

KRULL–GABRIEL DIMENSION OF DOMESTIC STRING ALGEBRAS

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Gena Puninski died on April 29, 2017. The other two authors dedicate this paper to his memory

ABSTRACT. We calculate the Krull–Gabriel dimension of the category of modules over any domestic string algebra, in particular showing that it is finite, thus confirming a conjecture of Schröer. We also compute the Cantor–Bendixson rank of each point of its Ziegler spectrum and determine the topology on this space.

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1. INTRODUCTION

The Gabriel dimension of a ring R is one measure of complexity of the category, $R\text{-Mod}$, of R -modules. It is computed by localising so as to make the finite length objects zero and then, successively and if necessary transfinitely, repeating the process on the quotient categories which arise. For finite-dimensional algebras it is too coarse, always giving the zero category at the first localisation.

A much finer dimension, the Krull–Gabriel dimension, $\text{KG}(R)$, of a ring, was introduced by Geigle in his thesis; see [7] (the terminology “Krull–Gabriel dimension” was introduced later). This is a variant of Gabriel dimension: it is applied with the (skeletally small) category, $R\text{-mod}$, of finitely presented modules in place of R ;

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hence the initial category to be localised is the category $(R\text{-mod}, \mathbf{Ab})$ of additive functors from $R\text{-mod}$ to the category \mathbf{Ab} of abelian groups. Furthermore it is the finitely presented finite length objects which are made zero at each stage.

This dimension also turns out [14, Lemma B.9] to be the m -dimension of the lattice of pp formulas for R -modules, equivalently of the lattice of pointed finite-dimensional modules. That dimension is a modification of the elementary Krull dimension, introduced by Garavaglia [6] and developed further in [30, §8] and [15, Ch. 10].

A result of Auslander (see [19, Prop. 7.2.8]) says that R has finite representation type if and only if $\text{KG}(R) = 0$. It is also known (see a remark in [8, p. 135] or use factorisable systems of morphisms; see [17, 0.2]) that this dimension is undefined (or “ $= \infty$ ”) for any wild algebra. But also many tame algebras have undefined KG-dimension. For instance, see [8, Cor. 4.2] or [10, 7.5] for the case for tubular algebras, and see [27, Prop. 2] for non-domestic string algebras.

It has been conjectured (see [15, 7.2.17]) that a finite-dimensional algebra R has Krull–Gabriel dimension if and only if it is (tame) domestic. There is some, admittedly rather limited, evidence to support this conjecture. A strong form of the conjecture (see [15, 9.1.15]) proposes that if R is a finite-dimensional algebra and if $\text{KG}(R)$ exists, then $\text{KG}(R)$ is finite.

Herzog [11] and Krause [13] proved that no finite-dimensional algebra has KG-dimension 1. Geigle [7] proved that if R is tame and hereditary, then $\text{KG}(R) = 2$ (and he also established this value for certain other classes of algebras, [8]). A few examples of algebras with larger Krull–Gabriel dimension were known. For instance, Burke and Prest [2] and Schröer [27] considered series of string algebras with KG-dimensions taking every finite value ≥ 3 . In a recent paper, [22], the third author showed that any 1-domestic (non-degenerate) string algebra has KG-dimension 3.

We prove the following result, conjectured by Schröer (see [15, 8.1.22], [29, p. 420]).

Theorem 1.1. *If R is a domestic string algebra, then the Krull–Gabriel dimension of R is finite. Indeed $\text{KG}(R) = n + 2$, where n is the maximal length of a path in the bridge quiver of R .*

The bridge quiver (see Section 3.2) is easily calculated from a presentation of R as a string algebra.

Since every non-domestic string algebra has KG-dimension ∞ , this completes the calculation of Krull–Gabriel dimension for string algebras.

Corollary 1.2. *If R is a string algebra, then the Krull–Gabriel dimension of R is finite if and only if R is of domestic (including finite) representation type.*

The Krull–Gabriel dimension of R also has a strong relationship with the transfinite powers of the radical of $R\text{-mod}$. For example, Krause proves that if an algebra R has finite Krull–Gabriel dimension $\text{KG}(R) = m$, then $\text{rad}_R^{\omega(m+1)} = 0$ (see [14, 8.14]). We obtain the following corollary (confirming, for string algebras, a general conjecture made by Schröer [29, p. 423]).

Corollary 1.3. *If R is a string algebra, then $\text{KG}(R) = m$ if and only if $\text{rad}_R^{\omega(m-1)} \neq 0$ and $\text{rad}_R^{\omega m} = 0$.*

We will make use of the Ziegler spectrum - a topology on the set of indecomposable pure-injectives - in particular, its Cantor–Bendixson analysis. Because there are no superdecomposable pure-injectives, the Cantor–Bendixson rank of a point is equal to its KG-dimension, and a mixture of topological and algebraic approximation arguments allows us to give a recipe which is in terms of the bridge quiver of R , for the rank of each module. One can see from the examples we use that this is easily applied in practice.

We first compute the ranks of string modules: neighbourhood bases of open sets for the 1- and 2-sided points already can be extracted from [20] and [23]. Then we extend this to find neighbourhood bases for Prüfer points. In these computations we will use the lattice, pp_R , of pp formulas, calculating the m-dimensions of certain intervals and hence the KG-dimensions of points. Having computed the dimensions of the Prüfer points, we will use elementary duality to obtain the ranks of the adic modules. By that stage we have enough information to compute the ranks of the generic points. Indeed, we give a complete description of the topology, in that we describe a neighbourhood basis of open sets for each point. We deduce, for example, that the Ziegler spectrum of a domestic string algebra is a T_0 space: there do not exist distinct points which are topologically indistinguishable.

This is a companion paper to [23], where the classification of indecomposable pure-injective modules was completed. After the heavy combinatorics of [23], we can reap the benefits of having that classification in hand, and the arguments in this paper are more direct and guided by our intuition.

Throughout K is an algebraically closed field, and the rings R considered will be K -algebras. We expect that none of the results depend on K being algebraically closed. The category of left R -modules is denoted $R\text{-Mod}$; $R\text{-mod}$ denotes the category of finitely presented modules.

We will use the notation pp_R for the lattice of pp formulas (in one free variable) for R -modules, equivalently for the lattice of pointed finitely presented modules.

In this paper we have to assume a fair bit of background, referring to other sources for many definitions, basic facts, and results. We do, however, define, or at least describe, the main ideas using a number of illustrative examples in order to convey the ideas. Sources on material specific to string algebras include the original paper [3], the very readable introductory sections in [26], and introductory sections of various papers, such as [23], of the second and third authors. For pure-injectivity, the Ziegler spectrum, and various other concepts and techniques arising from the model theory of modules, we use [19] as a comprehensive source, but there are many other accounts.

Most of the finite- and infinite-dimensional modules that we are interested in are described in terms of strings and bands. First we introduce the combinatorics of strings and bands, as well as the bridge quiver which shows how the bands are connected. Then we describe the associated modules and we recall the combinatorial description of the morphisms between them. We also describe the Ziegler spectrum, the topological space which has the indecomposable pure-injectives for its points. In Section 7 we introduce the ranks and dimensions, the computation of which is the main aim of this paper. That is accomplished in Section 9, and we give a ‘recipe’ for computing, not just the global rank but also the rank of every point of the Ziegler spectrum. It is easy to apply: first one computes the bridge quiver, and from that it is a simple matter to read off the ranks of points.

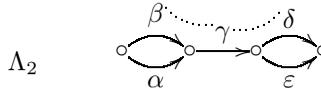
For each type of point, having determined how to compute the rank, we find a neighbourhood basis. Thus we obtain a complete description of the Ziegler spectrum, extending the results of [2], on modules over the algebras Λ_n , to all domestic string algebras.

2. THE ALGEBRAS

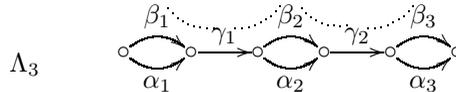
Throughout, R will be a **string algebra** (and usually domestic). That is, $R = KQ/I$ is the path algebra of a quiver with relations where:

- every vertex of Q has at most two incoming arrows and at most two outgoing arrows,
- for every arrow α there is at most one arrow β with $\beta\alpha \neq 0$ and at most one arrow γ with $\alpha\gamma \neq 0$, and
- I can be generated by monomials.

The Kronecker algebra $\circ \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} \circ$ is an example, as is the quiver below with relations $\gamma\beta = 0$ and $\delta\gamma = 0$:



More generally Λ_n is obtained by linking n Kronecker quivers in a similar way. Here is Λ_3 , the relations being $0 = \gamma_1\beta_1 = \beta_2\gamma_1 = \gamma_2\beta_2 = \beta_3\gamma_2$:

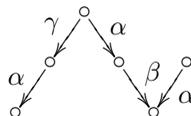


The next example has the quiver shown with relations $\beta^2 = \gamma^2 = 0$ and the longer relation $\beta\alpha\gamma = 0$ (which is shown by a dotted curve):



A **string** is a walk along arrows and inverse arrows in Q (avoiding zero relations and immediate backtracks), and a **band** is a walk which returns to its starting point and is **primitive** - not a proper power of any string - and (a technically useful restriction) has the form $\alpha c\beta^{-1}$ for some arrows α, β , and finite string c (we allow the empty string). We will consider finite strings and also (singly and doubly) infinite strings, describing a string of the form $u_1 u_2 \dots$, where each u_i is a direct or inverse arrow and $u_1 \dots u_n$ is a string for every n , as a **1-sided (infinite) string**, and one of the form $\dots u_{-1} u_0 u_1 \dots$ as a **2-sided string**.

An example of a string, $\alpha\gamma\alpha^{-1}\beta^{-1}\alpha$ over the algebra X_1 above, is shown by the following diagram (our convention is to draw direct arrows from the upper right to the lower left, and inverse arrows from the upper left to the lower right):



Taking a result for a definition, we will say that a string algebra is **domestic** if there are just finitely many bands (see [24, §11]). We say that two bands b and b' are **cyclically equivalent** if b' is a cyclic permutation of b , and **equivalent** if b' is cyclically equivalent to either b or b^{-1} (note that only one of these may occur). We will often use the term “band” for the equivalence class of a band and refer to a particular member of that class as a representative. If n is the number of equivalence classes of bands over R , then R is said to be **n -domestic**. For instance the algebra X_1 above is 1-domestic, with just the band $b = \alpha\gamma\alpha^{-1}\beta^{-1}$ up to equivalence, whereas Λ_2 has two bands (up to equivalence), $\alpha\beta^{-1}$ and $\varepsilon\delta^{-1}$.

Let $G_{2,3}$ denote the Gelfand–Ponomarev algebra whose quiver consists of one vertex and two loops α, β , with relations $\alpha\beta = \beta\alpha = \alpha^2 = \beta^3 = 0$. This algebra is not domestic - it is easy to see infinitely many primitive cyclic walks - or note that we have different bands, $\alpha\beta^{-1}$ and $\alpha\beta^{-2}$, beginning with the same letter (cf. Theorem 2.1 below).

In this paper we will deal only with domestic string algebras; equivalently for every arrow α , there is at most one band beginning with α . This and other required facts about the combinatorics of domestic string algebras can be found in [24, 26] and [21]; we state the following from [24, §11] for ease of reference. Band modules are defined in Section 4.3.

Theorem 2.1. *Suppose that R is a domestic string algebra and that b, b' are bands (hence each begins with a direct letter and ends with an inverse letter).*

- (1) *If b and b' have the same first letter, then $b = b'$.*
- (2) *If M is a b -band module, M' is a b' -band module, and they have an isomorphic simple submodule, then b is equivalent to b' . In particular if both b and b' contain the string $\beta^{-1}\alpha$, then b' is a cyclic permutation of b .*

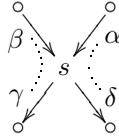
A **socle pair** of a band $b = \alpha \dots \beta^{-1}$ is the string $\beta^{-1}\alpha$ or a substring of b of the form $\delta^{-1}\gamma$. Such a pair defines, where the arrows meet, an element in the socle of the corresponding band modules (Section 4.3) and in any string module $C(w)$ (Section 4.2) when w contains the pair. Dually, a substring of b of the form $\delta\gamma^{-1}$ is a **top pair**, since it defines a simple submodule of the corresponding band module modulo its radical. So, by 2.1(2), over a domestic string algebra different bands have no socle pair, nor (by a dual argument) top pair, in common.

3. STRINGS, BANDS, AND BRIDGES

3.1. String orderings and m -dimension. At each vertex s of a quiver Q we partition the direct and inverse arrows entering s into two sets $H_{s,\pm 1}$ such that, when a walk arrives at s , it enters by a direct or inverse arrow from one set and leaves via a letter in the other set. More precisely we require that each set contain at most one direct arrow and at most one inverse arrow and that if $l_1, l_2 \in H_{s,i}$ are *letters* (that is, arrows or inverse arrows), then $l_1^{-1}l_2$ is not a string. The strings of length greater than one are placed into $H_{s,\pm 1}$ according to their first letters. We also put into $H_{s,i}$ a string $1_{s,i}$ of length 0 so as to be able to refer to walks which terminate at s .

For example, with reference to the diagram below where the relations are $\delta\alpha = 0 = \gamma\beta$, the letters pointing to s are $\alpha, \beta, \gamma^{-1}, \delta^{-1}$. If we put α into $H_{s,1}$, then it follows that $\gamma^{-1} \in H_{s,-1}$; also β must be put into $H_{s,-1}$, and then it follows that

$\delta^{-1} \in H_{s,1}$. In the diagram we show $H_{s,1}$ to the right and $H_{s,-1}$ to the left of s :



We totally order the strings in each of these sets by setting $c < d$, for $c, d \in H_{s,i}$, if $d = c\alpha d'$ for some d' and α or if $c = d\beta^{-1}c'$ for some c' and β or if $c = e\gamma^{-1}c'$ and $d = e\delta d'$ for some c', d', γ, δ . When it comes to looking at an occurrence of a particular vertex s in the diagrammatic representation of a string, one of the sets will be representing choices “to the right”, while the other set represents choices “to the left”. The meaning of this is purely local (to that occurrence of that vertex) but the terminology is useful. All this extends naturally to 1-sided infinite strings, each of these being fitted in, simply by comparing any particular finite string to a long-enough initial segment of the given infinite string; we use $\widehat{H}_{s,i}$ for these larger totally ordered sets.

Take, for instance, the Kronecker algebra \widetilde{A}_1 , and let s be the vertex where α ends. Then one could take $H_{s,1}$ to consist of all strings beginning (on the left) with α , together with $1_{s,1}$, in which case $H_{s,-1}$ consists of all strings beginning with β and $1_{s,-1}$. Each set H_i is ordered as a chain, for example $1_1 < \alpha$ in H_1 and $\beta\alpha^{-1} < \beta\alpha^{-1}\beta$ in H_{-1} . In fact H_1 is the following chain of type $\omega + \omega^*$, where ω^* denotes the ordering opposite to that of ω :

$$1_1 < \alpha\beta^{-1} < (\alpha\beta^{-1})^2 < \dots < (\alpha\beta^{-1})^n \alpha < \alpha\beta^{-1}\alpha < \alpha.$$

To place the infinite word $u = (\alpha\beta^{-1})^\infty$ in relation to these, note that the strings of the form $(\alpha\beta^{-1})^n \alpha$ are greater than u and $u > (\alpha\beta^{-1})^m$ for each m .

The m -dimension of a modular lattice L is defined by setting it to be 0 if L has finite length and, beyond that, by collapsing at each step of a possibly transfinite induction intervals of finite length; the reader may wish to look now at the precise definition in Section 8.1. For instance the m -dimension of the chain $\omega + \omega^*$ equals 1: at the first step of the m -dimension analysis we collapse together all the points in the copy of ω and all the points in the copy of ω^* , but nothing else, obtaining a 2-element lattice, which then collapses at the second step.

For a more complicated example, consider the following 1-domestic string algebra:

$$X_3 \quad \alpha \circlearrowleft \beta$$

with relations $\alpha^2 = \beta^2 = \alpha\beta = 0$. Let us include in H_1 (for the unique vertex s) all strings beginning with β or β^{-1} and 1_1 , so H_{-1} consists of all strings that start with α or α^{-1} and 1_{-1} . Note that H_1 contains the following descending chain of type ω^* :

$$\dots < \beta(\alpha^{-1}\beta)^3 < \beta(\alpha^{-1}\beta)^2 < \beta\alpha^{-1}\beta < \beta.$$

Each interval in this poset is a chain of type $\omega + \omega^*$: for instance the interval between $\beta\alpha^{-1}\beta$ and β consists of an ascending chain $\beta\alpha^{-1}\beta(\alpha\beta^{-1})^n$, $n = 1, \dots$, and, above it, a descending chain $\beta\alpha^{-1}\beta(\alpha\beta^{-1})^m\alpha$, $m = 1, \dots$. From this it is easily checked that the m -dimension of H_1 equals 2.

We define the **m-dimension** of a string $u \in \widehat{H}_i$ as the infimum of m -dimensions of intervals $[c, d]$ in H_i such that $c < u < d$ in \widehat{H}_i . For instance from the above description it follows that the string $u = (\alpha\beta^{-1})^\infty$ over \widetilde{A}_1 has m -dimension 1.

The consonance between this definition and that of the Krull–Gabriel dimension of string modules is Proposition 8.2 and Theorem 8.1.

For another example, consider the 1-sided (periodic) string $v = (\beta\alpha^{-1})^\infty \in \widehat{H}_1$ over X_3 . Note that v is the infimum of the descending chain of finite strings appearing above, with consecutive intervals of this chain having m-dimension 1. It is easily derived that the m-dimension of v is 2.

The calculation of the m-dimension of the chains H_i is straightforward (see [28, Thm. 4.3]).

3.2. The bridge quiver. Recall that R is a domestic string algebra.

It is easy to see that a band b cannot overlap b^{-1} in any string and b cannot touch a different band b' . Also two copies of b cannot overlap each other in a string, for otherwise b would contain a substring of the form $\alpha \dots \alpha$ which contradicts [21, 3.4]. Furthermore we have the following.

Lemma 3.1. *Let R be a domestic string algebra, let $b = \alpha \dots \beta^{-1}$ be a band, and let $c = d \dots e$ (where d, e are direct or inverse arrows) be a cyclic permutation of another band. Then b cannot overlap c in any string.*

Proof. Looking at socles and tops we see that b cannot be a subword of c or vice versa. Thus it suffices to consider the string

$$u = \alpha x d y \beta^{-1} z e,$$

where $b = \alpha x d y \beta^{-1}$, $c = d y \beta^{-1} z e$, and x, y, z denote strings of unspecified length. We claim that the following is a string:

$$v = \alpha x d y \beta^{-1} z e d y \beta^{-1}.$$

Otherwise v contains a forbidden path t as a subword whose letters are either all direct or all inverse. Note that the following underbraced and overbraced parts of v are strings: the first equals u , and the second is an initial part of c^2 :

$$\underbrace{\alpha x d y \beta^{-1} z e d y \beta^{-1}} \quad \text{and} \quad \overbrace{\alpha x d y \beta^{-1} z e d y \beta^{-1}}.$$

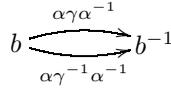
Thus t must protrude to the left and right of their common (braced twice) substring $d y \beta^{-1} z e$. But then $c = d y \beta^{-1} z e$ consists entirely of inverse letters, a contradiction.

Thus v is a string. Because R is domestic, it follows that $v = b^n$ for some n , so c is a substring of b^n . Then b and c have a common socle or top; hence c is equivalent to b , a contradiction. □

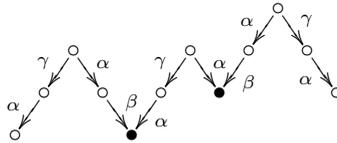
The bridge quiver of R shows the bands and the strings which directly connect them. Suppose that b and b' are different bands and u is a string such that bub' is again a string. Following [28, p. 664] say that u is a **bridge** from b to b' if u contains no band as a substring and neither can u be written as $u = u_1 u_2$ with both u_1, u_2 non-empty and such that $u_1 b'' u_2$ is a string for some band b'' .

Then the **bridge quiver**, $\mathcal{B} = \mathcal{B}(R)$, has for its vertices a chosen representative for each cyclic equivalence class of bands and an arrow, labelled c , from b to b' if c is a bridge from b to b' . Since R is domestic, it is easily checked that \mathcal{B} is finite and has no oriented cycles (cf. [26, §4.4]).

For instance, over X_1 , let us choose $b = \alpha\gamma\alpha^{-1}\beta^{-1}$ and b^{-1} as the vertices of \mathcal{B} . Then \mathcal{B} is the following quiver:



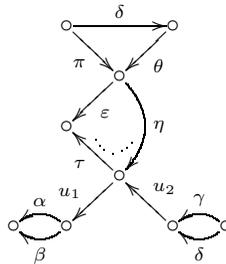
the bridge $b \xrightarrow{\alpha\gamma\alpha^{-1}} b^{-1}$ being visible in the following string:



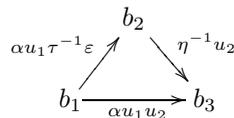
where the marked points show where the bands join the bridge.

For another example, if we choose $b = \alpha\beta^{-1}$, $b' = \varepsilon\delta^{-1}$ and their inverses for the vertices of the bridge quiver over Λ_2 , then the bridge quiver is the disjoint union of $b' \xrightarrow{\varepsilon\gamma} b$ and the inverse of this, $b^{-1} \xrightarrow{\gamma^{-1}\varepsilon^{-1}} b'^{-1}$.

To give one more example, consider the following 3-domestic string algebra:



where the relations are $\eta\pi = 0$, $\varepsilon\theta = 0$, $\tau\eta = 0$, $u_1\eta = 0$, $\tau u_2 = 0$, $\beta u_1 = 0$, and $u_2\delta = 0$. We choose as representatives of bands up to cyclic equivalence $b_1 = \alpha\beta^{-1}$, $b_2 = \pi\delta^{-1}\theta^{-1}$, $b_3 = \gamma\delta^{-1}$, and their inverses. Then the bridge quiver is the disjoint union of the diagram below and its “inverse”:



It may be that choices of representatives for bands can affect the shape of the bridge quiver, though we know no example, but the questions we investigate here are not sensitive to this. From now on we will assume that we have fixed representatives for cyclic equivalence classes of bands, hence vertices of the bridge quiver \mathcal{B} .

If w is a string, then (by Lemma 3.1) it can be uniquely written as $w = c_1 b_1^{k_1} c_2 \dots c_s b_s^{k_s} c_{s+1}$, where $b_i \in \mathcal{B}$, $k_i \geq 0$, possibly ∞ when $i = 1, s$, and where each c_i is a (possibly empty) string which contains no band. Though the c_i ($i \neq 1, s + 1$) may not be bridges, we may consider c_i as a string connecting b_{i-1} to b_i , hence the corresponding path $b_1 c_2 \dots c_s b_s$. We say that the **band-length** of w is $s - 1$. Note that if b'_i is another representative of the cyclic class of b_i , then b'_i is a substring of b_i^2 . Therefore (by enlarging k_i if necessary) we can get a string w' containing at least s bands from some other choice, \mathcal{B}' , of representative bands. So, for instance, the maximal possible band-length of a string does not depend on the choice of \mathcal{B} .

If b is a band, then the **right indent** of b is the maximal band-length of a string starting with b , and the **left indent** is the maximum possible band-length of a string starting with b^{-1} . For instance over X_1 , the right indent of the band $b = \alpha\gamma\alpha^{-1}\beta^{-1}$ equals 1, and its left indent is 0.

4. THE MODULES

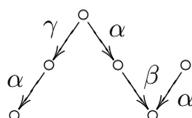
Since we compose actions of arrows from right to left it is natural to consider left R -modules.

4.1. Pure-injective modules and the Ziegler spectrum. Over an artin algebra, R , finite-length modules are pure-injective. More generally, a module over such a ring is **pure-injective** if it is a direct summand of a direct product of finite-dimensional modules (over general rings we cannot use this as the definition). The indecomposable pure-injective modules over domestic string algebras were classified in [23]. We will briefly introduce these modules; more details can be found in that paper.

Over any ring there is just a set of indecomposable pure-injectives up to isomorphism, and they were organised into a topological space by Ziegler. This space, the (left) **Ziegler spectrum**, ${}_R\mathbf{Zg}$, is (quasi)compact and has a basis of compact open sets (see [19, 5.1.22]). That basis can be described in various ways; here are two. Given any morphism $A \xrightarrow{f} B$ in $R\text{-mod}$ we set $(f) = \{N \in {}_R\mathbf{Zg} : (A, N)/\text{im}(f, N) \neq 0\}$, the set of indecomposable pure-injectives N , such that there is a morphism from A to N which does not factor through f . Also, given any pair $\varphi > \psi$ of pp formulas, set $(\varphi/\psi) = \{N \in {}_R\mathbf{Zg} : \varphi(N)/\psi(N) \neq 0\}$, the set of points on which the pp-pair φ/ψ is open (this is Ziegler’s original definition). The sets of the form (φ/ψ) are exactly those of the form (f) and form a basis of open sets for the topology on ${}_R\mathbf{Zg}$.

Theorem 4.1 (See [19, 5.3.36, 5.3.37]). *If R is an artin algebra, then the isolated, that is, open, points of ${}_R\mathbf{Zg}$ are exactly the finite-dimensional indecomposables, and, together, these are dense in ${}_R\mathbf{Zg}$.*

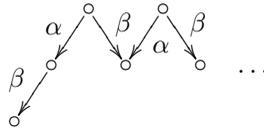
4.2. String modules: finite- and infinite-dimensional. To every string u , finite or infinite, we assign a **string module** $M(u)$. Roughly, this is obtained, from the corresponding walk through the quiver, by placing a 1-dimensional vector space at each vertex on the walk (including the starting vertex if there is one), then taking the direct sum of these vector spaces and equipping the result with an R -module structure, with the actions of the arrows given by u . We give an example, and a precise definition can be found in [3], [24]. The module $M(u)$ is indecomposable and it is also isomorphic to $M(u^{-1})$:



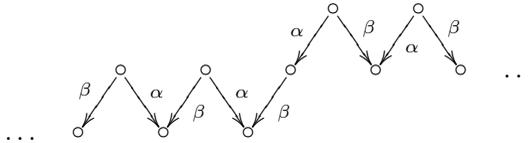
This represents the string module $M(u)$ where $u = \alpha\gamma\alpha^{-1}\beta^{-1}\alpha$, a basis for which is shown by small circles, with the action of the path algebra on these basis elements shown by the arrows (absence of an arrow acting on a basis element should be interpreted as the zero action).

Over a domestic string algebra ([24, §11, Prop. 1]), any infinite string w which is not completely periodic¹ is **almost periodic**, meaning eventually periodic in each direction but, if 2-sided, not completely periodic. Given such a string and looking to, say, the right, there is, by eventual periodicity in that direction, a map, the **shift endomorphism** of $M(w)$, which either moves basis elements on the right further to the right or moves them inwards (eventually annihilating them). In the first case we say that w is **expanding** to the right, otherwise **contracting** ([24, §3]). Correspondingly the associated string module $C(w)$ (see Section 4.4) will be formed using the direct product (if expanding) or direct sum (if contracting) of the spaces generated by the basis elements on the right. Similarly in the other direction, so a 2-sided string could be expanding in both directions, contracting in both directions, or mixed.

For example, the string $\beta(\alpha\beta^{-1})^\infty$ over X_3 is expanding, and so $C(w)$ is the direct product module and has its shift endomorphism annihilating the left-most basis element and moving every other one two places to the right:



For another example, the two-sided string $w = \infty(\beta\alpha^{-1})\beta(\alpha\beta^{-1})^\infty$ over X_3 is expanding on the right and contracting on the left. Thus the corresponding mixed module $C(w)$ is the submodule of the direct product module which consists of all sequences which are eventually 0 to the left:



All the finite-dimensional string modules and the modules $C(w)$ for w an infinite string are indecomposable pure-injectives [24].

4.3. Band modules: finite- and infinite-dimensional. To each band b we associate a family of indecomposable modules. The procedure is similar to that in defining string modules: take an indecomposable $K[T, T^{-1}]$ -module L , that is, a vector space V with an automorphism which is the action of T , deposit the vector space V at the first vertex and also at each vertex visited going once round the band b , then link these by identity maps (for the actions of the arrows) until the last step, where we link the last-deposited copy of V to the first-placed one using the action of T . This is similar to the procedure used to define string modules, but the vector spaces placed at vertices of a walk might have dimension > 1 and we “join the ends of the string together with a twist”. This **band module** (or b -band module if we wish to be more precise), denote it $M(b, L)$, is indecomposable, and if we replace the band by any cyclic permutation of it or by its inverse, then we obtain a module which is either isomorphic to this or to the module we would get if we started with the $K[T]$ -module given by V with the action of T^{-1} (the difference

¹If b is a band, then the completely periodic string $^\infty b^\infty$ is not associated to any indecomposable pure-injective. Indeed it is the image, under a representation embedding as in the following section, of $_{K[T]}K[T, T^{-1}]$, the pure-injective hull of which is far from indecomposable.

being whether the arrow which carries the action of T faces in the same or opposite direction). The indecomposable finite-dimensional $K[T, T^{-1}]$ -modules are factors of the ring by powers of maximal ideals and are n -fold self-extensions of simple modules, so the indecomposable finite-dimensional modules supported on a band are parametrised by $\text{maxspec}(K[T, T^{-1}]) \times \mathbb{N}$, where \mathbb{N} denotes the set of positive integers.

All the indecomposable finite-dimensional modules have now been listed, and there are no cases of isomorphism between them other than those already mentioned. The morphisms between these modules also have been completely described in combinatorial terms; see Section 5.

The description that we have given of the modules associated to a band b is essentially the description of a functor from $K[T, T^{-1}]$ -Mod to R -Mod. Up to equivalence, there are two such functors, the difference between them being the orientation of the arrow where we put the twist by T . It does not matter which we use since they have the same image. Any such functor is a representation embedding in the strong sense that it preserves indecomposability and reflects isomorphism ('strong' because the usual definition asks this just for finite-dimensional modules) and hence induces a homeomorphic embedding of Ziegler spectra.

Theorem 4.2 (See [19, 5.5.9]). *If $F : S\text{-Mod} \rightarrow R\text{-Mod}$ is a representation embedding (in the above strong sense), then F preserves pure-injectivity and induces a homeomorphic embedding of ${}_S Zg$ as a closed subset of ${}_R Zg$.*

We can deduce the following description of the infinite-dimensional indecomposable pure-injective band modules. To each band (up to equivalence) b and simple $K[T, T^{-1}]$ -module S we have its image, $M(b, S)$, a **quasisimple** module. We denote the n -fold self-extension of this module (which is the image of the n -fold self-extension of S) by $M(b, S[n])$. There is a ray of irreducible monomorphisms starting at that module, and the direct limit along this ray is the **Prüfer module** $\Sigma(b, S)$. Dually, there is the inverse limit of the coray of irreducible epimorphisms ending at that quasisimple, the **adic** module $\Pi(b, S)$. Finally there is a unique generic module $G = G_b$ associated to the band b , obtained by applying the above process to the $K[T, T^{-1}]$ -module $K(T)$. Each of these is an indecomposable pure-injective R -module which is the image, under a representation embedding as above, of the corresponding indecomposable pure-injective $K[T, T^{-1}]$ -module. See [5, 18, 25] and [19, §8.1] for more on these modules.

Theorem 4.2 also applies with S being the K -path algebra of some orientation of \widehat{A}_n , where n and the orientations of the arrows of \widehat{A}_n are chosen so that it is the obvious intermediary between $K[T, T^{-1}]$ -Mod and the b -band modules in R -Mod.

The relation between the structure of the category $K[T, T^{-1}]$ -mod and the infinite-dimensional points of ${}_{K[T, T^{-1}]} Zg$, as well as the description of that space, is well-known; similarly for $K\widehat{A}_n$. We will make considerable use of this knowledge, which can be found in, for example, [19, §8.1.2].

4.4. The list of modules. In summary, this is the complete (by [23]) list of isomorphism types of indecomposable pure-injective modules over a domestic string algebra, with parametrising sets as described earlier (we follow the notation of [24]):

- (1) finite-dimensional string modules;
- (2) finite-dimensional band modules;
- (3) infinite direct sum string modules;

- (4) infinite direct product string modules;
- (5) infinite mixed string modules;
- (6) Prüfer modules;
- (7) adic modules;
- (8) generic modules.

We will group these modules in the following way: (1) and (3)-(5) will be referred to as **string** modules; (2) and (6)-(8) will be referred to as **band** modules.

We will fix the following notation throughout the rest of the paper:

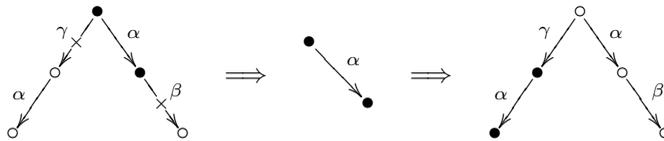
- $A(x)$ will be used for any indecomposable pure-injective module associated to the string or band x ;
- $B(x)$ will be used for any indecomposable band-type module associated to the band x ;
- $C(x)$ will denote the indecomposable pure-injective string-type module associated to the string x .
- We will use the notation $M(x)$ only to refer to the direct sum module associated to the string x .

5. THE MORPHISMS

5.1. Graph maps. A complete description of the maps between string and band modules has been developed by Krause and Crawley–Boevey in a series of papers [12], [5], and [4]. We recall this briefly. Let u be a (finite or infinite) string. Suppose that v is a substring of u which is closed under successors in u , meaning that at its endpoint(s) in u the next arrow(s), if there are any, point to rather than away from the end-vertices of v ; we say that v is an **image substring** of u . Clearly this occurrence of v as a substring of u gives a submodule of $C(u)$ which is the image of an embedding $C(v) \rightarrow C(u)$. For instance, if $u = \alpha\gamma\alpha^{-1}\beta^{-1}$ is a string over X_1 , then α is an image substring of u and α^{-1} is not.

Dually, if v is closed under predecessors in u , meaning that at its endpoint(s) in u the next arrow(s), if any, point away from the end-vertices of v - we say that v is a **factor substring** of u - then this occurrence of v as a substring of u defines a natural epimorphism from $C(u)$ to a copy of $C(v)$. For the example u just above, α^{-1} is a factor substring, whereas α is not.

A **graph map** from the string module $C(u)$ to the string module $C(u')$ is any map which is a composition of such an epimorphism with a monomorphism of the sort described, so it is obtained by matching a factor substring of u with an image substring of u' . Then (see the references) any morphism between string modules is a finite linear combination of graph maps. Here is an example over X_1 :



Morphisms to and from band modules have a similar description. Given a band module $B = M(b, S[n])$, a **graph map** to or from B is described combinatorially (and algebraically) similarly to the above, where we allow the string associated to B to be a large enough power of b (thus, a suitable-length, depending on the other module, segment of ${}^\infty b^\infty$). Then any morphism between B and a string module or a module associated to a different band (including b^{-1}) is a linear combination of

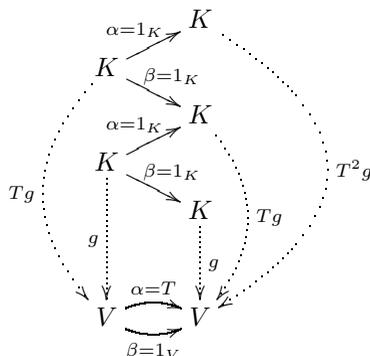
graph maps. The maps between B and modules associated to the same band are linear combinations of graph maps and maps which are compositions of irreducible maps (as seen in the Auslander–Reiten quiver).

Example 5.1. Consider the Kronecker algebra $\circ \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} \circ$ and the band $b = \beta\alpha^{-1}$.

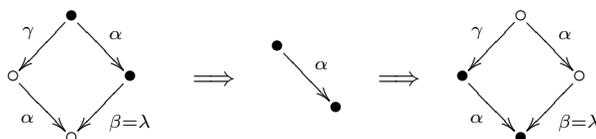
If L is an indecomposable finite-dimensional $K[T, T^{-1}]$ -representation, then the corresponding band module is given by the following, where V is the underlying vector space of L :

$$V \begin{matrix} \xrightarrow{\alpha=T} \\ \xleftarrow{\beta=1_V} \end{matrix} V.$$

Consider the string module $M(v)$ where $v = \beta\alpha^{-1}\beta\alpha^{-1}$. Fix some non-zero $g \in \text{Hom}_K(K, V)$, equivalently element of V , and map the left end of the string module $M(v)$ to this element in $M(b, L)$. This then determines a graph map as shown below, which will be an embedding iff the dimension of V is at least 3:



Here is an example of an endomorphism of a band module which is not the image of a morphism under one of the standard representation embeddings from \widetilde{A}_n -modules. In the following diagram of X_1 -modules, the first map is the obvious epimorphism and the second is the inclusion:



6. DUALITY

We will make use of a couple of dualities. One is the duality $(-)^* = \text{Hom}_K(-, K)$ between categories of finite-dimensional left and right R -modules. We need the following observations about the action of this on certain modules. Suppose that b is a band and that $M = M(b, S)$ is a quasisimple module. Consider the Hom-dual M^* of M . One can check that M^* is the quasisimple right module over R corresponding to the same “band” b (of course all the arrows are now reversed) with, depending on how one chooses to parametrise these right band modules, the same or inverse value of the parameter λ .

The other duality that we will use is elementary duality D , which is essentially the Auslander–Gruson–Jensen duality (see [19, §10.3]) between the categories of finitely presented functors but which induces dualities on other structures. For instance (see [19, §1.3]) D provides an anti-isomorphism between the lattices of right and left pp-formulas over R , taking a pp formula φ to its dual $D\varphi$. Furthermore (see [19, §5.4]) the map $(\varphi/\psi) \mapsto (D\psi/D\varphi)$ between basic open sets extends to open (and closed) sets of the right and left Ziegler spectra of R and, as such, defines an isomorphism between the lattices of open (respectively closed) subsets of these Ziegler spectra which preserves infinite unions (resp. infinite intersections). If, for example, KG-dimension is defined, then it is given a bijection $N \mapsto DN$ between points of the right and left Ziegler spectra (see [19, 5.4.20]).

In the case of domestic string algebras, from the non-existence [23, 5.6] of superdecomposable pure-injectives it follows [19, 5.3.24, 5.4.12, 5.4.18] that the points of the right and left Ziegler spectra are paired up, $N \rightarrow DN \rightarrow D^2N = N$, by elementary duality, which is a homeomorphism between these spaces and which, restricted to finite-dimensional indecomposables, is the usual duality $\text{Hom}_K(-, K)$. In the case of $K[T]$ -modules, the elementary dual of a Prüfer module is the adic module with the same parameter, and this coincides with its dual using, say, the duality $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. The latter duality applied to an adic module gives the direct sum of the corresponding Prüfer module with many copies of \mathbb{Q}/\mathbb{Z} , but the elementary dual of the adic is identifiable as the unique isolated point in the corresponding Ziegler-closed set (the Ziegler-closure of the adic and of the Prüfer are homeomorphic). Essentially the same applies over tame hereditary algebras. Simply from the topological information, the generic left and right modules are dual to each other.

General immediate consequences of there being the duality homeomorphism are that it restricts to a homeomorphism between the Ziegler-closure of any point and that of its dual and that a basis of open neighbourhoods of any point dualises to a basis of open neighbourhoods of its dual.

7. CANTOR–BENDIXSON RANK

The Cantor–Bendixson analysis of any space T runs as follows. At the first step we identify the isolated=open points of this space, give them CB rank 0, and then remove them, leaving the first derivative, T' , of T . In the case of the Ziegler spectrum of an artin algebra, the isolated points are exactly the finite-dimensional indecomposables (see [19, 5.3.33]). The input to the next step is T' , the points isolated in T' are assigned CB rank 1, and, inductively, we continue the procedure of forming derivatives and assigning ranks to points, transfinitely, by taking intersections at limit stages. If this process stabilises with a non-empty subspace (which must have no isolated points), then we assign CB-rank ∞ to these remaining points and say that the CB-rank of T is ∞ or undefined. Otherwise there is a least ordinal λ where we reach the empty space and which is not a limit if T is compact, in which case we set the CB-rank of T to be $\lambda - 1$. In the case of domestic string algebras since we have the condition (IC) (see [19, §5.3.2]), all the points of this maximal rank will be of finite length over their endomorphism rings (see [19, 5.3.22]), hence generic in Crawley-Boevey’s terminology, provided that R is not of finite representation type (recall that a **generic** module is one of finite endlength but not of finite length).

8. PP FORMULAS

Some of our arguments use pp formulas and the lattice they form. Exposition and summary of this material is easily available from various of the background references, so we just explain some of the more technical details here.

Suppose that s is a vertex of Q and let $e_s \in R$ be the corresponding primitive idempotent. If $d \in H_{s,1}$ is a string, then the property of being divisible by d (that is, $x \in dM$) can be expressed by a pp formula and includes the condition $x = e_s x$. A somewhat stronger condition is defined as follows. If there is an arrow γ such that $d\gamma^{-1}$ is a string, then the formula $(.d)(x)$ states that $x = e_s x$ and $x \in d\gamma^{-1}(0)$; otherwise $(.d)(x)$ says just that $x = e_s x$ and $x \in dM$. If c a string in $H_{s,-1}$ then the corresponding formulas $(.c)$ are defined similarly.

If $c \in H_{s,-1}$, $d \in H_{s,1}$, then $(c^{-1}.d)$ is the conjunction of $(.d)$ and $(.c)$. The special case $(1.d)$, on the left, is imposing the condition that $\zeta x = 0$ if there is a ζ such that ζd is a string; $(+1.d)$, on the left, is imposing the condition that x is divisible by the arrow ζ such that $\zeta^{-1}d$ is a string, if there is such a ζ (if there is no such ζ we make $(+1.d)$ the condition $x = 0$). See [20] or [9, Sec. 5.7], for instance, for this notation.

If c is a non-empty string which does not begin on a peak (that is, there is a string of the form $\gamma^{-1}c$), then we define $+c$ to be the result of adding a hook to the left of c , that is, $+c = c_1\gamma^{-1}c$, where c_1 contains only direct letters and is maximal possible such. Similarly c^+ is defined by adding a hook on the right, if possible (see [3, p. 162]).

To any non-zero element $m \in e_s M$ of a module element one may (see [23, p. 6]) associate a (1-sided) string $v = v(m)$ describing the divisibility of m by strings in $H_{s,1}$. Similarly a string $u = u(m)$ describes the divisibility of m by strings in $H_{s,-1}$, and these are combined into a 2-sided string $w(m) = u^{-1}.v$ (as mentioned earlier, informally we consider divisibility by v as divisibility to the right and divisibility by u as divisibility to the left).

8.1. pp formulas and m-dimension. Suppose that L is a modular lattice; we have in mind the lattice of pp formulas in one free variable. At the first stage of the m-dimension analysis of L we factor L by the congruence relation generated by the intervals of finite length: points a and b will be identified iff the interval $[a + b, a \wedge b]$ is of finite length (here $+$ denotes the sup operation). The result is again a modular lattice, and we repeat the process, transfinitely, where at limit stages we factor L by the union of the inverse images in L of the congruence relations generated so far. If the procedure stabilises with a non-trivial lattice (necessarily densely ordered), then we say that the **m-dimension** of L is ∞ or undefined. Otherwise, provided L has a top and a bottom, the first ordinal μ such that the μ -collapse of L is trivial is not a limit, and then we say that the **m-dimension** of L is $\mu - 1$. In particular m-dimension 0 exactly means finite length. See [19, §7.2] for details.

Clearly there is a rough parallel with the definition of Cantor–Bendixson rank. There is also a parallel process, which uses localisation, in the abelian category of finitely presented functors; see Section 8.2. Under certain conditions (see Theorem 8.3), including that of there being no superdecomposable pure-injectives, the CB-analysis of the Ziegler spectrum runs parallel with the m-dimension analysis on the lattice pp_R of all pp-formulas.

8.2. Krull–Gabriel dimension, m-dimension, and CB-rank. Krull–Gabriel (KG-) dimension is defined, in the first instance, on finitely presented functors from $R\text{-mod}$ to \mathbf{Ab} , but it can be extended to give a rank on points. In fact, the m-dimension and KG-dimension of an indecomposable pure-injective N are equal ([1, 4.2.8]). In our case both these dimensions also coincide with CB-rank, so there is no need to include a definition of KG-dimension here.

We recall just a little about m-dimension from [15], based on [30] (which in turn refined Garavaglia’s elementary Krull dimension [6]). We define (see [15, p. 214]) the **m-dimension** of an indecomposable pure-injective N in terms of m-dimension of intervals in the lattice of pp formulas which are open on N : $\text{mdim}(N) = \min\{\text{mdim}[\psi, \varphi] : \varphi(N) > \psi(N)\} = \min\{\text{mdim}[\psi, \varphi] : N \in (\varphi/\psi)\}$. Also let p be any non-zero pp-type realised in N and set $\text{mdim}(p) = \min\{\text{mdim}[\psi, \varphi] : \varphi \in p^+, \psi \in p^-\}$.

Theorem 8.1 ([30, 8.7]; see [15, 10.23]). *If p is the pp-type of a non-zero element of $N \in {}_R\text{Zg}$, then $\text{mdim}(p) = \text{mdim}(N)$.*

Proposition 8.2 ([1, 4.2.8]). *If N is an indecomposable pure-injective, then $\text{KGdim}(N) = \text{mdim}(N)$.*

If the isolation condition (IC) holds (as it does for modules over domestic string algebras since, by [23, 5.6], there is no superdecomposable pure-injective), then this equals the CB rank of N .

Theorem 8.3 ([30, 8.6]; see [15, 10.19] (assuming IC)). *For $N \in {}_R\text{Zg}$ we have $\text{mdim}(N) = \text{KGdim}(N) = \text{CB}(N)$.*

In fact, all this holds true when relativised to any closed subset X of the Ziegler spectrum and applies to any $N \in X$, with the lattice of pp formulas being replaced by its quotient, which is obtained by identifying formulas which agree on all modules in X , and with CB rank being measured in X . Then, still under the assumption IC, if \mathcal{U} is a basis of basic open neighbourhoods of N in X , $\text{mdim}_X(N) = \min\{\text{mdim}_X[\varphi, \psi] : (\varphi/\psi) \in \mathcal{U}\} = \text{CB}_X(N)$, where the subscripts indicate these relativisations ([15, p. 214, Exer. 3]).

Corollary 8.4. *Let R be a domestic string algebra.*

(1) *Let $\varphi > \psi$ be a pair of pp-formulas. The m-dimension of the interval $[\psi; \varphi]$ in the lattice of pp formulas equals the maximum of CB-ranks of points in the corresponding open subset (φ/ψ) of the Ziegler spectrum ${}_R\text{Zg}$.*

(2) *Suppose that N is an indecomposable pure-injective R -module with a chosen neighbourhood basis \mathcal{U} of open sets. Then the CB-rank of N is the minimum of the m-dimensions of intervals $[\psi; \varphi]$ where the pairs of pp-formulas (φ/ψ) run over \mathcal{U} .*

9. COMPUTING RANK FROM THE BRIDGE QUIVER

Here we compute the Cantor–Bendixson rank of each point of ${}_R\text{Zg}$, R a domestic string algebra, in a way that makes natural links between approximation through the bridge quiver and topological approximation. Since approximation by direct (and sometimes inverse) limits does imply topological approximation (see [19, 3.4.7, p. 114]) we can see some of this algebraically as well. Because there are no superdecomposable pure-injectives, the CB-rank of each point equals its KG dimension, which equals its m-dimension (Corollary 8.4(2)); sometimes, for brevity,

we will refer just to the ‘rank’ of a point. The arguments we use in this section have parallels which refer to the lattice of pp formulas and/or to the factorisation structure of morphisms in $R\text{-mod}$. Indeed, any of these three approaches can be used to establish the ‘recipes’ we give here for computing the rank of a point. The approach using topological approximation perhaps involves less technical detail since some of that gets hidden in the topology and the appeals to compactness.

The key to computing the rank of the module $A(w)$ associated to w is to consider whether the corresponding path in the bridge quiver is maximal or whether, and how far, it could be extended in either direction. When computing ranks of band modules we will need more detailed information, specifically whether paths in the bridge quiver leave a band via an arrow pointing away from or into the band. Note that it is possible (for example over X_3) for a path to include both a band and its inverse (but recall that for domestic algebras there are no cyclic paths in the bridge quiver).

As well as computing ranks, we give a neighbourhood basis for each infinite-dimensional point of the Ziegler spectrum of R .

9.1. Representation embeddings and topology. In computing these ranks we make heavy use of the fact that if N is a point in a topological space and U is an open set containing N , then the CB rank of N in the whole space is equal to the CB rank of N computed in the relative topology on U . This allows us to ignore some modules when computing ranks.

Recall that the finite-dimensional points of ${}_R\mathcal{Zg}$ are exactly those of rank 0. We ignore these finite-dimensional points from now on and work in the (closed) subset consisting of infinite-dimensional points. Since that set is the first Cantor–Bendixson derivative, ${}_R\mathcal{Zg}'$, of ${}_R\mathcal{Zg}$, we can compute ranks in this set and then just add 1 to get the rank of an infinite-dimensional point in the whole space. For 1-sided strings, the computation of rank is already in [20, §7], but we will recompute these here on our way to the ranks of other points.

Next, for each band b choose a representation embedding from $K[T, T^{-1}]\text{-Mod}$ to $R\text{-Mod}$ as in Section 4.3; this induces a homeomorphic embedding of ${}_{K[T, T^{-1}]}\mathcal{Zg}$ onto the closed subset of ${}_R\mathcal{Zg}$ ([16, Thm. 7]) consisting of b -band modules. Note that the images of these embeddings for different bands are disjoint.

Recall that we can alternatively use the path algebra of a suitably oriented quiver \widetilde{A}_n in place of $\widetilde{K}[T]$ (restricting to the closed subset of points where the action at each arrow of \widetilde{A}_n acts invertibly). There are just finitely many bands, so the union over all bands b of the images of these induced homeomorphic embeddings is closed and consists of all the indecomposable pure-injective band modules. Denote the complement of this, the set of all string modules, by U_1 . This then is an open subset of ${}_R\mathcal{Zg}$ and working in U_1 will allow us to ignore all band modules when computing ranks of string modules. It will also simplify consideration of band modules since we can fix one band b and then work in the open set consisting of the b -band modules and the string modules; that is, we can ignore the modules associated to other bands.

9.2. Ranks of string modules. In the remainder of this section we will work in $U = U_1 \cap {}_R\mathcal{Zg}'$, and “open set” will mean open subset of this set (recall that ${}_R\mathcal{Zg}'$ is the set of infinite-dimensional points of ${}_R\mathcal{Zg}$).

We show that the rank of an infinite-dimensional string module $C(w)$ is given in terms of the position in the bridge quiver of the infinite band(s) at the end(s) of w .

We recall the neighbourhood bases that were computed in [20] for 1-sided strings and in [23] for 2-sided strings.

Theorem 9.1 ([23, 6.2]). *Suppose that $w = u^{-1}v$ is a 2-sided non-periodic string over a domestic string algebra and $C(w)$ is the corresponding indecomposable pure-injective module. A basis of open sets in the Ziegler topology for $C(w)$ is given by the pairs $(c^{-1}.d) / (e^{-1}.d) + (c^{-1}.f)$, where $c \leq u < e$ in \widehat{H}_{-1} and $d \leq v < f$ in \widehat{H}_1 .*

In the case that the infinite word w is 1-sided, say $w = c^{-1}.v$ where c is finite, (see Section 8), c has a successor, c^+ in H_{-1} ; we will write ${}^+1$ for the inverse of this successor in the case that c is empty (rather, $c = 1_{-1}$). The next result is stated in [20] for general finite c in place of 1.

Theorem 9.2 ([20, 5.3]). *Suppose that v is a 1-sided non-periodic string over a domestic string algebra and $C(v)$ is the corresponding indecomposable pure-injective module. A basis of open sets in the Ziegler topology for $C(v)$ is given by the pairs $(1.d) / ({}^+1.d) + (1.f)$, where $d \leq v < f$ in \widehat{H}_1 .*

Making good choices in these results, we can get nicely described neighbourhood bases.

Corollary 9.3. *Let $w = {}^\infty b_0 c b^\infty$ be a 2-sided string, denote by w_n its image substring ${}^n b_0 c b^n$, and let U_n be the set of string modules $C(y)$ such that y contains w_n as an image substring. Then the U_n form a basis of open neighbourhoods of $C(w)$.*

Similarly if $w = c b^\infty$ with b a band and c finite, then let $w_n = c b^n$ and let U_n be the set of (1-sided) string modules $C(y)$ such that y contains w_n as an initial image substring. Then the U_n form a basis of open neighbourhoods of $C(w)$.

Proof. Split w_n as $u_n v_n$ where $u_n = {}^n b_0$ and $v_n = c b^n$. Denote by v_n° the word obtained from v_n by removing the last (inverse) letter, and by ${}^\circ u_n$ denote the word obtained from u_n by removing the first (direct) letter. Then the open set $((u_n.v_n)/({}^\circ u_n.v_n + u_n.v_n^\circ))$ has the form of a set as in Theorem 9.1. This open set separates $C(w)$ from all band modules; furthermore it consists of the string modules $M(w')$ where w' contains w_n as an image subword, for, if $C(w') \in ((u_n.v_n)/({}^\circ u_n.v_n + u_n.v_n^\circ))$, then there is a non-zero graph map from $M(w_n)$ to $C(w')$ which does not factor through the canonical map from $M(w_n)$ to the string module of any truncation. The first says that some factor substring of w_n occurs as an image substring of w ; the second says that there is such an occurrence where it is w_n , rather than a proper image string, which occurs.

The 1-sided case is similar, the relevant open set being $(1.w_n/{}^+1.w_n + 1.w_n^\circ)$. \square

Note that, given any 1-sided infinite string module, any of the open neighbourhoods in Theorem 9.2 serves to separate it from all 2-sided strings.

Now we compute the ranks of points.

Theorem 9.4. *If $w = c b^\infty$ is a 1-sided right-infinite string whose terminal band has right indent t , then the CB rank of $C(w)$ is $t + 1$.*

Proof. Consider the intersection of the open set U_n from Corollary 9.3 with U ; this consists of the modules $C(x)$ where x is a 1-sided (infinite) string with w_n as an initial image substring. If the right indent of b is 0, then the only point in this open set is $C(w)$ itself, which, therefore, has CB-rank 0 in $U_n \cap U$, hence has CB-rank 1 in the whole space. If the right indent of b is 1, so there are strings of the form $b^n u' b_1^\infty \neq b^\infty$, but no strings beginning with b and giving longer paths in the bridge quiver, then, by the case we just dealt with, each string module of the form $C(cb^n u' b_1^\infty)$ belongs to the open set U' and has rank 1. There are no other points in that open set apart from $C(w)$, which, therefore, has rank 2: it cannot have rank 1, since then we would have a (Ziegler-basic, hence) compact open set being a union of infinitely many isolated points, a contradiction. This argument continues inductively on the right indent of the band b (the cases 0 and 1 having just been treated), and so we obtain the result. \square

Now we move on to the 2-sided strings.

Theorem 9.5. *If w is a 2-sided infinite string whose terminal bands have indent s and t respectively, then the CB rank of $C(w)$ is $s + t + 2$.*

Proof. Say $w = {}^\infty b_0 u b_1^\infty$. Again take the open set U_n from Corollary 9.3 and consider its intersection with U . As before, we start with the case that the corresponding path from b_0 to b_1 is maximal in the bridge quiver and then work inwards.

In the case that the path between b_0 and b_1 is maximal, the only points in $U_n \cap U$, apart from $C(w)$, are the (infinitely many) 1-sided infinite string modules which contain $b_0^n u b_1^n$ as an image substring. By Theorem 9.4, these all have rank 1, so, arguing as in the 1-sided case, $C(w)$ has rank 2.

We continue inductively, first working in from the right side. Suppose that w has left indent 0 and right indent t . Then the set $U_n \cap U$ consists of: 1-sided string modules of the form ${}^\infty b_0 u b_1^n v$ with v a finite string - these have rank 1 by Theorem 9.4; 1-sided string modules of the form $v b_0^n u b_1^\infty$ with v finite - these have rank $\leq t + 1$ by Theorem 9.4; and the 2-sided string modules where the string has the form ${}^\infty b_0 u b_1^n \dots$ and has right indent $\leq t - 1$. Note that the value $t - 1$ is achieved among these 2-sided string modules; moreover, by induction, these 2-sided string modules have rank $\leq t - 1 + 2 = t + 1$, with the value $t + 1$ being achieved. Since $t \geq 1$ there are infinitely many points in $U_n \cap U$ with (maximal) rank $t + 1$ (because we have modules of the form ${}^\infty b_0 u b_1^k \dots$ for all $k \geq n$). Removing all these points from U' leaves only $C(w)$, which therefore has rank $t + 2$.

Now we can also move in from the left side: suppose first that the left indent, s , of w is 1 and the right indent is t . Then the 1-sided strings in $U_n \cap U$ have ranks up to and including $\max\{s = 1, t\} + 1$. The 2-sided strings have, by the above, ranks up to and including $t + 2$ - and there are infinitely many of these with rank $t + 2$ - and then there's $C(w)$, which, therefore, has rank $1 + t + 2 = s + t + 2$. Working in from the left as we did on the right but inducting on the left indent, we see that a 2-sided string with left indent s and right indent t has rank $s + t + 2$. \square

Example 9.6. For instance, let $R = \Lambda_2$ and let $w = (\varepsilon\delta^{-1})^\infty$. By choosing $b_1 = \varepsilon\delta^{-1}$ and $b_2 = \alpha\beta^{-1}$ we see that there is a path $b_1 \rightarrow b_2$ starting with b_1 in the bridge quiver and there are no others, so 1 is the maximal length of such paths and the indent of b_1 is 1. It follows that the CB-rank of the corresponding indecomposable pure-injective, the direct product module $C(w)$, is $1 + 1 = 2$. On the other hand, there is no bridge starting from b_2 ; hence the direct product module

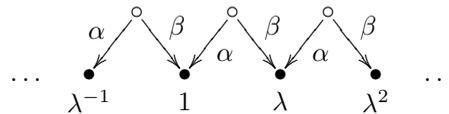
corresponding to the string b_2^∞ has CB-rank 1. Each terminating band of the 2-sided string $u = {}^\infty(\varepsilon\delta^{-1})\varepsilon\gamma(\alpha\beta^{-1})^\infty$ has indent 0, so the corresponding mixed module $C(u)$ has CB-rank $0 + 0 + 2 = 2$.

9.3. Ranks of band modules.

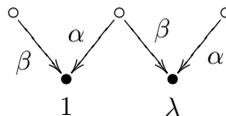
9.3.1. *Prüfer modules.* Let $b = \alpha \dots \beta^{-1}$ be a band and let $\Sigma(b, S)$ be a b -Prüfer module, with corresponding quasisimple module $M(b, S)$. Set $B_i = M(b, S[i])$. Fix an element $m \in \beta B_1 \cap \alpha B_1$ in the socle of B_1 . Consider the sequence of irreducible embeddings $B_i \rightarrow B_{i+1}$ and choose φ_i to be any pp formula which generates the pp-type of the image of m in B_i , so we have a descending sequence $\varphi_1 > \varphi_2 > \dots$ of pp formulas which together generate p^+ where p is the pp-type of m in $\Sigma(b, S)$. Note that each interval $[\varphi_{i+1}, \varphi_i]$ is simple and that any pp formula strictly below φ_i is less than or equal to φ_{i+1} . To see that, let $\theta < \varphi_i$. Then there is a morphism $f : B_i \rightarrow N$ where (N, n) is a free realisation of θ . By strictness of the inclusion, f is not a split embedding; hence it factors through the almost split map $g : B_i \rightarrow B_{i+1} \oplus B_{i-1}$ (the second term being absent if $i = 1$), say $f = hg$. Note that the image of m in the second component B_{i-1} is zero, so h takes $m \in B_{i+1}$ to n and hence $\varphi_{i+1} \geq \theta$, as claimed.

Let c be a cyclic permutation of b which starts with an inverse arrow and ends with a direct arrow. Write M_j for the string module $M(c^j)$ (so the M_j are the images of preinjective \widehat{A}_n -modules under a suitable representation embedding). In each M_j let n_j be the element in $\alpha M_j \cap \beta M_j$ which is the image of the composition $B_1 \rightarrow \Sigma(b, S) \rightarrow M_j$, the maps being the natural ones. Let ψ_j be any pp formula which generates the pp-type of n_j in M_j . Consider the epimorphism $M_{j+1} \rightarrow M_j$ which is obtained by factoring out the rightmost copy of c and mapping n_{j+1} to n_j , so we have the ascending chain $\psi_1 < \psi_2 < \dots$. Also note that $\varphi_i > \psi_j$ for every i, j (consider the morphism $B_i \rightarrow \Sigma(b, S) \rightarrow M_j$).

Example 9.7. To see all this more concretely, we can use that the direct product module $\overline{M} = \overline{M}({}^\infty b^\infty)$ has, as a direct summand, every Prüfer module (it is the dual of the image of $K[T, T^{-1}]$ under a suitable representation embedding); there is a unique, up to scalar multiple, embedding of the regular module B_1 into this module (see [12], or better [9, Sec. 6.3.2]). For example, take $b = \alpha\beta^{-1}$ over X_3 (in fact everything is happening over the Kronecker algebra) and let m be as above with $B_1 = M(\alpha\beta^{-1}, \lambda)$. Then m maps to the indicated infinite sum of elements in the socle of \overline{M} and the pp-type of m in \overline{M} is p :

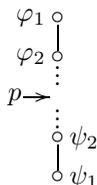


Continuing with the notation above, any occurrence of c^k as a factor substring of ${}^\infty b^\infty$ gives a map of \overline{M} onto the string module M_k . In the example c must be $\beta^{-1}\alpha$ and M_2 is shown below, with the (unique to scalar) image of m being the sum of the indicated socle elements. The formula ψ_k is chosen to generate the pp-type of n_k in M_k .



Theorem 9.8. *The open sets (φ_i/ψ_j) defined above form a neighbourhood basis of open sets of the Prüfer module $\Sigma(b, S)$.*

Proof. With notation as above, by [19, 5.1.21] a neighbourhood basis for $\Sigma(b, S)$ is given by sets of the form (φ/ψ) with $\varphi \in p^+$ and $\varphi > \psi \in p^-$. We have seen already that we can take φ to be one of the φ_i , so consider some pp formula in p^- with ψ below each φ_i ; we must show that $\psi < \psi_j$ for some j .



Since ψ is a sum of pp formulas realised in indecomposable modules, we can assume that ψ has an indecomposable free realisation, (N, n) say. Since $\psi < \varphi_i$ there is a morphism $f : (B_i, m) \rightarrow (N, n)$. If N were a b' -band module where $b' \neq b, b^{-1}$, then, since $f(m) \neq 0$, b' would contain a socle pair of the form $\beta^{-1}\alpha$, contradicting Theorem 2.1(2). If N were a b - or b^{-1} -band module, then (some component of) f would be induced by a matching between a factor substring and image substring of ${}^\infty b^\infty$ (and/or its inverse). But, again, since the image of m is non-zero, that is impossible.

So N must be a string module, and then we can use the description of morphisms from band modules to string modules (see [9, §6.3.2] for a clear description): any morphism from B_i to a string module is a linear combination of compositions of irreducible maps and maps which factor through $\Sigma(b, S)$, hence through the product module $\overline{M}({}^\infty b^\infty)$ seen above. Any composition of graph maps which annihilates m can be ignored, and the rest are embeddings. So any component of f is essentially the composition of a map from some B_{i+t} to \overline{M} followed by a graph map induced by a factor substring u of ${}^\infty b^\infty$. Taking a copy of c^k containing u as a substring, we see that the latter map factors through the canonical map to some M_k . Therefore, choosing j large enough, f itself will factor through M_j and so $\psi_j \geq \psi$, as claimed. □

We also describe this basis as sets of points of the spectrum.

Corollary 9.9. *Define $U_{i,j} = \{M(b, S[m]) \mid m \geq i\} \cup \{\Sigma(b, S)\} \cup \mathcal{Z}_j$, where \mathcal{Z}_j is the set of all string modules $C(y)$ such that y contains as an image substring a finite factor string of the form $x'c^jx''$ of ${}^\infty b^\infty$, where the string c is as in the proof of Theorem 9.8 (that is, all “long enough” finite factor strings of ${}^\infty b^\infty$). Then the $U_{i,i}$ form a basis of open neighbourhoods of $\Sigma(b, S)$.*

Proof. It is not difficult to see that $U_{i,j}$ is open (cf. the last part of the proof of Corollary 9.15). It suffices to show that $U_{i,j} \subseteq (\varphi_i/\psi_{j-1})$; clearly both subsets contain no b' -band where $b' \neq b, b^{-1}$, and they do coincide on the set of b -band modules, so it suffices to check the inclusion just of their intersections with string modules. Let $N = C(y) \in U_{i,j}$ be a string module. There is an obvious morphism from B_i to N sending the socle element m to a linear combination n of at least j different ‘adjacent’ canonical basis elements in $\alpha N \cap \beta N$. Then n will satisfy φ_i , but, since clearly this map does not factor through M_{j-1} , n does not satisfy ψ_{j-1} . □

Suppose that b is a band and bc is a string; we will say that bc **ascends from** b if $b^\infty < bc$ and bc **descends from** b if $bc < b^\infty$ (the remaining case is that bc is a substring of b^∞). For instance, the string $\beta\alpha^{-1}\beta\alpha$ over X_3 will ascend from the band $b = \beta\alpha^{-1}$.

Theorem 9.10. *Let b be a band. Then the rank of each b -Prüfer module is $s + t + 1$ where s is the indent of b via strings that ascend from b and t is the indent of b via strings that ascend from b^{-1} .*

Proof. Because the intervals $[\varphi_{i+1}, \varphi_i]$ are simple, by the definition of m -dimension it suffices to prove that each interval $[\psi_i, \psi_{i+1}]$ has m -dimension $s + t$. By Corollary 8.4(1) this will be the maximum value of CB-rank of points in the corresponding open set (ψ_{j+1}/ψ_j) . We show that this interval contains no band modules.

Any module in (ψ_{j+1}/ψ_j) contains an element in the intersection $\text{im}(\alpha) \cap \text{im}(\beta)$, so any band module would be a b - or b^{-1} -band module. But then there would be a map from, say, B_1 to M_{j+1} to a b - or b^{-1} -band module, with the image of $m \in B_1$ being non-zero. Hence there would be a factor substring of ${}^\infty b^\infty$ which includes the string $\beta^{-1}\alpha$ or $\alpha\beta^{-1}$ equal to an image substring of ${}^\infty b^\infty$; this (consider the ends of this substring) is impossible.

The string modules in (ψ_{j+1}/ψ_j) are obtained by taking factor substrings d of c^{j+1} (which are not also factor substrings of c^j , so which are obtained by factoring out only “small” submodules at either side) and then embedding these as image substrings of infinite strings w , which, therefore, ascend from b . By the computation of ranks in Section 9.2, the points of maximal rank in this open set will have the form ${}^\infty b_1 d' b_2^\infty$, where b_2 is the nearest band to b on a path witnessing that the maximal indent of b via ascending paths and b_1 is similarly a nearest neighbouring band on a maximal band-length path that ascends from b^{-1} (and d' contains some d as above as an image substring). By Theorem 9.5 this module has rank $(s - 1) + (t - 1) + 2 = s + t$. In the case where there is an ascending path just to one side, say from b , we use Theorem 9.4 to obtain the value $(s - 1) + 1 = s$ for the m -dimension of the interval and hence the value s for the rank of $\Sigma(b, S)$. In the case that there is no path ascending from b , the interval contains only finitely many finite-dimensional points, so has rank 0.

Thus in each case, the maximal rank of points in (ψ_{j+1}/ψ_j) , and hence the m -dimension of $[\psi_j, \psi_{j+1}]$ is $s + t$. □

Example 9.11. Let R be the path algebra of Λ_3 . Let $b_i = \beta_i\alpha_i^{-1}$ ($i = 1, 2, 3$) be representatives of the bands. Then the bridge quiver consists of $b_1 \xrightarrow{\gamma_1^{-1}\alpha_2^{-1}} b_2 \xrightarrow{\gamma_2^{-1}\alpha_3^{-1}} b_3$ and its inverse. Because there is no ascending path from b_1 , the rank of any b_1 -Prüfer module is 1. There is an ascending path to the left of b_2 (that is, for b_2^{-1}) of length 1, so we obtain $1 + 0 + 1 = 2$ for the rank of each b_2 -Prüfer point. Finally there is an ascending path of length 2 to the left of b_3 , so the corresponding Prüfer points have rank $2 + 0 + 1 = 3$.

9.3.2. Adic modules. The ranks of adic modules could be calculated by a similar, though dual, argument, using the coray of epimorphisms in the tube with quasisimple module $M(b, S)$. It is quicker to make use of elementary duality. We will use that the adic left R -module $\Pi(b, S)$ is the elementary dual of a Prüfer right R -module attached to a certain band b^* for right R -modules. For finite-dimensional modules (and some others) elementary duality is simply $\text{Hom}_K(-, K)$ -duality. Given a right

R -module L , we will write $L^* = \text{Hom}_K(L, K)$ for its Hom-dual left module, and the same for left-to-right Hom-duality.

First observe that right modules over R , which has been presented as a string algebra by a quiver Q with relations, are representations of the quiver which is obtained from Q by reversing all arrows and, correspondingly, all relations; that is, they are left modules over the opposite algebra to R , and so everything that we have proved applies in this way to right R -modules. Note also that the “opposite” of a band b for left R -modules is a cyclic permutation of a ‘dual’ band, b^* , for right R -modules. Furthermore, the parametrisation of quasisimple b -band modules is, depending on our choice of b^* and of how to parametrise b^* -band modules, either the same as or inverse to that for b -band modules, meaning that $M(b, S)^* = M(b^*, S)$ or $= M(b^*, S^{-1})$, where by S^{-1} we mean the image of the simple $K[T, T^{-1}]$ -module where T acts as the inverse of the action on the simple module S . We will set $M(b, S)^* = M(b^*, S^*)$ as a notation to cover either case. In the next result we compute the elementary dual pairs of indecomposable pure-injective right and left modules.

Proposition 9.12. *For a domestic string algebra R , elementary duality gives a homeomorphism between the right and left Ziegler spectra of R , with pairing between infinite-dimensional points being as follows (for finite-dimensional points, the pairing is just $\text{Hom}_K(-, K)$ -duality).*

- (1) *The b -Prüfer modules are dual to the b^* -adic modules, and the b -adic modules are dual to the b^* -Prüfer modules.*
- (2) *If w is a contracting 1-sided or 2-sided string, then $C(w)$ is dual to the expanding string module $C(w^*)$; and if w is an expanding 1-sided or 2-sided string, then $C(w)$ is dual to the contracting string module $C(w^*)$.*
- (3) *If w is a mixed 2-sided string, then $C(w)$ is dual to the mixed string module $C(w^*)$.*
- (4) *The b -generic module is elementary dual to the b^* -generic module.*

Proof. (1) and (4) follow from the corresponding result for \widetilde{A}_n -modules.

(2) For 1-sided strings this follows from (1) (considering the tubes parametrised by $0, \infty$). If w is a contracting 2-sided string, then the module $C(w)$ is the direct limit of the graph map monomorphisms between the 1-sided string modules $C(w_n)$ where w_n is any increasing sequence of image substrings of w , so is in the Ziegler-closure of these points, indeed is easily seen to be an accumulation point of minimal rank. Dualising, we have that the elementary dual is the accumulation point of minimal rank in the Ziegler-closure of the $C(w_n)^* = C(w_n^*)$, and it is easily checked that $C(w^*)$ has this property and therefore is the dual of $C(w)$.

(3) Consider a 2-sided mixed string w . The corresponding indecomposable pure-injective module $C(w)$ is the direct limit of the obvious system of embeddings between 1-sided product modules $C(w_n)$ where the w_n form an increasing sequence of expanding 1-sided image substrings. Since we have already computed the elementary duals of the $C(w_n)$, we can proceed as in (2) and deduce that $C(w_n^*)$ is the elementary dual of $C(w_n)$. □

Using this and Corollary 9.9 we obtain neighbourhood bases for adic modules.

Corollary 9.13. *The adic module $\Pi(b, S)$ has, for a neighbourhood basis, the open sets $V_{ij} = \{M(b, S[m]) \mid m \geq i\} \cup \{\Pi(b, S)\} \cup \mathcal{W}_j$, where \mathcal{W}_j is the set of all string modules $C(y)$ such that y has, as a factor substring, a finite string of the form*

$x'b^jx''$ of ${}^\infty b^\infty$, which is an image substring of ${}^\infty b^\infty$. (Of course we may limit to $j = i$ for aesthetic reasons.)

It remains only to observe that, because the arrows turn around in going from left to right modules, the description of the rank of $\Sigma(b^*, S^*)$ in terms of arrows ascending from the band b^* dualises to the same description of the rank of $\Pi(b, S)$ but in terms of arrows which descend from the band b . We therefore obtain the following result for calculating the rank of adic modules using the above and the fact that the CB-ranks elementary-dual pairs of modules are equal.

Theorem 9.14. *Let b be a band. Then the rank of each b -adic module is $s + t + 1$ where s is the indent of b via strings that descend from b and t is the indent of b via strings that descend from b^{-1} .*

Continuing Example 9.11, there is a path of length 2 descending from b_1 ; therefore the rank of every b_1 -adic module is $0 + 2 + 1 = 3$. Similarly the rank of each b_2 -adic module is 2 and the rank of each b_1 -adic module is 1.

9.3.3. *Generic modules.* Suppose that b is a band; first we describe a neighbourhood basis of the b -generic.

Corollary 9.15. *The sets U which satisfy the following conditions form a basis of open neighbourhoods of the b -generic G_b : U contains all the infinite-dimensional b -band modules, all but finitely many finite-dimensional b -band modules, and no b' -band modules for $b' \neq b, b^{-1}$. Furthermore there is an n such that the string modules in U are exactly the $C(y)$ where y contains b^n as an image subword.*

Proof. Choose an open set containing G_b ; by what we have already mentioned we can reduce it to an open set U containing G_b and no b' -band module over any other band b' . The intersection of U with the image of a representation embedding giving the b -band modules is the homeomorphic image of an open neighbourhood of the generic in the initial ring. It is known from [18] and [25] that any such set must contain all the infinite-dimensional points and all but finitely many finite-dimensional points. So, it remains to determine the intersection of U with the set of string modules.

We claim that there is an n such that U contains every string module $C(y)$ where y contains b^n as an image substring. Suppose, for a contradiction, that for each n there is a string x_n which contains b^n as an image substring but such that $C(x_n)$ is not in U . We must show that G_b is in the closure of the $C(x_n)$. It is easy to see that there is some band b_0 and finite string c such that, for each n , $b_0^n c b^n$ occurs as an image substring of some $x_{m(n)}$. Therefore, by Corollary 9.3, the module $C({}^\infty b_0 c b^\infty)$ is in the closure of this set. First, one can check that the closure of $C({}^\infty b_0 c b^\infty)$ consists only of that module plus generic points, for we have computed neighbourhood bases of all but the generic points and, by inspection, none of them but G_{b_0} and G_b are in the closure of this point.

Choose a socle pair $\alpha^{-1}\beta$ for b . Let φ be the pp formula whose solution set is $\text{im}(\alpha) \cap \text{im}(\beta)$. This solution set in $C({}^\infty b_0 c b^\infty)$ is, note, infinite-dimensional. Indeed, since there is a contracting or expanding endomorphism of $C({}^\infty b_0 c b^\infty)$ (see Section 4.2) acting on the b^∞ part of $C({}^\infty b_0 c b^\infty)$, that solution set is of infinite

length over the endomorphism ring of $C(\infty b_0 c b^\infty)$ and so must have an infinite (ascending or descending) chain of pp-definable subgroups. This implies (by compactness in, say, the model-theoretic sense)² that one of the points in the Ziegler-closure of M_0 also has non-zero solution set for φ . Those points all are generic, and the only generic point which satisfies this is G_b , as required.

Thus we have shown that, for any open set U containing G_b , there is n such that every string module containing b^n as an image substring is in U . It remains to note that the set of string modules containing b^n as an image substring is open; the proof of Corollary 9.3 applied with b^n in place of w_n there shows exactly this. Therefore this set, together with the intersection of U with the b -band modules, is an open set (being the union of an open set with an open subset of the complement of that set) and contains G_b . Furthermore, we have just shown that every open set containing G_b contains such a set, so this does give a basis. \square

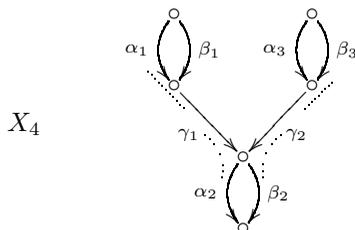
Finally we compute the CB-rank of each generic module. We will see that the rank of a generic module is determined not just by the ranks of modules supported on the same band, but also by the 2-sided strings which pass through b , that is, which contain b as a substring. Consider the band $b_2 = \beta_2 \alpha_2^{-1}$ over Λ_3 from Example 9.11. The ranks of the b_2 -Prüfers and b_2 -adics are 2, but, as we will see below, the rank of the b_2 -generic is 4. This is because there is a 2-sided string passing through b_2 such that the corresponding mixed module has rank 3.

Theorem 9.16. *If b is a band, then the rank of the associated generic G_b is $s+t+2$ where s , respectively t , is the left, respectively right, indent of b .*

Proof. Consider a band b_0 adjacent to b on a longest path containing b in the bridge quiver. Let w_0 be a 2-sided infinite string of the form ${}^\infty b_0 c_0 b^\infty$. The corresponding string module $M_0 = C(w_0)$ has rank $s-1+t+2 = s+t+1$. We have shown already in the proof of Corollary 9.15 that G_b is in the closure of this module, hence has rank at least $s+t+2$.

But now observe that every point in an open neighbourhood U , as in Corollary 9.15, of G_b , apart from G_b itself, has been assigned a CB-rank by the results already proved, and $s+t+1$ is the maximal value attained. Hence the CB-rank of G_b is exactly $s+t+2$, as claimed. \square

Example 9.17. As an example consider the following 3-domestic string algebra:



with short relations $\gamma_1 \alpha_1 = \alpha_2 \gamma_1 = \beta_2 \gamma_2 = \gamma_2 \beta_3 = 0$.

²This can be said in terms of the pp-sort φ evaluated at M with its induced model-theoretic structure: since it has infinite endlength there is a non-isolated point in its Ziegler-closure. Alternatively, and more directly, one can produce an irreducible pp-type which is not realised in M .

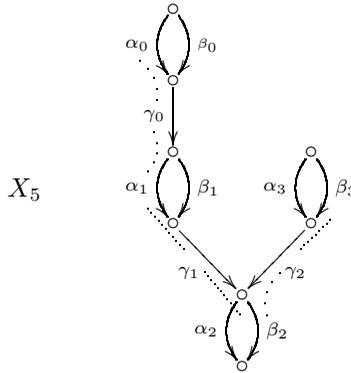
Choose $b_i = \alpha_i\beta_i^{-1}$, $i = 1, 2, 3$, and their inverses as the vertices of the bridge quiver. The bridge quiver of X_4 consists of the path

$$b_1 \xrightarrow{\gamma_1^{-1}\beta_2^{-1}} b_2 \xrightarrow{\alpha_2\gamma_2} b_3$$

and its inverse.

Since the left indent of b_1 is 0 and the right indent is 2, the rank of G_{b_1} is $0 + 2 + 2 = 4$. Similarly the ranks of G_{b_2} and G_{b_3} are 4.

Example 9.18. To get an example where the generics do not all have the same rank, consider the string algebra X_5 :



with short relations as indicated.

Choose $\alpha_i\beta_i^{-1}$ and their inverses for $i = 0, \dots, 3$ as the vertices of the bridge quiver. Then the bridge quiver is the following directed graph and its inverse:

$$\begin{array}{ccccc}
 b_2 & \xrightarrow{\alpha_2\gamma_1} & b_1^{-1} & \xrightarrow{\beta_1\gamma_0} & b_0^{-1} \\
 & \searrow & & & \\
 & & & & b_3
 \end{array}$$

From the bridge quiver we see that the rank of G_{b_2} is $0 + 2 + 2 = 4$ and G_{b_0}, G_{b_1} also have rank 4. On the other hand G_{b_3} has rank $1 + 0 + 2 = 3$.

In particular we have the following theorem confirming a conjecture of Schröer.

Theorem 9.19. *Let R be a domestic string algebra. The Krull–Gabriel dimension of the category of R -modules is $n + 2$ where n is the maximal length of a path in the bridge quiver of R .*

The following is immediate.

Corollary 9.20. *If R is a domestic string algebra, then its Ziegler spectrum is a T_0 space.*

Moreover, by comparing Theorem 9.19 with [28, Thm. 3], we obtain the following as a direct corollary.

Corollary 9.21. *Let R be a string algebra and let $m \in \mathbb{N}$. Then $\text{KG}(R) = m$ if and only if $\text{rad}_R^{\omega(m-1)} \neq 0$ and $\text{rad}_R^{\omega m} = 0$.*

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