}[(\gamma_3 f + \gamma_1 p_1)(-\frac{p_2}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \alpha_3 p_2] \tag{A30}$$

$$Tr\mathbf{M} = \frac{L_{11}}{f}[\gamma_3(\hat{\gamma}\cdot\vec{p}) + \alpha_3(\hat{\alpha}\cdot\vec{p})] + \\ + L_{22} + \frac{L_{12}}{f}[(\hat{\gamma}\cdot\vec{p})\frac{\gamma_1 p_2}{\hat{\gamma}\cdot\vec{p}} - \gamma_1 p_2] \tag{A31}$$

$$Tr\mathbf{M} = L_{11}\frac{f}{f} + L_{22} + L_{12}\cdot 0 = \boxed{L_{11} + L_{22}} \tag{A32}$$

In a similar way we compute the determinant of $\mathbf{M}$:

$$Det\mathbf{M} = \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} = \\ = [L_{11}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)][L_{22} + L_{12}(-\tfrac{\alpha_3 p_2}{f})] + \\ - [L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})][L_{11}(-\tfrac{\alpha_3 p_2}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{22}(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + L_{12}(-\tfrac{\alpha_3 p_2}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{21}(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)] \tag{A33}$$

$$Det\mathbf{M} = L_{11}L_{22}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{22}L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{12}^2(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)(-\tfrac{\alpha_3 p_2}{f}) + \\ + L_{11}L_{12}(-\tfrac{\alpha_3 p_2}{f}) + \\ - L_{11}L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)(-\tfrac{\alpha_3 p_2}{f}) + \\ - L_{12}^2(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)(-\tfrac{\alpha_3 p_2}{f}) + \\ - L_{22}L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(-\tfrac{\hat{\beta}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ - L_{12}L_{21}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A34}$$

$$Det\mathbf{M} = L_{11}L_{22}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) + \\ + L_{11}L_{12}(-\tfrac{\alpha_3 p_2}{f}) - L_{11}L_{12}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 + \\ - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1)(-\tfrac{\alpha_3 p_2}{f}) - L_{12}L_{21}(\alpha_1 - \tfrac{\alpha_3 p_1}{f})(\alpha_1 - \tfrac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) \tag{A35}$$

As we have already found that:

$$(\alpha_1 - \frac{\alpha_3 p_1}{f})(\alpha_1 - \frac{\hat{\alpha}\cdot\vec{p}}{\hat{\gamma}\cdot\vec{p}}\gamma_1) = 1 \tag{A36}$$

the previous equation becomes:

$$Det\mathbf{M} = L_{11}L_{22} - L_{12}L_{21} + L_{11}L_{12}(-\frac{\alpha_3 p_2}{f}) + \\ - L_{11}L_{12}(-\frac{\alpha_3 p_2}{f}) = \boxed{L_{11}L_{22} - L_{12}L_{21}} \tag{A37}$$

Therefore the two matrices:

$$\mathbf{L} = \begin{pmatrix} \left.\frac{\partial V_\alpha^\pi}{\partial X_\alpha^\pi}\right|_{\vec{X}^\pi = \vec{P}^\pi} & \left.\frac{\partial V_\alpha^\pi}{\partial X_\beta^\pi}\right|_{\vec{X}^\pi = \vec{P}^\pi} \\ \left.\frac{\partial V_\beta^\pi}{\partial X_\alpha^\pi}\right|_{\vec{X}^\pi = \vec{P}^\pi} & \left.\frac{\partial V_\beta^\pi}{\partial V_\beta^\pi}\right|_{\vec{X}^\pi = \vec{P}^\pi} \end{pmatrix}$$

and

$$\mathbf{M} = \begin{pmatrix} \left.\frac{\partial v_1}{\partial x_1}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_1}{\partial x_2}\right|_{\vec{x}=\vec{p}} \\ \left.\frac{\partial v_2}{\partial x_1}\right|_{\vec{x}=\vec{p}} & \left.\frac{\partial v_2}{\partial x_2}\right|_{\vec{x}=\vec{p}} \end{pmatrix}$$

have the same trace and the same determinant, and, of course, the same eigenvalues.

Appendix B
Singular point location

We want to study the evolution of the position of the singular point of the 2-D motion field produced by the 3-D motion of a plane with normal vector $\hat{\gamma}$ translating with uniform speed $\vec{T}$ and undergoing a linear deformation with center $\vec{P}(t)$ and matrix $\mathbf{L}$. The solution can be simplified remembering that at each time t, we can replace the 3-D motion with the equivalent 2-D linear deformation on the plane $\hat{\gamma}\cdot\vec{X} = d(t)$. This field is

easily written in the reference system of the plane, $(O^\pi, \hat{\alpha}, \hat{\beta})$ and the equations for the singular point of this equivalent field on the plane π are:

$$V_\alpha^\pi = L_{11}(X_\alpha^\pi - P_\alpha^\pi) + L_{12}(X_\beta^\pi - P_\beta^\pi) + \\ + E_{tr}(X_\alpha^\pi - Q_\alpha^\pi) = 0$$

$$V_\beta^\pi = L_{21}(X_\alpha^\pi - P_\alpha^\pi) + L_{22}(X_\beta^\pi - P_\beta^\pi) + \\ + E_{tr}(X_\beta^\pi - Q_\beta^\pi) = 0 \tag{B1}$$

where $E_{tr} = -\frac{\hat{\gamma}\cdot\vec{T}}{d}$ and $\vec{Q}^\pi = -\frac{1}{E_{tr}}(\vec{T}\cdot\hat{\alpha}, \vec{T}\cdot\hat{\beta})$. The singular point of the 2-D motion field on the image plane is the projection of the singular point of the equivalent 2-D field (B1). The singular point of the equivalent 2-D motion (B1) on the plane π is:

$$X_\alpha^* = \frac{(Det\mathbf{L})P_\alpha^\pi + E_{tr}^2 Q_\alpha^\pi + L_{11}E_{tr}P_\alpha^\pi + L_{22}E_{tr}Q_\alpha^\pi + L_{12}E_{tr}(P_\beta^\pi - Q_\beta^\pi)}{(Det\mathbf{L}) + E_{tr}((Tr\mathbf{L}) + E_{tr})}$$

$$X_\beta^* = \frac{(Det\mathbf{L})P_\beta^\pi + E_{tr}^2 Q_\beta^\pi + L_{11}E_{tr}Q_\beta + L_{22}E_{tr}P_\beta^\pi + L_{21}E_{tr}(P_\alpha^\pi - Q_\alpha^\pi)}{(Det\mathbf{L}) + E_{tr}((Tr\mathbf{L}) + E_{tr})} \tag{B2}$$

Now it is useful to write the coordinates of the singular point (B2) in the reference system $(O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$. As we have taken the 3D position of O^π as $\vec{O} = d(\hat{\gamma}\cdot\hat{e}_1, \hat{\gamma}\cdot\hat{e}_2, \hat{\gamma}\cdot\hat{e}_3)$, the 3D coordinates of the point $(X_\alpha^\pi, X_\beta^\pi)$ of π are:

$$X_i = X_\alpha^\pi \alpha_i + X_\beta^\pi \beta_i + d\gamma_i \qquad i = 1, 2, 3 \tag{B3}$$

and the position of the singular point is given by:

$$X_i^* = X_\alpha^* \alpha_i + X_\beta^* \beta_i + d\gamma_i \qquad i = 1, 2, 3 \tag{B4}$$

Replacing X_α^* and X_β^* with the expressions of (B2), we have:

$$X_i^* = \frac{N_i}{(Det\mathbf{L}) + E_{tr}((Tr\mathbf{L}) + E_{tr})} \tag{B5}$$

where N_i is defined as:

$$N_i = (Det\mathbf{L})P_i + E_{tr}^2 Q_i + E_{tr}\{L_{11}[(\vec{P}\cdot\hat{\alpha})\alpha_i + \\ + (\vec{Q}\cdot\hat{\beta})\beta_i + (\vec{P}\cdot\hat{\gamma})\gamma_i] + L_{22}[(\vec{Q}\cdot\hat{\alpha})\alpha_i + \\ + (\vec{P}\cdot\hat{\beta})\beta_i + (\vec{P}\cdot\hat{\gamma})\gamma_i] + L_{12}(\vec{P}\cdot\hat{\beta} - \vec{Q}\cdot\hat{\beta})\alpha_i + \\ + L_{21}(\vec{P}\cdot\hat{\alpha} - \vec{Q}\cdot\hat{\alpha})\beta_i\} \tag{B6}$$

Remembering the perspective projection formula:

$$x_i^* = f\frac{X_i^*}{X_3^*} \tag{B7}$$

we have:

$$x_1^* = f\frac{N_1}{N_3}$$
$$x_2^* = f\frac{N_2}{N_3} \tag{B8}$$

Now we introduce the temporal dependences for $\vec{P}$, E_{tr} and $\vec{Q}$ as:

$$P_i(t) = P_i(0) + T_i t \tag{B9}$$

$$E_{tr}(t) = -\frac{(\hat{\gamma}\cdot\vec{T})}{d(0) + (\hat{\gamma}\cdot\vec{T})t} \tag{B10}$$

$$\vec{Q}(t) = \frac{d(0) + (\hat{\gamma}\cdot\vec{T})t}{(\hat{\gamma}\cdot\vec{T})}\vec{T} \tag{B11}$$

In order to simplify the calculations it is convenient to set, as in the text, the time origin so that $d(0) = 0$. In this case for $t = 0$ the plane passes through the optical center. The equations become:

$$E_{tr}(t) = -\frac{1}{t} \tag{B12}$$

$$\vec{Q}(t) = \vec{T}t \tag{B13}$$

introducing (B9),(B12) and (B13) into eqn. (B6) we have:

$$N_i = \frac{1}{t}\{T_i[(Det\mathbf{L})t^2 - (Tr\mathbf{L}) + 1] + (Det\mathbf{L})P_i(0)t+$$
$$+[L_{12}(\vec{T}\cdot\hat{\beta})\alpha_i + L_{21}(\vec{T}\cdot\hat{\alpha})\beta_i]t - L_{11}(\vec{P}(0)\cdot\hat{\alpha})\alpha_i+$$
$$-L_{22}(\vec{P}(0)\cdot\hat{\beta})\beta_i + L_{12}(\vec{P}(0)\cdot\hat{\beta})\alpha_i+$$
$$+L_{21}(\vec{P}\cdot\hat{\alpha})\beta_i \tag{B14}$$

and the position of the singular point becomes:

$$x_1^* = f\left\{\frac{T_1}{T_3} + \frac{\Theta_1 t+\Phi_1}{T_3^2[(Det\mathbf{L})t^2-(Tr\mathbf{L})t+1]+T_3(Det\mathbf{L})P_3(0)t}\right\}$$
$$x_2^* = f\left\{\frac{T_2}{T_3} + \frac{\Theta_2 t+\Phi_2}{T_3^2[(Det\mathbf{L})t^2-(Tr\mathbf{L})t+1]+T_3(Det\mathbf{L})P_3(0)t}\right\} \tag{B15}$$

where

$$\Theta_1 = T_3[(Det\mathbf{L})P_1(0) + L_{12}(\vec{T}\cdot\hat{\beta})\alpha_1+ +L_{21}(\vec{T}\cdot\hat{\alpha})\beta_1] - T_1P_3(0) \tag{B16}$$

$$\Phi_1 = -T_3[L_{11}(\vec{P}(0)\cdot\hat{\alpha})\alpha_1 + L_{22}(\vec{P}(0)\cdot\hat{\beta})\beta_1+ +L_{12}(\vec{P}(0)\cdot\hat{\beta})\alpha_1 + L_{21}(\vec{P}(0)\cdot\hat{\alpha})\beta_1] \tag{B17}$$

$$\Theta_2 = T_3[(Det\mathbf{L})P_2(0) + L_{12}(\vec{T}\cdot\hat{\beta})\alpha_2+ +L_{21}(\vec{T}\cdot\hat{\alpha})\beta_2] - T_2P_3(0) \tag{B18}$$

$$\Phi_2 = -T_3[L_{11}(\vec{P}(0)\cdot\hat{\alpha})\alpha_2 + L_{22}(\vec{P}(0)\cdot\hat{\beta})\beta_2+ +L_{12}(\vec{P}(0)\cdot\hat{\beta})\alpha_2 + L_{21}(\vec{P}(0)\cdot\hat{\alpha})\beta_2] \tag{B19}$$

From (B15) it is evident that, fot $t \to \infty$, i.e. when the plane is far from the collision, the position of the singular point is $\vec{x}^* = (T_1/T_3, T_2/T_3)$.

The curve described by the evolution of the point is found eliminating t from the system. With a few steps we have (we drop the stars in the following):

$$t = \frac{\Phi_1(x_2T_3 - fT_2) - \Phi_2(x_1T_3 - fT_1)}{\Theta_2(x_1T_3 - fT_1) - \Theta_1(x_2T_3 - fT_2)} \tag{B20}$$

From the first of eqs. (B15) and (B20), dividing by $(x_1T_3 - fT_1)$ we find a second-order polynomial equation describing the trajectory of the singular point:

$$T_3(Det\mathbf{L})[\Phi_1(x_2T_3 - fT_2) - \Phi_2(x_1T_3 - fT_1)]^2+$$
$$+[(Det\mathbf{L}) - T_3(Tr\mathbf{L})][\Phi_1(x_2T_3 - fT_2)+$$
$$-\Phi_2(x_1T_3 - fT_1)][\Theta_2(x_1T_3 - fT_1) - \Theta_1(x_2T_3+$$
$$-fT_2)] + [\Theta_2(x_1T_3 - fT_1) - \Theta_1(x_2T_3 - fT_2)]^2+$$
$$-(\Theta_2^2\Phi_1 - \Theta_1\Theta_2\Phi_2)(x_1T_3 - fT_1)f - (\Theta_1^2\Phi_2+$$
$$-\Theta_1\Theta_2\Phi_1)(x_2T_3 - fT_2) = 0 \tag{B21}$$

This equation can be simplified with a change of variables: $y_1 = (x_1 - f\frac{T_1}{T_3})$, $y_2 = (x_2 - f\frac{T_2}{T_3})$:

$$(Det\mathbf{L})[(\Phi_1y_2) - (\Phi_2y_1)]^2 + [(Det\mathbf{L})+$$
$$-T_3(Tr\mathbf{L})][(\Phi_1y_2) - (\Phi_2y_1)][(\Theta_2y_1) - (\Theta_1y_2)]+$$
$$+[(\Theta_2y_1) - (\Theta_1y_2)]^2 - (\Theta_2^2\Phi_1 - \Theta_1\Theta_2\Phi_2)y_1+$$
$$-(\Theta_1^2\Phi_2 - \Theta_1\Theta_2\Phi_1)y_2 = 0 \tag{B22}$$

In this way we have proved that the curve described by the singular point is a conic.

Now we can see in more detail the equations of the conics in the three cases of Fig. 3, with the translation perpendicular to the image plane ($T_1 = T_2 = 0$, $T_3 = T$) with the deforming plane parallel to the image plane and a deformation described just by one elementary component. In this case $\gamma_1 = \gamma_2 = 0$, $P(0)\cdot\hat{\gamma} = 0$ and we can choose the reference systems so as to have $\hat{\alpha} \equiv \hat{e}_1$, $\hat{\beta} \equiv \hat{e}_2$. In the case of a pure expansion (i.e. $L_{11} = L_{22} = E$, $L_{12} = L_{21} = 0$) eqn. (B15) becomes:

$$x_1 = f\frac{EP_1(0)}{T(Et-1)} \tag{B23}$$
$$x_2 = f\frac{EP_2(0)}{T(Et-1)}$$

If $Et \neq 1$ we can write:

$$T(Et-1)x_1 - EP_1(0)f = 0$$
$$T(Et-1)x_2 - EP_2(0)f = 0 \qquad \text{(B24)}$$

and we obtain:

$$t = \frac{EP_1(0)f + Tx_1}{TEx_1} = \frac{EP_2(0)f + Tx_2}{TEx_2} \qquad \text{(B25)}$$

The equation of the curve is the straight line:

$$P_1(0)x_2 - P_2(0)x_1 = 0 \qquad \text{(B26)}$$

In the case of a pure rotation (i.e. $L_{11} = L_{22} = 0$ and $L_{12} = -L_{21} = \omega$) the system becomes:

$$x_1 = f\frac{\omega^2(P_1(0)\omega t - P_2(0)}{T(\omega^2 t^2 + 1))}$$
$$x_2 = f\frac{\omega^2(P_2(0)\omega t + P_1(0)}{T(\omega^2 t^2 + 1))} \qquad \text{(B27)}$$

multiplying the first equation by x_2, the second by x_1 and subtracting the two equations we obtain:

$$t = \frac{1}{\omega}\frac{P_1(0)x_1 + P_2(0)x_2}{P_1(0)x_2 - P_2(0)x_1} \qquad \text{(B28)}$$

from eqs. (??) and (??) we have:

$$x_1T^2(P_1(0)x_1 + P_2(0)x_2)^2 + x_1T^2(P_1(0)x_2 +$$
$$-P_2(0)x_1)^2 - TP_1(0)f(P_1(0)x_1 +$$
$$+P_2(0)x_2)(P_1(0)x_2 - P_2(0)x_1) +$$
$$+\omega P_2(0)Tf(P_1(0)x_2 - P_2(0)x_1)^2 = 0 \qquad \text{(B29)}$$

and after a few steps we obtain the equation of a circle:

$$Tx_1^2 + Tx_2^2 + \omega f P_2(0)x_1 - \omega f P_1(0)x_2 = 0 \qquad \text{(B30)}$$

When $L_{11} = -L_{22} = S_1$ and $L_{12} = L_{21} = 0$, the singular point is located at:

$$x_1 = f\frac{-S_1TP_1(0)(S_1t+1)}{T^2(1-S_1^2t^2)} \qquad \text{(B31)}$$
$$x_2 = f\frac{-S_1TP_2(0)(S_1t-1)}{T^2(1-S_1^2t^2)}$$

and, after a few steps, the curve is found to be the hyperbola:

$$2Tx_1x_2 - S_1P_2(0)fx_1 + S_1P_1(0)fx_2 = 0 \qquad \text{(B32)}$$

FIGURE LEGENDS

Fig. 1: The reference system $(O^\pi, \hat{\alpha}, \hat{\beta})$ solid to the plane π with normal unit vector $\hat{\gamma}$ and the reference system $(O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ solid to the imaging device, with O coinciding with the optical center and the image plane coinciding with the plane $X_3 = f$.

Fig. 2: Linear deformations (A: expansion; C: rotation; E: shear) and their projections (B, D and F respectively) on the image plane. The normal vector to the plane $\hat{\gamma}$ is (0.92,0,0.40).

Fig. 3: A: trajectories of the singular point on the image plane in the case of a translation and an expansion (solid line, see eqn. B26), a rotation (broken line, see eqn. (B30)) and a shear (dotted line, see eqn. (B32)). The translating plane is always parallel to the image plane. B: allowed trajectories of the singular point in the Tr**M** and Det**M** plane.

Fig. 4: Simulation of the 2D motion field on the image plane of moving and deforming planes. A: 2D motion field on the image of an expanding and translating plane with center of deformation $\vec{P}^\pi = (148, 148)$ pixel and eigenvalue $E = 1$ frame^{-1}. The moving plane is perpendicular to the optical axis with distance from the optical center $d = 3000$ pixel, and translates with $\vec{T} = (0, 0, -500)$ pixel frame^{-1}. B: Same as A, but with $d = 500$ pixel. C: evolution of the position of the singular point on the image plane for the expanding and translating plane in A and B. D: 2D motion field on the image of a rotating and translating plane with center of deformation $\vec{P}^\pi = (148, 148)$ pixel and angular speed $\omega = 1$ rad frame^{-1}. The moving plane is perpendicular to the optical axis with distance from the optical center $d = 3000$ pixel, and translates with $\vec{T} = (0, 0, -500)$ pixel frame^{-1}. E: same as in D but with $d = 500$ pixel. F: evolution of the position of the singular point on the image plane for the rotating and translating plane in D and E. G: 2D motion field on the image of a deforming and translating plane with center of deformation $\vec{P}^\pi = (148, 148)$ pixel and component of shear $S_1 = 1$ frame $^{-1}$. The moving plane is perpendicular to the optical axis with distance from the optical center $d = 3000$ pixel, and translates with $\vec{T} = (0, 0, -500)$ pixel frame^{-1}. H: same as in G but with $d = 500$ pixel. I: evolution of the position of the singular point on the image plane for the deforming and translating plane in G and H. In C, F and I the arrows point toward the collisions.

Fig. 5: A and B: two images (256×256) of an expanding tiger. C and D: two images of the same tiger undergoing a shear deformation.

Fig. 6: Examples of the sparse optical flows obtained with the algorithm of De Micheli et al., 1993 for the expanding tiger (Fig. 5 A and B) and for the deforming tiger (Fig. 5 C and D) respectively. Images were convolved with a Gaussian filter $\frac{1}{\sqrt{2\sigma}} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$ with $\sigma = 1.5$ pixels and the control parameters d_H and c_H were set equal to 0.4 and 0.3 respectively, so as to obtain a sparse but reliable optical flow (see De Micheli et al, 1993 for further details). C and D are the best linear vector field through the sparse optical flows shown in A and B respectively, obtained with a least squares fit. From these estimation of the optical flow, the deformation components were obtained from eqs. (33–35)

Fig. 7: A reproduces an image of synthetic deforming clouds. B and C are the sparse optical flow (computed with the technique of De Micheli et al., 1993 with the values of $1.5, 0.4$ and 0.3 for σ, d_H and c_H respectively) and its best linear approximation respectively. D reproduces a plot of the time-to-collision against number of frame. This values are obtained by filtering the values computed with eqn. (30) with the filter: $\overline{ttc}(k) = \frac{a}{a+1}[ttc(k-1) - 1] + \frac{1}{a+1} ttc(k)$ with a depending on the measurement error (see De Micheli et al., 1993 for further details).

Fig. 8: A and B reproduce two images of a black drop of liquid deforming in water. C, D are the optical flows computed with the technique of De Micheli et al., 1993 ($\sigma = 1.5, d_H = 0.4, c_H = 0.3$) and Campani and Verri, 1992 (after a gausssian filtering with $\sigma = 1.5$ pixels and with patches of 41×41 pixels) respectively. E represents the contour of the drop at the two frames of A and B with the superimposed optical flow computed at

the edges of the first frame with the technique of Campani and Verrin and magnificated by a factor equal to the number of frames (12) separating the two images. F reproduces the coefficient of expansion obtained from the best linear approximation of the sparse optical flow computed with the technique of De Micheli et al., 1993.

Fig. 9: A and B reproduce two images of a stirring magnet deforming a mixture of water and dark particles. C is an example of an optical flow computed with a correlation-based technique. This technique consists of the comparison with a suitable distance function, of the grey level pattern in a patch of 41×41 pixels around each pixel of the frame with other patterns of the same size in the successive frame centered in a set of points appropriately shifted from the original position. The shift minimizing the distance function is taken as the true motion. D is the angular velocity against the distance from the center of rotation. The angular velocity at the distance ρ was computed as the average value of the rotational component of the optical flow in the annulus inside the two circles with radius $\rho \pm 5$ pixels.

Fig. 10: A and B reproduce two frames of an eraser rubber squeezed by two clamps. C and D reproduce the subsampled optical flow obtained from the entire image (computed with the technique of Campani and Verri (1990) over patches of 41×41 pixels) and a dense flow relative to the region near the singular point (the black dot). The analysis of the optical flow around the singular point provides an estimate of the deformation parameters.

Fig. 11: A and B reproduce two frames of an elastic textile expanded along y direction. C and D reproduce the optical flow obtained from the entire image and a dense flow relative to the region near the singular point (black dot). The optical flow was computed as in the case of Fig. 10.

Table 1: Estimations of the components of deformation obtained from the sparse optical flows relative to the sequences of Fig. 5. Eqs. (30) and (33–35) were used. σ is the standard deviation.

Table 2: The recovery of the components of deformation for different values of the angle Θ between the deforming plane and the image plane. Eqs. (30) and (33–35) were used. σ is the standard deviation.

Table 3: Estimation of the true time to collision and components of deformation for a translating and deforming plane. The average estimated expansion, rotation and shear were $2.08 \cdot 10^{-2}$ frame^{-1} , $1.74 \cdot 10^{-2}$ frame^{-1} and $1.98 \cdot 10^{-2}$ frame^{-1} respectively, while the true values were $1.98 \cdot 10^{-2}$ frame^{-1} , $1.74 \cdot 10^{-2}$ frame^{-1} and $1.98 \cdot 10^{-2}$ frame^{-1}.