A system of first order Hamilton-Jacobi equations related to an optimal debt management problem

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Abstract. The paper studies a system of first order Hamilton-Jacobi equations with discontinuous coefficients, arising from a model of deterministic optimal debt management in infinite time horizon, with exponential discount and currency devaluation. The existence of an equilibrium solution is obtained by a suitable concatenation of backward solutions to the system of Hamilton-Jacobi equations. A detailed analysis of the behavior of the solution as the debt-ratio-income tends to infinity is provided.

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1. Introduction

Consider a system of Hamilton-Jacobi equations

$$\begin{cases} rV \;=\; H(x,V',p)+\dfrac{\sigma^2x^2}{2}\cdot V'', \\ (r+\lambda+v(x))\cdot p-(r+\lambda) \;=\; H_\xi(x,V',p)\cdot p'+\dfrac{\sigma^2x^2}{2}\cdot p'', \\ v(x) \;=\; \underset{w\in[0,+\infty]}{\operatorname{argmin}}\left\{c(w)-wxV'(x)\right\}, \end{cases} \tag{1}$$

with the boundary conditions

$$V(0)=0,\qquad V(x^*)=B\qquad\text{and}\qquad p(0)=1,\qquad p(x^*)=\theta(x^*),$$

motivated by an optimal debt management problem in infinite time horizon with exponential discount. As in [4, 5, 6, 8, 12, 13], this modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders. Here, $H_\xi(x,\xi,p)$ is the partial derivative of H with respect to the variable

ξ, the independent variable x is the debt-to-income ratio, x^* is a threshold of the debt-to-income ratio where the borrower must declare bankruptcy, B is a given constant, accounting for the bankruptcy cost, the salvage function $\theta \in [0,1]$ determines the fraction of capital that can be recovered by lenders when bankruptcy occurs, and

- V is the value function for the borrower who is a sovereign state that can decide the devaluation rate of its currency $v \in [0,+\infty[$ and the fraction of its income $u \in [0,1]$ which is used to repay the debt,
- p is the discounted rate at which the lenders buy bonds to offset the possible loss of part of their investment.

Since p is determined by the expected evolution of the debt-to-income ratio at all future times, it depends globally on the entire feedback controls u and v. This leads to a highly nonstandard optimal control problem, and a "solution" must be understood as a Nash equilibrium, where the strategy implemented by the borrower represents the best reply to the strategy adopted by the lenders, and conversely.

For the stochastic model ($\sigma > 0$), the authors proved in [12] the existence of an equilibrium solution (V_σ, p_σ) as a steady state of an auxiliary parabolic system. The proof requires a careful analysis to construct an invariant domain and apply a fixed-point result to derive the existence of a steady state for the auxiliary parabolic system. Moreover, they also established the upper (lower) bound of discounted bond price p_σ and the expected total optimal cost for servicing the debt V_σ. Here, a natural question is trying to understand whether a solution exists and its structure remains unchanged in the deterministic case ($\sigma = 0$). A classical approach for a solution in this case is the vanishing viscosity method. More precisely, one studies the limit $(V_\sigma, p_\sigma) \to (V, p)$ as the diffusion coefficient $\sigma \to 0+$ and show that the limits (V_σ, p_σ) yields a solution to (1) with $\sigma = 0$. However, this is a highly nontrivial problem and still remains open.

The present paper aims to provide a direct study to the deterministic case ($\sigma = 0$) of the system (1) by looking at the corresponding system of differential inclusions

$$\begin{cases} V'(x) & \in & \{F^-(x,V(x),p(x)), F^+(x,V(x),p(x))\} \\ p'(x) & \in & \{G^-(x,V(x),p(x)), G^+(x,V(x),p(x))\} \end{cases} \tag{2}$$

where $F^\pm(x,V,p)$ solves the equation $rV = H(x,\xi,p)$ with variable ξ for a given x, V, p, and

$$G^\pm(x,\eta,p) \;=\; \frac{(r+\lambda+v^*(x,F^\pm(x,\eta,p)))p-(r+\lambda)}{H_\xi(x,F^\pm(x,\eta,p),p)}.$$

In Theorem 3.5, we first construct a solution (V,p) of (1) with boundary conditions by a suitable concatenation of backward solutions of (2), and then determine an equilibrium solution to the corresponding differential game with deterministic dynamics. Moreover, we show that there exists a point

$x_1 \in [0, x^*[$ such that if the debt-ratio-income is less than x_1, then the optimal strategy will reach a steady state, otherwise bankruptcy in finite time is unavoidable. In our construction, the main technical difficulties in the analysis stem from the fact that, the system (2) is not monotone and $F^{\pm}$ are just Hölder continuous at points where H_ξ vanishes. Moreover, $p(\cdot)$ may well have many discontinuities. At these points, backward solutions is not necessarily unique and does not allow a detail analysis. Thereafter, in Proposition 4.1, using the analysis of sub- and super-solutions, we study an asymptotic behaviour of (V, p) as the maximum debt-to-income threshold x^* is pushed to $+\infty$. Consequently,

- if the salvage rate decay sufficiently slowly, i.e., the lenders can still recover a sufficiently high fraction of their investment after the bankruptcy, then the best choice for the borrower is to implement a "Ponzi-like trivial strategy", where no effort is done to actually repay the debt, which is serviced by starting new and new loans.
- otherwise, if the salvage rate $\theta(x^*)$ decays sufficiently fast, then Ponzi-like trivial strategy is no longer an optimal solution for the borrower;
- for sufficiently large initial debt-to-income and bankruptcy threshold and recovery fraction after bankruptcy, the optimal strategy for the borrower will use currency devaluation v to deflate the debt-to-income.

The remainder of the paper is organized as follows. In Section 2, we provide a more detailed description of the model and the system of Hamilton-Jacobi equations satisfied by (V, p), and study basic properties of H. In Section 3, we construct a solution to (1) with $\sigma = 0$, and then derive an equilibrium solutions to the model of optimal debt management. In Section 4, we perform a detailed analysis of the behavior of the optimal feedback controls as $x^* \to \infty$. We close by an appendix which contains some concepts of convex analysis and collect some further technical results related to the Hamiltonian function.

2. Model derivation and system of Hamilton-Jacobi equations

2.1. A deterministic optimal debt management problem

In this subsection, we shall recall our deterministic optimal debt management problem with currency devaluation and exponential discount in [12]. Here, the borrower is a sovereign state, that can decide to devaluate its currency, and its total income $Y(t)$ and total debt $X(t)$ are governed by the control dynamics

$$\begin{cases} \dot{Y}(t) & = & (\mu + v(t))Y(t), \\ \dot{X}(t) & = & -\lambda X(t) + \dfrac{(\lambda + r)X(t) - U(t)}{p(t)}, \end{cases} \tag{1}$$

where r is the interest rate paid on bonds, λ is the rate at which the borrower pays back the principal, $\mu > r$ is the average growth rate of the economy, and

- $U(t)$ is the rate of payments that the borrower chooses to make to the lenders at time t;
- $v(t) \geq 0$ is the devaluation rate at time t, regarded as an additional control.

We define the debt-to-income ratio $x \doteq \frac{X}{Y}$, and set $u \doteq \frac{U}{Y}$. The system (1) yields

$$\dot{x}(t) \;=\; \left(\frac{\lambda + r}{p(t)} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p(t)}. \tag{2}$$

In this model, the borrower is forced to declare bankruptcy when the debt-ratio-income x reaches threshold x^*. The bankruptcy time is denoted by

$$T_b \;\doteq\; \inf\{t > 0 : x(t) = x^*\} \;\in\; \mathbb{R} \cup \{+\infty\}. \tag{3}$$

Throughout the following, we consider a control in feedback form, so that

$$(u, v) = (u^*(x), v^*(x)) \in [0, 1] \times [0, +\infty[\text{ for } x \in [0, x^*].$$

Given an initial size x_0 of the debt-to-income ratio, the borrower wants to find a pair of optimal controls (u, v) which minimizes his total expected cost, exponentially discounted in time:

$$\text{minimize } J[x^*, u^*, v^*] \;\doteq\; \int_0^{T_b} e^{-rt}[L(u^*(x(t))) + c(v^*(x(t)))]\, dt + e^{-rT_b} B \tag{4}$$

where $c(v)$ is the social cost resulting from devaluation, $L(u)$ is the cost to the borrower for putting income towards paying the debt, and B is the cost of bankruptcy. We shall assume the following structural conditions on the cost functions L, c:

(A1) *The implementing cost function L is twice continuously differentiable for $u \in [0, 1[$, and satisfies*

$$L(0) \;=\; 0, \quad L'(u) \;>\; 0, \quad L''(u) \;>\; 0 \text{ and } \lim_{u \to 1^-} L(u) \;=\; +\infty.$$

(A2) *The social cost c is twice continuously differentiable for $v \in [0, +\infty[$, and satisfies*

$$c(0) \;=\; 0, \quad c'(v) \;>\; 0, \quad c''(v) \;>\; 0 \text{ and } \lim_{v \to \infty} c(v) \;=\; +\infty.$$

To complete the model, we need an equation determining the discounted bond price $p(\cdot)$ in the evolution equation (2) for $x(\cdot)$. Without the presence of the devaluation of currency ($v = 0$), when a foreign investor buys a bond of unit nominal value, he will receive a continuous stream of payments with intensity $(r + \lambda)e^{-\lambda t}$. If bankruptcy never occurs, the payoff for a foreign investor (exponentially discounted in time) is

$$\Psi \;=\; \int_0^{\infty} e^{-rt} \cdot (r + \lambda)e^{-\lambda t} dt \;=\; 1.$$

Otherwise, the lenders recover only a fraction $\theta(x^*) \in [0,1]$ of their outstanding capital. In this case, taking account of the presence of the devaluation of currency, the payoff for a foreign investor will be

$$\begin{aligned}\Psi &= \int_0^{T_b} (r+\lambda)\cdot \exp\left\{-\int_0^t (r+\lambda+v^*(x(s))ds\right\} dt \\ &\qquad + \exp\left\{-\int_0^{T_b} (r+\lambda+v^*(x(s)))ds\right\}\cdot \theta(x^*).\end{aligned}$$

If the outstanding capital is recovered in full (i.e., $\theta(x^*)=1$) and $v=0$, then again $\Psi=1$. In general, however, $\theta(x^*)<1, v\neq 0$, and thus $\Psi<1$. As in [12], to offset this possible loss, the investors buy a bond with unit nominal value at a discounted price

$$\begin{aligned}p^*(x) &= \int_0^{T_b} (r+\lambda)\cdot \exp\left\{-\int_0^t (r+\lambda+v^*(x(s)))ds\right\} dt \\ &\qquad + \exp\left\{-\int_0^{T_b} (r+\lambda+v(x^*(s)))ds\right\}\cdot \theta(x^*) \qquad (5)\end{aligned}$$

for every initial debt-to-income ratio $x\in[0,x^*]$. We then set $p(t)=p^*(x(t))$.

2.2. System of first order Hamilton-Jacobi equations

The control system (2)–(5) is not standard. Indeed, the discount price p in (5) depends on the debt-to-income ratio not only at the present time t but also at all future times. Here, we are mainly interested in constructing optimal controls (u^*,v^*) in feedback form.

Definition 2.1 (Equilibrium solution in feedback form). A couple of piecewise Lipschitz continuous functions $(u^*(\cdot),v^*(\cdot))$ and l.s.c. $p^*(\cdot)$ provide an *equilibrium solution* to the debt management problem (2)-(4), with continuous value function $V^*(\cdot)$, if

(i) Given the price $p^*=p^*(x)$, one has that V^* is the value function and $(u^*(x),v^*(x))$ is the optimal feedback control, in connection with the deterministic control problem

$$\text{minimize:} \int_0^{T_b} e^{-rt}[L(u(t))+c(v(t))]\,dt + e^{-rT_b}B, \qquad (6)$$

subject to

$$\dot{x}(t) = \left(\frac{\lambda+r}{p^*(x)}-\lambda-\mu-v(t)\right)x(t)-\frac{u(t)}{p^*(x)}, \qquad x(0) = x_0, \qquad (7)$$

where the time T_b is determined by (3).

(ii) Given the feedback control $(u^*(x),v^*(x))$ in (7), for every $x_0\in[0,x^*]$ one has

$$p^*(x_0) = \int_0^{T_b}(r+\lambda)\exp\left\{-\int_0^t \big(\lambda+r+v^*(x(s)\big)\,ds\right\}dt +$$

$$+\exp\left\{-\int_0^{T_b}(r+\lambda+v^*(x(t)))\,dt\right\}\cdot\theta(x^*). \tag{8}$$

Under the assumptions **(A1)-(A2)**, the Hamiltonian function

$$H:[0,x^*]\times\mathbb{R}\times[0,1]\to\mathbb{R},$$

associated to the dynamics (2) and to the cost functions L, c, is defined by

$$H(x,\xi,p) := \min_{u\in[0,1]}\left\{L(u)-u\,\frac{\xi}{p}\right\}+\min_{v\geq 0}\left\{c(v)-vx\xi\right\}+\left(\frac{\lambda+r}{p}-\lambda-\mu\right)x\,\xi. \tag{9}$$

The Debt Management Problem leads to the following implicit system of first order ODEs satisfied by the value function V and the discounted rate p

$$\begin{cases} rV(x) \;=\; H(x,V'(x),p(x)) \\ (r+\lambda+v(x))p(x)-(r+\lambda) \;=\; H_\xi(x,V'(x),p(x))\cdot p'(x) \\ v(x) \;=\; \underset{\omega\in[0,+\infty]}{\operatorname{argmin}}\{c(\omega)-\omega xV'(x)\} \end{cases} \tag{10}$$

with the boundary conditions

$$V(0)=0,\qquad V(x^*)=B\qquad\text{and}\qquad p(0)=1,\qquad p(x^*)=\theta(x^*). \tag{11}$$

In the following, we shall construct a solution (V^*,p^*) to (10)-(11) and show that (p^*,V^*) and the associated feedback controls (u^*,v^*) give an equilibrium solution to the debt management problem (2)-(4).

Since the optimal debt management problem has been modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders, the equilibrium solution in Definition 2.1 can be viewed as a Nash equilibrium. In particular, when the discount bond price p is given, V will be the value function of an optimal control problem and thus will be a solution of the first equation in (10) in the viscosity sense.

2.3. Basic properties of H and normal form of the system

Let us first present some basic properties of the Hamiltonian function and the normal form of the system (10) which will be used to provide a semi-explicit formula for the optimal feed back strategy (u^*,v^*). For any fixed $x\geq 0$, $p\in]0,1]$, $\xi\geq 0$, we denote by

$$u^*(\xi,p)\doteq\underset{u\in[0,1]}{\operatorname{argmin}}\left\{L(u)-u\frac{\xi}{p}\right\},\qquad v^*(x,\xi)\doteq\underset{v\in[0,+\infty]}{\operatorname{argmin}}\left\{c(v)-vx\xi\right\}.$$

Set $c'_{\max} := \sup_{v\geq 0} c'(v)$. By Lemma 5.3 and Lemma 5.4 in the Appendix, one has that

$$u^*(\xi,p) \;=\; \begin{cases} 0 & \text{if } 0\leq\xi<pL'(0) \\ (L')^{-1}(\xi/p) & \text{if } \xi\geq pL'(0)>0 \end{cases}$$

and

$$v^*(x,\xi) = \begin{cases} 0, & \text{if } 0 \le x\xi \le c'(0), \\ (c')^{-1}(x\xi), & \text{if } 0 < c'(0) < x\xi < c'_{\max}, \\ +\infty, & \text{if } x\xi \ge c'_{\max}, \end{cases}$$

and the gradient of the Hamiltonian function $H(\cdot)$ can be expressed in terms of $u^*(\xi,p)$ and $v^*(x,\xi)$ at any point $(x,\xi,p) \in [0,+\infty[\times[0,+\infty[\times]0,1]$ with $x\xi < c'_{\max}$ by

$$\begin{cases} H_x(x,\xi,p) = \Big[(\lambda+r) - p(\lambda+\mu+v^*(x,\xi))\Big] \cdot \dfrac{\xi}{p} \\ H_\xi(x,\xi,p) = \dfrac{1}{p} \cdot \Big[x\big((\lambda+r) - p(\lambda+\mu+v^*(x,\xi))\big) - u^*(\xi,p)\Big] \\ H_p(x,\xi,p) = (u^*(\xi,p) - x(\lambda+r)) \cdot \dfrac{\xi}{p^2}. \end{cases} \tag{12}$$

From Lemma 5.5 and Lemma 5.6, the map $\xi \mapsto H(x,\xi,p)$ is concave with $H(x,0,p) = 0$, $\lim_{\xi\to+\infty} H(x,\xi,p) = -\infty$ for $x > 0$, and reaches its maximum value at a unique point $\xi^\sharp(x,p)$ such that

$$0 \le H^{\max}(x,p) := \max_{\xi\ge 0} H(x,\xi,p) = H(x,\xi^\sharp(x,p),p) < \infty.$$

Moreover, for any fixed $x \in]0,x^*]$, $p \in]0,1]$, and η satisfying $0 \le r\eta < H^{\max}(x,p)$, the equation $r\eta = H(x,\xi,p)$ in the unknown variable ξ admits exactly two distinct real solutions $\{F^-(x,\eta,p), F^+(x,\eta,p)\}$ with

$$0 < F^-(x,\eta,p) < \xi^\sharp(x,p) < F^+(x,\eta,p).$$

Extend the definition of $\eta \mapsto F^\pm(x,\eta,p)$ by setting

$$F^\pm\left(x, \frac{1}{r} H^{\max}(x,p), p\right) = \xi^\sharp(x,p),$$

we also have that the continuous maps $\eta \mapsto F^-(x,\eta,p)$ and $\eta \mapsto F^+(x,\eta,p)$ are respectively strictly increasing and strictly decreasing in the interval $\left[0, \dfrac{H^{\max}(x,p)}{r}\right]$.

Definition 2.2 (Normal form of the system). Given $(x,p) \in]0,x^*]\times]0,1]$ such that $0 < r\eta \le H^{\max}(x,p)$ we define the maps

$$G^\pm(x,\eta,p) = \frac{(r+\lambda+v^*(x,F^\pm(x,\eta,p)))p - (r+\lambda)}{H_\xi(x,F^\pm(x,\eta,p),p)}. \tag{13}$$

Notice that if $rV(x) > H^{\max}(x,p)$, then the first equation of (10) has no solution. Otherwise, if $0 < rV(x) < H^{\max}(x,p)$ this equation splits into

$$\begin{cases} V'(x) = F^-(x,V(x),p(x)), \\ p'(x) = G^-(x,V(x),p(x)), \end{cases} \quad \text{or} \quad \begin{cases} V'(x) = F^+(x,V(x),p(x)), \\ p'(x) = G^+(x,V(x),p(x)). \end{cases}$$

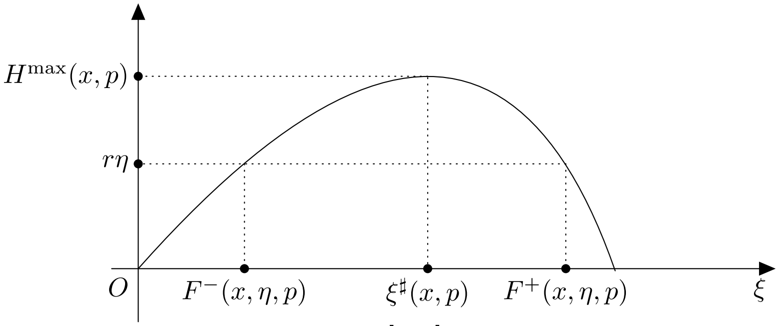

FIGURE 1. For $x \geq 0$, $p \in]0,1]$, the function $\xi \mapsto H(x,\xi,p)$ has a unique global maximum $H^{\max}(x,p)$ attained at $\xi = \xi^\sharp(x,p)$. For $0 < r\eta \leq H^{\max}$, this defines the values $F^-(x,\eta,p) \leq \xi^\sharp(x,p) \leq F^+(x,\eta,p)$. Moreover, $F^\pm\left(x, \frac{1}{r}H^{\max}(x,p), p\right) = \xi^\sharp(x,p)$.

Remark 2.3. Recalling (2) and (59), we observe that

- The value $V'(x) = F^+(x, V(x), p) \geq \xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot{x}(t) < 0$. The total debt-to-ratio is decreasing.
- The value $V'(x) = F^-(x, V(x), p) \leq \xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot{x}(t) > 0$. The total debt-to-ratio is increasing.
- When $rV(x) = H^{\max}(x,p)$, then the value
$$V'(x) \;=\; F^+(x, V(x), p) \;=\; F^-(x, V'(x), p) \;=\; \xi^\sharp(x,p)$$
corresponds to the unique control strategy such that $\dot{x}(t) = 0$.

In general, the map $\eta \mapsto F(x,\eta,p)$ is not Lipschitz in a neighborhood of $\xi^\sharp(x,p)$. Here, we shall complete this section by providing the Hölder continuity of the map $\eta \mapsto F(x,\eta,p)$ which is needed to construct our concatenating backward solution.

Lemma 2.4. *Given* $x_1 \in]0, x^*]$, $p_1 \in]0,1]$, *there exists a constant* $C = C(x_1, p_1)$ *such that*
$$|F^-(x,\eta_2,p) - F^-(x,\eta_1,p)| \;\leq\; C \cdot |\eta_2 - \eta_1|^{1/2}, \tag{14}$$
for all $x \in [x_1, x^*]$, $p \in [p_1, 1]$, $0 < \eta_1, \eta_2 \leq \frac{1}{r} H^{\max}(x,p)$.

Proof. Fixed any $(x,p) \in [x_1, x^*] \times [p_1, 1]$, it holds
$$\eta \;=\; \frac{1}{r} \cdot H(x, F^-(x,\eta,p), p) \quad \forall \eta \in \left[0, \frac{H^{\max}(x,p)}{r}\right].$$

Moreover, by Lemma 5.4, set $\bar{\xi}(x,p) = \frac{1}{x} \cdot \min\{xpL'(0), c'(0)\}$, we have that

$$\begin{cases} H_\xi(x,\xi,p) = & \frac{1}{p} \cdot \Big[x\big((\lambda + r) - p(\lambda + \mu)\big)\Big] > 0, \text{ if } \xi \le \bar{\xi}(x,p) \\ H_{\xi\xi}(x,\xi,p) \le & -\frac{1}{p} \cdot \min\left\{\frac{1}{pL''(u^*(\xi,p))}, \frac{x^2 p}{c''(v^*(x,\xi))}\right\} < 0, \\ & \qquad\qquad \text{if } \bar{\xi}(x,p) < \xi \le \xi^\sharp(x,p). \end{cases} \tag{15}$$

Let $\bar{\eta}(x,p) \in \left[0, \frac{1}{r} H^{\max}(x,p)\right]$ be such that

$$\bar{\eta}(x,p) \;=\; \frac{1}{r} \cdot H(x, \min\{\bar{\xi}(x,p), \xi^\sharp(x,p)\}, p).$$

By the continuity of the map $\eta \mapsto F^-(x,\eta,p)$, we only need to prove (14) in two cases.

- If $0 < \eta_1 < \eta_2 \le \bar{\eta}(x,p)$ then

  $$\xi_1 \;:=\; F^-(x,\eta_1,p) \;\le\; F^-(x,\eta_2,p) \;:=\; \xi_2 \;\le\; \bar{\xi}(x,p).$$

  From the first equation of (15), we have

  $$\begin{aligned} \eta_2 - \eta_1 =& \frac{1}{r} \cdot [H(x,\xi_2,p) - H(x,\xi_1,p)] \\ =& \frac{1}{rp} \cdot \Big[x\big((\lambda + r) - p(\lambda + \mu)\big)\Big] \cdot (\xi_2 - \xi_1) \\ \ge& \frac{x_1(r-\mu)}{r} \cdot (\xi_2 - \xi_1), \end{aligned}$$

  and this yields (14).
- If $\bar{\eta}(x,p) < \eta_1 < \eta_2 \le \frac{1}{r} H^{\max}(x,p)$ then

  $$\bar{\xi}(x,p) \;<\; \xi_1 \;:=\; F^-(x,\eta_1,p) \;\le\; F^-(x,\eta_2,p) \;:=\; \xi_2 \;\le\; \xi^\sharp(x,p).$$

  From the second equation of (15), we estimate

  $$\begin{aligned} \eta_2 - \eta_1 =& \frac{1}{r} \cdot [H(x,\xi_2,p) - H(x,\xi_1,p)] \\ =& \frac{1}{r} \cdot \left[\int_0^1 H_\xi(x, \xi_2 - s(\xi_2 - \xi_1), p)\, ds\right] \cdot (\xi_2 - \xi_1) \\ \ge& \frac{1}{r} \cdot \left[\int_0^1 H_\xi(x, \xi_2 - s(\xi_2 - \xi_1), p) ds - H_\xi(x,\xi_2,p)\right] \cdot (\xi_2 - \xi_1) \\ \ge& \frac{1}{2rp} \cdot \min\left\{\frac{1}{pL''(u^*(\xi_3,p))}, \frac{x^2 p}{c''(v^*(x,\xi_3))}\right\} \cdot (\xi_2 - \xi_1^2) \end{aligned} \tag{16}$$

  for some $\xi_3 \in [\xi_1, \xi_2]$. The increasing property of $u^*(\cdot,p)$ and $v^*(x,\cdot)$ yields

  $$\begin{aligned} u^*(\xi,p) \le& u^*(\xi^\sharp(x,p),p) := u^\sharp(x,p), \\ v^*(x,\xi_3) \le& v^*(x,\xi^\sharp(x,p)) := v^\sharp(x,p). \end{aligned}$$

Recalling (59) and (62), we have that

$$v^\sharp(x,p) \ \le \ \frac{\lambda+r}{p} - (\lambda+\mu) \ \le \ \frac{\lambda+r}{p_1}, \qquad \xi^\sharp(x,p) \ \le \ \frac{1}{x}\cdot c'\left(\frac{\lambda+r}{p_1}\right).$$

Thus, (60) and (9) imply

$$\begin{aligned} L(u^\sharp(x,p)) + c(v^\sharp(x,p)) \le & H^{\max}(x,p) \le \frac{(\lambda+r)x}{p}\cdot\xi^\sharp(x,p) \\ \le & \frac{\lambda+r}{p_1}\cdot c'\left(\frac{\lambda+r}{p_1}\right), \end{aligned}$$

and

$$\begin{aligned} u^*(\xi,p) \le & L^{-1}\left(\frac{\lambda+r}{p_1}\cdot c'\left(\frac{\lambda+r}{p_1}\right)\right) < 1, \\ v^*(x,p) \le & c^{-1}\left(\frac{\lambda+r}{p_1}\cdot c'\left(\frac{\lambda+r}{p_1}\right)\right), \end{aligned}$$

and (16) yields (14).

The proof is complete. □

3. An equilibrium solution to the Debt Management Problem

In this section, we will provide a detailed analysis on the existence of a solution to the system of Hamilton-Jacobi equation (10) with boundary conditions (11) which yields an equilibrium solution to the Debt Management Problem (2)-(4). A solution to will be constructed in the next following subsections.

3.1. Constant strategies

We begin our analysis from the control strategies keeping the debt-to-income ratio constant in time, i.e., such that the corresponding solution $x(\cdot)$ of (2) is constant. In this case, there is no bankruptcy risk, i.e., $T_b = +\infty$.

Definition 3.1 (Constant strategies). Let $\bar{x} > 0$ be given. We say that a pair $(\bar{u},\bar{v}) \in [0,1[\times[0,+\infty[$ is a constant strategy for $\bar{x}$ if

$$\begin{cases} \left[\left(\dfrac{\lambda+r}{\bar{p}} - \lambda - \mu - \bar{v}\right)\bar{x} - \dfrac{\bar{u}}{\bar{p}}\right] \ = \ 0, \\ \bar{p} \ = \ \dfrac{r+\lambda}{r+\lambda+\bar{v}}, \end{cases}$$

where the second relation comes from taking $T_b = +\infty$ in (3).

From these equations, if a couple $(\bar{u},\bar{v}) \in [0,1[\times[0,+\infty[$ is a constant strategy then it holds $(r+\lambda)(r-\mu)\bar{x} = (r+\lambda+\bar{v})\bar{u}$. In this case, the borrower will never go bankrupt and thus the cost of this strategy in (4) is computed by

$$\begin{aligned}\frac{1}{r}\cdot\Big[L(\bar u)+c(\bar v)\Big] &= \frac{1}{r}\cdot\left[L\left(\frac{(r+\lambda)(r-\mu)\bar x}{r+\lambda+\bar v}\right)+c\left(\bar v\right)\right]\\ &= \frac{1}{r}\cdot\left[L\left((r-\mu)\bar x\cdot\bar p\right)+c\left(\left(1-\frac{1}{\bar p}\right)(r+\lambda)\right)\right].\end{aligned}$$

Notice that if $\bar x(r-\mu)>1$, since $0\le\bar u<1$ we must have $\bar v>1$ and $\bar p<1$, in particular if the debt-to-income ratio (DTI) is sufficiently large, every constant strategy needs to implement currency devaluation. A more precise estimate will be provided in Proposition 3.4.

We are now interested in the minimum cost of a strategy keeping the debt constant. To this aim, we first characterize the cost of a constant strategy in terms of the variables x,p.

Lemma 3.2. *Given any* $(x,p)\in]0,+\infty[\times]0,1]$*, we have*

$$\begin{aligned}H^{\max}(x,p) \;=\; \min\Big\{L(u)+c(v):\; u\in[0,1],\,v\ge 0,\\ u=\big[(\lambda+r)-(\lambda+\mu+v)p\big]\cdot x\Big\}. \quad (17)\end{aligned}$$

Moreover, $(\hat u,\hat v)$ *realizes the minimum in the right hand side of* (17) *if and only if*

$$\begin{cases} c(\hat v)+px\hat v\xi^\sharp(x,p) &= \min\limits_{\zeta\ge 0}\left\{px\xi^\sharp(x,p)\zeta+c(\zeta)\right\},\\ L(\hat u)+\hat u\xi^\sharp(x,p) &= \min\limits_{u\in[0,1]}\left\{\xi^\sharp(x,p)u+L(u)\right\}.\end{cases}$$

Proof. Set

$$\Lambda(v):=f(v)+g(\beta v),\ \text{where } f(\zeta):=\begin{cases}c(\zeta), & \text{for } \zeta\ge 0,\\ +\infty, & \text{for } \zeta<0,\end{cases}$$

$$C(x,p):=\big[(\lambda+r)-(\lambda+\mu)p\big]\cdot x,$$

$$g(\zeta):=\begin{cases}L(C(x,p)+\zeta), & \text{if } C(x,p)+\zeta\in[0,1],\\ +\infty, & \text{if } C(x,p)+\zeta\notin[0,1],\end{cases}$$

and $\beta:=-xp$. By standard argument in convex analysis (see e.g. Theorem 4.2 and Remark 4.2 p. 60 of [9]), denoted by f°, g° the convex conjugates of f,g respectively (see in Definition (5.1)), we have

$$\inf_{v\in\mathbb{R}}\Lambda(v) \;=\; \sup_{\nu\in\mathbb{R}}\;\left[-f^\circ(\beta\nu)-g^\circ(-\nu)\right]$$

$$\begin{aligned}
&= \sup_{\nu\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)+xp\nu\zeta\right\} + \min_{C(x,p)+\zeta\in[0,1]}\left\{L(C+\zeta)+\nu\zeta\right\}\right]\\
&= \sup_{\nu\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)+xp\nu\zeta\right\} + \min_{u\in[0,1]}\left\{L(u)+\nu u\right\} - C\nu\right]\\
&= \sup_{\xi\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)-x\xi\zeta\right\} + \min_{u\in[0,1]}\left\{L(u)-u\cdot\frac{\xi}{p}\right\} + \frac{C(x,p)}{p}\cdot\xi\right]\\
&= \sup_{\xi\in\mathbb{R}} H(x,\xi,p) = H^{\max}(x,p).
\end{aligned}$$

Moreover, since $\sup_{\xi\in\mathbb{R}} H(x,\xi,p)$ is attained only at $\xi=\xi^\sharp(x,p)$ according to the strict concavity of $\xi\mapsto H(x,\xi,p)$, $(\hat{u},\hat{v})$ realizes the minimum in the right hand side of (17) if and only if

$$\begin{cases} f(\hat{v})+f^\circ(\beta\xi^\sharp(x,p))-\beta\hat{v}\xi^\sharp(x,p) &= 0,\\ g(\beta\hat{v})+g^\circ(-\xi^\sharp(x,p))+\beta\hat{v}\xi^\sharp(x,p) &= 0,\end{cases}$$

which implies $\hat{v}\geq 0$, $C(x,p)-px\hat{v}\in[0,1]$, and

$$\begin{cases} c(\hat{v})+px\hat{v}\xi^\sharp(x,p) &= \min_{\zeta\geq 0}\{px\xi^\sharp(x,p)\zeta+c(\zeta)\},\\ L(C(x,p)-px\hat{v})-px\hat{v}\xi^\sharp(x,p) &= \min_{\nu\in\mathbb{R}}\left\{\xi^\sharp(x,p)\nu+L(C(x,p)+\nu)\right\}.\end{cases}$$

The second relation can be rewritten as

$$L(\hat{u})+\hat{u}\xi^\sharp(x,p) = \min_{u\in[0,1]}\left\{\xi^\sharp(x,p)u+L(u)\right\},$$

and this complete the proof. □

Formula (17) allows us to give a simpler characterization of the minimum cost of a strategy keeping the debt-to-income ratio constant in time. Indeed, given $x\in[0,x^*]$, we select $(u(x),v(x))$ keeping the debt-to-income ratio constant in time. This defines uniquely a value $p=p(x)$ by Definition 3.1 and imposes a relation between $u(x)$ and $v(x)$. Then we take the minimum over all the costs of such strategies, i.e., the right hand side of formula (17). This naturally leads to the following definition.

Definition 3.3 (Optimal cost for constant strategies). Given $x\in[0,x^*]$, we define

$$W(x) = \frac{1}{r}\cdot H^{\max}(x,p_c(x)),$$

where

$$\begin{cases} p_c(x) &= \dfrac{r+\lambda}{r+\lambda+v_c(x)},\\ v_c(x) &= \operatorname*{argmin}_{v\geq 0}\left[L\left(\dfrac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right)+c(v)\right].\end{cases} \tag{18}$$

For every $x \in [0, x^*]$, $W(x)$ denotes the minimum cost of a strategy keeping the debt-to-income ratio constant in time. The next results proves that if the debt-to-income ratio is sufficiently small, the optimal strategy keeping it constant does not use the devaluation of currency.

Proposition 3.4 (Non-devaluating regime for optimal const. strategies). *Let $x_c \geq 0$ be the unique solution of the following equation in x*

$$(r+\lambda)c'(0) \;=\; (r-\mu)xL'\left((r-\mu)x\right).$$

Then

- *for all $x \in [0, \min\{x_c, x^*\}]$ we have*

$$W(x) = \frac{1}{r}\cdot L((r-\mu)x), \qquad\qquad p_c(x) = 1,$$

- *for all $x \in]\min\{x_c, x^*\}, x^*]$ we have*

$$W(x) \;=\; \frac{1}{r}\left[L\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v_c(x)}\right)+c(v_c(x))\right],$$

$$p_c(x) \;=\; \frac{r+\lambda}{r+\lambda+v_c(x)} < 1,$$

where $v_c(x) > 0$ solves the following equation in v

$$c'(v) \;=\; \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2}\cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right).$$

- *for every $x \in]0, x^*[$ we have*

$$W'(x) \;=\; \frac{r-\mu}{r}p_c(x)L'(p_c(x)(r-\mu)x) < \xi^\sharp(x, p_c(x)). \tag{19}$$

Proof. Given $x \in]0, x^*[$, we define the convex function

$$\Lambda^x(v) := \begin{cases} \dfrac{1}{r}\cdot\left[L\left(\dfrac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right)+c(v)\right], \\ \qquad\qquad\qquad\qquad \text{if } v \geq 0,\ \dfrac{(r+\lambda)(r-\mu)x}{r+\lambda+v} \in [0,1], \\ +\infty, \text{otherwise.} \end{cases}$$

We compute

$$\frac{d}{dv}\Lambda^x(v) \;=\; \frac{1}{r}\cdot\left[c'(v) - L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right)\frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2}\right],$$

which is monotone increasing and satisfies $\displaystyle\lim_{v\to+\infty}\frac{d}{dv}\Lambda^x(v) = +\infty$,

$$\frac{d}{dv}\Lambda^x(v) \;\geq\; \frac{d}{dv}\Lambda^x(0) \;=\; \frac{1}{r}\cdot\left[c'(0) - L'\left((r-\mu)x\right)\frac{(r-\mu)x}{r+\lambda}\right].$$

Two cases may occur:

- If $\frac{d}{dv}\Lambda^x(0) \geq 0$, we have that $v = 0$ realizes the minimum of Λ^x on $[0, +\infty[$. This occours when $x \in [0, \min\{x_c, x^*\}]$ where x_c is the unique solution of
$$(r+\lambda)c'(0) \;=\; (r-\mu)xL'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda}\right),$$
and it implies $W(x) = \frac{1}{r} \cdot L((r-\mu)x)$ and $p_c(x) = 1$.
- If we have $\min\{x_c, x^*\} < x \leq x^*$, then there exists a unique point $v_c(x) > 0$ such that $\frac{d}{dv}\Lambda^x(v_c(x)) = 0$, and this point is characterized by
$$c'(v_c(x)) \;=\; \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v_c(x))^2} \cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v_c(x)}\right).$$
The remaining statements follows noticing that for $\min\{x_c, x^*\} < x \leq x^*$ we have
$$\begin{aligned} W'(x) &= \frac{\partial \Lambda^x}{\partial x}(v_c(x)) + \frac{\partial}{\partial v}\Lambda^x(v_c(x)) \cdot v_c'(x) \;=\; \frac{\partial \Lambda^x}{\partial x}(v_c(x)) \\ &= \frac{r-\mu}{r} p_c(x) L'(p_c(x)(r-\mu)x), \end{aligned}$$
and deriving the explicit expression of $W(x)$ for $[0, \min\{x_c, x^*\}]$ yields the same formula. Notice that, by (61), we have
$$\begin{aligned} &\xi^\sharp(x, p_c(x)) = p_c(x)L'\left(\left[(\lambda+r) - (\lambda+\mu+v^\sharp(x,p_c(x)))p_c(x)\right] \cdot x\right) \\ &= p_c(x)L'\left(\left[(\lambda+r) - (\lambda+\mu+v^\sharp(x,p_c(x))) \cdot \frac{\lambda+r}{\lambda+r+v^\sharp(x,p_c(x))}\right] \cdot x\right) \\ &= p_c(x)L'(p_c(x)(r-\mu)\cdot x) > W'(x), \end{aligned}$$
where we used the fact that L' is strictly increasing and, since the argument of L' must be nonnegative, we have
$$\frac{\lambda+r}{\lambda+\mu+v^\sharp(x,p_c(x))} \;\geq\; p_c(x),$$

and the proof is complete. □

3.2. Existence of an equilibrium solution.

We are now ready to establish an existence result of a equilibrium solution to the debt management problem (2) - (4). Before going to state our main theorem, we recall from Proposition 3.4 that v_c is the unique solution to
$$c'(v) \;=\; \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2} \cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right),$$
and
$$\begin{aligned} p_c(x^*) &= \frac{r+\lambda}{r+\lambda+v_c(x^*)} \;<\; 1, \\ W(x^*) &= \frac{1}{r}\left[L\left(\frac{(r+\lambda)(r-\mu)x^*}{r+\lambda+v_c(x^*)}\right) + c(v_c(x^*))\right]. \end{aligned} \tag{20}$$

Theorem 3.5. *Assume that the cost functions L and c satisfies the assumptions* ***(A1)-(A2)****, and moreover*

$$W(x^*) > B \qquad \text{and} \qquad \theta(x^*) \leq p_c(x^*). \tag{21}$$

Then the debt management problem (2) - (4) *admits an equilibrium solution* (u^*, v^*, p^*) *associated to Lipschitz continuous value functions* V^* *in feedback form such that* p^* *is decreasing,* V^* *is strictly increasing and*

$$V^*(x) \leq W^*(x) \qquad \forall x \in [0, x^*].$$

Toward the proof of this theorem, we first study basic properties of the backward solutions of the system of implicit ODEs (10). In fact, an equilibrium solution will be constructed by a suitable concatenation of backward solutions.

3.2.1. Backward solutions. Recalling that

$$G^-(x, \eta, p) = \frac{(r + \lambda + v^*(x, F^-(x, \eta, p)))p - (r + \lambda)}{H_\xi(x, F^-(x, \eta, p), p)},$$

we first define the backward solution to the system (10) starting from x^*.

Definition 3.6 (Backward solution for x^*). Let $x \mapsto (Z(x, x^*), q(x, x^*))$ be the backward solution of the system of ODEs

$$\begin{cases} Z'(x) &= F^-(x, Z(x), q(x)), \\ q'(x) &= G^-(x, Z(x), q(x)), \end{cases} \qquad \text{with} \qquad \begin{cases} Z(x^*) &= B, \\ q(x^*) &= \theta(x^*). \end{cases} \tag{22}$$

with $H_\xi(x, F^-(x, Z(x), q(x)), q(x)) \neq 0$.

The following Lemma states some basic properties of the backward solution. In particular, the backward solution $Z(\cdot, x^*)$, starting from B at x^* with $W(x^*) < B$, survives backward at least until the first intersection with the graph of $W(\cdot)$. Moreover, in this interval is monotone increasing and positive. In the same way, $q(\cdot, x^*)$ is always in $]0, 1]$.

Proposition 3.7. *[Basic properties of the backward solution] Assume that*

$$W(x^*) > B \qquad \text{and} \qquad \theta(x^*) < \frac{r + \lambda}{r + \lambda + v^*(x^*, F^-(x^*, B, \theta(x^*)))}. \tag{23}$$

Then the backward solution of (22) *is well-defined in* $]x_W^*, x^*]$ *where*

$$x_W^* := \begin{cases} 0 & \text{if } Z(x, x^*) < W(x) \text{ for all } x \in]0, x^*[, \\ \sup\{x \in]0, x^*[: \ Z(x, x^*) \geq W(x)\} & \text{otherwise.} \end{cases}$$

Moreover,

(i). $Z(\cdot, x^*)$ *is strictly monotone increasing in* $]x_W^*, x^*[$*, and* $Z(x, x^*) > 0$ *for all* $x \in]x_W^*, x^*]$*;*

(ii). $q(x, x^*) \in]0, 1]$ *for all* $x \in]x_W^*, x^*]$.

Proof.

1. Denote by $I_{x^*} \subseteq [0, x^*]$ the maximal domain of the backward equation (22), define $y(x)$ to be the maximal solution of

$$\begin{cases} \dfrac{dy}{dx}(x) &= \dfrac{1}{H_\xi\left(x, Z'(x,x^*), q(x,x^*)\right)}, \\ y(x^*) &= 0, \end{cases}$$

and let J_{x^*} the intersection of its domain with $[0, x^*]$. From (23), it holds that

$$H_\xi(x, F^-(x^*, B, \theta(x^*)), \theta(x^*)) \;>\; 0$$

and

$$\begin{aligned} &G^-(x^*, F^-(x^*, B, \theta(x^*)), \theta(x^*)) \\ &\qquad = \frac{[r + \lambda + v^*(x^*, F^-(x^*, B, \theta(x^*))] \cdot \theta(x^*) - (r+\lambda)}{H_\xi(x, F^-(x^*, B, \theta(x^*)), \theta(x^*))} < 0. \end{aligned}$$

Thus, the maximal domain I_x contains $]x^* - \delta, x^*]$ for some $\delta > 0$, and $q(\cdot, x^*)$ is non-increasing in I_x. To prove $I_x \supseteq]x^*_W, x^*]$, we assume by a contradiction that there exists $x_2 \in J_{x^*} \cap]x^*_W, x^*[$ such that $H_\xi(x_2, Z'(x_2, x^*), q(x_2, x^*)) = 0$. Then

$$\xi^\sharp(x_2, q(x_2, x^*)) = Z'(x_2, x^*), \qquad Z(x_2, x^*) = \frac{1}{r} \cdot H^{\max}(x_2, q(x_2, x^*)),$$

and

$$u^\sharp(x_2, q(x_2, x^*)) \;=\; \left[(\lambda + r) - (\lambda + \mu + v^\sharp(x_2, q(x_2, x^*)))q(x_2, x^*)\right] \cdot x_2 \,.$$

Since $q(x_2, x^*) \le \dfrac{r + \lambda}{r + \lambda + v^\sharp(x_2, Z'(x_2, x^*))}$, we estimate

$$\begin{aligned} H^{\max}(x_2, q(x_2, x^*)) =& L(u^\sharp(x_2, q(x_2, x^*))) + c(v^\sharp(x_2, q(x_2, x^*))) \\ =& L\left(\left[(\lambda + r) - (\lambda + \mu + v^\sharp(x_2, q(x_2, x^*)))q(x_2, x^*)\right] \cdot x_2\right) \\ & + c(v^\sharp(x_2, q(x_2, x^*))) \\ \ge& L\left(\frac{r + \lambda(r - \mu)x_2}{\lambda + \mu + v^\sharp(x_2, q(x_2, x^*))}\right) + c(v^\sharp(x_2, q(x_2, x^*))) \\ \ge& H^{\max}(x_1, p_c(x_2)). \end{aligned}$$

Thus,

$$Z(x_2, x^*) \;=\; \frac{1}{r} \cdot H^{\max}(x_2, q(x_2, x^*)) \;\ge\; \frac{1}{r} \cdot H^{\max}(x_2, p_c(x_2)) \;=\; W(x_2),$$

and this yields a contradiction.

2. By construction, $y(\cdot)$ is strictly monotone and invertible in $]x^*_W, x^*]$, let $x = x(y)$ be its inverse, from the inverse function theorem we get

$$\begin{cases} \dfrac{d}{dy}\; Z(x(y), x^*) &= Z'(x(y), x^*) \cdot H_\xi\left(x(y), Z'(x(y), x^*), q(x(y), x^*)\right), \\ \dfrac{d}{dy}\; q(x(y), x^*) &= q'(x(y), x^*) \cdot H_\xi\left(x(y), Z'(x(y), x^*), q(x(y), x^*)\right). \end{cases}$$

Since the map $\xi \mapsto H(x,\xi,q)$ is concave, it holds

$$H_\xi(x,0,q(x,x^*)) \;\geq\; H_\xi(x,\xi,q(x,x^*)) \;\geq\; H_\xi\left(x,Z'(x,x^*),q(x,x^*)\right),$$

for all $\xi \in [0,Z'(x,x^*)]$. Thus,

$$\begin{aligned} rZ(x(y),x^*) &= H\left(x(y),Z'(x(y),x^*),q(x(y),x^*)\right) \\ &= \int_0^{Z'(x(y),x^*)} H_\xi(x,\xi,q(x(y),x^*))\,d\xi \\ &\geq Z'(x(y),x^*)\cdot H_\xi(x,Z(x(y),x^*),q(x(y),x^*)) \\ &= \frac{d}{dy}Z(x(y),x^*), \end{aligned}$$

and this implies that

$$Z(x,x^*) \;\geq\; Be^{ry(x)} \;>\; 0 \qquad \forall x \in]x_W^*,x^*].$$

With a similar argument for $q(\cdot,x^*)$, we obtain

$$\left[r+\lambda+v^*(x(y),Z'(x(y),x^*)\right]\cdot q(x(y),x^*)-(r+\lambda) \;=\; \frac{d}{dy}q(x(y),x^*)).$$

Hence,

$$\begin{aligned} (r+\lambda)(q(x(y),x^*)-1) &\leq \frac{d}{dy}q(x(y),x^*) \\ &\leq \left[r+\lambda+v^*(x(y),Z'(x(y),x^*)\right]\cdot q(x(y),x^*), \end{aligned}$$

and this yields

$$q(x,x^*)\leq 1 \qquad \text{and} \qquad q(x,x^*)\geq \theta(x^*)\cdot e^{(r+\lambda+v^*(x,Z'(x,x^*))y(x)}>0$$

for all $x\in I_{x^*}\cap[0,x^*]$. In particular, $q(x,x^*)\in]0,1]$ for all $x\in]x_W^*,x^*]$. □

As far as the graph of $Z(\cdot,x^*)$ intersects the graph of $W(\cdot)$, $Z(\cdot,x^*)$ is no longer optimal. The following lemma investigate the local behavior of $Z(\cdot,x^*)$ and $W(\cdot)$ near to an intersection of their graphs.

Lemma 3.8 (Comparison between optimal constant strategy and backward solution). *Let $I\subseteq]0,x^*[$ be an open interval, $(Z,q):I\to[0,+\infty[\times]0,1[$ be a backward solution, and $\bar{x}\in\bar{I}$. If*

$$\lim_{\substack{x\to\bar{x}\\ x\in I}} Z(x) \;=\; W(\bar{x})$$

then $p_c(\bar{x})\geq \limsup_{I\ni x\to\bar{x}} q(x)$ and $W'(x)<F^-(x,W(x),p_c(x))$.

Proof. Let $\{x_j\}_{j\in\mathbb{N}}\subseteq I$ be a sequence converging to $\bar{x}$ and $q_{\bar{x}}\in[0,1]$ be such that $q_{\bar{x}}=\limsup_{x\to\bar{x}+}q(x)=\lim_{j\to\infty}q(x_j)$. We have

$$\begin{aligned} H^{\max}(x,p_c(x)) &= \lim_{j\to+\infty} H\left(x_j,Z'(x_j),q(x_j)\right) \\ &\leq \lim_{j\to+\infty} H^{\max}(x_j,q(x_j)) = H^{\max}(\bar{x},q_{\bar{x}}). \end{aligned}$$

From 5.6 (4), it holds that $p_c(\bar{x}) \geq q_{\bar{x}}$. By Proposition 3.4, we have $W'(\bar{x}) < \xi^\sharp(\bar{x}, p_c(\bar{x}))$, and so

$$H(\bar{x}, W'(\bar{x}), p_c(\bar{x})) \ < \ H^{\max}(\bar{x}, p_c(\bar{x})) \ = \ rW(\bar{x}).$$

Thus, by applying the strictly increasing map $F^-(\bar{x}, \cdot, p_c(x))$ on both sides, we obtain $W'(x) < F^-(x, W(x), p_c(x))$. □

The functions $F^-(x, Z, q)$ and $G^-(x, Z, q)$ are smooth for $H_\xi(x, Z, q) \neq 0$ but only Hölder continuous with respect to Z near to the surface

$$\Sigma \ = \ \left\{(x, Z, q) \in \mathbb{R}^3 \ : \ H_\xi(x, Z, q) = 0\right\} .$$

Given any $x_0 \in [0, x^*)$, the definition of the solution of the Cauchy problem

$$\begin{cases} Z'(x) &= F^-(x, Z(x), q(x)), \\ q'(x) &= G^-(x, Z(x), q(x)), \end{cases} \qquad \text{with} \qquad \begin{cases} Z(x_0) &= W(x_0)\,, \\ q(x_0) &= p_c(x_0). \end{cases} \tag{24}$$

requires some care. For any $\varepsilon > 0$, we denote by $Z_\varepsilon(\cdot, x_0), q_\varepsilon(\cdot, x_0)$ the backward solution to (24) with the terminal data

$$Z_\varepsilon(x_0, x_0) \ = \ W(y_0) - \varepsilon \qquad \text{and} \qquad q_\varepsilon(x_0, x_0) \ = \ p_c(x_0).$$

With the same argument in the proof of Proposition 3.7, the solution is uniquely defined on a maximal interval $[a_\varepsilon(x_0), x_0]$ such that $Z_\varepsilon(\cdot, x_0)$ is increasing, $q_\varepsilon(\cdot, x_0)$ is decreasing and

$$Z_\varepsilon(a_\varepsilon(x_0), x_0) \ = \ W(a_\varepsilon(x_0)), \qquad\qquad q_\varepsilon(a_\varepsilon(x_0), x_0) \ \leq \ p_c(a_\varepsilon(x_0)).$$

Let $x^\flat$ be the unique solution to the equation

$$c'(0) \ = \ x \cdot L'((r - \mu)x)\,. \tag{25}$$

It is clear that $0 < x^\flat < x_c$ where x_c is defined in Proposition 3.4 as the unique solution to the equation

$$(r + \lambda)c'(0) \ = \ (r - \mu)xL'\left((r - \mu)x\right).$$

Two cases are considered:

CASE 1: For any $x_0 \in]0, x^\flat]$, we claim that

$$a_\varepsilon(x_0) \ = \ 0, \qquad q_\varepsilon(x, x_0) \ = \ 1 \qquad \forall x \in [0, x_0]\,,$$

and $Z_\varepsilon(\cdot, x_0)$ solves backward the following ODE

$$Z'(x) \ = \ F^-(x, Z(x), 1), \qquad Z(x_0) \ = \ W(x_0) - \varepsilon \tag{26}$$

for $\varepsilon > 0$ sufficiently small. Indeed, let Z_1 be the unique backward solution of (26). From (61), it holds

$$F^-(x, W(x), 1) \ = \ \xi^\sharp(x, 1) \ = \ L'((r-\mu)x) \ > \ \frac{r - \mu}{r} \cdot L'((r-\mu)x) \ = \ W'(x)$$

for all $x \in]0, x^\flat]$. As in [5], a contradiction argument yields

$$0 \ < \ Z_1(x) \ < \ W(x) \qquad \forall x \in]0, x_0]\,.$$

Thus, Z_1 is well-defined on $[0,x_0]$ and $Z_1(0)=0$. On the other hand, it holds

$$Z'(x_1) \;=\; F^-(x,Z(x),1) \;\leq\; \xi^\sharp(x,1) \;=\; L'((r-\mu)x) \;\leq\; L'((r-\mu)x^\flat)$$

for all $x\leq x^\flat$ and (25) implies that

$$v^*(x,Z_1'(x)) \;=\; 0 \qquad \forall x\in[0,x^\flat]\,.$$

Therefore, $(Z_1(x),1)$ solves (24) and the uniqueness yields

$$Z_\varepsilon(x,x_0) \;=\; Z_1(x) \qquad \text{and} \qquad q_\varepsilon(x,x_0) \;=\; 1 \qquad \forall x\in[0,x_0]\,.$$

Thanks to the monotone increasing property of the map $\xi\to F^-(x,\xi,1)$, a pair
$(Z(\cdot,x_0),q(\cdot,x_0))$ denoted by

$$q(x,x_0) \;=\; 1 \qquad \text{and} \qquad Z(x,x_0) \;=\; \sup_{\varepsilon>0} Z_\varepsilon(x,x_0) \qquad \forall x\in[0,x_0]$$

is the unique solution of (24). If the initial size of the debt is $\bar{x}\in[0,x_0]$ we think of $Z(\bar{x},x_0)$ is as the expected cost of (6)-(7) with $p(\cdot,x_0)=1$, $x(0)=x_0$ achieved by the feedback strategies

$$u(x,x_0) \;=\; \operatorname*{argmin}_{w\in[0,1]}\left\{L(w)-\,Z'(x,x_0)\cdot w\right\}, \qquad v(x,x_0) \;=\; 0 \qquad (27)$$

for all $x\in[0,x_0]$. With this strategy, the debt has the asymptotic behavior $x(t)\to x_0$ as $t\to\infty$.

CASE 2: For $x_0\in(x^\flat,x^*_W]$, system of ODEs (24) does not admit a unique solution in general since it is not monotone. The following lemma will provide the existence result of (24) for all $x_0\in(x^\flat,x^*_W]$.

Lemma 3.9. *There exists a constant $\delta_\flat>0$ depending only on $x^\flat$ such that for any $x_0\in\left(x^\flat,x^*_W\right)$, it holds*

$$x_0-a_\varepsilon(x_0) \;\geq\; \delta_{x^\flat} \qquad \forall\varepsilon\in(0,\varepsilon_0)$$

for some $\varepsilon_0>0$ sufficiently small.

Proof. From (19) and (61), it holds

$$\inf_{x\in[x^\flat,x^*_W]}\left\{\xi^\sharp(x,p_c(x))-W'(x)\right\} \;=\; \delta_{1,\flat} \;>\; 0.$$

In particular, we have

$$F^-(x_0,W(x_0),p_c(x_0))-W'(x_0) \;=\; \delta_{1,\flat}.$$

By continuity of the map $\eta\mapsto F^-(x_0,\eta,p_c(x_0))$ on $[0,W(x)]$, we can find a constant $\varepsilon_1>0$ sufficiently small such that

$$F^-(x_0,\eta,p_c(x_0)) \;\geq\; W'(x_0)+\frac{\delta_{1,\flat}}{2} \quad \forall\xi\in[W(x_0)-\varepsilon_1,W(x_0)].$$

On the other hand, the continuity of W' yields

$$\delta_{2,\flat} \;=\; \sup\left\{s\geq 0: W'(x_0-\tau) \;<\; W'(x_0)+\frac{\delta_{1,\flat}}{4} \quad \forall\tau\in[0,s]\right\}>0.$$

For a fixed $\varepsilon \in (0, \varepsilon_1)$, denote by

$$x_1 := \inf \left\{ s \in (0, x_0] : F^-\big(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\big) > W'(x) \ \forall x \in (s, x_0] \right\}.$$

If $x_1 > x_0 - \delta_{2,\bar{x}}$ then it holds

$$F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_1, x_0)\big) = W'(x_1) \le W'(x_0) + \frac{\delta_{1,\flat}}{4} \tag{28}$$

and there exists $x_2 \in (x_1, x_0]$ such that

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) = W'(x_0) + \frac{\delta_{1,\flat}}{2} \tag{29}$$

and

$$F^- \left(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\right) \le W'(x_0) + \frac{\delta_{1,\flat}}{2} \qquad \forall x \in [x_1, x_2]. \tag{30}$$

Recalling that $(x, \eta, p) \mapsto F^-(x, \eta, p)$ is defined by $H(x, F^-(x, \eta, p), p) = r\eta$, by the implicit function theorem, set $\xi = F^-(x, \eta, p)$, we have

$$\begin{aligned}
\frac{\partial}{\partial p} F^-(x, \eta, p) &= -\frac{H_p(x, \xi, p)}{H_\xi(x, \xi, p)} \\
&= \frac{\xi}{p} \cdot \frac{u^*(x, \xi, p) - x(\lambda + r)}{u^*(x, \xi, p) - x(\lambda + r) + xp(\lambda + \mu + v^*(x, \xi))} \\
&= \left(1 + \frac{x(\lambda + \mu + v^*(x, \xi))}{H_\xi(x, \xi, p)}\right) \frac{\xi}{p} > \frac{F^-(x, \eta, p)}{p} > 0.
\end{aligned}$$

Since $q_\varepsilon(\cdot, x_0)$ is decreasing, it holds

$$F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_1, x_0)\big) \ge F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big),$$

and (28)-(29) yield

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) - F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big) \ge \frac{\delta_{1,\flat}}{4}.$$

On the other hand, from (12) it follows that the map $x \to F^-(x, \eta, p)$ is monotone decreasing and thus

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) - F^-\big(x_2, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big) \ge \frac{\delta_{1,\flat}}{4}. \tag{31}$$

Observe that the map $\eta \to F^-(x, \eta, p)$ is Hölder continuous due to Lemma 2.4. More precisely, there exist a constant $C_{x^\flat} > 0$ such that

$$\left|F^-(x, \eta_2, p) - F^-(x, \eta_1, p)\right| \le C_{x^\flat} \cdot \left|\eta_2 - \eta_1\right|^{\frac{1}{2}}$$

for all $\eta_1, \eta_2 \in (0, W(x)]$, $x \in [\bar{x}, x^*]$, $p \in [\theta(x^*), 1]$. Thus, (31) implies that

$$|Z_\varepsilon(x_2, x_0) - Z_\varepsilon(x_1, x_0)| \ge \frac{\delta_{1,\flat}^2}{16 C_{x^\flat}^2}.$$

Recalling (30), we have

$$Z'_\varepsilon(x, x_0) = F^- \left(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\right) \le W'(x_0) + \frac{\delta_{1,x^\flat}}{2} \qquad \forall x \in [x_1, x_2],$$

and this yields

$$|x_2 - x_1| \ \geq \ \frac{\delta^2_{1,x^\flat}}{8C^2_{x^\flat}\left[2W'(x_0) + \delta_{1,x^\flat}\right]}.$$

Therefore,

$$x_0 - a_\varepsilon(x_0) \ \geq \ \delta_{x^\flat} \ := \ \min\left\{\delta_{1,x^\flat}, \frac{\delta^2_{1,x^\flat}}{8C^2_{x^\flat}\left[2W'(x_0) + \delta_{1,x^\flat}\right]}\right\} \ > \ 0,$$

and the proof is complete. □

Remark 3.10. In general, the backward Cauchy problem (24) may admit more than one solution.

As a consequence of Lemma 3.9, there exists a sequence $\{\varepsilon_n\}_{n\geq 0} \to 0+$ such that the sequence of backwards solutions $\{(Z_{\varepsilon_n}(\cdot, x_0), q_{\varepsilon_n}(\cdot, x_0))\}_{n\geq 1}$ converges to $(Z(\cdot, x_0), q(\cdot, x_0))$ which is a solution of (24). With the same argument in the proof of Proposition 3.7, we can extend backward the solution $(Z(\cdot, x_0), q(\cdot, x_0))$ until $a(x_0)$ such that

$$\lim_{x\to a(x_0)+} Z(a(x_0), x_0) \ = \ W(a(x_0)),$$

and Lemma 3.8 yields $\lim_{x\to a(x_0)+} q(a(x_0), x_0) \leq p_c(a(x_0))$. If the initial size of the debt is $\bar{x} \in [a(x_0), x_0]$ we think of $Z(\bar{x}, x_0)$ is as the expected cost of (6)-(7) with $p(\cdot, x_0)$, $x(0) = x_0$ achieved by the feedback strategies

$$\begin{cases} u(x, x_0) & = \ \underset{w\in[0,1]}{\operatorname{argmin}}\left\{L(w) - \dfrac{Z'(x, x_0)}{p(x, x_0)} \cdot w\right\}, \\ v(x, x_0) & = \ \underset{v\geq 0}{\operatorname{argmin}}\left\{c(v) - vxZ'(x, x_0)\right\}. \end{cases} \tag{32}$$

With this strategy, the debt has the asymptotic behavior $x(t) \to x_0$ as $t \to \infty$.

3.2.2. Construction of an equilibrium solution. We are now ready to construct an solution to the system of Hamilton-Jacobi equation (10) with boundary conditions (11). By induction, we define a family of back solutions as follows:

$$x_1 \ := \ x^*_W, \qquad (Z_1(x), q_1(x)) \ = \ (Z(x, x^*), q(x, x^*)) \qquad \forall x \in [x_1, x^*]$$

and

$$x_{n+1} \ := \ a(x_n), \qquad (Z(x, x_n), q(x, x_n)) \qquad \forall x \in [x_{n+1}, x_n].$$

From Case 1 and Lemma 3.9, there exists a natural number $N_0 < 1 + \dfrac{x^* - x^\flat}{\delta_{x^\flat}}$ such that our construction will be stop in N_0 step, i.e.,

$$x_{N_0} > 0, \qquad a(x_{N_0}) = 0 \qquad \text{and} \qquad \lim_{x\to a(x_{N_0})} Z(x, x_{N_0}) = 0.$$

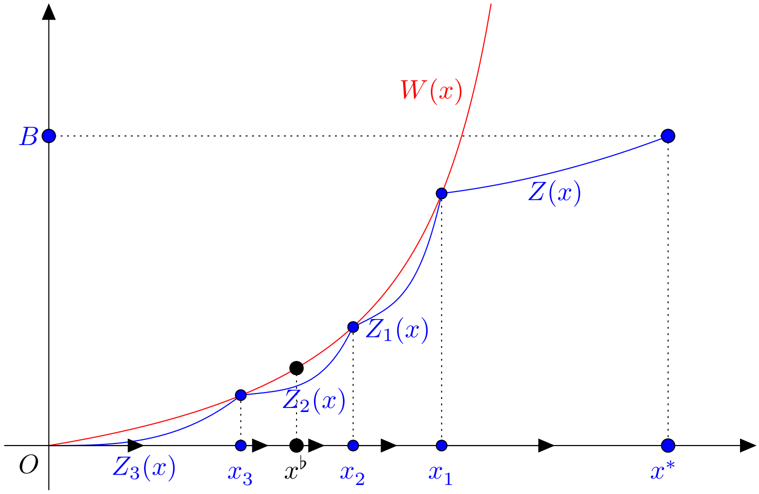

FIGURE 2. Construction of a solution: starting from (x^*, B) we solve backward the system until the first touch with the graph of W at $(x_1, W(x_1))$. Then we restart by solving backward the system with the new terminal conditions $(W(x_1), p_c(x_1))$, until the next touch with the graph of W at $(x_2, W(x_2))$ and so on. In a finite number of steps we reach the origin. If a touch occurs at $x_{n_0} < x^\flat$ then the backward solution from x_{n_0} reaches the origin with $q \equiv 1$. Given an initial value $\bar{x}$ of the debt-to-income ratio, if $0 \leq x_{n+1} < \bar{x} < x_n < x_1$ the the optimal strategy let the debt-to-income ratio increase asymptotically to x_n (no bankruptcy), while if $x_1 < \bar{x} < x^*$ then the optimal strategy let the debt-to-income ratio increase to x^*, thus providing bankruptcy in finite time.

We will show that a feedback equilibrium solution to the debt management problem is obtained as follows

$$(V^*(x), p^*(x)) = \begin{cases} (Z(x, x^*), q(x, x^*)), & \forall x \in (x_W, x^*], \\ (Z(x, x_k), q(x, x_k)), & \forall x \in (a(x_k), x_k], \\ & k \in \{1, 2, \dots, N_0\}, \end{cases} \tag{33}$$

and

$$\begin{cases} u^*(x) & = \underset{w \in [0,1]}{\operatorname{argmin}} \left\{ L(w) - \dfrac{(V^*)'(x)}{p^*(x)} \cdot w \right\}, \\ v^*(x) & = \underset{v \geq 0}{\operatorname{argmin}} \left\{ c(v) - vx(V^*)'(x) \right\}. \end{cases} \tag{34}$$

Proof of Theorem 3.5. From the monotone increasing property of the maps $\xi \mapsto v^*(x^*, \xi)$, $\eta \mapsto F^-(x^*, \eta, \theta(x^*))$ and $p \mapsto F^-(x^*, W(x^*), p)$, we have

$$\theta(x^*) \cdot (r + \lambda + v^*(x^*, F^-(x^*, B, \theta(x^*))) < p_c(x^*) \cdot (r + \lambda + v^*(x^*, F^-(x^*, W(x^*), p_c(x^*))) = r + \lambda,$$

and it yields (23). By Proposition 3.7 and Lemma 3.9, a pair $V^*(\cdot), p^*(\cdot)$ in (33) is well-defined on $[0, x^*]$. In the remaining steps, we show that V^*, p^*, u^*, v^* provide an equilibrium solution. Namely, they satisfy the properties (i)-(ii) in Definition 2.1.

1. To prove (i) in Definition 2.1, let $V(\cdot)$ be the value function for the optimal control problem (6)-(7). For any initial value, $x(0) = x_0 \in [0, x^*]$, the feedback controls u^* and v^* in (34) yield the cost $V^*(x_0)$. This implies

$$V(x_0) \le V^*(x_0).$$

To prove the converse inequality we need to show that, for any measurable control $u : [0, +\infty[\mapsto [0, 1]$ and $v : [0, +\infty[\to [0, +\infty[$, calling $t \mapsto x(t)$ the solution to

$$\dot{x}(t) = \left(\frac{\lambda + r}{p^*(x(t))} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p^*(x(t))}, \qquad x(0) = x_0, \tag{35}$$

it holds

$$\int_0^{T_b} e^{-rt}[L(u(x(t))) + c(v(x(t)))]\, dt + e^{-rT_b} B \ge V^*(x_0) \tag{36}$$

where

$$T_b = \inf \left\{t \ge 0;\ x(t) = x^*\right\}$$

is the bankruptcy time (possibly with $T_b = +\infty$).

For $t \in [0, T_b]$, consider the absolutely continuous function

$$\phi^{u,v}(t) := \int_0^t e^{-rs} \cdot [L(u(s)) + c(v(s))]\ ds + e^{-rt} V^*(x(t)).$$

At any Lebesgue point t of $u(\cdot)$ and $v(\cdot)$, recalling that (V^*, p^*) solves the system (10), we compute

$$\begin{aligned}
\frac{d}{dt}\phi^{u,v}(t) =& e^{-rt} \cdot \Big[L(u(t)) + c(v(t)) - rV^*(x(t)) + (V^*)'(x(t)) \cdot \dot{x}(t)\Big] \\
=& e^{-rt} \cdot \Big[L(u(t)) + c(v(t)) - rV^*(x(t)) \\
& + (V^*)'(x(t)) \left(\left(\frac{\lambda + r}{p^*(x(t))} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p^*(x(t))}\right)\Big] \\
\ge& e^{-rt} \cdot \Big[\min_{\omega \in [0,1]} \left\{L(\omega) - \frac{(V^*)'(x(t))}{p^*(x(t))}\,\omega\right\}
\end{aligned}$$

$$+\min_{\zeta\in[0+\infty[}\{c(\zeta)-(V^*)'(x(t))x(t)\,\zeta\}$$

$$+\left(\frac{\lambda+r}{p^*(x(t))}-\lambda-\mu\right)x(t)(V^*)'(x(t))-rV^*(x(t))\Big]$$

$$=e^{-rt}\cdot\Big[H\left(x(t),(V^*)'(x(t)),p^*(x(t))\right)-rV^*(x(t))\Big]=0.$$

Thus,

$$V^*(x_0)=\phi^{u,v}(0)\le\lim_{t\to T_b-}\phi^{u,v}(t)=\int_0^{T_b}e^{-rt}\cdot[L(u(t))+c(v(t))]\,dt+e^{-rT_b}B,$$

and this yields (36).

2. It remains to check (ii) in Definition 2.1. The case $x_0=0$ is trivial. Two remain cases will be considered.

CASE 1: If $x_0\in]x_1,x^*]$ then $x(t)>x_1$ for all $t\in[0,T_b]$. This implies

$$\dot{x}(t)=H_\xi(x(t),Z(x(t),x^*),q(x(t),x^*)).$$

From the second equation in (10) it follows

$$\frac{d}{dt}\,p(x(t))\;=\;p'(x(t))\dot{x}(t)\;=\;(r+\lambda+v^*(x(t)))p(x(t))-(r+\lambda),$$

Thus, for every $t\in[0,T_b]$ it holds

$$p(x(0))=p(x(t))\cdot\int_0^t e^{-(r+\lambda+v^*(x(\tau)))}\,d\tau+\int_0^t(r+\lambda)\int_0^\tau e^{-(r+\lambda+v^*(x(s)))}\,ds\,d\tau.$$

By letting $t\to T_b$, we obtain

$$p(x_0)=\int_0^{T_b}(r+\lambda)\int_0^\tau e^{-(r+\lambda+v^*(x(s)))}\,ds\,d\tau+\theta(x^*)\cdot\int_0^{T_b}e^{-(r+\lambda+v^*(x(\tau)))}\,d\tau.$$

CASE 2: Assume that $x_0\in[a(x_k),x_k[$ for some $k\in\{1,2,...,N_0\}$. In this case, $T_b=+\infty$ and $x(t)\in[a_{x_k},x_k[$ such that

$$\lim_{t\to+\infty}x(t)\;=\;x_k.$$

With a similar computation, we obtain

$$p(x_0)\;=\;\theta(x^*)\cdot\int_0^\infty e^{-(r+\lambda+v^*(x(\tau)))}\,d\tau,$$

proving (ii). □

4. Dependence on x^*

In this section, we study the behavior of the total cost for servicing when the maximum size x^* of the debt-ratio-income, at which bankruptcy is declared, becomes very large. It turns out that a crucial role in the asymptotic behavior of V as $x^* \to +\infty$ is played by the decay rate of the salvage rate $\theta(x^*)$ as $x^* \to +\infty$, which represents the fraction of the investment that can be recovered by the investors after the bankruptcy (and the unitary bond discounted price at the bankruptcy threshold). More precisely, the following proposition show that

- if the salvage rate decay sufficiently slowly, i.e., the lenders can still recover a sufficiently high fraction of their investment after the bankruptcy, then the best choice for the borrower is to implement the Ponzi-like trivial strategy;
- otherwise, if the salvage rate $\theta(x^*)$ decays sufficiently fast, then Ponzi-like trivial strategy is no longer an optimal solution for the borrower.

Proposition 4.1. *Let* $(V(x,x^*), p(x,x^*))$ *be constructed in Theorem* 3.5. *The following holds:*

(i) if $\limsup_{s\to+\infty} \theta(s)s = R < +\infty$ *then*

$$\liminf_{x^*\to+\infty} V(x,x^*) \ \geq\ B\cdot\left(1-\frac{R}{x}\right)^{\frac{r}{r+\lambda}} \tag{37}$$

for all

$$x \ \geq\ \frac{1}{r-\mu}\cdot\max\left\{4, \frac{4B}{L'(0)}, \frac{4C_1B}{c'(0)}, 2C_1c^{-1}(rB)\right\}.$$

(ii) if $\lim_{s\to+\infty} \theta(s)s = +\infty$ *then*

$$\limsup_{x^*\to\infty} V(x,x^*) \ =\ 0 \qquad \forall x\in[0,x^*[. \tag{38}$$

Proof. **1.** We first provide an upper bound on $v(\cdot,x^*)$. From (10) and (9), we estimate

$$\begin{aligned} H(x,\xi,p) \ &\geq\ \min_{v\geq 0}\{c(v)-x\xi v\} + [(r-\mu)x-1]\cdot\frac{\xi}{p} \\ &\geq\ \min_{v\geq 0}\{c(v)-x\xi v\} + \frac{(r-\mu)x}{2}\cdot\frac{\xi}{p} \ :=\ K(x,\xi,p) \end{aligned}$$

for all $\xi, p > 0$ and $x \geq \dfrac{2}{r-\mu}$. We compute

$$K_\xi(x,\xi,p) \ =\ \frac{(r-\mu)x}{2p} - xv_K$$

where

$$v_K \ =\ \begin{cases} 0 & \text{if } \ 0\leq x\xi < c'(0), \\ (c')^{-1}(x\xi) & \text{if } \ x\xi \geq c'(0) > 0. \end{cases}$$

This implies that the maximum of K is achieved for $v_K = \dfrac{r-\mu}{2p}$ and its value is

$$\max_{\xi\geq 0} K(x,\xi,p) = K(x,\xi_K,p) = c\left(\frac{r-\mu}{2p}\right), \text{ with } \xi_K = \frac{c'(v_K)}{x}.$$

Thus, the monotone increasing property of the map $\xi \to H(x,\xi,p(x,x^*))$ on the interval
$\left[0,\xi^\sharp(x,p(x,x^*))\right]$ implies that

$$F^-(x,V(x,x^*),p(x,x^*)) < \xi_K \quad \Longrightarrow \quad v(x,x^*) \leq \frac{r-\mu}{2p(x,x^*)}. \tag{39}$$

provided that $c\left(\dfrac{r-\mu}{2p(x,x^*)}\right) \geq rB$. From (10)) and (9), it follows

$$\begin{aligned} rB &\geq -xV'(x,x^*)v(x,x^*) + [(r-\mu)x - u(x,x^*)]\cdot\frac{V'(x,x^*)}{p(x,x^*)} \\ &\geq \left[\frac{(r-\mu)x}{2} - 1\right]\cdot\frac{V'(x,x^*)}{p(x,x^*)} \geq \frac{(r-\mu)x}{4}\cdot\frac{V'(x,x^*)}{p(x,x^*)}. \end{aligned}$$

Thus, if

$$\begin{cases} p(x,x^*) \leq & \min\left\{\frac{r-\mu}{2c^{-1}(rB)}, \frac{(r-\mu)c'(0)}{4B}\right\} \\ x \geq & \max\left\{\frac{4}{r-\mu}, \frac{4B}{(r-\mu)L'(0)}\right\} \end{cases} \tag{40}$$

then

$$\frac{V'(x,x^*)}{p(x,x^*)} \leq \frac{4B}{(r-\mu)x} \leq L'(0) \quad \Longrightarrow \quad u(x,x^*) = 0, \tag{41}$$

and

$$V'(x,x^*)x \leq \frac{4B}{r-\mu}\cdot p(x,x^*) \leq c'(0) \quad \Longrightarrow \quad v(x,x^*) = 0. \tag{42}$$

In this case, from (10), (9) and (12), it holds

$$(r+\lambda)(p(x,x^*)-1) = \left(\frac{\lambda+r}{p(x,x^*)} - \lambda - \mu\right)xp'(x,x^*).$$

Thus,

$$p(x,x^*) = \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

provided that (40) holds.

2. Assume that

$$\limsup_{s\in[0,+\infty)} \theta(s)s = R < +\infty,$$

there exists a constant $C_1 < +\infty$ such that $\sup_{s\in[0,+\infty)} \theta(s)s = C_1$. Since $p(\cdot,x^*)$ is increasing, it holds

$$p(x,x^*) \leq \frac{\theta(x^*)x^*}{x} \leq \frac{C_1}{x} \qquad \text{if (40) holds}. \tag{43}$$

Denote by

$$M \ := \ \frac{1}{r-\mu}\cdot \max\left\{4, \frac{4B}{L'(0)}, \frac{4C_1B}{c'(0)}, 2C_1c^{-1}(rB)\right\},$$

we then have

$$u(x,x^*) \ = \ v(x,x^*) \ = \ 0 \qquad \forall x\in[M,x^*], x^*\geq M\,.$$

From (10), (9) and (12), (V,p) solves the system of ODEs

$$\begin{cases} V'(x,x^*) & = \ \dfrac{rp}{[(\lambda+r)-(\lambda+\mu)p(x,x^*)]x}\cdot V \\ p'(x,x^*) & = \ (\lambda+r)\cdot\dfrac{p(x,x^*)(p(x,x^*)-1)}{[(\lambda+r)-(\lambda+\mu)p(x,x^*)]\,x} \end{cases} \tag{44}$$

for all $x\in[M,x^*]$ with $x^*\geq M$. Solving the above system of ODEs (see in Section 5 of [5]), we obtain that

$$V(x,x^*) = B\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}, \; p(x,x^*) = \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

for all $x\geq[M,x^*]$. Thus,

$$\liminf_{x^*\to+\infty} V(x,x^*) \ \geq \ B\cdot\left(1-\frac{R}{x}\right)^{\frac{r}{r+\lambda}} \qquad \forall x\geq M,$$

and this yields (37).

3. We are now going to prove (ii). Assume that

$$\limsup_{s\to+\infty} \ \theta(s)s \ = \ +\infty\,. \tag{45}$$

Set

$$\gamma := \min\left\{\frac{r-\mu}{2c^{-1}(rB)}, \frac{(r-\mu)c'(0)}{4B}\right\}, \; M_2 := \max\left\{\frac{4}{r-\mu}, \frac{4B}{(r-\mu)L'(0)}\right\}.$$

For any $x^*>M_2$, denote by

$$\tau(x^*) \ := \ \begin{cases} x^* & \text{if} \qquad \theta(x^*)\geq\gamma\,, \\ \inf\left\{x\geq M_2 \;\middle|\; p(x,x^*)\leq\gamma\right\} & \text{if} \qquad \theta(x^*)<\gamma\,. \end{cases}$$

From (40)–(42), the decreasing property of p yields

$$p(x,x^*) \ \geq \ \gamma \qquad \forall x\in[M_2,\tau(x^*)[\tag{46}$$

and

$$p(x,x^*) \ < \ \gamma \qquad \Longrightarrow \qquad u(x,x^*) \ = \ v(x,x^*) \qquad \forall x\in[\tau(x^*),x^*]\,.$$

As in the step 2, for any $x\in[\tau(x^*),x^*]$, we have

$$V(x,x^*) = B\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}, \; p(x,x^*) \ = \ \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

This implies that

$$V(x,x^*) \;=\; B\cdot\left(\frac{p(x,x^*)x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \;\leq\; B\cdot\left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \tag{47}$$

for all $x\in[\tau(x^*),x^*]$. On the other hand, for any $x\in[M_2,\tau(x^*)]$, from (10), (9) and (46), it holds

$$rV(x,x^*) \;\leq\; \frac{r+\lambda}{p(x,x^*)}xV'(x,x^*) \;\leq\; \frac{(r+\lambda)x}{\gamma}\cdot V'(x,x^*)\,.$$

This implies that for all $x\in[M_2,\tau(x^*)]$ it holds

$$V(x,x^*)\leq V(\tau(x^*),x^*)\cdot\left(\frac{x}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}}\leq B\cdot\left(\frac{x}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}}. \tag{48}$$

For any fix $x_0\geq M_2$, we will prove that

$$\limsup_{x^*\to+\infty}\; V(x_0,x^*) \;=\; 0. \tag{49}$$

Two cases are considered:

- If $\limsup_{x^*\to+\infty}\tau(x^*)=+\infty$ then (48) yields

$$\lim_{x^*\to+\infty}V(x_0,x^*) \;\leq\; \liminf_{x^*\to+\infty}\; B\cdot\left(\frac{x_0}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}} \;=\; 0.$$

- If $\limsup_{x^*\to+\infty}\tau(x^*)<+\infty$ then

$$\tau(x^*) \;<\; M_3 \qquad \forall x^* \;>\; 0$$

  for some $M_3>0$. Recalling (47) and (45), we obtain that

$$\lim_{x^*\to\infty}\;V(x_0,x^*) \;\leq\; \lim_{x^*\to\infty}V(x_0+M_3,x^*) \;\leq\; \lim_{x^*\to\infty}B\cdot\left(\frac{x_0+M_3}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \;=\; 0.$$

Thus, (49) holds and the increasing property of $V(\cdot,x^*)$ yields (38). □

We complete this section by showing that for sufficiently large initial debt-ratio-income and bankruptcy threshold and recovery fraction after bankruptcy, the optimal strategy for the borrower will use currency devaluation to deflate the debt-ratio-income. For simplicity, let us consider x^* and B^* sufficiently large such that

$$x^* \;>\; \frac{L'(0)+Br}{L'(0)\cdot(r-\mu)} \qquad\text{and}\qquad B\;\geq\;\frac{2(r-\mu)c'(0)}{r}. \tag{50}$$

In this case, the following holds:

Proposition 4.2 (Devaluating strategies). *Let* $x\mapsto(V(x,x^*),p(x,x^*))$ *be an equilibrium solution of* (10) *with boundary conditions* (11). *If*

$$\theta(x^*)x^* \;>\; \frac{2(r+\lambda)c'(0)}{r-\mu}\cdot\left(\frac{1}{rB}+\frac{1}{L'(0)}\right) \tag{51}$$

then the function

$$v^*(x,x^*) \;=\; \operatorname*{argmin}_{\omega\geq 0}\{c(\omega)-\omega x V'(x,x^*)\}$$

is not identically zero.

Proof. Set $M := \dfrac{L'(0)+Br}{L'(0)\cdot(r-\mu)}$. Assume by a contradiction that $v^*(x,x^*) = 0$ for all $x\in[M,x^*]$. In particular, we have

$$0 \;\leq\; xV'(x,x^*) \;\leq\; c'(0) \qquad x\in[M,x^*]. \tag{52}$$

The system (10) in $[M,x^*]$ reduces to

$$\begin{cases} rV(x) \;=\; \tilde{H}(x,V'(x),p(x)) \\ (r+\lambda)(p(x)-1) \;=\; \tilde{H}_\xi(x,V'(x),p(x))\cdot p'(x) \end{cases} \tag{53}$$

with

$$\tilde{H}(x,\xi,p) \;=\; \min_{u\in[0,1]}\left\{L(u)-\frac{u}{p}\,\xi\right\}+\left(\frac{\lambda+r}{p}-\lambda-\mu\right)x\,\xi.$$

Since $r>\mu$ and $p\in[0,1]$, it holds

$$\tilde{H}(x,\xi,p) \;\geq\; -\frac{\xi}{p}+(\lambda+r-p(\lambda+\mu))\,x\,\frac{\xi}{p} \;\geq\; ((r-\mu)x-1)\cdot\frac{\xi}{p}$$

and (53) yields

$$rB \;\geq\; rV(x,x^*) \;\geq\; ((r-\mu)x-1)\cdot\frac{V'(x,x^*)}{p(x,x^*)}.$$

Thus, for $x\in[M,x^*]$ we obtain

$$\frac{V'(x,x^*)}{p(x,x^*)} \;\leq\; \frac{rB}{(r-\mu)x-1}\leq L'(0),$$

which immediately implies

$$u^*(x,x^*) \;:=\; \operatorname*{argmin}_{u\in[0,1]}\left\{L(u)-u\cdot\frac{V'(x,x^*)}{p(x,x^*)}\right\} \;=\; 0.$$

Hence, $(V(\cdot,x^*),p(\cdot,x^*))$ solves (10) on $[M,x^*]$ and

$$V(x,x^*) = B\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}} \;\geq\; B\cdot\left(1-\frac{r}{r+\lambda}\cdot p(x,x^*)\right), \tag{54}$$

$$\begin{aligned} p(x,x^*) =& \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}} \\ \geq& \frac{\theta(x^*)x^*}{x}\cdot\left(1-\frac{r-\mu}{r+\lambda}\cdot p(x,x^*)\right), \end{aligned} \tag{55}$$

for all $x\in[M,x^*]$. From the above inequality, we derive

$$p(x,x^*) \;\geq\; \frac{(r+\lambda)\theta(x^*)x^*}{(r+\lambda)x+(r-\mu)\theta(x^*)x^*}.$$

Thus, (52) and (54) imply

$$\begin{aligned} c'(0) \geq xV'(x,x^*) \;&=\; rp(x,x^*)\cdot\frac{V(x,x^*)}{(\lambda+r)-(\lambda+\mu)p(x,x^*)} \\ &\geq \frac{rp(x,x^*)B}{r+\lambda}\cdot\frac{r+\lambda-rp(x,x^*)}{(\lambda+r)-(\lambda+\mu)p(x,x^*)} \;\geq\; \frac{rp(x,x^*)B}{r+\lambda} \\ &\geq \frac{rB\theta(x^*)x^*}{(r+\lambda)x+(r-\mu)\theta(x^*)x^*} \end{aligned}$$

for all $x\in[M,x^*]$. In particular, choose $x=M$ and recall (50), we get

$$M \;\geq\; \frac{rB-(r-\mu)c'(0)}{(r+\lambda)c'(0)}\cdot\theta(x^*)x^* \;\geq\; \frac{rB}{2(r+\lambda)c'(0)}\cdot\theta(x^*)x^*$$

and it contradicts (51). □

5. Appendix

We first introduce now some concepts of convex analysis, referring the reader to [9] and [14] for a comprehensive introduction to the subject.

Definition 5.1. [Convex conjugate and subdifferential] We recall that the convex conjugate $\Lambda^\circ:\mathbb{R}^d\to\mathbb{R}\cup\{\pm\infty\}$ of a map $\Lambda:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ is the lower semicontinuous convex function defined by

$$\Lambda^\circ(z^*) \;=\; \sup_{z\in\mathbb{R}^d}\Big\{\langle z^*,z\rangle-\Lambda(z)\Big\}.$$

Let $\Lambda:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ be proper (i.e., not identically $+\infty$), convex, lower semicontinuous functions, $x\in\operatorname{dom}\Lambda:=\{x\in\mathbb{R}^d:\,\Lambda(x)\in\mathbb{R}\}$. We define the *subdifferential in the sense of convex analysis* of Λ at x by setting

$$\partial\Lambda(x) \;:=\; \{v_x\in\mathbb{R}^d:\,\Lambda(y)-\Lambda(x)\geq\langle v_x,y-x\rangle \text{ for all } y\in\mathbb{R}^d\}.$$

The following result provide a list of some properties of the subdifferential in the sense of convex analysis.

Lemma 5.2 (Properties of the subdifferential). *Let $\Lambda,\Gamma:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ be proper (i.e., not identically $+\infty$), convex, lower semicontinuous functions,*

1. *If Λ is classically (Fréchet) differentiable at x, then $\partial\Lambda(x)=\{\Lambda'(x)\}$;*
2. *$z^*\in\partial\Lambda(z)$ if and only if $z\in\partial\Lambda^\circ(z^*)$;*
3. *$\Lambda(x_0)=\min\limits_{x\in\mathbb{R}^d}\Lambda(x)$ if and only if $0\in\partial\Lambda(x_0)$;*
4. *$z^*\in\partial\Lambda^\circ(z)$ if and only if $\Lambda(z)+\Lambda^\circ(z^*)=\langle z^*,z\rangle$. In this case $z^*\in\operatorname{dom}\Lambda^\circ$;*
5. *$\lambda\geq 0$ we have $\partial(\lambda\Lambda)(z)=\lambda\partial\Lambda(z)$;*
6. *if there exists $z\in\operatorname{dom}(\Lambda)\cap\operatorname{dom}(\Gamma)$ such that Λ is continuous at z then $\partial(\Lambda+\Gamma)(x)=\partial\Lambda(x)+\partial\Gamma(x)$ for all $x\in\operatorname{dom}(\Lambda)\cap\operatorname{dom}(\Gamma)$;*

7. *let* $\bar{y} \in \mathbb{R}^m$, $\Lambda : \mathbb{R}^m \to \mathbb{R}^d$ *be a linear map,* Γ *be continuous and finite at* $\Lambda(\bar{y})$*; Then* $\partial(\Gamma \circ \Lambda)(y) = \Lambda^T \partial\Gamma(\Lambda y)$ *for all* $y \in \mathbb{R}^m$*, where* $\Lambda^T : \mathbb{R}^d \to \mathbb{R}^m$ *is the adjoint of* Λ.

Lemma 5.3. *If* ***(A1)-(A2)*** *hold then* $L^\circ, c^\circ : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *are continuously differentiable in the interior of their domains, and*

$$\operatorname{int dom} L^\circ = \mathbb{R}, \qquad \operatorname{int dom} c^\circ =]-\infty, c'_{\max}[\quad \text{with} \quad c'_{\max} = \sup_{w \geq 0} c'(w).$$

Moreover,

$$(L^\circ)'(\rho) = \begin{cases} 0, & \text{if } \rho \leq L'(0), \\ (L')^{-1}(\rho), & \text{if } \rho > L'(0), \end{cases}$$

$$(c^\circ)'(\rho) = \begin{cases} 0, & \text{if } \rho \leq c'(0), \\ (c')^{-1}(\rho), & \text{if } \rho \in]c'(0), c'_{\max}[. \end{cases}$$

Proof. Recalling the assumptions $\mathbf{(A1)} - \mathbf{(A2)}$ on L, c, the equations

$$L^\circ(\rho_1) + L(u) \;=\; u\rho_1, \qquad\qquad c^\circ(\rho_2) + c(v) \;=\; v\rho_2,$$

admits as unique solutions

$$u(\rho_1) = \begin{cases} 0, & \text{if } \rho_1 < L'(0), \\ (L')^{-1}(\rho_1), & \text{if } \rho_1 \geq L'(0), \end{cases}$$

$$v(\rho_2) = \begin{cases} 0, & \text{if } \rho_2 < c'(0), \\ (c')^{-1}(\rho_2), & \text{if } c'_{\max} > \rho_2 \geq c'(0). \end{cases}$$

The result now follows from Theorem 23.5, Theorem 25.1, and Theorem 26.3 in [14]. □

Let L°, c° be the convex conjugate of L and c. According to (9), we have that

$$-H(x,\xi,p) \;\doteq\; L^\circ\left(\frac{\xi}{p}\right) + c^\circ(x\xi) - \left(\frac{\lambda + r}{p} - \lambda - \mu\right) x\,\xi, \tag{56}$$

and the map $\xi \mapsto -H(x,\xi,p)$ is convex and lower semicontinuous. Moreover, given $x > 0$, $p \in]0,1]$, $\xi \geq 0$, such that $x\xi < c'_{\max}$ we denote by $u^*(\xi,p) \in [0,1]$ and $v^*(x,\xi) \in [0,+\infty[$ the unique elements of $\partial L^\circ\left(\frac{\xi}{p}\right)$ and $\partial c^\circ(x\xi)$, respectively, provided by Lemma 5.3. The function $\rho \mapsto (c')^{-1}(\rho)$ is strictly increasing in $]c'(0), c'_{\max}[$ and

$$\lim_{\rho \to [c'_{\max}]^-} (c')^{-1}(\rho) = \sup_{\rho < c'_{\max}} (c')^{-1}(\rho) = +\infty,$$

Therefore, by monotonicity, we extend the definition of $v^*(x,\xi)$ by setting $v^*(x,\xi) = +\infty$ if $x\xi \geq c'_{\max}$.

As a consequence of Lemma 5.3, the following holds:

Lemma 5.4. *Assume **(A1)-(A2)**, and let H be defined as in* (9)*. Then H is continuous differentiable in the interior of its domain, and its gradient at points $(x,\xi,p)\in[0,+\infty[\times[0,+\infty[\times]0,1]$ with $x\xi<c'_{\max}$ can be expressed in terms of $u^*(\xi,p):=(L^\circ)'(\xi/p)$ and $v^*(x,\xi):=(c^\circ)'(x\xi)$ by*

$$\begin{cases} H_x(x,\xi,p) &= \Big[(\lambda+r)-p(\lambda+\mu+v^*(x,\xi))\Big]\cdot\dfrac{\xi}{p},\\ H_\xi(x,\xi,p) &= \dfrac{1}{p}\cdot\Big[x\big((\lambda+r)-p(\lambda+\mu+v^*(x,\xi))\big)-u^*(\xi,p)\Big],\\ H_p(x,\xi,p) &= (u^*(\xi,p)-x(\lambda+r))\cdot\dfrac{\xi}{p^2}, \end{cases} \tag{57}$$

and

$$\begin{cases} u^*(\xi,p) &= \underset{u\in[0,1]}{\operatorname{argmin}}\left\{L(u)-u\,\dfrac{\xi}{p}\right\},\\ v^*(x,\xi) &= \underset{v\geq 0}{\operatorname{argmin}}\left\{c(v)-vx\xi\right\}. \end{cases}$$

Moreover, for all $x>0$, $0<p\leq 1$, it holds

$$\begin{aligned} \nabla u^*(\xi,p) &= \frac{(1,-L'(u^*(\xi,p)))}{pL''(u^*(\xi,p))} \qquad \text{if } \xi>pL'(0), \\ \nabla v^*(x,\xi) &= \frac{(\xi,x)}{c''(v^*(x,\xi))} \qquad \text{if } x\xi>c'(0),\\ \lim_{\xi\to[c'_{\max}]^-} v^*(x,\xi) &= +\infty. \end{aligned} \tag{58}$$

Lemma 5.5. *Let the assumptions **(A1)-(A2)** hold. Then*

1. *for all $\xi\geq 0$ and $p\in]0,1]$, the function H satisfies*

$$\left(\frac{(\lambda+r)x-1}{p}-(\lambda+\mu+v^*(x,\xi))x\right)\xi\leq H(x,\xi,p) \leq\left(\frac{\lambda+r}{p}-(\lambda+\mu)\right)x\xi,$$

$$\frac{(\lambda+r)x-1}{p}-(\lambda+\mu+v^*(x,\xi))x\leq H_\xi(x,\xi,p)\leq\left(\frac{\lambda+r}{p}-(\lambda+\mu)\right)x,$$

2. *for every $x,p>0$ the map $\xi\mapsto H(x,\xi,p)$ is concave and satisfies*

$$H(x,0,p)\;=\;0,\qquad H_\xi(x,0,p)\;=\;\left(\frac{\lambda+r}{p}-(\lambda+\mu)\right)x.$$

Proof. The concavity of $\xi\mapsto H(x,\xi,p)$ for every $x,p>0$ is immediate from the definition of H in (9). The equalities in item (2) are immediate from Lemma 5.3. The upper bound on $H(x,\xi,p)$ follows from the positivity of L° and c°. Its lower estimate comes from the second part of Lemma 5.3, in particular from the upper estimate on $L^\circ(\cdot)$. By concavity, the map $\xi\mapsto H_\xi(x,\xi,p)$ is monotone decreasing, thus $H_\xi(x,\xi,p)\leq H_\xi(x,0,p)$, which proves the upper bound on $H_\xi(x,\xi,p)$ together with item (2). Its lower estimate for $H_\xi(x,\xi,p)$ comes from Lemma 5.4 and $u^*\in[0,1]$. $\square$

The following Lemma will catch some relevant properties of $H(\cdot)$ needed to study the system (10).

Lemma 5.6. *Let $x \geq 0$ and $0 < p \leq 1$ be fixed, and set*

$$H^{\max}(x,p) \doteq \max_{\xi \geq 0} H(x,\xi,p).$$

Then

1. *there exists $\xi^\sharp(x,p) > 0$ such that, given $\eta > 0$, the equation $r\eta = H(x,\xi,p)$ admits*
   - *no solutions $\xi \in [0,+\infty)$ if $r\eta > H^{\max}(x,p)$,*
   - *$\xi^\sharp(x,p)$ as unique solution if $r\eta = H^{\max}(x,p)$,*
   - *exactly two distinct solutions $\{F^-(x,\eta,p), F^+(x,\eta,p)\}$ with*
   $$0 \ < \ F^-(x,\eta,p) \ < \ \xi^\sharp(x,p) \ < \ F^+(x,\eta,p)$$
   if $0 < r\eta < H^{\max}(x,p)$,
2. *we extend the definition of $\eta \mapsto F^\pm(x,\eta,p)$ by setting*
   $$F^\pm\left(x, \frac{1}{r} H^{\max}(x,p), p\right) \ = \ \xi^\sharp(x,p),$$
   thus for fixed $x > 0$, $p \in]0,1]$, the maps $\eta \mapsto F^-(x,\eta,p)$ and $\eta \mapsto F^+(x,\eta,p)$ are respectively strictly increasing and strictly decreasing in $\left[0, \dfrac{H^{\max}(x,p)}{r}\right]$.
3. *for all $0 < \eta < H^{\max}(x,p)/r$ with $x > 0$ and $p \in]0,1]$, we have*
   $$\frac{\partial}{\partial \eta} F^\pm(x,\eta,p) \ = \ \frac{r}{H_\xi(x, F^\pm(x,\eta,p), p)},$$
4. *The map $p \mapsto H^{\max}(x,p)$ is strictly decreasing on $]0,1]$ for every fixed $x \in]0,x^*[$.*

Proof. Since for all fixed $x > 0$, $0 < p \leq 1$ we have that $\xi \mapsto H(x,\xi,p)$ is the minimum of a family of affine functions of ξ, we have that the map $\xi \mapsto H(x,\xi,p)$ is concave down. Recalling (12), and the monotonicity properties of $u^*(\cdot,p)$ and $v^*(x,\cdot)$, since

- $H_\xi(x,\xi,p) = H_\xi(x,0,p) > 0,$ for all $\xi \in [0, \min\{pL'(0), c'(0)/x\}]$,
- $\xi \mapsto H_\xi(x,\xi,p),$ is strictly decreasing for all $\xi > \min\{pL'(0), c'(0)/x\}$,
- $\displaystyle\lim_{\xi \to [c'_{\max}]^-} H_\xi(x,\xi,p) \leq \lim_{\xi \to [c'_{\max}]^-} \left(\frac{(r+\lambda)x}{p} - (c')^{-1}(x\xi)\right) = -\infty.$

we have that $\xi \mapsto H_\xi(x,\xi,p)$ vanishes in at most one point in $]0, c'_{\max}[$, so $\xi \mapsto H(x,\xi,p)$ reaches its maximum value $H^{\max}(x,p)$ on $[0,+\infty)$ at a unique point $\xi^\sharp(x,p)$, moreover it is strictly increasing for $0 < \xi < \xi^\sharp(x,p)$ and strictly decreasing for $\xi > \xi^\sharp(x,p)$, with $\xi^\sharp(x,p) \geq \min\{pL'(0), c'(0)/x\}$. We define

- the strictly increasing map $\eta \mapsto F^-(x,\eta,p)$, for $0 < \eta < \dfrac{1}{r} \cdot H^{\max}(x,p)$, to be the inverse of $\xi \mapsto \dfrac{1}{r} \cdot H(x,\xi,p)$ for $0 < \xi < \xi^\sharp(x,p)$;

- the strictly decreasing map $\eta \mapsto F^+(x,\eta,p)$, for $0<\eta<\frac{1}{r}\cdot H^{\max}(x,p)$, to be the inverse of $\xi \mapsto \frac{1}{r}\cdot H(x,\xi,p)$ for $\xi>\xi^\sharp(x,p)$.

This proves (1) and (2). Now, set

$$u^\sharp(x,p) \doteq u^*(\xi^\sharp(x,p),p), \qquad v^\sharp(x,p) \doteq v^*(x,\xi^\sharp(x,p)).$$

From (57), it holds

$$u^\sharp(x,p) = \Big[(\lambda+r)-(\lambda+\mu+v^\sharp(x,p))p\Big]\cdot x, \tag{59}$$

$$H^{\max}(x,p) = L(u^\sharp(x,p))+c(v^\sharp(x,p)). \tag{60}$$

Moreover,

- if $\xi^\sharp(x,p)\geq pL'(0)$ then

$$\xi^\sharp(x,p) = pL'(u^\sharp(x,p)) = pL'\left(\Big[(\lambda+r)-(\lambda+\mu+v^\sharp(x,p))p\Big]\cdot x\right), \tag{61}$$

- if $\xi^\sharp(x,p)\geq \dfrac{c'(0)}{x}$ then

$$\xi^\sharp(x,p) = \frac{c'(v^\sharp(x,p))}{x}. \tag{62}$$

Conversely, for any fixed $x\geq 0$ and $0<p\leq 1$, if

$$u^*(\xi,p) = x\Big((\lambda+r)-p(\lambda+\mu+v^*(x,\xi))\Big),$$

then $\xi=\xi^\sharp(x,p)$, $v^*(x,\xi)=v^\sharp(x,p)$ and $u^*(\xi,p)=u^\sharp(x,p)$. Indeed, this follows from the fact that $H_\xi(x,\xi,p)=0$ iff $\xi=\xi^\sharp(x,p)$.

Consider the equation $\eta = H(x,\xi,p)/r$ for a given $\eta>0$, and, noticing that, given $0<\xi<\xi^\sharp(x,p)$ we have

$$F^-(x,\eta,p) = F^-\left(x,\frac{1}{r}H(x,\xi,p),p\right) = \xi \qquad \forall 0<\xi<\xi^\sharp(x,p),$$

$$F^+(x,\eta,p) = F^+\left(x,\frac{1}{r}H(x,\xi,p),p\right) = \xi \qquad \forall \xi^\sharp(x,p)>\xi,$$

and so (3) follows from the Inverse Function Theorem. To prove item (4), we notice that

$$\frac{d}{dp}H^{\max}(x,p) = \frac{d}{dp}H(x,\xi^\sharp(x,p),p) = H_p(x,\xi^\sharp(x,p),p).$$

Recalling (12), we have

$$\begin{aligned} H_p(x,\xi^\sharp(x,p),p) &= \Big[u^\sharp(x,p)-(r+\lambda)x\Big]\cdot\frac{\xi^\sharp(x,p)}{p} \\ &= -(\lambda+\mu+v^\sharp(x,p))x\xi^\sharp(x,p) < 0, \end{aligned}$$

since for $x,p\neq 0$ we have $\xi^\sharp(x,p)>0$. □

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