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First Integrals and Symmetries of Nonholonomic Systems

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Abstract

In nonholonomic mechanics, the presence of constraints in the velocities breaks the well-understood link between symmetries and first integrals of holonomic systems, expressed by Noether's Theorem. However, there is a known special class of first integrals of nonholonomic systems generated by vector fields tangential to the group orbits, called *horizontal gauge momenta*, that suggests that some version of this link still holds. In this paper we prove that, under certain conditions on the symmetry group and the system, the (nonholonomic) momentum map is conserved along the nonholonomic dynamics, thus extending Noether's Theorem to the nonholonomic framework. Our analysis leads to a constructive method, with fundamental consequences to the integrability of some nonholonomic systems as well as their hamiltonization. We apply our results to three paradigmatic examples: the snakeboard, a solid of revolution rolling without sliding on a plane, and a heavy homogeneous ball that rolls without sliding inside a convex surface of revolution.

1. Introduction

1.1. Symmetries and first integrals

The existence of first integrals plays a fundamental role in the study of dynamical systems and influences many aspects of their behavior, in particular their integrability. It is well-known that in holonomic systems with symmetries (described by a suitable action of a Lie group), Noether's Theorem ensures that the components of the momentum map are first integrals of the dynamics. When constraints in the velocities are imposed and *nonholonomic systems* [11,23,48,50] are considered, this conservation property no longer holds. In fact, the presence of constraints in the velocites prevents the system from being variational (and thus lagrangian/hamiltonian) and hence Noether's Theorem cannot be applied. Therefore the presence of symmetries does not necessarily lead to first integrals (see

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[11,12,18,20,22,23,30,32,42,44,46,48,54,57]); in particular, the components of the momentum map need not be conserved by the dynamics.

Indeed the situation with nonholonomic systems with symmetries is quite subtle: only certain components of the momentum map may be conserved and, in general, it is not clear which ones or how many. However, many examples (see e.g., Table 1) admit first integrals linear in the momenta that are generated by vector fields that are not infinitesimal generators of the symmetry action, but are still tangential to the group orbits [7,9,28,31,57]; these are the so-called *horizontal gauge momenta* [9]. The goal of this paper is to set a theoretical framework that unifies there types of examples by proposing a systematic way to determine the total amount of horizontal gauge momenta and to write a general method to compute them.

The research of a possible link between the presence of symmetries and the existence of first integrals in nonholonomic systems—if any exists—has been an active field of research in the last thirty years [9,10,12,18,22,30,32,42,44,46,54,57], and it dates back at least to the fifties with the work of Agostinelli [1] and fifteen years later with the works of Iliev [40,41]. More recently, new tools and techniques, with a strong relation with the symmetries of the system, have been introduced in order to understand the dynamical and geometrical aspects of nonholonomic systems, such as nonholonomic momentum map, momentum equations, and gauge momenta. However, so far, the study of horizontal gauge momenta for different examples has been done case by case without—a priori—knowing if they exist or how many one could find. It is well known that first integrals are often found by integration of a system of PDEs or, in special cases (see for example Table 1), a system of ODEs, but a theoretical explanation of how and why a system of ODEs is sufficient to construct first integrals was not known.

1.2. Main results of the paper

Given a nonholonomic system on a manifold Q with a symmetry described by the (free and proper) action of a Lie group G, we consider functions of type $J_\xi = \langle J, \xi \rangle$, where J is the canonical momentum map and ξ is a section of the bundle $Q \times \mathfrak{g} \to Q$, with the property that the infinitesimal generator of each $\xi(q)$, $q \in Q$, is tangential to the constraint distribution. When J_ξ is a first integral of the nonholonomic dynamics, we say that J_ξ is a horizontal gauge momentum. Our nonholonomic Noether Theorem (Theorem 3.14) gives the theory and tools to predict the exact amount of horizontal gauge momenta and also to establish them by construction. First, denoting by k the rank of the distribution S, given by the intersection of the constraint distribution D with the tangent bundle to the G-orbits, Theorem 3.14 asserts the existence of k (functionally independent and G-invariant) horizontal gauge momenta, when certain hypotheses are satisfied. The importance of this first part resides in the fact that we are able to infer the exact amount of horizontal gauge momenta without the need of computing them. Moreover, this number is nothing but the rank of the vertical part of the constraints. Second, we write an explicit *system of linear ordinary differential equations*, that admits global solutions that give rise to the k horizontal gauge momenta. This system

Table 1. Nonholonomic systems and related horizontal gauge momenta with respect to the symmetry

System	Symmetry	rank(S)	♯ horizontal gauge momenta
Nonholonomic oscillator	$\mathbb{T}^2$	1	**1**
Vertical Disk	$SE(2) \times S^1$	2	**2**
Tippe–top	$SE(2) \times S^1$	2	**2**
Falling disk	$SE(2) \times S^1$	2	**2**
Snakeboard	$SE(2) \times S^1$	2	**2**
Body of revolution	$SE(2) \times S^1$	2	**2**
Ball in a cylinder	$SO(3) \times S^1$	2	**2**
Ball in a cup/cap	$SO(3) \times S^1$	2	**2**
Ball in a surface of revolution	$SO(3) \times S^1$	2	**2**

is constructed using the theoretical information of the nonholonomic system, and hence it has a "general character" in the sense that, given a nonholonomic system (satisfying certain hypotheses) we can always write this system. Therefore we show a unified way of computing the horizontal gauge momenta for many examples. We also conclude that, in these cases, the components of the *nonholonomic momentum map* [3,12] (in a suitable basis) are conserved along the dynamics, addressing the fundamental question raised in [3,12]. These results are based on an intrinsic and global formulation of the *momentum equation* that characterizes the horizontal gauge momenta.

Table 1 shown the evidence for the generality of our result, showing some of the classical examples of nonholonomic systems that fit into the scheme of Theorem 3.14 emphasizing the relation between the rank(S) and the number of horizontal gauge symmetries. We use our method to study in detail four of these examples: the nonholonomic oscillator, the snakeboard, a solid of revolution rolling on a plane and a heavy homogeneous ball rolling on a surface of revolution (in particular, the last two examples are paradigmatic of a large class of nonholonomic systems with symmetry) thus unifying many works in nonholonomic mechanics, as e.g. [11,17,21–23,28,38,50,56,57].

Even though Theorem 3.14 applies to many important examples, not every nonholonomic system is expected to admit horizontal gauge momenta. Therefore, we study particular examples of nonholonomic systems that do not satisfy the hypotheses of the Theorem, showing that as a consequence of this failure, the system admits less than k (and even none) horizontal gauge momenta and clarifying the need of each of the hypotheses.

The fact that we know the exact number of horizontal gauge momenta and have a systematic way of constructing them has fundamental consequences on the geometry and dynamics of nonholonomic systems; see e.g. [5,17,23,26,27,37,38,56]. Under the hypotheses of Theorem 3.14 we first show that the reduced dynamics is integrable by quadratures and, if some compactness conditions are satisfied, it is indeed periodic (Theorem 4.4). From a more geometric point of view, if the reduced dynamics is periodic, we have that the reduced space inherits the structure of an S^1-principal bundle outside the equilibria. Second, we prove

(Theorem 4.5) the *hamiltonization* of these nonholonomic systems (see also [8, 37]), were, precisely the existence of $k = \text{rank}(S)$ horizontal gauge momenta and the fact that $\dim(Q/G) = 1$ guarantee the existence of a *Poisson* bracket on the reduced space $\mathcal{M}/G$ that describes the reduced dynamics. This bracket is constructed using a *dynamical gauge transformation by a 2-form* that we also show to be related to the *momentum equation*. Third, when the reduced dynamics is periodic, we can obtain information on the complete dynamics (Theorem 4.12). In particular, if the symmetry group G is compact, the reconstructed dynamics is quasi-periodic on tori of dimension at most $r + 1$, where r is the rank of the symmetry group G, and the phase space inherits the structure of a torus bundle. If the symmetry group is not compact, the situation is less simple, but still understood: the complete dynamics is either quasi-periodic or diffeomorphic to $\mathbb{R}$, and whether one or the other case is more frequent (or generic) depends on the symmetry group (see Sect. 4.2, Appendix B and [2,33]).

1.3. Outline of the paper

The paper is organized as follows: in Sect. 2 we recall the basic aspects and notations of nonholonomic systems and horizontal gauge momenta. In Sect. 3 we present an intrinsic formulation of the momentum equation and the main result of the paper, Theorem 3.14. The results of this Section are illustrated with the example of the nonholonomic oscillator. The fundamental consequences of Theorem 3.14, integrability and hamiltonization, are studied in Sect. 4. Finally, in Sect. 5 we first apply our techniques and results to three paradigmatic examples outlined in bold in Table 1. We also study different cases where the hypotheses of Theorem 3.14 are not satisfied. The paper is complemented by two appendices: Appendix A recalls basic definitions regarding almost Poisson brackets and gauge tranformations, and Appendix B presents basic facts about reconstruction theory. Throughout the work, we assume that all objects (functions, manifolds, distributions, etc) are smooth. Unless stated otherwise, we consider Lie group actions that are free and proper or we confine our analysis to the submanifold where the action is free and proper. In this article, there are two types of comments, labelled **Remark** and **Observation** respectively; the Remarks are central comments to the problems studied in this paper, while the Observations are comments that complement the ideas but are not central. Finally, whenever possible, summation over repeated indices is understood.

2. Initial Setting: Nonholonomic Systems and Horizontal Gauge Momenta

2.1. Nonholonomic systems with symmetries

A nonholonomic system is a mechanical system on a configuration manifold Q with (linear) constraints in the velocities. The permitted velocities are represented by a nonintegrable constant-rank distribution D on Q. A nonholonomic system, denoted by the pair (L, D), is given by a manifold Q, a lagrangian function $L : TQ \to \mathbb{R}$ of mechanical type, i.e., $L = \frac{1}{2}\kappa - U$ for κ and U the kinetic and potential energy respectively, and a nonintegrable distribution D on Q. We now write the equations of motion of such systems following [10].

Since the lagrangian L is of mechanical type, the Legendre transformation $Leg : TQ \to T^*Q$ defines the submanifold $\mathcal{M} := Leg(D)$ of T^*Q. Moreover, since Leg is linear on the fibers, $\tau_{\mathcal{M}} := \tau|_{\mathcal{M}} : \mathcal{M} \to Q$ is also a subbundle of $\tau : T^*Q \to Q$, where τ denotes canonical projection. Then, if Ω_Q denotes the canonical 2-form on T^*Q and H the hamiltonian function induced by the lagrangian L, we denote by $\Omega_{\mathcal{M}} := \iota^*\Omega_Q$ and $H_{\mathcal{M}} := \iota^*H$ the 2-form and the hamiltonian on $\mathcal{M}$, where $\iota : \mathcal{M} \to T^*Q$ is the natural inclusion. We define the (noningrable) distribution $\mathcal{C}$ on $\mathcal{M}$ given, at each $m \in \mathcal{M}$, by

$$\mathcal{C}_m := \{v_m \in T_m\mathcal{M} \ : T\tau_{\mathcal{M}}(v_m) \in D_q \text{ for } q = \tau_{\mathcal{M}}(m)\}. \tag{2.1}$$

The nonholonomic dynamics is then given by the integral curves of the vector field X_{nh} on $\mathcal{M}$, taking values in $\mathcal{C}$ (i.e., $X_{\text{nh}}(m) \in \mathcal{C}_m$) such that

$$\mathbf{i}_{X_{\text{nh}}}\Omega_{\mathcal{M}}|_{\mathcal{C}} = dH_{\mathcal{M}}|_{\mathcal{C}}, \tag{2.2}$$

where $\Omega_{\mathcal{M}}|_{\mathcal{C}}$ and $dH_{\mathcal{M}}|_{\mathcal{C}}$ are the point-wise restriction of the forms to $\mathcal{C}$. It is worth noticing that the 2-section $\Omega_{\mathcal{M}}|_{\mathcal{C}}$ is nondegenerate and thus we have a well defined vector field X_{nh} satisfying (2.2), called the *nonholonomic vector field*.

On the hamiltonian side we will denote a nonholonomic system by the triple $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$.

Symmetries of a nonholonomic system. We say that an action of a Lie group G on Q defines a *symmetry* of the nonholonomic system (L, D) if it is free and proper and its tangent lift leaves L and D invariant.

Let $\mathfrak{g}$ be the Lie algebra associated to the Lie group G. At each $q \in Q$, we denote by $V_q \subset T_qQ$ the tangent space to the G-orbit at q, that is $V_q := \text{span}\{\eta_Q(q) : \eta \in \mathfrak{g}\}$, where $\eta_Q(q)$ denotes the infinitesimal generator of η at q.

The lift of the G-action to the cotangent bundle T^*Q leaves also the submanifold $\mathcal{M} \subset T^*Q$ invariant, hence there is a well defined G-action on $\mathcal{M}$ denoted by $\Psi : G \times \mathcal{M} \to \mathcal{M}$. The hamiltonian function $H_{\mathcal{M}}$ and the 2-section $\Omega_{\mathcal{M}}|_{\mathcal{C}}$ are G-invariant and we say that $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ is a *nonholonomic system with a G-symmetry*. We denote by $\mathcal{V}_m \subset T_m\mathcal{M}$ the tangent space to the G-orbit at $m \in \mathcal{M}$ (i.e., $\mathcal{V}_m = \{\eta_{\mathcal{M}}(m) : \eta \in \mathfrak{g}\}$).

Definition 2.1. ([12]) A nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry verifies the *dimension assumption* if, for each $q \in Q$,

$$T_qQ = D_q + V_q. \tag{2.3}$$

Equivalently, the dimension assumption can be stated as $T_m\mathcal{M} = \mathcal{C}_m + \mathcal{V}_m$ for each $m \in \mathcal{M}$.

At each $q \in Q$, we define the distribution S over Q whose fibers are $S_q := D_q \cap V_q$ and the distribution $\mathfrak{g}_S$ over Q with fibers

$$(\mathfrak{g}_S)_q = \{\xi^q \in \mathfrak{g} \ : \ \xi_Q(q) \in S_q\}, \tag{2.4}$$

where $\xi_Q(q) := (\xi^q)_Q(q)$. Due to the dimension assumption (2.3), $\mathfrak{g}_S \to Q$ is a vector subbundle of $Q \times \mathfrak{g} \to Q$ and, if the action is free then $\text{rank}(S) = \text{rank}(\mathfrak{g}_S)$ (see [5]). During this article, we denote by $\Gamma(\mathfrak{g}_S)$ the *sections* of the bundle $\mathfrak{g}_S \to Q$.
Reduction by symmetries. If $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ is a nonholonomic system with a G-symmetry, the nonholonomic vector field X_{nh} is G-invariant, i.e., $T\Psi_g(X_{\text{nh}}(m)) = X_{\text{nh}}(\Psi_g(m))$ with $\Psi_g : \mathcal{M} \to \mathcal{M}$ the G-action on $\mathcal{M}$ and $g \in G$, and hence it can be reduced to the quotient space $\mathcal{M}/G$. More precisely, denoting by $\rho : \mathcal{M} \to \mathcal{M}/G$ the orbit projection, the reduced dynamics on $\mathcal{M}/G$ is described by the integral curves of the vector field

$$X_{\text{red}} := T\rho(X_{\text{nh}}). \tag{2.5}$$

Splitting of the tangent bundle. The dimension assumption ensures the existence of a *vertical complement* W *of the constraint distribution* D (see [4]), that is, W is a distribution on Q so that

$$TQ = D \oplus W \qquad \text{where} \qquad W \subset V. \tag{2.6}$$

A vertical complement W also induces a splitting of the vertical space $V = S \oplus W$. Moreover, there is a one to one correspondence between the choice of an Ad-invariant subbundle $\mathfrak{g}_W \to Q$ of $\mathfrak{g} \times Q \to Q$ such that, at each $q \in Q$,

$$(\mathfrak{g} \times Q)_q = (\mathfrak{g}_S)_q \oplus (\mathfrak{g}_W)_q, \tag{2.7}$$

and the choice of a G-invariant vertical complement of the constraints W.

Observation 2.2. If the G-action is free, the existence of a G-invariant vertical complement W is guaranteed by choosing $W = S^\perp \cap V$, where $S^\perp$ denotes the orthogonal complement of S with respect to the (G-invariant) kinetic energy metric (however W does not have to be chosen in this way). In the case of non-free actions, as anticipated in the Introduction, we restrict our study to the submanifold $\widetilde{Q}$ of Q where the action is free (see Examples 5.2 and 5.3).[1] □

Next, we pull back the decomposition (2.6) to $\mathcal{M}$. From (2.6) and (2.1) we obtain the corresponding decomposition on $T\mathcal{M}$,

$$T\mathcal{M} = \mathcal{C} \oplus \mathcal{W} \qquad \text{with} \qquad \mathcal{W} \subset \mathcal{V}, \tag{2.8}$$

where, at each $m \in \mathcal{M}$, $\mathcal{W}_m = \{(\xi^q)_{\mathcal{M}}(m) \ : \ \xi^q \in (\mathfrak{g}_{\mathcal{W}})_q \text{ for } q = \tau_{\mathcal{M}}(m)\}$. We define the distribution $\mathcal{S} = \mathcal{C} \cap \mathcal{V}$ or equivalently, for each $m \in \mathcal{M}$,

$$\mathcal{S}_m = \{(\xi^q)_{\mathcal{M}}(m) \ : \ \xi^q \in (\mathfrak{g}_S)_q \text{ for } q = \tau_{\mathcal{M}}(m)\}.$$

[1] If the action is not free, it can be proven that for compact Lie groups G (or the product of a compact Lie group and a vector space), the dimension assumption guarantees that it is always possible to choose a G-invariant vertical complement W, [5].

2.2. Horizontal gauge momenta

Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry and recall that $\Theta_{\mathcal{M}}$ is the Liouville 1-form restricted to $\mathcal{M}$ (i.e., $\Theta_{\mathcal{M}} := \iota^*\Theta_Q$). It is well known that for a nonholonomic system, an element η of the Lie algebra does not necessarily induce a first integral of the type $\mathcal{J}_\eta$ (see [30] for a discussion of this fact).

Definition 2.3. ([9,28]) A function $\mathcal{J} \in C^\infty(\mathcal{M})$ is a *horizontal gauge momentum* if there exists $\zeta \in \Gamma(\mathfrak{g}_S)$ such that $\mathcal{J} = \mathcal{J}_\zeta := \mathbf{i}_{\zeta_{\mathcal{M}}}\Theta_{\mathcal{M}}$ and also $\mathcal{J}$ is a first integral of the nonholonomic dynamics $X_{\rm nh}$, i.e., $X_{\rm nh}(\mathcal{J}) = 0$. In this case, the section $\zeta \in \Gamma(\mathfrak{g}_S)$ is called *horizontal gauge symmetry*.

We are interested in looking for horizontal gauge momenta of a given nonholonomic system with symmetries satisfying the dimension assumption. Looking for a horizontal gauge momentum $\mathcal{J}$ is equivalent to look for the corresponding horizontal gauge symmetry.

Observation 2.4. The original definition of *horizontal gauge momentum* introduced in [9] (and later in [28,31]) was not exactly as in Definition 2.3 but given in local coordinates. □

Definition 2.5. [12] The *nonholonomic momentum map* $J^{\rm nh} : \mathcal{M} \to \mathfrak{g}_S^*$ is the bundle map over the identity, given, for each $m \in \mathcal{M}$ and $\xi \in \mathfrak{g}_S|_m$, by

$$\langle J^{\rm nh}, \xi\rangle(m) = \mathbf{i}_{\xi_{\mathcal{M}}}\Theta_{\mathcal{M}}(m). \tag{2.9}$$

Hence, if ζ is a horizontal gauge symmetry, the corresponding horizontal gauge momentum can be seen as a function of the type $\langle J^{\rm nh}, \zeta\rangle \in C^\infty(\mathcal{M})$ that is a first integral of $X_{\rm nh}$.

Observe that the existence of a horizontal gauge momentum, implies the existence of a global section on $\mathfrak{g}_S \to Q$. Hence, in order to prove that a nonholonomic system admits exactly k horizontal gauge symmetries, we have to assume the triviality of the bundle $\mathfrak{g}_S \to Q$, that is, $\mathfrak{g}_S \to Q$ admits a global basis of sections that we denote by

$$\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}. \tag{2.10}$$

The basis $\mathfrak{B}_{\mathfrak{g}_S}$ induces functions $J_1, ..., J_k$ on $\mathcal{M}$ (linear on the fibers) defined by

$$J_i := \langle J^{\rm nh}, \xi_i\rangle = i_{(\xi_i)_{\mathcal{M}}}\Theta_{\mathcal{M}} \qquad \text{for } i = 1, ..., k. \tag{2.11}$$

The functions $J_1, ..., J_k$ are the components of the nonholonomic momentum map in the basis (2.10). Moreover, if there exists a basis $\mathfrak{B}_{\rm HGS} = \{\zeta_1, ..., \zeta_k\}$ of $\Gamma(\mathfrak{g}_S)$ given by horizontal gauge symmetries, then the corresponding components of the nonholonomic momentum map are the horizontal gauge momenta $\{\mathcal{J}_1, ..., \mathcal{J}_k\}$ and, in this case, we have that each component of the nonholonomic momentum map is conserved along the nonholonomic dynamics.

Proposition 2.6. *A nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying the dimension assumption admits, at most, $k = rank(S)$ (functionally independent) horizontal gauge momenta.*

Proof. Consider $\xi_1, \xi_2 \in \Gamma(\mathfrak{g}_S)$. It is easy to see that if $J_1 = \mathbf{i}_{(\xi_1)_\mathcal{M}}\Theta_\mathcal{M}$ and $J_2 = \mathbf{i}_{(\xi_2)_\mathcal{M}}\Theta_\mathcal{M}$ are functionally independent functions then ξ_1, ξ_2 are linearly independent.

If $\mathcal{J} \in C^\infty(\mathcal{M})$ is a horizontal gauge momentum with ζ its associated horizontal gauge symmetry, then $\mathcal{J}$ and ζ can be written, with respect to the basis (2.10), as

$$\mathcal{J} = f^i J_i \quad \text{and} \quad \zeta = f^i \xi_i, \quad \text{for } f^i \in C^\infty(Q). \tag{2.12}$$

We call the functions f^i, $i = 1, ..., k$ the *coordinate functions of $\mathcal{J}$* with respect to the basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$.

From now, if not otherwise stated, we assume the following conditions on the symmetry given by the action of the Lie group G.

Conditions $\mathcal{A}$. We say that a nonholonomic system with a G-symmetry satisfies *Conditions $\mathcal{A}$* if

($\mathcal{A}$1) the dimension assumption (2.3) is fulfilled;
($\mathcal{A}$2) the bundle $\mathfrak{g}_S \longrightarrow Q$ is trivial;
($\mathcal{A}$3) the action of G on Q is proper and free.

A section ξ of the bundle $Q \times \mathfrak{g} \to Q$ is *G-invariant* if $[\xi, \eta] = 0$ for all $\eta \in \mathfrak{g}$. As a consequence of Conditions $\mathcal{A}$ we obtain the following Lemma.

Lemma 2.7. *Consider a nonholonomic system with a G-symmetry satisfying Conditions $\mathcal{A}$, then*

(i) *there exists a global basis $\mathfrak{B}_{\mathfrak{g}_S}$ of $\Gamma(\mathfrak{g}_S)$ given by G-invariant sections.*
(ii) *Let $\xi \in \Gamma(\mathfrak{g}_S)$. The function $J_\xi = \mathbf{i}_{\xi_\mathcal{M}}\Theta_\mathcal{M}$ is G-invariant if and only if $\xi \in \Gamma(\mathfrak{g}_S)$ is G-invariant.*
(iii) *Let $\rho_Q : Q \to Q/G$ be the orbit projection associated to the G-action on Q. If $X \in \mathfrak{X}(Q)$ is ρ_Q-projectable, then $[X, \xi_Q] \in \Gamma(V)$, for $\xi \in \Gamma(Q \times \mathfrak{g} \to Q)$.*

Proof. Items (ii) and (iii) were already proven in [8, Lemma 3.8]. To prove item (i) observe that items ($\mathcal{A}$2) and ($\mathcal{A}$3) imply that S admits a global basis of G-invariant sections $\{Y_1, ..., Y_k\}$, i.e., $[Y_i, \nu_Q] = 0$ for all $\nu \in \mathfrak{g}$. Since the action is free, we conclude that, for $(\xi_i)_Q = Y_i$ we have that $[\xi_i, \nu] \in \Gamma(Q \times \mathfrak{g})$ is the zero section and thus ξ_i are G-invariant. □

Under Conditions $\mathcal{A}$, we guarantee the existence of a global G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S}$ of sections of $\mathfrak{g}_S \to Q$ with associated G-invariant functions J_i (defined as in (2.12)). Hence, $\mathcal{J}$ is a G-invariant horizontal gauge momentum if and only if the corresponding coordinate functions f^i in (2.12) are G-invariant as well.

3. A Nonholonomic Noether Theorem and the Conservation of the Nonholonomic Momentum Map

3.1. An intrinsic formulation of the momentum equation

In order to achieve our goal of giving a precise estimate of the number of (functionally independent) horizontal gauge momenta of a nonholonomic system, we write a *momentum equation*. Let $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ be a nonholonomic system with a G-symmetry satisfying Conditions $\mathcal{A}$. First, we consider a G-invariant decomposition (or a principal connection)

$$TQ = H \oplus V \qquad \text{so that} \qquad H \subset D. \tag{3.13}$$

We denote by $A : T\mathcal{M} \to \mathfrak{g}$ the connection 1-form such that Ker$A = H$. Since the vertical space V is also decomposed as $V = S \oplus W$, the connection A can be written as $A = A_S + A_W$, where, for each $X \in TQ$, $A_W : TQ \to \mathfrak{g}$ is given by

$$A_W(X) = \eta \qquad \text{if and only if} \qquad \eta_Q = P_W(X),$$

and $A_S : TQ \to \mathfrak{g}$ is given by

$$A_S(X) = \xi \qquad \text{if and only if} \qquad \xi_Q = P_S(X), \tag{3.14}$$

where $P_W : TQ \to W$ and $P_S : TQ \to S$ are the corresponding projections associated to decomposition

$$TQ = H \oplus S \oplus W. \tag{3.15}$$

Second, we see that each map A_S and A_W defines a corresponding 2-form on Q in the following way (see [4]): on the one hand, the *W-curvature* on Q is a $\mathfrak{g}$-valued 2-form defined, for each $X, Y \in TQ$, as

$$K_W(X, Y) = d^D A_W(X, Y) = dA_W(P_D(X), P_D(Y)) = -A_W([P_D(X), P_D(Y)]),$$

with $P_D : TQ = D \oplus W \to D$ the projection to the first factor. On the other hand, after the choice of a global basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ of $\mathfrak{g}_S \to Q$, the $\mathfrak{g}$-valued 1-form A_S on Q can be written as $A_S = Y^i \otimes \xi_i$, where Y^i are 1-forms on Q such that $Y^i|_H = Y^i|_W = 0$ and $Y^i((\xi_j)_Q) = \delta_{ij}$ for all $i = 1, ..., k$ (recall that the sum over repeated indexes is understood). Then the corresponding $\mathfrak{g}$-valued 2-form is given, for each $X, Y \in TQ$, by

$$(d^D Y^i) \otimes \xi^i(X, Y) = dY^i(P_D(X), P_D(Y)) \otimes \xi^i.$$

Recalling that $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ is the canonical projection, we define the $\mathfrak{g}$-valued 2-forms $\bar{\sigma}_{\mathfrak{g}_S}$ and $\sigma_{\mathfrak{g}_S}$ on Q and $\mathcal{M}$, respectively, by

$$\begin{aligned} \bar{\sigma}_{\mathfrak{g}_S} &:= K_W + d^D Y^i \otimes \xi_i, \\ \sigma_{\mathfrak{g}_S} &:= \tau_{\mathcal{M}}^* \bar{\sigma}_{\mathfrak{g}_S}. \end{aligned} \tag{3.16}$$

Equivalently, $\sigma_{\mathfrak{g}_S}$ is given by $\sigma_{\mathfrak{g}_S} = \mathcal{K}_{\mathcal{W}} + d^{\mathcal{C}} \mathcal{Y}^i \otimes \xi_i$, where $\mathcal{K}_{\mathcal{W}} = \tau_{\mathcal{M}}^* K_W$, $\mathcal{Y}^i = \tau_{\mathcal{M}}^* Y^i$ and $d^{\mathcal{C}} \mathcal{Y}^i(\mathcal{X}, \mathcal{Y}) = d\mathcal{Y}^i(P_{\mathcal{C}}(\mathcal{X}), P_{\mathcal{C}}(\mathcal{Y}))$ for $\mathcal{X}, \mathcal{Y} \in T\mathcal{M}$, and $P_{\mathcal{C}} : T\mathcal{M} \longrightarrow \mathcal{C}$ the projection associated to decomposition (2.8).

Definition 3.1. Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions $\mathcal{A}$ and denote by $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ a global basis of $\Gamma(\mathfrak{g}_S)$.The 2-form $\langle J, \sigma_{\mathfrak{g}_S}\rangle$ on $\mathcal{M}$ is defined by

$$\begin{aligned}\langle J, \sigma_{\mathfrak{g}_S}\rangle &:= \langle J, \mathcal{K}_{\mathcal{W}}\rangle + \langle J, d^{\mathcal{C}}\mathcal{Y}^i \otimes \xi^i\rangle,\\ &:= \langle J, \mathcal{K}_{\mathcal{W}}\rangle + J_i\, d^{\mathcal{C}}\mathcal{Y}^i,\end{aligned}$$

where $J : \mathcal{M} \to \mathfrak{g}^*$ is the canonical momentum map restricted to $\mathcal{M}$ and $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

The 2-form $\langle J, \sigma_{\mathfrak{g}_S}\rangle$ already appeared in [8] for a specific choice of the basis $\mathfrak{B}_{\mathfrak{g}_S}$ (see Sect. 4.1).

Lemma 3.2. *Assume that Conditions $\mathcal{A}$ are satisfied, then*

(i) The $\mathfrak{g}$-valued 2-forms $\bar{\sigma}_{\mathfrak{g}_S}$ and $\sigma_{\mathfrak{g}_S}$ depend on the chosen basis $\mathfrak{B}_{\mathfrak{g}_S}$.
(ii) If the basis $\mathfrak{B}_{\mathfrak{g}_S}$ is G-invariant, then the 2-form $\langle J, \sigma_{\mathfrak{g}_S}\rangle$ is G-invariant as well.

Proof. It is straightforward to see that the $\mathfrak{g}$-valued 2-forms $\bar{\sigma}_{\mathfrak{g}_S}$ and $\sigma_{\mathfrak{g}_S}$ depend directly on the chosen basis $\mathfrak{B}_{\mathfrak{g}_S}$. Item (ii) is proven in [8, Lemma 3.8].

Proposition 3.3. (Momentum equation) *Let us consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions $\mathcal{A}$, and let $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ...\xi_k\}$ be a (global) basis of $\Gamma(\mathfrak{g}_S)$ with associated momenta $J_1, ..., J_k$ as in* (2.11)*. The function $\mathcal{J} = f^i J_i$, for $f^i \in C^\infty(Q)$, is a horizontal gauge momentum if and only if the coordinate functions f^i satisfy the momentum equation*

$$f^i \langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{Y}_i, X_{\rm nh}) + J_i X_{\rm nh}(f^i) = 0, \tag{3.17}$$

where $\mathcal{Y}_i := (\xi_i)_{\mathcal{M}}$.

Proof. First, from Lemma 2.7 observe that if $\mathcal{X}$ is a vector field on $\mathcal{M}$ that is $T\rho$-projectable, then $[\mathcal{Y}_i, \mathcal{X}] \in \Gamma(\mathcal{V})$ for $i = 1, ..., k$. Thus, using (3.16),

$$\begin{aligned}\sigma_{\mathfrak{g}_S}(\mathcal{Y}_i, \mathcal{X}) &= [d^{\mathcal{C}}\tau_{\mathcal{M}}^* A_W + d^{\mathcal{C}}\tau_{\mathcal{M}}^* Y^j \otimes \xi_j](\mathcal{Y}_i, \mathcal{X})\\ &= -\tau_{\mathcal{M}}^* A_W([\mathcal{Y}_i, \mathcal{X}]) - \tau_{\mathcal{M}}^* Y^j([\mathcal{Y}_i, \mathcal{X}]) \otimes \xi_j\\ &= -\tau_{\mathcal{M}}^* A([\mathcal{Y}_i, \mathcal{X}]).\end{aligned}$$

Second, by the definition of the canonical momentum map $J : \mathcal{M} \to \mathfrak{g}^*$, we get that

$$\langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{Y}_i, \mathcal{X}) = -\langle J, \tau_{\mathcal{M}}^* A([\mathcal{Y}_i, \mathcal{X}])\rangle = -\mathbf{i}_{[\mathcal{Y}_i, \mathcal{X}]}\Theta_{\mathcal{M}}.$$

Then, recalling that $\Omega_{\mathcal{M}} = -d\Theta_{\mathcal{M}}$ and using that $\Theta_{\mathcal{M}}(\mathcal{X})$ is an invariant function, we observe that

$$\begin{aligned}&(\Omega_{\mathcal{M}} + \langle J, \sigma_{\mathfrak{g}_S}\rangle)(\mathcal{Y}_i, \mathcal{X})\\ &\quad = -\mathcal{Y}_i(\Theta_{\mathcal{M}}(\mathcal{X})) + \mathcal{X}(J_i) + \Theta_{\mathcal{M}}([\mathcal{Y}_i, \mathcal{X}]) - \mathbf{i}_{[\mathcal{Y}_i, \mathcal{X}]}\Theta_{\mathcal{M}} = dJ_i(\mathcal{X}).\end{aligned}$$

Now, $\mathcal{J} = f^i J_i$ is a first integral of X_{nh} if and only if $0 = d\mathcal{J}(X_{\text{nh}}) = f^i dJ_i(X_{\text{nh}}) + J_i X_{\text{nh}}(f^i)$ which is equivalent, for $X_{\text{nh}} = \mathcal{X}$, to

$$\begin{aligned} 0 &= f^i(\Omega_{\mathcal{M}} + \langle J, \sigma_{\mathfrak{g}_S}\rangle)(\mathcal{Y}_i, X_{\text{nh}}) + J_i X_{\text{nh}}(f^i) \\ &= -f^i dH_{\mathcal{M}}(\mathcal{Y}_i) + f^i \langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{Y}_i, X_{\text{nh}}) + J_i X_{\text{nh}}(f^i). \end{aligned}$$

Using the G-invariance of the hamiltonian function $H_{\mathcal{M}}$ we get (3.17). □

Observation 3.4. From the proof of Proposition 3.3, we observe that the *momentum equation* can be equivalently written as $0 = f^i \Theta_{\mathcal{M}}([(\xi_i)_{\mathcal{M}}, X_{\text{nh}}]) - J_i X_{\text{nh}}(f^i)$.□

As a consequence of Proposition 3.3 (or more precisely Remark 3.4), we recover the well-known result that horizontal symmetries generate first integrals [10,12]. Recall that a *horizontal symmetry* is an element $\eta \in \mathfrak{g}$ such that $\eta_Q \in \Gamma(D)$ (see e.g. [11]).

Corollary 3.5. (Horizontal symmetries) *Let $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ be a nonholonomic system with a G-symmetry satisfying Conditions $\mathcal{A}$. If the bundle $\mathfrak{g}_S \to Q$ admits a horizontal symmetry η, then the function $\langle J, \eta\rangle$ is a horizontal gauge momentum for the nonholonomic system. Hence if there is global basis of horizontal symmetries of $\mathfrak{g}_S$, then the nonholonomic system admits $k = rank\,(\mathfrak{g}_S)$ horizontal gauge momenta.*

Proof. If η_1 is a horizontal symmetry, then let $\mathfrak{B}_{\mathfrak{g}_S} = \{\eta_1, \xi_2, ..., \xi_k\}$ a basis of $\Gamma(\mathfrak{g}_S)$. A section $\zeta = f^1\eta_1 + f^i\xi_i$ is a horizontal gauge symmetry if $J_1 X_{\text{nh}}(f_1) + f^i\Theta_{\mathcal{M}}([X_{\text{nh}}, (\xi_i)_{\mathcal{M}}]) + J_i X_{\text{nh}}(f^i) = 0$, since $[X_{\text{nh}}, \eta_1] = 0$. Then we see that $f^1 = 1$ and $f^i = 0$ for $i = 2, ..., k$ is a solution of the momentum equation and hence η_1 is a horizontal gauge symmetry. As a consequence, if the bundle $\mathfrak{g}_S \to Q$ admits a basis of horizontal symmetries, then the nonholonomic admits k horizontal gauge momenta. □

A set of solutions $(f^1, ..., f^k)$ of the momentum equation (3.17) may depend on $\mathcal{M}$ and not only on Q. Based on the fact that equation (3.17) is quadratic in the fibers, we show next that it is equivalent to a system of partial differential equations for the functions f^i on the manifold Q.

3.2. The "strong invariance" condition on the kinetic energy

We now introduce and study an invariance property, called strong invariance, that involves the kinetic energy, the constraints and the G-symmetry. This condition is crucial to state our main result in Theorem 3.14.

Definition 3.6. Consider a Riemannian metric κ on a manifold Q and a distribution $S \subseteq V \subseteq TQ$ on Q, where V is the vertical space with respect to a G-action on Q. The metric κ is called *strong invariant on S* (or *S-strong invariant*) if, for all G-invariant sections $Y_1, Y_2, Y_3 \in \Gamma(S)$, it holds that

$$\kappa(Y_1, [Y_2, Y_3]) = -\kappa(Y_3, [Y_2, Y_1]).$$

Remark 3.7. It is important to remark the tensorial character of this condition. In fact, it is enough to check the commuting condition on a G-invariant basis of the distribution S to assert that the metric is S-invariant. More precisely, let $\mathfrak{B} = \{Y_1, ..., Y_k\}$ be a G-invariant basis of sections on S, such that $\kappa(Y_j, [Y_i, Y_l]) = -\kappa(Y_l, [Y_i, Y_j])$, for all $i, j, l \in \{1, ..., k\}$. If Y_a, Y_b, Y_c are G-invariant sections on S, then $Y_a = f_a^j Y_j, Y_b = f_b^i Y_i$ and $Y_c = f_c^l Y_l$, for f_a^j, f_b^i, f_c^l G-invariant functions on Q and hence,

$$\begin{aligned}\kappa(Y_a, [Y_b, Y_c]) =& f_a^j \kappa(Y_j, f_b^i f_c^l [Y_i, Y_l] + f_b^i Y_i(f_c^l) Y_l - f_c^l Y_l(f_b^i) Y_i)\\ =& f_a^j f_b^i f_c^l \kappa(Y_j, [Y_i, Y_l]) = -\kappa(Y_c, [Y_b, Y_a]).\end{aligned}$$

First we observe that, for a Riemannian metric κ, being G-invariant is weaker than being strong invariant on the whole tangent bundle as the following example shows:

Example 3.8. **The case $Q = G$ with a strong invariant metric on TG.** Consider a Lie group G acting on itself with the left action and let κ_G be a Riemannian metric on it. In this case, the metric being G-invariant is equivalent to being left invariant, while being strong invariant on TG is equivalent to being bi-invariant. In fact, if the metric is strong invariant on TG then $\kappa_G([Y_i, Y_j], Y_l) = -\kappa_G(Y_j, [Y_i, Y_l])$ for all $Y_i \in \mathfrak{X}(G)$ such that $[Y_i, \eta^R] = 0$ for all $\eta \in \mathfrak{g}$ and η^R the corresponding right-invariant vector field on G (we are using that the infinitesimal generator associated to the left action is the corresponding right invariant vector field on G). Then, the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ defined by

$$\langle \eta_1, \eta_2 \rangle = \kappa_G(\eta_1^L, \eta_2^L)(e), \qquad \text{for } \eta_i \in \mathfrak{g}$$

is ad-invariant and hence the metric κ_G turns out to be bi-invariant on G.

Example 3.9. **A nonholonomic system with a strong invariant kinetic energy on the vertical distribution V.** Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry. If the kinetic energy metric κ is strong invariant on V then it induces a bi-invariant metric on the Lie group G. This case only may occur when the group of symmetries G is compact or a product of a compact Lie group with a vector space. In order to prove this, we first observe the following:

Lemma 3.10. *The kinetic energy metric satisfies $\kappa([Y_i, Y_j], Y_l) = -\kappa(Y_j, [Y_i, Y_l])$ for all G-invariant $Y_i \in \Gamma(V)$ if and only if $\kappa([(\eta_a)_Q, (\eta_b)_Q], (\eta_c)_Q) = -\kappa((\eta_b)_Q, [(\eta_a)_Q, (\eta_c)_Q])$ for all $\eta_i \in \mathfrak{g}$.*

Proof. The vertical distribution V admits a basis of G-invariant sections $\{Y_1, ..., Y_n\}$. For $\eta \in \mathfrak{g}$, there are functions $g^j \in C^\infty(Q)$, $j = 1, ..., n$ so that $\eta_Q = g^j Y_j$ and hence $0 = [Y_i, \eta_Q] = g^j [Y_i, Y_j] + Y_i(g^j) Y_j$. Then we obtain that

$$\begin{aligned}\kappa([(\eta_a)_Q, (\eta_b)_Q], (\eta_c)_Q) &= g_a^i g_b^j g_c^l \kappa([Y_i, Y_j], Y_l) + g_a^i g_c^l \kappa(Y_i(g_b^j) Y_j, Y_l)\\ &\quad - g_b^j g_c^l \kappa(Y_j(g_a^i) Y_i, Y_l)\\ &= -g_a^i g_b^j g_c^l \kappa([Y_i, Y_j], Y_l).\end{aligned}$$

Conversely, we write $Y_i = g_i^a \eta_a$ and we repeat the computation. □

As a direct consequence of Lemma 3.10, if the kinetic energy is strong invariant on V, then $\kappa([(\eta_a)_Q, (\eta_b)_Q], (\eta_c)_Q) = -\kappa((\eta_b)_Q, [(\eta_a)_Q, (\eta_c)_Q])$ for all $\eta_i \in \mathfrak{g}$. Hence, for each $q \in Q$, there is an *ad*-invariant inner product on $\mathfrak{g}$ defined, at each $\eta_1, \eta_2 \in \mathfrak{g}$ by

$$\langle \eta_1, \eta_2 \rangle_q = \kappa((\eta_1)_Q(q), (\eta_2)_Q(q)).$$

Therefore, there exists a *family* of bi-invariant metrics κ_G^q on G defined by $\kappa_G^q(\eta_1^L(g), \eta_2^L(g)) = \langle \eta_1, \eta_2 \rangle_q$.

Example 3.11. **The symmetry group G is abelian.** Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry, and let G be an abelian Lie group, then the Lie algebra $\mathfrak{g}$ is also abelian and the kinetic energy metric satisfies $\kappa([(\eta_1)_Q, (\eta_2)_Q], (\eta_3)_Q) = 0$ for all $\eta_i \in \mathfrak{g}$. Following Example 3.9, we have also that $\kappa([Y_1, Y_2], Y_3) = 0$ for all G-invariant sections Y_i on V and hence the kinetic energy is trivially strong invariant on V.

Example 3.12. **Horizontal symmetries.** Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions $\mathcal{A}$ and with the bundle $\mathfrak{g}_S \to Q$ admitting a global basis of G-invariant horizontal symmetries $\{\eta_1, ..., \eta_k\}$ of the bundle $\mathfrak{g} \times Q \to Q$. Then the vector space generated by the constant sections η_i is an abelian subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and the kinetic energy metric is strong invariant on S.

3.3. Nonholonomic Noether Theorem

In this section we state the main result of this article: a nonholonomic version of Noether Theorem which involves two fundamental results. Under certain hypotheses, we first predict the exact amount of horizontal gauge momenta that a nonholonomic system with symmetry admits, without the need of computing them. Second, we establish a systematic way of constructing these horizontal gauge momenta.

Given a nonholonomic system with a G-symmetry we add a fourth condition to Conditions $\mathcal{A}$:

Condition ($\mathcal{A}4$). The G-symmetry satisfies that the manifold Q/G has dimension 1.

Condition ($\mathcal{A}4$) can be equivalently formulated saying that the rank of any horizontal space H defined as in (3.13) is 1. We stress that this case is meaningful, since it is met, for example, in all the systems listed in Table 1.

Definition 3.13. Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry and let H be defined as in (3.13). We say that H is *S-orthogonal* if it is given by

$$H := S^{\perp} \cap D,$$

where the orthogonal space to S is taken with respect to the kinetic energy metric.

The S-orthogonality condition implies that H is a G-invariant distribution, while Condition $(\mathcal{A}4)$ guarantees that it is trivial, and thus H always admits a (G-invariant) global generator X_0, that is also ρ_Q-projectable, with respect to the projection on the G-orbits $\rho_Q : Q \to Q/G$.

Now, let $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ be a nonholonomic system with a G-symmetry satisfying Conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ (that is, the G-symmetry satisfies Conditions $\mathcal{A}$ and Condition $(\mathcal{A}4)$). Then there is a global G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ of sections of $\mathfrak{g}_S$ and, as usual, we denote by $Y_i := (\xi_i)_Q$ the corresponding sections on S. Taking into consideration also the generator X_0 of H, the set $\{X_0, Y_1, ..., Y_k\}$ defines a global basis of $D = H \oplus S$. Following splitting (3.15), we also consider a (possible non global) basis $\{Z_1, ..., Z_N\}$ of the vertical complement W and we denote by $(v^0, v^1, ..., v^k, w^1, ..., w^N)$ the coordinates on TQ associated to the basis

$$\mathfrak{B}_{TQ} = \{X_0, Y_1, ..., Y_k, Z_1, ...Z_N\}, \tag{3.18}$$

(for short we write the coordinates (v^0, v^i, w^a) associated to the basis $\mathfrak{B}_{TQ} = \{X_0, Y_j, Z_a\}$). If $\mathfrak{B}_{T^*Q} = \{X^0, Y^i, Z^a\}$ is the basis of T^*Q dual to $\mathfrak{B}_{TQ}$, we denote by (p_0, p_i, p_a) the induced coordinates on T^*Q. The constraint submanifold $\mathcal{M}$ is described as

$$\mathcal{M} = \{(q, p_0, p_i, p_a) \in T^*Q \ : \ p_a = \kappa_{a0}v^0 + \kappa_{aj}v^j\},$$

where $p_0 = \kappa_{00}v^0 + \kappa_{0i}v^i$, and $p_i = \kappa_{i0}v^0 + \kappa_{ij}v^j$, with $\kappa_{00} = \kappa(X_0, X_0)$, $\kappa_{0i} = \kappa(X_0, Y_i)$, $\kappa_{ij} = \kappa(Y_i, Y_j)$, $\kappa_{0a} = \kappa(X_0, Z_a)$ and $\kappa_{aj} = \kappa(Z_a, Y_j)$. We can then define the dual basis

$$\mathfrak{B}_{T^*\mathcal{M}} = \{\mathcal{X}^0, \mathcal{Y}^i, \mathcal{Z}^a, dp_0, dp_i\} \qquad \text{and} \qquad \mathfrak{B}_{T\mathcal{M}} = \{\mathcal{X}_0, \mathcal{Y}_i, \mathcal{Z}_a, \partial_{p_0}, \partial_{p_i}\} \tag{3.19}$$

of $T^*\mathcal{M}$ and $T\mathcal{M}$ respectively, where $\mathcal{X}^0 = \tau_{\mathcal{M}}^* X^0$, $\mathcal{Y}^i = \tau_{\mathcal{M}}^* Y^i$, $\mathcal{Z}^a = \tau_{\mathcal{M}}^* Z^a$. Observe that, by the G-invariance of p_0 and p_i, we have that $\mathcal{Y}_i = (\xi_i)_{\mathcal{M}}$ and $J_i = \mathbf{i}_{\mathcal{Y}_i}\Theta_{\mathcal{M}} = p_i$.

Next, we state a nonholonomic version of Noether Theorem. Recall that $\rho_Q : Q \to Q/G$ denotes the orbit projection of a G-action on the configuration manifold Q. Moreover, a G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ of the bundle $\mathfrak{g}_S \to Q$, defined in (2.4), induces the G-invariant functions R^i_j on Q and functions $\bar{R}^i_j$ the functions on Q/G, given by

$$R^i_j := \kappa^{il}[\kappa(Y_l, [Y_j, X_0]) + \kappa(X_0, [Y_j, Y_l])] \qquad \text{and} \qquad \rho_Q^* \bar{R}^i_j = R^i_j, \tag{3.20}$$

where $Y_i = (\xi_i)_Q$ and κ^{il} are the elements of the matrix $[\kappa|_S]^{-1}$, which is the inverse of the restriction of the kinetic energy to S (the matrix with elements κ_{il}).

Theorem 3.14. (Nonholonomic Noether Theorem) *Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ and with a S-orthogonal horizontal space H. Moreover assume that the kinetic energy metric is strong invariant on S and that*

$$\kappa(X_0, [Y, X_0]) = 0$$

for X_0 a ρ_Q-projectable vector field on Q taking values in H and for all $Y \in \Gamma(S)$. Then

(i) the system admits $k = rank(S)$ G-invariant (functionally independent) horizontal gauge momenta;
(ii) the k (functionally independent) G-invariant horizontal gauge momenta can be written as

$$\mathcal{J}_l = f_l^i J_i,$$

for $J_i = \mathbf{i}_{(\xi_i)_\mathcal{M}}\Theta_\mathcal{M}$ with $\xi_i \in \mathfrak{B}_{\mathfrak{g}_S}$ and f_l^i G-invariant functions on Q such that the corresponding functions $\bar{f}_l = (\bar{f}_l^1, ..., \bar{f}_l^k)$ for $l = 1, ..., k$ on Q/G, so that $f_l^i = \rho_Q^* \bar{f}_l^i$, are the k solutions of the linear system of ordinary differential equations on Q/G

$$\bar{R}_j^i \bar{f}^j - \bar{X}_0(\bar{f}^i) = 0, \tag{3.21}$$

where $\bar{R}_j^i$ are the functions defined in (3.20) and $\bar{X}_0$ is the globally defined vector field on Q/G such that $T\rho_Q(X_0) = \bar{X}_0$.

Proof. The proof is divided in two steps. In Step 1 we write the *momentum equation* (3.17) in (global) coordinates, defined by the basis $\mathfrak{B}_{TQ}$ in (3.18) and in Step 2 we derive the system (3.21) from this coordinate version of the momentum equation.

*Step*1. Let us consider the G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ in (2.10) with $Y_i = \overline{(\xi_i)_Q}$ for $i = 1, ..., k$ and the basis $\mathfrak{B}_{TQ}$ and $\mathfrak{B}_{T^*Q}$ in (3.18). First, observe that the 2-form $\langle J, \sigma_{\mathfrak{g}_S}\rangle$ is semi-basic with respect to the bundle $\tau_\mathcal{M} : \mathcal{M} \to Q$. Let us denote by $\mathcal{X}_1$, $\mathcal{X}_2$ any element in the subset $\{\mathcal{X}_0, \mathcal{Y}_1, ..., \mathcal{Y}_k\}$ of the basis $\mathfrak{B}_{T\mathcal{M}}$ in (3.19), and by $X_1 := T\tau_\mathcal{M}(\mathcal{X}_1)$ and $X_2 := T\tau_\mathcal{M}(\mathcal{X}_2)$ the corresponding elements in the basis of $\mathfrak{B}_{TQ}$. Then we have

$$\begin{aligned}
\langle J, \mathcal{K}_\mathcal{W}\rangle(\mathcal{X}_1, \mathcal{X}_2) &= p_a\, dZ^a(X_1, X_2) = -p_a Z^a([X_1, X_2]) \\
&= -(\kappa_{0a} v^0 + \kappa_{ja} v^j) Z^a([X_1, X_2]), \\
\langle J, d^\mathcal{C}\mathcal{Y}^i \otimes \xi_i\rangle(\mathcal{X}_1, \mathcal{X}_2) &= p_i d\mathcal{Y}^i(\mathcal{X}_1, \mathcal{X}_2) \\
&= -p_i Y^i([X_1, X_2]) = -(\kappa_{0i} v^0 + \kappa_{ij} v^j) Y^i([X_1, X_2]), \\
&= -\kappa_{ij} v^j Y^i([X_1, X_2]),
\end{aligned}$$

since $\kappa_{0i} = 0$ by the S-orthogonality of H. Using that $[X_1, X_2] \in \Gamma(V)$ (observe that $[Y_i, Y_j] \in \Gamma(V)$ since V is integrable, and $[X_0, Y_i] \in \Gamma(V)$ since X_0 is ρ_Q-projectable, see Lemma 2.7) then $[X_1, X_2] = Z^a([X_1, X_2])Z_a + Y^j([X_1, X_2])Y_j$ and thus

$$\langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{X}_1, \mathcal{X}_2) = -v^0\kappa(X_0, [X_1, X_2]) - v^j\kappa(Y_j, [X_1, X_2]).$$

Second, using that $T\tau_\mathcal{M}(X_{\text{nh}}(q, p)) = v^0 X_0 + v^i Y_i$ (recall that X_{nh} is a *second order equation*) and also recalling that the functions f^i are G-invariant on Q, we obtain that the *momentum equation* in Proposition 3.3 is written as

$$0 = f^i v^0 \langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{Y}_i, \mathcal{X}_0) + f^i v^j \langle J, \sigma_{\mathfrak{g}_S}\rangle(\mathcal{Y}_i, \mathcal{Y}_j) + p_i v^0 X_0(f^i).$$

Putting together the last two equations we obtain that a function $\mathcal{J} \in C^\infty(\mathcal{M})$ of the form $\mathcal{J} = f^i J_i$ is a G-invariant horizontal gauge momentum of the nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ if and only if the coordinate functions $f^i \in C^\infty(Q)^G$ satisfy

$$v^l v^j \Big(f^i \kappa(Y_j, [Y_i, Y_l]) \Big) + (v^0)^2 \Big(f^i \kappa(X_0, [Y_i, X_0]) \Big) + v^0 v^j P_{0j} = 0, \quad (3.22)$$

where $P_{0j} := f^i(\kappa(Y_j, [Y_i, X_0]) + \kappa(X_0, [Y_i, Y_j])) - \kappa_{ij} X_0(f^i)$.

Note that if the horizontal distribution H is not chosen to be S-orthogonal, then the *momentum equation* (3.22) is modified in the second term.

In order to obtain the simplest form of the coordinate version of the *momentum equation*, we require the orthogonality condition between H and S.

*Step*2. Next we show how the system (3.21) arises as a consequence of (3.22). Since (3.22) is a second order polynomio in the variables (v^0, v^i), it is zero when its associated matrix is skew-symmetric, that is when

(*a*) $\kappa(Y_j, [Y_i, Y_l]) = -\kappa(Y_l, [Y_i, Y_j])$, for all $i, j, l = 1, ..., k$,
(*b*) $f^i \kappa(X_0, [Y_i, X_0]) = 0$,
(*c*) $P_{0j} = 0$, for all $j = 1, ..., k$.

First we observe that items (*a*) and (*b*) are trivially satisfied by the hypotheses of the theorem (item (*a*) is just the definition of strong invariance). Second, we prove that item (*c*) determines the system of ordinary differential equations (3.21) defining the G-invariant functions f^i on Q.

Let us define the matrix $[N]$ with entries $N_{lj} = \kappa(Y_l, [Y_j, X_0]) + \kappa(X_0, [Y_j, Y_l])$ and $[\kappa|_S]$ the kinetic energy matrix restricted to S (which is symmetric and invertible with elements κ_{li}). Then, the condition $P_{0j} = 0$ is written in matrix form as $[N]f = [\kappa|_S]X_0(f)$ for $f = (f^1, ..., f^k)^t$, which is equivalent to $R.f = X_0(f)$ for R the matrix with entries $R^i_j = [\kappa|_S]^{il} N_{lj}$. Therefore, item (*c*) is satisfied if and only if the functions $f = (f^1, ..., f^k)$ are a solution of the linear system of differential equations defined on Q

$$R^i_j f^j - X_0(f^i) = 0, \quad \text{for each } i = 1, ..., k. \quad (3.23)$$

Since $X_0 \in \Gamma(H)$ is ρ_Q-projectable, then there is a (globally defined) vector field $\bar{X}_0$ on Q/G such that $T\rho_Q(X_0) = \bar{X}_0$. Moreover, R^i_j are also G-invariant functions (κ, X_0 and Y_i are G-invariant), and thus we conclude that the system (3.23) is well defined on Q/G. That is, (3.23) represents a (globally defined) linear system of k ordinary differential equations for the functions $(\bar{f}^1, ..., \bar{f}^k)$ on Q/G, that is written as

$$\bar{R}^i_j \bar{f}^j - \bar{X}_0(\bar{f}^i) = 0, \quad \text{for each } i = 1, ..., k. \quad (3.24)$$

where $\bar{R}^i_j$ are the corresponding functions on Q/G (recall that $\dim(Q/G) = 1$). The system (3.24) admits k independent solutions $\bar{f}_l = (\bar{f}^1_l, ..., \bar{f}^k_l)$ for $l = 1, ..., k$. Moreover, $f_l = (f^1_l, ..., f^k_l)$ with $f^i_l = \rho^*_Q(\bar{f}^i_l)$ are k independent solutions of (3.23) and hence $\mathcal{J}_l = f^i_l J_i$ are (functionally independent) G-invariant horizontal gauge momenta for $l = 1, ..., k$.

It is important to note that item (*c*) is the only item determining the functions f^i, while the other two items are intrinsic conditions imposed on the nonholonomic system. □

Remark 3.15. Based on the fact that $\dim(Q/G) = 1$, the system $\bar{R}^i_j \bar{f}^j - \bar{X}_0(\bar{f}^i) = 0$ defines a system of ordinary differential equations on Q/G that admits k independent solutions, and that induces the desired horizontal gauge momenta. This system of ordinary differential equations is also general, in the sense that it is constructed by the characteristic elements of a nonholonomic system with symmetries as the kinetic energy, the constraint manifold and the symmetry group. In Sect. 5, we will write explicitly this system for different examples that, up to now, where studied individually.

In the light of Definition 2.5, we can state

Corollary 3.16. (Conservation of the nonholonomic momentum map) *Given a nonholonomic system satifying hypotheses of Theorem 3.14, the k horizontal gauge symmetries form a basis of $\Gamma(\mathfrak{g}_S)$, and then the associated components of the nonholonomic momentum map are conserved along the nonholonomic dynamics.*

Observation 3.17. The momentum equation (3.22) does not depend on the potential energy function but only on the G-invariance of it. As a consequence, the horizontal gauge momentum $\mathcal{J}$, defined from Theorem 3.14, is a first integral of $(\mathcal{M}, \Omega_\mathcal{M}|_\mathcal{C}, H_\mathcal{M} = \frac{1}{2}\kappa|_\mathcal{M} + U)$ for *any G-invariant* potential energy function U on Q. Such a property, called weak-Noetherinity, has been first observed and studied in [28,29,31]. □

Corollary 3.18. *Consider a nonholonomic system $(\mathcal{M}, \Omega_\mathcal{M}|_\mathcal{C}, H_\mathcal{M})$ with a G-symmetry satisfying Conditions ($\mathcal{A}$1)-($\mathcal{A}$4) and with a strong invariant kinetic energy on S. If the horizontal space H, defined in (3.13), is orthogonal to the vertical space V (with respect to the kinetic energy metric), then the system admits automatically $k = rank(S)$ G-invariant (functionally independent) horizontal gauge momenta.*

Proof. If $V^\perp = H$ then H is S-orthogonal and also $\kappa(X_0, [Y_i, X_0]) = 0$ for all $i = 1, ..., k$. Thus we are under the hypotheses of Theorem 3.14. □

Observation 3.19. Since it is not always possible to choose $H = V^\perp$ with $H \subset D$, in some examples we have to check that $\kappa(X_0, [X_0, Y]) = 0$ for all $Y \in \Gamma(S)$. This condition is equivalently written as $\kappa(X_0, [X_0, Y_i]) = 0$ for all $i = 1, ..., k$ where $Y_i = (\xi_i)_Q$ with ξ_i elements of the G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S}$ in (2.10), which is identically expressed as $(\pounds_{X_0}\kappa)(X_0, Y_i) = 0$ or $\kappa(\nabla_{X_0} Y_i, X_0) = 0$ for ∇ the Levi-Civita connection associated to the kinetic energy metric. □

Guiding Example: nonholonomic oscillator. The nonholonomic oscillator describes a particle in $Q = S^1 \times \mathbb{R} \times S^1$ with a Lagrangian given by $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(y)$ and constraints in the velocities $\dot{z} = y\dot{x}$. The constraint distribution is given by $D = \text{span}\{Y := \partial_x + y\partial_z, \partial_y\}$. The Lie group $G = S^1 \times S^1$ acts on Q so that $V = \text{span}\{\partial_x, \partial_z\}$ and leaves D and L invariant. Then $S = \text{span}\{Y\}$ and the kinetic energy metric is trivially strong invariant on S since $\text{rank}(S) = 1$ (in fact, it is strong invariant on V, see Example 3.11). Moreover, we see that $V^\perp = \text{span}\{\partial_y\} \subset D$ and hence defining the horizontal space $H := V^\perp$, Corollary 3.18 guarantees the existence of one G-invariant horizontal gauge momentum.

Next, we will follow Theorem 3.14 to compute the horizontal gauge momentum $\mathcal{J}$ for this example. Let us consider the basis $\mathcal{B}_{TQ} = \{X_0 = \partial_y, Y = \partial_x + y\partial_z, \partial_z\}$ of TQ with coordinates (v^0, v^Y, v^z). Observe that this basis induces the vertical complement of the constraints $W = \text{span}\{\partial_z\}$. Then on T^*Q we have the dual basis $\mathcal{B}_{T^*Q} = \text{span}\{dy, dx, \epsilon := dz - ydx\}$ with coordinates (p_0, p_Y, p_z). The constraint submanifold $\mathcal{M}$ is given by $\mathcal{M} = \{x, y, z, p_0, p_Y, p_z) : p_z = \frac{y}{1+y^2} p_Y\}$.

Recall that G acts on Q defining a principal bundle $\rho_Q : Q \to Q/G$ so that $\rho_Q(x, y, z) = y$. The Lie algebra of the symmetry group is $\mathfrak{g} = \mathbb{R}^2$ and $\mathfrak{g}_S = \text{span}\{\xi = (1, y)\}$ while $\mathfrak{g}_W = \text{span}\{(0, 1)\}$. Following (2.11), the element $\xi \in \Gamma(\mathfrak{g}_S)$ defines the function $J_\xi := \langle J^{\text{nh}}, \xi\rangle = p_Y$ and the horizontal gauge momentum will be written as $\mathcal{J} = f(y)p_Y$ (f is already considered as a G-invariant function on Q).

The momentum equation from Proposition (3.3): The function $\mathcal{J}$ is a horizontal gauge momenta if and only if f satisfies that $f(y)\langle J, \sigma_{\mathfrak{g}_S}\rangle(\xi_\mathcal{M}, X_{\text{nh}}) + p_Y X_{\text{nh}}(f) = 0$. Since $d^\mathcal{C} dx = 0$ then $\langle J, d^\mathcal{C} dx \otimes \xi\rangle = 0$ and thus the momentum equation remains $f(y)\langle J, \mathcal{K}_\mathcal{W}\rangle(\xi_\mathcal{M}, X_{\text{nh}}) + p_Y f'(y) = 0$.

The differential equation of Theorem 3.14: Next, we write the momentum equation in coordinates as it is expressed (3.21). Since $\text{rank}(S) = 1$, the ordinary differential equation to be solved, for $f = f(y)$, is

$$R_Y^Y f - f' = 0, \qquad \text{for } R_Y^Y = \frac{1}{\kappa(Y,Y)}\kappa(Y, [Y, \partial_y]) = -\frac{y}{1+y^2}.$$

Therefore, the solution $f(y) = \frac{1}{\sqrt{1+y^2}}$ gives the (already known) horizontal gauge momenta $\mathcal{J} = \frac{1}{\sqrt{1+y^2}} p_Y$ (which in canonical coordinates gives $\mathcal{J} = \sqrt{1 + y^2}\, p_x$).

We finally, we study the existence of horizontal gauge momenta for the case when $\text{rank}(H) = 0$ (that is, Condition $\mathcal{A}4$ does not hold but instead $Q/G = \{x\}$). We need observe that there are still $k = \text{rank}(S) = \text{rank}(D)$ horizontal gauge symmetries.

Proposition 3.20. *A nonholonomic system on a Lie group G for which the left action is a symmetry and the kinetic energy metric is strong invariant on D, has $k = $ rank(D) horizontal gauge momenta (G-invariant and functionally independent).*

Proof. In this case, $TQ = V$ which means that $Q \simeq G$ and the only G-invariant functions are the constant functions. Consider a G-invariant basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ of $\Gamma(\mathfrak{g}_S)$. We have to check that the momentum equation (3.17) is satisfied only for constant functions. In fact, $\zeta = f^i\xi_i$ is a horizontal gauge symmetry if and if $f_i \in C^\infty(Q)^G$ and

$$f^i v^l v^j \kappa(Y_j, [Y_i, Y_l]) = 0, \tag{3.25}$$

for $(\xi_i)_Q = Y_i$. Since the kinetic energy is strong invariant on $D = S$, then $f^1 = 1$ and $f^i = 0$ for $i > 1$ is a solution of (3.25) and hence ξ_1 is a horizontal gauge symmetry. Therefore, the sections of the basis $\mathfrak{B}_{\mathfrak{g}_S}$ are k horizontal gauge symmetries. □

As illustrative examples, see the vertical disk and the Chaplygin sleigh in [28] and [11], respectively.

4. Existence of Horizontal Gauge Momenta and Related Consequences on the Dynamics and Geometry of the Systems

4.1. Integrability and hamiltonization of the reduced dynamics

As we saw in Sect. 2.1, a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry can be reduced to the quotient manifold $\mathcal{M}/G$ and the reduced dynamics is given by integral curves of the vector field X_{red} on $\mathcal{M}/G$ defined in (2.5). Moreover, since the hamiltonian function $H_{\mathcal{M}}$ on $\mathcal{M}$ is G-invariant as well, it descends to a *reduced hamiltonian function* H_{red} on the quotient $\mathcal{M}/G$, i.e., $H_{\mathcal{M}} = \rho^* H_{\text{red}}$, and as expected, it is a first integral of X_{red}. The following Lemma will be used in the subsequence subsections.

Lemma 4.1. *If* $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ *is a nonholonomic system with a* G*-symmetry satisfying Conditions* $(\mathcal{A}1)$*,* $(\mathcal{A}2)$ *and* $(\mathcal{A}4)$ *then* $\dim(\mathcal{M}/G) = k + 2$*, where* $k = \operatorname{rank}(S)$.

Proof. From (3.15), we have that $D = H \oplus S$ and thus we observe that $\operatorname{rank}(D) = k + 1$, since $\operatorname{rank}(H) = \dim(Q/G) = 1$ and $\operatorname{rank}(S) = k$. Then $\dim(\mathcal{M}) = \dim(Q) + \operatorname{rank}(D)$ and hence, since G acts on T^*Q by the lifted action, $\dim(\mathcal{M}/G) = \dim(Q/G) + \operatorname{rank}(D) = k + 2$. □

Integrability of the reduced system In this Section, we recall the concept of 'broad integrability' and we show that the reduced dynamics X_{red} on $\mathcal{M}/G$ of a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying the hypotheses of Theorem 3.14, is integrable by quadratures or geometric integrable,[2] and if some compactness hypothesis is satisfied it is also 'broadly integrable'. In order to perform our analysis we identify broad integrability, which extends complete, or better non-commutative, integrability outside the Hamiltonian framework, with quasi-periodicity of the dynamics. We base our analysis on the characterization of quasi-periodicity outside the hamiltonian framework, introduced in [13] (see also [25,34,58]).

Definition 4.2. A vector field X on a manifold M of dimension n, is called *broad integrable*, if

(i) there exists a submersion $F = (f_1, \ldots, f_{n-d}) : M \longrightarrow \mathbb{R}^{n-d}$ with compact and connected level sets, whose components $f_1, \ldots, f_{n-d}$ are first integrals of X, i.e. $X(f_i) = 0$, for all $i = 1, \ldots, n-d$;
(ii) there exists d linearly independent vector fields, $Y_1, \ldots, Y_d$ on M tangent to the level sets of the first integrals (i.e., $Y_\alpha(f_i) = 0$ for all $\alpha = 1, \ldots, d$ and for all $i = 1, \ldots, n-d$) that pairwise commute and commute with X.[3]

As in the hamiltonian case, being broad integrable, has important consequences in the characterization of the dynamics and the geometry of the phase space.

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[2] We recall that integrability by quadratures is also called *geometric integrability*, see [52].
[3] The vector fields $Y_1, \ldots, Y_d$ are also called *dynamical symmetries* of X.

Theorem 4.3. ([13,34,58]) *Let M be a manifold of dimension n. If the vector field X on M is broad integrable, then*

(i) for each $c \in \mathbb{R}^{n-d}$, the level sets $F^{-1}(c)$ of F on M are diffeomorphic to d-dimensional tori;
(ii) the flow of X is conjugated to a linear flow on the fibers of F. Precisely, for each $c \in \mathbb{R}^{n-d}$, there exists a neighbourhood $\mathcal{U}$ of $F^{-1}(c)$ in M and a diffeomorphism

$$\begin{aligned} \Phi : \mathcal{U} &\longrightarrow F(\mathcal{U}) \times \mathbb{T}^d \\ m &\longrightarrow \Phi(m) = (F(m), \varphi(m)) \end{aligned}$$

which conjugate the flow of X on $\mathcal{U}$ to the linear flow

$$\dot{F} = 0, \qquad \dot{\varphi} = \omega(F);$$

on $F(\mathcal{U}) \times \mathbb{T}^d$, for certain functions $\omega_i : F(\mathcal{U}) \longrightarrow \mathbb{R}$.

Now, we go back to our nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry. If we assume that the hypotheses of Theorem 3.14 are satisfied, then the nonholonomic system admits $k = \mathrm{rank}(S)$ (functionally independent) G-invariant horizontal gauge momenta. This fact, plus recalling that H_{red} is a first integral of X_{red} and the fact that reduced manifold $\mathcal{M}/G$ has dimension $k+2$, ensures that the reduced dynamics X_{red} is integrable by quadratures. Moreover, if the joint level sets of the first integrals are connected and compact the reduced dynamics satisfies the hypotheses of Theorem 4.3 and it is then broad integrable on circles. We can summarize these integrability issues as follows:

Theorem 4.4. *Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions ($\mathcal{A}$1)-($\mathcal{A}$4). If the hypotheses of Theorem 3.14 are fulfilled, then*

(i) The vector field X_{red} admits $k+1$ (functionally independent) first integrals $\{\bar{\mathcal{J}}_1, \dots, \bar{\mathcal{J}}_k, H_{\mathrm{red}}\}$ on $\mathcal{M}/G$, where H_{red} is the reduced hamiltonian;
(ii) The map $F = (\bar{\mathcal{J}}_1, \dots, \bar{\mathcal{J}}_k, H_{\mathrm{red}}) : \mathcal{M}/G \longrightarrow \mathbb{R}^{k+1}$ is a surjective submersion. The non equilibrium orbits of the reduced dynamics X_{red} are given by the joint level sets of $(\bar{\mathcal{J}}_1, \dots, \bar{\mathcal{J}}_k, H_{\mathrm{red}})$, and hence the reduced dynamics is integrable by quadratures;
(iii) If the map $F = (\bar{\mathcal{J}}_1, \dots, \bar{\mathcal{J}}_k, H_{\mathrm{red}}) : \mathcal{M}/G \longrightarrow \mathbb{R}^{k+1}$ is proper, then the reduced dynamics is broad integrable and the reduced phase space inherits the structure of a S^1-principal bundle.

Hamiltonization The non-hamiltonian character of a nonholonomic system can also be seen by the fact that the dynamics is not described by a symplectic form or a Poisson bracket. More precisely, as we have seen in Sect. 2.1, the restriction of the 2-form $\Omega_{\mathcal{M}}$ on the distribution $\mathcal{C}$ is nondegenerate and hence it allows to define the *nonholonomic bracket* $\{\cdot, \cdot\}_{\mathrm{nh}}$ on functions on $\mathcal{M}$ (see [39,45,55]), given, for each $f \in C^\infty(\mathcal{M})$, by

$$X_f = \{\cdot, f\}_{\mathrm{nh}} \text{ if and only if } \mathbf{i}_{X_f} \Omega_{\mathcal{M}}|_{\mathcal{C}} = df|_{\mathcal{C}}, \tag{4.26}$$

where $(\cdot)|_{\mathcal{C}}$ denotes the point-wise restriction to $\mathcal{C}$. The nonholonomic bracket is an *almost Poisson bracket* on $\mathcal{M}$ (see Appendix A for more details) with characteristic distribution given by the nonintegrable distribution $\mathcal{c}$ and we say that it *describes the dynamics* since the nonholonomic vector field $X_{\rm nh}$ is hamiltonian with respect to the bracket and the hamiltonian function $H_{\mathcal{M}}$, i.e.,

$$X_{\rm nh} = \{\cdot, H_{\mathcal{M}}\}_{\rm nh}. \tag{4.27}$$

In this framework, we use the triple $(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}})$ to define a nonholonomic system.

If the nonholonomic system admits a G-symmetry, then the nonholonomic bracket $\{\cdot,\cdot\}_{\rm nh}$ is G-invariant and it defines an almost Poisson bracket $\{\cdot,\cdot\}_{\rm red}$ on the quotient space $\mathcal{M}/G$ given, for each $\bar{f}, \bar{g} \in C^\infty(\mathcal{M}/G)$, by

$$\{\bar{f}, \bar{g}\}_{\rm red} \circ \rho(m) = \{\bar{f}\circ\rho, \bar{g}\circ\rho\}_{\rm nh}(m), \qquad m \in \mathcal{M}, \tag{4.28}$$

where $\rho : \mathcal{M} \to \mathcal{M}/G$ is, as usual, the orbit projection (see Appendix A). The reduced bracket $\{\cdot,\cdot\}_{\rm red}$ describes the reduced dynamics $X_{\rm red}$ (defined in (2.5)) since

$$X_{\rm red} = \{\cdot, H_{\rm red}\}_{\rm red}.$$

The *hamiltonization problem* studies whether the reduced dynamics $X_{\rm red}$ is hamiltonian with respect to a Poisson bracket on the reduced space $\mathcal{M}/G$ (that might be a different bracket from $\{\cdot,\cdot\}_{\rm red}$).

One of the most important consequences of Theorem 3.14 is related with the hamiltonization problem as the following theorem shows.

Theorem 4.5. *If a nonholonomic system* $(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}})$ *with a* G*-symmetry verifying Conditions* $(\mathcal{A}1)$*-*$(\mathcal{A}4)$ *satisfies the hypotheses of Theorem* 3.14*, then there exists a rank 2-Poisson bracket* $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$ *on* $\mathcal{M}/G$ *describing the reduced dynamics as follows:*

$$X_{\rm red} = \{\cdot, H_{\rm red}\}_{\rm red}^{B_{\rm HGM}},$$

for $H_{\rm red} : \mathcal{M}/G \to \mathbb{R}$ *being the reduced hamiltonian.*

The problem of finding the bracket $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$, once k horizontal gauge momenta exist, was already studied in [8,37]). However here, in the light of the techniques introduced to prove Theorem 3.14, we take a different path to put in evidence the role played by the *momentum equation*. More precisely, first we study how different choices of a (global G-invariant) basis $\mathfrak{B}_{\mathfrak{g}_S}$ of $\Gamma(\mathfrak{g}_S)$ generate different rank 2-Poisson brackets on $\mathcal{M}/G$. If the nonholonomic system admits k (functionally independent G-invariant) horizontal gauge symmetries then there will be a rank 2-Poisson bracket $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$ that describes the dynamics which is defined by choosing the basis of $\Gamma(\mathfrak{g}_S)$ given by the horizontal gauge symmetries. Then we show how $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$ depends on the system of differential equations (3.21). For the basic definitions regarding Poisson brackets, bivector fields and gauge transformations see Appendix A.

Let us consider a 2-form B on $\mathcal{M}$ that is semi-basic with respect to the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$. The gauge transformation of $\{\cdot,\cdot\}_{\rm nh}$ by the 2-form B gives the almost Poisson bracket $\{\cdot,\cdot\}_B$ defined, at each $f \in C^\infty(\mathcal{M})$, by

$$\mathbf{i}_{X_f}(\Omega_{\mathcal{M}} + B)|_{\mathcal{C}} = df|_{\mathcal{C}} \quad \text{if and only if} \quad X_f = \{\cdot, f\}_B.$$

If the 2-form B is G-invariant, then the bracket $\{\cdot,\cdot\}_B$ is also G-invariant and it can be reduced to an almost Poisson bracket $\{\cdot,\cdot\}^B_{\rm red}$ on the quotient manifold $\mathcal{M}/G$ given, at each $\bar f, \bar g \in C^\infty(\mathcal{M}/G)$, by

$$\{\bar f, \bar g\}^B_{\rm red} \circ \rho(m) = \{\bar f \circ \rho, \bar g \circ \rho\}_B(m), \tag{4.29}$$

where $m \in \mathcal{M}$, Diag. (A.51) (see also [6,36]).

Let $\mathfrak{B}_{\mathfrak{g}_S}$ be a global G-invariant basis of $\Gamma(\mathfrak{g}_S)$ and recall from (2.11) the associated G-invariant momenta J_i.

Proposition 4.6. *Consider a nonholonomic system $(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions $(\mathcal{A}1)$-$(\mathcal{A}3)$. Given a (global G-invariant) basis $\mathfrak{B}_{\mathfrak{g}_S}$ of $\Gamma(\mathfrak{g}_S)$, the associated 2-form $B_\sigma = \langle J, \sigma_{\mathfrak{g}_S}\rangle$ induces a gauge transformation of the nonholonomic bracket $\{\cdot,\cdot\}_{\rm nh}$ so that*

(i) the gauge related bracket $\{\cdot,\cdot\}_{B_\sigma}$ on $\mathcal{M}$ is G-invariant;

(ii) The induced reduced bracket $\{\cdot,\cdot\}^{B_\sigma}_{\rm red}$ on $\mathcal{M}/G$ is Poisson with symplectic leaves given by the common level sets of the momenta $\bar J_i$, where $\bar J_i \in C^\infty(\mathcal{M}/G)$ so that $\rho^* \bar J_i = J_i$. In particular, if Condition $(\mathcal{A}4)$ is satisfied, then the Poisson bracket $\{\cdot,\cdot\}_{B_\sigma}$ has 2-dimensional leaves.

Proof. (i) By construction, we see that the 2-form $\langle J, \sigma_{g_S}\rangle$ is semi-basic with respect to the bundle $\mathcal{M} \to Q$ and, by Lemma 3.2, it is G-invariant as well. Therefore, the gauge transformation by the 2-form $\langle J, \sigma_{g_S}\rangle$ defines a G-invariant almost Poisson bracket $\{\cdot,\cdot\}_{B_\sigma}$.

(ii) The G-invariant bracket $\{\cdot,\cdot\}_{B_\sigma}$ induces, on the quotient space $\mathcal{M}/G$, an almost Poisson bracket $\{\cdot,\cdot\}^{B_\sigma}_{\rm red}$. It is shown in [8, Proposition 3.9] that $\{\cdot,\cdot\}^{B_\sigma}_{\rm red}$ is a Poisson bracket with symplectic leaves given by the common level sets of the momenta $\bar J_i \in C^\infty(\mathcal{M}/G)$.[4] □

Note that the reduced nonholonomic vector field $X_{\rm red}$ might not be tangential to the foliation of the bracket $\{\cdot,\cdot\}^{B_\sigma}_{\rm red}$.

Definition 4.7. We say that a nonholonomic system $(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}})$ with a G-symmetry is *hamiltonizable by a gauge transformation* if there exists a G-invariant 2-form B so that $\{\cdot,\cdot\}^B_{\rm red}$ is Poisson[5] and

$$X_{\rm red} = \{\cdot, H_{\rm red}\}^B_{\rm red} \tag{4.30}$$

for $H_{\rm red} : \mathcal{M} \to \mathbb{R}$ being the reduced hamiltonian.

[4] In the notation of [8], B_σ corresponds to the 2-form B_1 but for any G-invariant basis of $\Gamma(\mathfrak{g}_S)$. The bracket $\{\cdot,\cdot\}^{B_\sigma}_{\rm red}$ is denoted by $\{\cdot,\cdot\}^1_{\rm red}$ in the cited reference.

[5] More generally, the bracket $\{\cdot,\cdot\}^B_{\rm red}$ can be conformally Poisson.

Definition 4.8. [6] A gauge transformation by a 2-form B of the nonholonomic bracket $\{\cdot,\cdot\}_{\text{nh}}$ is *dynamical* if B is semi-basic with respect to the bundle $\mathcal{M} \to Q$ and $\mathbf{i}_{X_{\text{nh}}} B = 0$. That is, if B induces a bracket $\{\cdot,\cdot\}_B$ that describes the nonholonomic dynamics: $X_{\text{nh}} = \{\cdot, H_{\mathcal{M}}\}_B$.

Therefore, once we know that different 2-forms of the type B_σ (recall that different choices of basis of $\Gamma(\mathfrak{g}_S)$ define different 2-forms) produce different Poisson brackets on the reduced space, we need to find the one that is *dynamical*, if it exists.

Observe that if the system admits k (G-invariant) horizontal gauge momenta, then we have a preferred basis $\mathfrak{B}_{\text{HGS}} = \{\zeta_1, ..., \zeta_k\}$ of $\Gamma(\mathfrak{g}_S)$ given by the horizontal gauge symmetries. Let us denote by σ_{HGS} the 2-form $\sigma_{\mathfrak{g}_S}$ (defined in (3.16)), computed with respect to the basis $\mathfrak{B}_{\text{HGS}}$, and by $B_{\text{HGS}} := \langle J, \sigma_{\text{HGS}}\rangle$. The proof of Theorem 4.5 is based on the following two facts: on the one hand, B_{HGS} defines a dynamical gauge transformation and on the other hand (by Proposition 4.6) the resulting reduced bracket $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$ is Poisson.

Proof of Theorem 4.5. Under the hypotheses of Theorem 3.14, the nonholonomic system admits k G-invariant horizontal gauge momenta $\{\mathcal{J}_1, ..., \mathcal{J}_k\}$ with the corresponding G-invariant horizontal gauge symmetries that generate a basis $\mathfrak{B}_{\text{HGS}} = \{\zeta_1, ...\zeta_k\}$ of $\Gamma(\mathfrak{g}_S)$. Following [8, Theorem 3.7] and, in particular [8, Corollary 3.13] since $\text{rank}(H) = 1$, the 2-form $B_{\text{HGS}} = \langle J, \sigma_{\text{HGS}}\rangle$ associated to the basis $\mathfrak{B}_{\text{HGS}}$ induces a dynamical gauge transformation and hence the induced reduced bracket $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$ describes the reduced dynamics: $X_{\text{red}} = \{\cdot, H_{\text{red}}\}_{\text{red}}^{B_{\text{HGM}}}$. This bracket is then Poisson with symplectic leaves defined by the common level sets of the horizontal gauge momenta $\{\mathcal{J}_1, ..., \mathcal{J}_k\}$ (Proposition 4.6). □

The following diagrams compare Proposition 4.6 with Theorem 4.5. The first diagram illustrates the case when we perform a gauge transformation by a 2-form B_σ (associated to the choice of a basis $\mathfrak{B}_{\mathfrak{g}_S}$ of $\Gamma(\mathfrak{g}_S)$, Proposition 4.6) while the second one illustrates the case when the 2-form is B_{HGS} (associated to the basis $\mathfrak{B}_{\text{HGS}}$ given by horizontal gauge momenta, Theorem 4.5). In both cases, we obtain that the resulting reduced brackets $\{\cdot,\cdot\}_{\text{red}}^{B_\sigma}$ and $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$ are Poisson. However, $\{\cdot,\cdot\}_{\text{red}}^{B_\sigma}$ might not describe the reduced dynamics since B_σ is not necessarily dynamical. On the other hand, B_{HGS} is always dynamical and thus the reduced bracket $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$ describes the dynamics: $X_{\text{red}} = \{\cdot, H_{\text{red}}\}_{\text{red}}^{B_{\text{HGM}}}$.

$$\begin{array}{ccc}
(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}}) & \xrightarrow[\text{by } B_\sigma]{\text{gauge transf}} & (\mathcal{M}, \{\cdot,\cdot\}_{B_\sigma}) \\
\Big\downarrow \text{reduction} & & \Big\downarrow \\
(\mathcal{M}/G, \{\cdot,\cdot\}_{\rm red}, H_{\rm red}) & & (\mathcal{M}/G, \{\cdot,\cdot\}_{\rm red}^{B_\sigma})
\end{array}$$

$$\begin{array}{ccc}
(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}}) & \xrightarrow[\text{by } B_{\rm HGS}]{\textit{dynamical}\text{ gauge transf}} & (\mathcal{M}, \{\cdot,\cdot\}_{B_{\rm HGM}}, H_{\mathcal{M}}) \\
\Big\downarrow \text{reduction} & & \Big\downarrow \\
(\mathcal{M}/G, \{\cdot,\cdot\}_{\rm red}, H_{\rm red}) & & (\mathcal{M}/G, \{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}, H_{\rm red})
\end{array}$$

Remark 4.9. Under the hypotheses of Theorem 4.5, the functions $\{H_{\mathcal{M}}, \mathcal{J}_1, ..., \mathcal{J}_k\}$ are in involution with respect to the bracket $\{\cdot,\cdot\}_{B_{\rm HGM}}$, where $\{\mathcal{J}_1, ..., \mathcal{J}_k\}$ are the horizontal gauge momenta defined by Theorem 3.14. In addition, also the reduced functions $\{H_{\rm red}, \bar{\mathcal{J}}_1, ..., \bar{\mathcal{J}}_k\}$ on $\mathcal{M}/G$ are in involution with respect to the reduced bracket $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$. However these functions are not necessarily in involution with respect to the brackets $\{\cdot,\cdot\}_{\rm nh}$ and $\{\cdot,\cdot\}_{\rm red}$, respectively.

In many cases, the horizontal gauge momenta cannot be explicitly written, instead they are defined in terms of the solutions of the system of differential equations (3.21). Next Theorem gives the formula to write explicitly the dynamical gauge transformation $B_{\rm HGS}$ (and as a consequence the Poisson bracket $\{\cdot,\cdot\}_{\rm red}^{B_{\rm HGM}}$) in a chosen basis $\mathfrak{B}_{\mathfrak{g}_S}$ that is not necessarily given by the horizontal gauge symmetries. Examples 5.2 and 5.3 make explicit the importance of the following formula:

Theorem 4.10. *Consider a nonholonomic system described by the triple* $(\mathcal{M}, \{\cdot,\cdot\}_{\rm nh}, H_{\mathcal{M}})$ *with a G-symmetry verifying Conditions* $(\mathcal{A}1)$-$(\mathcal{A}4)$. *Let* $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ *be a global G-invariant basis of* $\Gamma(\mathfrak{g}_S)$ *and* X_0 *a* ρ*-projectable vector field on* Q *generating the S-orthogonal horizontal space* H. *If the hypotheses of Theorem* 3.14 *are satisfied, then the 2-form* $B_{\rm HGS}$ *is written with respect to the basis* $\mathfrak{B}_{\mathfrak{g}_S}$ *as*

$$\begin{aligned} B_{\rm HGS} := \langle J, \sigma_{\rm HGS}\rangle &= \langle J, \mathcal{K}_{\mathcal{W}}\rangle - \langle J, R^i_j \mathcal{X}^0 \wedge \mathcal{Y}^j \otimes \xi_i\rangle + \langle J, d\mathcal{Y}^i \otimes \xi_i\rangle, \\ &= p_a d^{\mathcal{C}} \varepsilon^a - J_i R^i_j \mathcal{X}^0 \wedge \mathcal{Y}^j + J_i d^{\mathcal{C}} \mathcal{Y}^i \end{aligned} \tag{4.31}$$

for R^i_j *and* J_i *being the functions defined in* (3.20) *and* (2.11), *respectively, and* $\mathcal{X}^0 = \tau^*_{\mathcal{M}} X^0$, $\mathcal{Y}^i = \tau^*_{\mathcal{M}} Y^i$, $\varepsilon^a = \tau^*_{\mathcal{M}} \epsilon^a$ *being the corresponding forms on* $\mathcal{M}$.

Proof. In order to prove formula (4.31), consider the basis $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1, ..., \xi_k\}$ (not necessarily given by horizontal gauge symmetries), and define the corresponding functions J_i as in (2.11). If we denote by F the fundamental matrix of solutions of the system of ordinary differential equations (3.21) (i.e., the columns of F are the independent solutions $(f_l^1, ..., f_l^k)$) and by R the $k \times k$-matrix with entries R^i_j, then

$$R.F = X_0(F) \quad \text{and} \quad \mathcal{J} = F^T \mathbf{J}, \quad \text{where} \quad \mathcal{J} = \begin{pmatrix} \mathcal{J}_1 \\ \vdots \\ \mathcal{J}_k \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} J_1 \\ \vdots \\ J_k \end{pmatrix}. \tag{4.32}$$

Moreover, let us denote by $\mathcal{Y}^i_{\text{HGS}}$ the 1-forms on $\mathcal{M}$ such that $\mathcal{Y}^i_{\text{HGS}}((\zeta_l)_{\mathcal{M}}) = \delta_{il}$ and $\mathcal{Y}^i_{\text{HGS}}|_{\mathcal{H}} = \mathcal{Y}^i_{\text{HGS}}|_{\mathcal{W}} = 0$. Then if $\mathcal{Y}_{\text{HGS}} = (\mathcal{Y}^1_{\text{HGS}}, ..., \mathcal{Y}^k_{\text{HGS}})^T$ we have that $\mathcal{Y}_{\text{HGS}} = F^{-1}\mathcal{Y}$ where $\mathcal{Y} = (\mathcal{Y}^1, ..., \mathcal{Y}^k)^T$. Hence

$$\begin{aligned} \langle J, d^{\mathcal{C}}\mathcal{Y}^i_{\text{HGS}} \otimes \zeta_i \rangle = \mathcal{J}^T. {}^{\mathcal{C}}\mathcal{Y}_{\text{HGS}} &= \mathbf{J}^T F d^{\mathcal{C}}(F^{-1}\mathcal{Y}) \\ &= \mathbf{J}^T F X_0(F^{-1})\mathcal{X}^0 \wedge \mathcal{Y} + \mathbf{J}^T F F^{-1} d^{\mathcal{C}}\mathcal{Y} \\ &= -\mathbf{J}^T F(F^{-1} X_0(F) F^{-1})\mathcal{X}^0 \wedge \mathcal{Y} + \mathbf{J}^T d^{\mathcal{C}}\mathcal{Y} \\ &= -\mathbf{J}^T R\mathcal{X}^0 \wedge \mathcal{Y} + \mathbf{J}^T d^{\mathcal{C}}\mathcal{Y} \\ &= -J_i R^i_j \mathcal{X}^0 \wedge \mathcal{Y}^j + \langle J, d^{\mathcal{C}}\mathcal{Y}^i \otimes \xi_i \rangle. \end{aligned}$$

Finally, we conclude, using Definition 3.1, that

$$B_{\text{HGS}} = \langle J, \mathcal{K}_{\mathcal{W}} \rangle + \langle J, d^{\mathcal{C}}\mathcal{Y}^i_{\text{HGS}} \otimes \zeta_i \rangle = p_a d^{\mathcal{C}}\varepsilon^a - J_i R^i_j \mathcal{X}^0 \wedge \mathcal{Y}^j + J_i d^{\mathcal{C}}\mathcal{Y}^i.$$

Observation 4.11. Following Example 3.12 and Corollary 3.5, a nonholonomic system with a G-symmety that admits a basis of $\mathfrak{g}_S \to Q$ given by G-invariant horizontal symmetries is hamiltonizable without the need of a gauge transformation (i.e., $B_{\text{HGS}} = 0$ in this case, see [8]) □

4.2. Horizontal gauge momenta and broad integrability of the complete system

In the previous subsections we have studied the dynamics and the geometry of the reduced system. Under the hypotheses of Theorem 3.14 the reduced dynamics is integrable by quadratures, and if the joint level sets of the first integrals are connected and compact the reduced dynamics consists of periodic orbits or equilibria. Moreover the reduced system is hamiltonizable via a rank-2 Poisson structure, whose (global) Casimirs are the k horizontal gauge momenta. In this Section we aim to obtain information on the dynamics and geometry of the complete system. We will then focus in the case in which the reduced dynamics is periodic and, by using techniques of reconstruction theory, we will see that if the symmetry group G is compact, then the dynamics of the complete systems is quasi-periodic on tori of dimension at most *rank* $G + 1$, where *rank* G denotes the rank of the group, i.e. the dimension of the maximal abelian subgroup of G. If the symmetry group G is not compact, the complete dynamics can be either quasi-periodic on tori or an unbounded copy of $\mathbb{R}$, depending on the symmetry group. Some details on these aspects are reviewed in Appendix B, but see also [2,33]. We thus show how the broad integrability of the complete dynamics of these type of systems is deeply related to their symmetries, that are able to produce, not only the right amount of dynamical symmetries, but also the complementary number of first integrals. We will then apply these results to the example of a heavy homogeneous ball that rolls without sliding inside a convex surface of revolution (see Sect. 5.3). This case presents a periodic dynamics in the reduced space, and a broadly integrable complete dynamics on tori of dimension at most three, thus re-obtaining the results in [26,38].

We say that a G-invariant subset $\mathcal{P}$ of $\mathcal{M}$ is a *relative periodic orbit for* $X_{\rm nh}$, if it invariant by the flow and its projection on $\mathcal{M}/G$ is a periodic orbit of $X_{\rm red}$. Now, we can summarize these results as follows:

Theorem 4.12. *Let us consider a nonholonomic system* $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ *with a* G*-symmetry satisfying Conditions* $(\mathcal{A}1)$-$(\mathcal{A}4)$. *Assume that the hypotheses of Theorem 3.14 are fulfilled, and that the reduced dynamics is periodic, then*

(i) *if the group* G *is compact, the flow of* $X_{\rm nh}$ *on a relative periodic orbit* $\mathcal{P}$ *is quasi-periodic with at most* $rank\, G+1$ *frequencies and the phase space if fibered in tori of dimension up to rank* $G+1$.
(ii) *if* G *is non-compact, the flow of* $X_{\rm nh}$ *on a relative periodic orbit* $\mathcal{P}$ *is either quasi-periodic, or a copy of* $\mathbb{R}$, *that leaves every compact subset of* $\mathcal{P}$ *as* $t\to\pm\infty$.[6]

Proof. To prove this result we combine the results on integrability of the reduced system given by Theorem 4.4 with the results on reconstruction theory from periodic orbits recalled in Appendix B.

More precisely, we confine ourselves to the subspace of the reduced space $\mathcal{M}/G$ in which the dynamics is periodic. Then, if the symmetry group is compact, the reconstructed dynamics is generically quasi-periodic on tori of dimension $d+1$, where r is the rank of the group [23,35,38,43]. The phase space, or at least a certain region of it, has the structure of a $\mathbb{T}^{d+1}$ fiber bundle, (see [26] for details on the geometric structure of the phase space in this case). On the other hand if the group is not compact, the reconstructed orbits are quasi-periodic or a copy of $\mathbb{R}$ that 'spirals' toward a certain direction. □

5. Examples

5.1. The snakeboard

The snakeboard is a derivation of the skateboard where the rider is allowed to generate a rotation in the axis of the wheels creating a torque so that the board spins about a vertical axis, see [12,49]. We denote by r the distance from the center of the board to the pivot point of the wheel axes, by m the mass of the board, by $\mathbb{J}$ the inertial of the rotor and by $\mathbb{J}_1$ the inertia of each wheel. Following [12] we assume that the parameters are chosen such that $\mathbb{J}+2\mathbb{J}_1+\mathbb{J}_0=mr^2$, where $\mathbb{J}_0$ denotes the inertia of the board. The snakeboard is then modelled on the manifold $Q=SE(2)\times S^1\times S^1$ with coordinates $q=(\theta,x,y,\psi,\phi)$, where (θ,x,y)

[6] A dynamical behaviour that leaves every compact subset of $\mathcal{P}$ as $t\to\pm\infty$ is usually called *drift* or *drifting motion* (see [33] for a discussion on this fact).

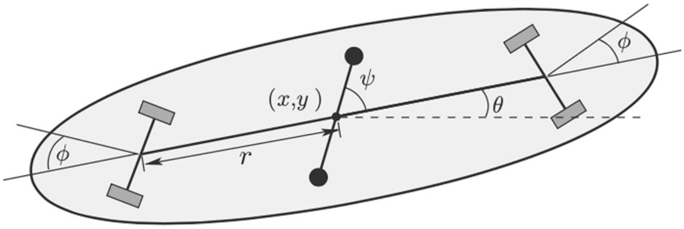

Fig. 1. The snakeboard

represent the orientation and position of the board, ψ is the angle of the rotor with respect to the board, and ϕ is the angle of the front and back wheels with respect to the board (in this simplified model they are assumed to be equal).

The Lagrangian is given by

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\mathbb{J}\dot{\psi}^2 + \mathbb{J}\dot{\psi}\dot{\theta} + \mathbb{J}_0\dot{\phi}^2.$$

The nonholonomic constraints impose that the front and back wheels roll without sliding and hence the constraint 1-forms are defined to be

$$\begin{aligned}\omega^1 &= -\sin(\theta + \phi)\, dx + \cos(\theta + \phi)\, dy - r\cos\phi\, d\theta,\\ \omega^2 &= -\sin(\theta - \phi)\, dx + \cos(\theta - \phi)\, dy + r\cos\phi\, d\theta.\end{aligned} \tag{5.33}$$

Note that ω^1 and ω^2 are independent whenever $\phi \neq \pm\pi/2$. Therefore, we define the configuration manifold Q so that $q = SE(2) \times S^1 \times (-\pi/2, \pi/2)$. The constraint distribution D is given by

$$D = \text{span}\{Y_\theta := \sin\phi\partial_\theta - r\cos\phi\cos\theta\partial_x - r\cos\phi\sin\theta\partial_y,\ \partial_\psi,\ \partial_\phi\}. \tag{5.34}$$

The existence of horizontal gauge momenta. The system is invariant with respect to the free and proper action on Q of $G = SE(2) \times S^1$ given by

$$\begin{aligned}&\Phi((\alpha, a, b; \beta), (\theta, x, y, \psi, \phi))\\ &\quad = (\theta + \alpha, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \psi + \beta, \phi),\end{aligned}$$

and hence $V = \text{span}\{\partial_\theta, \partial_\psi, \partial_x, \partial_y\}$ and $S = \text{span}\{Y_\theta, \partial_\psi\}$ (see [12,15]). First, we observe that $[Y_\theta, \partial_\psi] = 0$ and hence the kinetic energy metric is trivially strong invariant on S. Second, $H := \text{span}\{\partial_\phi\}$ and it is straightforward to check that $V^\perp = H$. Then, by Corollary 3.18(i) the system admits 2 (functionally independent) G-invariant horizontal gauge momenta.

The computation of the of horizontal gauge momenta. Let us consider the adapted basis to $TQ = D \oplus W$, given by $\mathfrak{B}_{TQ} = \{Y_\theta, \partial_\psi, \partial_\phi, Z_1, Z_2\}$, where

$$Z_1 := \frac{1}{2\cos\phi}\left(-\sin\theta\partial_x + \cos\theta\partial_y - \frac{1}{r}\partial_\theta\right) \quad \text{and}$$
$$Z_2 := \frac{1}{2\cos\phi}\left(-\sin\theta\partial_x + \cos\theta\partial_y + \frac{1}{r}\partial_\theta\right).$$

Denoting by $(p_\theta, p_\psi, p_\phi, p_1, p_2)$ the coordinates on T^*Q associated to the dual basis

$$\mathfrak{B}_{T^*Q} = \{\alpha_\theta := -\tfrac{1}{r\cos\phi}(\cos\theta dx + \sin\theta dy), d\psi, d\phi, \omega^1, \omega^2\},$$

we obtain that

$$\mathcal{M} = \left\{(q; p_\theta, p_\psi, p_\phi, p_1, p_2) \ : \ p_1 = -p_2 = -\tfrac{1}{2}\left(\tfrac{(mr^2-\mathbb{J})\sin\phi}{r\cos\phi\,\Delta}p_\theta + \tfrac{mr\cos\phi}{\Delta}p_\psi\right)\right\},$$

where $\Delta = \Delta(\phi) = mr^2 - \mathbb{J}\sin^2\phi$ (recall that $\Delta(\phi) > 0$, since $mr^2 > \mathbb{J}$).

We consider the global basis of $\mathfrak{g}_S$ given by $\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1 = (\sin\phi, -r\cos\phi\cos\theta + y, -r\cos\phi\sin\theta - x; 0), \xi_2 = (0,0,0;1)\}$, and we observe that $(\xi_1)_Q = Y_\theta$ and $(\xi_2)_Q = \partial_\psi$. Following (2.11), $J_1 = \langle J^{\rm nh}, \xi_1\rangle = p_\theta$ and $J_2 = \langle J^{\rm nh}, \xi_2\rangle = p_\psi$.

The function $\mathcal{J} = f_\theta(\phi)p_\theta + f_\psi(\phi)p_\psi$ is a horizontal gauge momentum if and only if $R.f = f'$ where R is the 2×2 matrix given in (3.21), $f = (f_\theta, f_\psi)^t$ and $f' = (f'_\theta, f'_\psi)$ for $f'_\theta = \frac{d}{d\phi}f_\theta$ (analogously for f'_ψ). In our case, using that $\{Y_\theta, \partial_\psi\}$ is a basis of S and $X_0 = \partial_\phi$, we obtain

$$R = [\kappa|_S]^{-1}N, \qquad \text{for } [\kappa|_S] = \begin{pmatrix} mr^2 & \mathbb{J}\sin\phi \\ \mathbb{J}\sin\phi & \mathbb{J}\end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 0 \\ -\mathbb{J}\cos\phi & 0\end{pmatrix}.$$

Hence, we arrive to the linear system

$$\frac{\cos\phi}{\Delta}\begin{pmatrix} \mathbb{J}\sin\phi & 0 \\ -mr^2\cos\phi & 0\end{pmatrix}\begin{pmatrix} f_\theta \\ f_\psi\end{pmatrix} = \begin{pmatrix} f'_\theta \\ f'_\psi\end{pmatrix}, \tag{5.35}$$

which admits 2 independent solutions: $f_1 = (f_1^\theta, f_1^\psi)$, with $f_1^\theta = \frac{1}{\sqrt{2\Delta}}$, $f_1^\psi = -f_\theta^1\sin\phi$, and $f_2 = (0,1)$. Therefore the horizontal gauge momenta can be written as

$$\mathcal{J}_1 = \frac{1}{\sqrt{2\Delta}}\,(p_\theta - p_\psi\ \sin\phi) \qquad \text{and} \qquad \mathcal{J}_2 = p_\psi. \tag{5.36}$$

Remarks 5.1. (i) On the one hand, since ξ_2 is a horizontal symmetry, it is expected to have $\mathcal{J}_2 = p_\psi$ conserved (Cor. 3.5). On the other hand, the horizontal gauge momentum $\mathcal{J}_1$ is realized by a non-constant section ζ_1, and we recover the expression of $\mathcal{J}_1$ given in [15] [7].
(ii) The horizontal gauge momenta (5.36) can also be obtained from the *momentum equation* in Proposition 3.3, which in case is written as $f_\theta\langle J, \sigma_{\mathfrak{g}_S}\rangle(Y_\theta, X_{\rm nh}) + f_\psi\langle J, \sigma_{\mathfrak{g}_S}\rangle(\partial_\psi, X_{\rm nh}) + p_\theta X_{\rm nh}(f_\theta) + p_\psi X_{\rm nh}(f_\psi) = 0$.

Hamiltonization and integrability. The system descends to the quotient manifold $\mathcal{M}/G$ equipped with coordinates $(\phi, p_\phi, p_\theta, p_\psi)$. The G-invariant horizontal gauge momenta $\mathcal{J}_1, \mathcal{J}_2$ in (5.36) and the hamiltonian function $H_\mathcal{M}$, also descend to functions $\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2$ and $H_{\rm red}$ on $\mathcal{M}/G$.
Integrability. Since the reduced space $\mathcal{M}/G$ is 4-dimensional, Theorem 4.4 guarantees that the reduced dynamics is integrable by quadratures. We observe that the

[7] We thank the referee for bringing this reference to our knowledge.

reduced system is not periodic, thus we cannot say anything generic on the complete dynamics nor on the geometry of the phase space.
Hamiltonization. Theorem 4.5 guarantees that the system is Hamiltonizable. In order to write the Poisson bracket on $\mathcal{M}/G$ that describes the dynamics, we compute the 2-form $B_{\rm HGS}$ in terms of the basis $\mathfrak{B}_{TQ} = \{Y_1 := Y_\theta, Y_2 := \partial_\psi, X_0 := \partial_\phi, \partial_x, \partial_y\}$ using Theorem 4.10. Let us denote by R^i_j the elements of the matrix R in (5.35), and using Thm. 4.10 (note that $R^1_2 = R^2_2 = 0$) we have that

$$B_{\rm HGS} = \langle J, \mathcal{K}_\mathcal{W}\rangle - p_\theta R^1_1 d\phi \wedge d\theta - p_\psi R^2_1 d\phi \wedge d\theta + p_\theta d\alpha_\theta.$$

Following Sect. 3.1 we compute

$$\begin{aligned}\langle J, \mathcal{K}_\mathcal{W}\rangle|_\mathcal{C} &= \iota^*(p_1)d\omega^1 + \iota^*(p_2)d\omega^2|_\mathcal{C} \\ &= -\left(\tfrac{(mr^2-\mathbb{J})\sin\phi}{\cos\phi\,\Delta}p_\theta + \tfrac{mr^2\cos\phi}{\Delta}p_\psi\right) d\phi \wedge \alpha_\theta|_\mathcal{C}.\end{aligned}$$

Finally, substituting R^1_1, R^2_1 and using that $d\alpha_\theta|_\mathcal{C} = \tan\phi\, d\phi \wedge \alpha_\theta$ we obtain that $B_{\rm HGS} = 0$.

As a consequence of Theorem 4.5 the reduced bracket $\pi_{\rm red}$ which is given by

$$\pi_{\rm red} = \partial_\phi \wedge \partial_{p_\phi} + \tfrac{\cos\phi}{\Delta}(\mathbb{J}\sin\phi\; p_\theta - mr^2 p_\psi)\partial_{p_\phi} \wedge \partial_{p_\theta},$$

is a Poisson bracket on $\mathcal{M}/G$ with $\bar{\mathcal{J}}_1$ and $\bar{\mathcal{J}}_2$ playing the role of Casimirs. The reduced nonholonomic vector field is then $X_{\rm red} = \{\cdot, H_{\rm red}\}_{\rm red}$.

Observation 5.2. The G-symmetry considered in this paper is different than the one considered in [4,8], therefore the reduced bracket obtained here is not the same as the one presented in these citations. Moreover, in [4,8], the snakeboard was described by a twisted Poisson bracket (with a 4-dimensional foliation) while here, we show that the snakeboard can be described by a rank 2-Poisson bracket. □

5.2. Solids of Revolution

Let $\mathcal{B}$ be a strongly convex body of revolution, i.e., a body which is geometrically and dynamically symmetric under rotations about a given axis ([5,23]). Let us assume that the surface $\mathbf{S}$ of $\mathcal{B}$ is invariant under rotations around a given axis, which in our case is chosen to be e_3. Then its principal moments of inertia are $\mathbb{I}_1 = \mathbb{I}_2$ and $\mathbb{I}_3$.

The position of the body in $\mathbb{R}^3$ is given by the coordinates $(g, \mathbf{x})$ where $g \in SO(3)$ is the orientation of the body with respect to an inertial frame (e_x, e_y, e_z) and $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ is the position of the center of mass. Denoting by $\mathbf{m}$ the mass of the body, the lagrangian $L : T(SO(3) \times \mathbb{R}^3) \to \mathbb{R}$ is given by

$$L(g, \mathbf{x}; \boldsymbol{\Omega}, \dot{\mathbf{x}}) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{1}{2}\mathbf{m}||\dot{\mathbf{x}}||^2 + \mathbf{mg}\langle \mathbf{x}, e_3\rangle,$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity in body coordinates, $\langle\cdot,\cdot\rangle$ represents the standard pairing in $\mathbb{R}^3$ and $\mathbf{g}$ the constant of gravity.

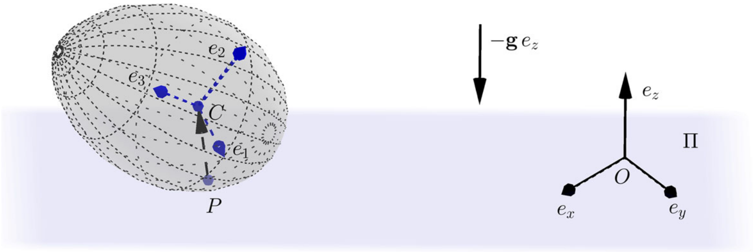

Fig. 2. Solid of revolution rolling on a horizontal plane

Let s be the vector from the center of mass of the body to a fixed point on the surface $\mathbf{S}$. If we denote by $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ the third row of the matrix $g \in SO(3)$, then s can be written as $s : S^2 \to \mathbf{S}$ so that

$$s(\boldsymbol{\gamma}) = (\varrho(\gamma_3)\gamma_1, \varrho(\gamma_3)\gamma_2, \zeta(\gamma_3)) = \varrho(\gamma_3)\boldsymbol{\gamma} - L(\gamma_3)e_3,$$

where $\varrho = \varrho(\gamma_3)$, $\zeta = \zeta(\gamma_3)$ are the smooth functions defined in [23], and $L = L(\gamma_3) = \varrho(\gamma)\gamma_3 - \zeta$. The configuration space is described as

$$Q = \{(g, \mathbf{x}) \in SO(3) \times \mathbb{R}^3 \ : \ z = -\langle \boldsymbol{\gamma}, s \rangle\},$$

and it is diffeomorphic to $SO(3) \times \mathbb{R}^2$. The nonholonomic constraint describing the rolling without sliding are written as

$$\boldsymbol{\Omega} \times s + \mathbf{b} = 0,$$

where $\mathbf{b} = g^t \dot{\mathbf{x}}$ (with g^t the transpose of g).

Let us consider the (local) basis of TQ given by $\{X_1^L, X_2^L, X_3^L, \partial_x, \partial_y\}$, where X_i^L are the left invariant vector fields on $SO(3)$ and we denote the corresponding coordinates on TQ by $(\boldsymbol{\Omega}, \dot{x}, \dot{y})$. Then the constraint distribution D is given by $D = \mathrm{span}\{X_1, X_2, X_3\}$, where

$$X_i := X_i^L + (\boldsymbol{\alpha} \times s)_i \partial_x + (\boldsymbol{\beta} \times s)_i \partial_y + (\boldsymbol{\gamma} \times s)_i \partial_z,$$

for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the first and second rows of the matrix $g \in SO(3)$. The constraints 1-forms are

$$\epsilon^1 = dx - \langle \boldsymbol{\alpha}, s \times \boldsymbol{\lambda} \rangle \quad \text{and} \quad \epsilon^2 = dy - \langle \boldsymbol{\beta}, s \times \boldsymbol{\lambda} \rangle,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ are the (Maurer-Cartan) 1-forms on $SO(3)$ dual to the left invariant vector fields $\{X_L^1, X_L^2, X_L^3\}$.

The symmetries. The Lagrangian and the constraints are invariant with respect to the action of the special Euclidean group $SE(2)$ acting on Q, at each $(g; x, y) \in Q$, by

$$\Psi((h; a, b)), (g; x, y)) = (\tilde{h}.g; h.(x, y)^t + (a, b)^t),$$

where $h \in SO(2)$ is an orthogonal 2×2 matrix and $\tilde{h} = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in SO(3)$. The symmetry of the body makes also the system invariant with respect to the right S^1-action on Q given by $\Psi_{S^1}(h_\theta, (g, x, y)) = (g\tilde{h}_\theta^{-1}, h_\theta(x, y)^t)$, where we identify $\theta \in S^1$ with the orthogonal matrix $h_\theta \in SO(2)$.

Therefore, the symmetry group of the system is the Lie group $G = S^1 \times SE(2)$, with associated Lie algebra $\mathfrak{g} \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$. The vertical space V is given by

$$V = \text{span}\{(\eta_1)_Q = -X_3^L - y\partial_x + x\partial_y,$$
$$(\eta_2)_Q = \langle \boldsymbol{\gamma}, \mathbf{X}^L \rangle - y\partial_x + x\partial_y,\ (\eta_3)_Q = \partial_x,\ (\eta_4)_Q = \partial_y\},$$

where η_i are the canonical Lie algebra elements in $\mathfrak{g}$ and $\mathbf{X}^L = (X_1^L, X_2^L, X_3^L)$. We observe that the action is not free, since $(\eta_i)_Q(g, x, y)$ are not linearly independent at $\gamma_3 = 1$. We check that the dimension assumption (2.3) is satisfied: $TQ = D + V$. Let us choose $W = \text{span}\{\partial_x, \partial_y\}$ as vertical complement of the constraints and then the basis of TQ adapted to the splitting (2.6) is $\mathbf{B}_{TQ} = \{X_1, X_2, X_3, \partial_x, \partial_y\}$, with dual basis given by $\mathbf{B}_{T^*Q} = \{\lambda_1, \lambda_2, \lambda_3, \epsilon^1, \epsilon^2\}$. The associated coordinates on T_q^*Q are $(\mathbf{M}, K_1, K_2)$ for $\mathbf{M} = (M_1, M_2, M_3)$ and the submanifold $\mathcal{M}$ of T^*Q is then described by

$$\mathcal{M} = \{(g, x, y; \mathbf{M}, K_1, K_2)\ :\ K_1 = \mathbf{m}\langle \boldsymbol{\alpha}, s \times \boldsymbol{\Omega}\rangle, \quad K_2 = \mathbf{m}\langle \boldsymbol{\beta}, s \times \boldsymbol{\Omega}\rangle\}, \tag{5.37}$$

where $\mathbf{M} = \mathbb{I}\boldsymbol{\Omega} + ms \times (\boldsymbol{\Omega} \times s)$. The horizontal gauge momenta are functions on $\mathcal{M}$ linear in the coordinates M_i.

The existence of horizontal gauge momenta. First, we observe that the G-action satisfies Conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ outside $\gamma_3 = \pm 1$ and thus, in what follows, we will work on the manifolds $\widetilde{Q} \subset Q$ and $\widetilde{\mathcal{M}} \subset \mathcal{M}$ defined by the condition $\gamma_3 \neq \pm 1$. Second, we consider the splitting

$$T\widetilde{Q} = H \oplus S \oplus W, \tag{5.38}$$

where $S = D \cap V = \text{span}\{Y_1 := X_3, Y_2 := \langle \gamma, \mathbf{X}\rangle\}$, with $\mathbf{X} = (X_1, X_2, X_3)$ and H is generated by $X_0 = \gamma_1 X_2 - \gamma_2 X_1$ (observe that $H = S^\perp \cap D$). Now, we check that the kinetic energy is strong invariant on S: in this case, it is enough to see that $\kappa([Y_1, Y_2], Y_1) = 0$ and $\kappa([Y_1, Y_2], Y_2) = 0$. These two facts are easily verified using simply that $[X_i^L, X_j^L] = X_k^L$ for i, j, k cyclic permutations of 1, 2, 3. In the same way, we also check that $\kappa(X_0, [Y_i, X_0]) = 0$, for $i = 1, 2$. Therefore, by Theorem 3.14, we conclude that the system admits $2 = \text{rank}(S)$ G-invariant (functionally independent) horizontal gauge momenta $\mathcal{J}_1, \mathcal{J}_2$ on $\widetilde{\mathcal{M}}$ (recovering the results in [17,23]).

The computation of the 2 horizontal gauge momenta. In order to compute the horizontal gauge momenta, we consider the basis $\mathfrak{B}_{\mathfrak{g}_S}$ of $\Gamma(\mathfrak{g}_S \to \widetilde{Q})$, defined by

$$\mathfrak{B}_{\mathfrak{g}_S} = \{\xi_1 := (1; 0, (h_1, h_2)), \xi_2 := (0; 1, (g_1, g_2))\},$$

where $h_1 = h_1(g, x, y) = y + \varrho\beta_3$, $h_2 = h_2(g, x, y) = -x - \varrho\alpha_3$ and $g_1 = g_1(g, x, y) = y - L\beta_3$, $g_2 = g_2(g, x, y) = -x + L\alpha_3$. The components of the nonholonomic momentum map, in the basis $\mathfrak{B}_{\mathfrak{g}_S}$, are given by

$$J_1 = \langle \mathcal{J}^{\text{nh}}, \xi_1 \rangle = \mathbf{i}_{(\xi_1)_\mathcal{M}} \Theta_\mathcal{M} = -M_3 \quad \text{and}$$
$$J_2 = \langle \mathcal{J}^{\text{nh}}, \xi_2 \rangle = \mathbf{i}_{(\xi_2)_\mathcal{M}} \Theta_\mathcal{M} = \langle \boldsymbol{\gamma}, \mathbf{M} \rangle,$$

where we are using that $(\xi_1)_Q = Y_1$ and $(\xi_2)_Q = Y_2$, see (2.11). Then, a function $\mathcal{J} = f_1 J_1 + f_2 J_2$ is a horizontal gauge momentum if and only if the coordinate functions (f_1, f_2) satisfy the *momentum equation* (3.17)

$$f_1 \langle J, \sigma_{\mathfrak{g}_S} \rangle (\mathcal{Y}_1, X_{\text{nh}}) + f_2 \langle J, \sigma_{\mathfrak{g}_S} \rangle (\mathcal{Y}_2, X_{\text{nh}}) - M_3 X_{\text{nh}}(f_1) + \langle \boldsymbol{\gamma}, \mathbf{M} \rangle X_{\text{nh}}(f_2) = 0.$$

That is, considering the basis, $\mathfrak{B}_{T\widetilde{Q}} = \{X_0, \; Y_1, \; Y_2, \; \partial_x, \; \partial_y\}$, the G-invariant coordinate functions $(f_1 = f_1(\gamma_3), f_2 = f_2(\gamma_3))$ are the solutions of the system of ordinary differential equations (defined on $\widetilde{Q}/G$)

$$R \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \bar{X}_0(f_1) \\ \bar{X}_0(f_2) \end{pmatrix}, \qquad \text{for } R = [\kappa|_S]^{-1}[N], \tag{5.39}$$

where $\bar{X}_0 = T\rho_{\widetilde{Q}}(X_0) = (1-\gamma_3^2)\partial_{\gamma_3}$, the matrix $[N]$ has elements $N_{lj} = \kappa(Y_l, [Y_i, X_0]) - \kappa(X_0, [Y_i, Y_l])$ that in this case gives

$$[N] = m(1-\gamma_3^2) \begin{pmatrix} -\varrho A & \varrho(B - \langle \boldsymbol{\gamma}, s \rangle) \\ LA - \varrho \langle \boldsymbol{\gamma}, s \rangle & -LB \end{pmatrix}$$

for $A = \varrho'(1-\gamma_3^2) - \varrho\gamma_3$ and $B = L'(1-\gamma_3^2) - L\gamma_3 - \langle \boldsymbol{\gamma}, s \rangle$ (with $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\gamma_3}(\cdot)$) and

$$[\kappa|_S] = \begin{pmatrix} \mathbb{I}_3 + m\varrho^2(1-\gamma_3^2) & -\mathbb{I}_3\gamma_3 - Lm\varrho(1-\gamma_3^2) \\ -\mathbb{I}_3\gamma_3 - Lm\varrho(1-\gamma_3^2) & \langle \boldsymbol{\gamma}, \mathbb{I}\boldsymbol{\gamma} \rangle + L^2 m (1-\gamma_3^2) \end{pmatrix}.$$

The system (5.39) admits two independent solutions $\bar{f}_1 = (\bar{f}_1^1, \bar{f}_1^2)$ and $\bar{f}^2 = (\bar{f}_2^1, \bar{f}_2^2)$ on $\widetilde{Q}/G$ and therefore we conclude that the two (G-invariant) horizontal gauge momenta $\mathcal{J}_1$ and $\mathcal{J}_2$ are

$$\mathcal{J}_1 = -f_1^1 M_3 + f_1^2 \langle \boldsymbol{\gamma}, \mathbf{M} \rangle \quad \text{and} \quad \mathcal{J}_2 = -f_2^1 M_3 + f_2^2 \langle \boldsymbol{\gamma}, \mathbf{M} \rangle, \tag{5.40}$$

where $f_j^i = \rho_{\widetilde{Q}}^* \bar{f}_j^i$ for $i, j = 1, 2$.

Observations 5.3. (i) For $f = (f^1, f^2)$, the system (5.39) is equivalently written as $(1-\gamma_3^2)^{-1} R f = f'$. Therefore, we recover the system of ordinary differential equations from [5,17,23] (and [9] for the special case of the Tippe-Top and of the rolling disk).

(ii) The G-invariant horizontal gauge momenta $\mathcal{J}_1, \mathcal{J}_2$ descend to the quotient $\widetilde{\mathcal{M}}/G$ as functions $\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2$ that are functionally independent. It has been proven in [23] that the functions $\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2$ can be extended to the whole *differential space* $\mathcal{M}/G$. In this case, it makes sense to talk about $2 = \text{rank}(\mathfrak{g}_S)$ horizontal gauge momenta.

□

Integrability and hamiltonization. The nonholonomic dynamics X_{nh} defined on $\widetilde{\mathcal{M}}$ can be reduced to $\widetilde{\mathcal{M}}/G$ obtaining the vector field X_{red} (see (2.5)). Using the basis $\mathfrak{B}_{T\widetilde{Q}} = \{X_0, Y_1, Y_2, \partial_x, \partial_y\}$ and its dual basis of $T^*\widetilde{Q}$

$$\mathfrak{B}_{T^*\widetilde{Q}} = \left\{ X^0 := \frac{\gamma_1\lambda_2 - \gamma_2\lambda_1}{1-\gamma_3^2},\ Y^1 := \gamma_3 \frac{\gamma_1\lambda_1 + \gamma_2\lambda_2}{1-\gamma_3^2} - \lambda_3, \right.$$
$$\left. Y^2 := \frac{\gamma_1\lambda_1 + \gamma_2\lambda_2}{1-\gamma_3^2},\ \epsilon^1,\ \epsilon^2 \right\}, \tag{5.41}$$

we denote by $(v^0, v^1, v^2, v^x, v^y)$ and $(p_0, p_1, p_2, K_1, K_2)$ the associated coordinates on $T\widetilde{Q}$ and $T^*\widetilde{Q}$, respectively. The reduced manifold $\widetilde{\mathcal{M}}/G$ is represented by the coordinates $(\gamma_3, p_0, p_1, p_2)$.

Integrability. Theorem 4.4 guarantees that the reduced system on $\widetilde{\mathcal{M}}/G$ admits three functionally independent first integrals, namely two horizontal gauge momenta $\bar{\mathcal{J}}_1$ and $\bar{\mathcal{J}}_2$, and the reduced energy H_{red}. Since $\dim(\widetilde{\mathcal{M}}/G) = 4$, the reduced dynamics is integrable by quadratures. However, the reduced dynamics is not generically periodic, and therefore we can say nothing generic on the complete dynamics or on the geometry of the phase space.

Hamiltonization. Even though the hamiltonization of this example has been studied in [5,37], here we see it as a direct consequence of Theorem 3.14. That is, since this nonholonomic system satisfies the hypotheses of Theorem 3.14, it is *hamiltonizable by a gauge transformation* (Definition 4.7). The reduced bracket $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$ on $\widetilde{\mathcal{M}}/G$ defines a rank-2 Poisson structure, with 2-dimensional leaves given by the common level sets of $\bar{\mathcal{J}}_1$ and $\bar{\mathcal{J}}_2$, that describes the (reduced) dynamics.

In what follows we show how the 2-form B_{HGM}, inducing the dynamical gauge transformation that defines $\{\cdot,\cdot\}_{\text{red}}^{B_{\text{HGM}}}$, depends directly on the ordinary system of differential equations (5.39). Consider the basis $\mathfrak{B}_{T\widetilde{Q}}$ and $\mathfrak{B}_{T^*\widetilde{Q}}$ given in (5.41) and following Theorem 4.10,

$$B_{\text{HGM}} = \langle J, \sigma_{\text{HGM}}\rangle = \langle J, \mathcal{K}_{\mathcal{W}}\rangle - J_i\, R_1^i \mathcal{X}^0 \wedge \mathcal{Y}^1 - J_i\, R_2^i \mathcal{X}^0 \wedge \mathcal{Y}^2 + J_i d\mathcal{Y}^i,$$

where $\mathcal{X}^0 = \tau_{\widetilde{\mathcal{M}}}^* X^0$ and $\mathcal{Y}^i = \tau_{\widetilde{\mathcal{M}}}^* Y^i$ for $i = 1, 2$ are the corresponding 1-forms on $\widetilde{\mathcal{M}}$. Using (5.37) we have that (see [5]),

$$\begin{aligned}\langle J, \mathcal{K}_{\mathcal{W}}\rangle|_{\mathcal{C}} &= K_1\, d\epsilon^1|_{\mathcal{C}} + K_2\, d\epsilon^2|_{\mathcal{C}} \\ &= m\varrho\langle \boldsymbol{\gamma}, s\rangle\langle \boldsymbol{\Omega}, d\boldsymbol{\lambda}\rangle - m(\varrho^2\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + \varrho' c_3)\langle \boldsymbol{\gamma}, d\boldsymbol{\lambda}\rangle \\ &\quad + m(\varrho L\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + L' c_3) d\lambda_3|_{\mathcal{C}}.\end{aligned}$$

Now, recalling the definition of X^0, Y^1 and Y^2 in $\mathfrak{B}_{T^*Q}$ (5.41), we compute the term

$$\begin{aligned}J_i\, R_1^i \mathcal{X}^0 \wedge \mathcal{Y}^1 + J_i\, R_2^i \mathcal{X}^0 \wedge \mathcal{Y}^2 &= (1-\gamma_3^2)^{-1}(v^l N_{l1}\langle \boldsymbol{\gamma}, d\boldsymbol{\lambda}\rangle + v^l N_{l2}\, d\lambda_3), \\ &= -m(\varrho^2\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + \varrho' c_3)\langle \boldsymbol{\gamma}, d\boldsymbol{\lambda}\rangle \\ &\quad + m(\varrho L\langle \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle + L' c_3) d\lambda_3.\end{aligned}$$

where we use that $v^1 = (1-\gamma_3^2)^{-1}(\langle \boldsymbol{\gamma}, \boldsymbol{\Omega}\rangle\gamma_3 - \Omega_3)$ and $v^2 = (1-\gamma_3^2)^{-1}(\langle \boldsymbol{\gamma}, \boldsymbol{\Omega}\rangle - \gamma_3\Omega_3)$. Finally, since $dY^i = 0$ for $i = 1, 2$, we obtain that

$$B_{\text{HGM}} = m\varrho\langle \boldsymbol{\gamma}, s\rangle\langle \boldsymbol{\Omega}, d\boldsymbol{\lambda}\rangle,$$

recovering the dynamical gauge transformation from [5,37]. For the explicit formulas for the brackets, see [5].

Observations 5.4. (i) Since the G-action on $\mathcal{M}$ is proper but not free, the quotient $\mathcal{M}/G$ is a stratified differential space, [5,23] with a 4 dimensional regular stratum given by $\widetilde{\mathcal{M}}/G$ and a 1-dimensional singular stratum, associated to S^1-isotropy type, that is described by the condition $\gamma_3 = \pm 1$. Moreover, the relation between the coordinates on $T^*\widetilde{Q}$ relative to the basis $\mathbf{B}_{T^*\widetilde{Q}}$ and $\mathfrak{B}_{T^*\widetilde{Q}}$ is

$$p_0 = \gamma_1 M_2 - \gamma_2 M_1, \quad p_1 = \gamma_1 M_1 + \gamma_2 M_2, \quad p_2 = M_3,$$

Therefore, adding $p_3 = M_1^2 + M_2^2$, we conclude that the coordinates $(\gamma_3, p_0, p_1, p_2, p_3)$ on $\mathcal{M}/G$ are the same coordinates used in [21,23].
(ii) It is straightforward to write the equations of motion on $\widetilde{\mathcal{M}}/G$ in the variables $(\gamma_3, p_0, p_1, p_2)$ for the reduced hamiltonian H_{red} recovering the equations in [21,23]. We stress that there is no need to compute them to find the horizontal gauge momenta, nor to study the integrability or the hamiltonization of the system.
(iii) The Routh sphere, the ellipsoid rolling on a plane and the falling disk [14,21,23], are seen as particular cases of this example.

□

5.3. A homogeneous ball on a surface of revolution

Let us consider the holonomic system formed by a homogeneous sphere of mass $\mathbf{m}$ and radius $r > 0$, which center C is constrained to belong to a convex surface of revolution Σ (i.e., the ball rolls on the surface $\tilde{\Sigma}$, see Fig. 3). The surface Σ is obtained by rotating about the z-axis the graph of a convex and smooth function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Thus, Σ is described by the equation $z = \phi(x^2 + y^2)$. To guarantee smoothness and convexity of the surface, we assume that ϕ verifies that $\phi'(0^+) = 0$, $\phi'(s) > 0$ and $\phi''(s) > 0$, when $s > 0$. To ensure that the ball has only one contact point with the surface we ask the curvature of $\phi(s)$ to be at most 1/r. The configuration manifold Q is $\mathbb{R}^2 \times SO(3)$ with coordinates (x, y, g) where G is the orthogonal matrix fixing the attitude of the sphere and (x, y) are the coordinates of C with respect to a reference frame with origin O and z-axis coinciding with the figure axis of Σ.

Let us denote by $n = n(x, y)$ the outward normal unit vector to Σ with components (n_1, n_2, n_3) given by

$$\frac{n_1}{n_3} = 2x\phi', \quad \frac{n_2}{n_3} = 2y\phi' \quad \text{and} \quad n_3 = -\frac{1}{\sqrt{1 + 4(x^2 + y^2)(\phi')^2}}.$$

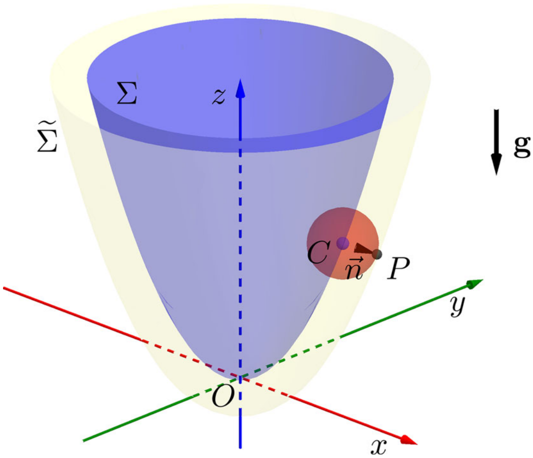

Fig. 3. The homogeneous ball on a convex surface of revolution

If $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the ball in the space frame, then the Lagrangian of the holonomic system on TQ is

$$L(x, y, g, \dot{x}, \dot{y}, \omega) = \frac{\mathbf{m}}{2n_3^2}\left((1 - n_2^2)\dot{x}^2 + 2n_1 n_2\,\dot{x}\dot{y} + \dot{y}^2(1 - n_1^2)\right) + \frac{1}{2}\langle \mathbb{I}\omega, \omega\rangle - \mathbf{mg}\phi, \quad (5.42)$$

where $\mathbf{g}$ denotes the gravity acceleration and $\mathbb{I}$ the moment of inertia of the sphere with respect to its center of mass.

Geometry of the constrained system. The ball rotates without sliding on the surface $\widetilde{\Sigma}$, and hence the nonholonomic constraints equations are

$$\dot{x} = -r\,(\omega_2 n_3 - \omega_3 n_2)\,, \qquad \dot{y} = -r\,(\omega_3 n_1 - \omega_1 n_3)\,.$$

We denote by $\{X_1^R, X_2^R, X_3^R\}$ the right invariant vector fields on $SO(3)$ and by $\{\rho_1, \rho_2, \rho_3\}$ the right Maurer-Cartan 1-forms, that form a basis of $T^*SO(3)$ dual to $\{X_1^R, X_2^R, X_3^R\}$. Then the constraint 1-forms are given by

$$\epsilon^1 = dx - r\,(n_2\rho_3 - n_3\rho_2)\,, \quad \epsilon^2 := dy - r\,(n_3\rho_1 - n_1\rho_3)\,.$$

The constraint distribution D defined by the annihilator of ϵ^1 and ϵ^2 has fiber, at $q = (x, y, g)$, given by

$$D_q = \text{span}\left\{Y_x := \partial_x - \frac{1}{rn_3}(n_2 X_n - X_2^R),\ Y_y := \partial_y + \frac{1}{rn_3}(n_1 X_n - X_1^R),\ X_n\right\}, \quad (5.43)$$

where $X_n := n_1\, X_1^R + n_2\, X_2^R + n_3\, X_3^R$. Consider the basis of TQ

$$\mathbf{B}_{TQ} = \left\{Y_x, Y_y, X_n, Z_1, Z_2\right\}, \quad (5.44)$$

where $Z_1 := \frac{1}{rn_3}X_2^R - \frac{n_2}{rn_3}X_n$ and $Z_2 := -\frac{1}{rn_3}X_1^R + \frac{n_1}{rn_2}X_n$ with associated coordinates $(\dot{x}, \dot{y}, \omega_n, \omega^1, \omega^2)$, for $\omega_n = n \cdot \omega = n_i\omega_i$, the normal component of the angular velocity ω. The dual frame of (5.44) is

$$\mathbf{B}_{T^*Q} = \left\{dx, dy, \rho_n, \epsilon^1, \epsilon^2, \right\}, \quad (5.45)$$

where $\rho_n = n_i \rho_i$, with associated coordinates $(p_x, p_y, p_n, M_1, M_2)$ on T^*Q. The manifold $\mathcal{M} = \kappa^\sharp(D)$ is given by

$$\mathcal{M} = \left\{(x, y, g; p_x, p_y, p_n, M_1, M_2) \ : \ M_1 = \tfrac{-I}{I+mr^2} p_x, \ M_2 = \tfrac{-I}{I+mr^2} p_y\right\}.$$

The symmetries. Consider the action Ψ of the Lie group $G = SO(2) \times SO(3)$ on the manifold Q given, at each $(x, y, g) \in Q$ and $(h_\theta, h) \in SO(2) \times SO(3)$, by

$$\Psi_{(h_\theta,h)}(x, y, g) = (h_\theta(x, y)^t, \tilde{h}_\theta g h),$$

where $\tilde{h}_\theta$ is the 3×3 rotational matrix of angle θ with respect to the z-axis. In other words, $SO(3)$ acts on the right on itself and $SO(2)$ acts by rotations about the figure axis of the surface Σ. The Lagrangian (5.42) and the constraints (5.43) are invariant with respect to the lift of this action to TQ given by $\Psi_{(h_\theta,h)}(x, y, g, \dot{x}, \dot{y}, \omega) = (h_\theta(x, y)^t, \tilde{h}_\theta g h, h_\theta(\dot{x}, \dot{y})^t, \omega)$. The invariance of the kinetic energy and the constraints D ensures that Ψ restricts to an action on $\mathcal{M}$, that leaves the equations of motion invariant.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the infinitesimal generators

$$(1; \mathbf{0})_Q = -y\partial_x + x\partial_y + X_3^R \quad \text{and}$$
$$(0; \mathbf{e}_i)_Q = \alpha_i\, X_1^R + \beta_i\, X_2^R + \gamma_i\, X_3^R, \ \text{for } i = 1, 2, 3,$$

where $\mathbf{e}_i$ denotes the i-th element of the canonical basis of $\mathbb{R}^3$ and, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$ $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ the rows of the matrix $g \in SO(3)$. Observe that $(1; \mathbf{0})_Q$ is an infinitesimal generator of the $SO(2)$-action and the others are infinitesimal generators of the $SO(3)$-action. We then underline that the G-symmetry satisfies the dimension assumption and it is proper and free whenever $(x, y) \neq (0, 0)$ (note that the rank of V is 3 for $(x, y) = (0, 0)$ and it is 4 elsewhere, showing that the action is not even locally free).

Let us denote by $\widetilde{Q} \subset Q$ and $\widetilde{\mathcal{M}} \subset \mathcal{M}$ the manifolds where the G-action is free, i.e. $(x, y) \neq (0, 0)$. The vertical distribution $S = D \cap V$ on $\widetilde{Q}$ has rank 2 with fibers

$$S_q = \text{span}\{Y_1 := -yY_x + xY_y, Y_2 := X_n\}.$$

The bundle $\mathfrak{g}_S \to Q$ has a global basis $\mathfrak{B}_{\mathfrak{g}_S}$ of sections given by

$$\mathfrak{B}_{\mathfrak{g}_S} = \left\{\xi_1 := \left(1; \frac{x}{r\,n_3}, \frac{y}{r\,n_3}, 0\right), \xi_2 := (0; n\, g)\right\},$$

and we check that $(\xi_1)_Q = Y_1$ and $(\xi_2)_Q = Y_2$. Finally we observe that $\widetilde{Q}/G$ has dimension 1 ($\rho_{\widetilde{Q}} : \widetilde{Q} \to \widetilde{Q}/G$ is given by $\rho_{\widetilde{Q}}(x, y, g) = x^2 + y^2$) and hence the G-symmetry satisfies Conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ on $\widetilde{Q}$.

The existence of horizontal gauge momenta. Using the basis (5.44) and the definition of S, we consider the decomposition

$$T\widetilde{Q} = H \oplus S \oplus W,$$

where W is a vertical complement of the constraints given by $W := \text{span}\{Z_1, Z_2\}$ and $H := S^\perp \cap D$ is generated by $X_0 := xY_x + yY_y$. As in Example 5.2, in this case, it is enough (and straightforward using that $n_3(x, y)$ is rotational invariant and that $[X_1^R, X_2^R] = -X_3^R$ for all cyclic permutations) to check that $\kappa([Y_1, Y_2], Y_1) = 0$ and $\kappa([Y_1, Y_2], Y_2) = 0$ to guarantee that the kinetic energy is strong invariant on S. Finally, we also see that $\kappa(X_0, [Y_i, X_0]) = 0$, for $i = 1, 2$. Therefore, following Theorem 3.14, the system admits two G-invariant (functionally independent) horizontal gauge momenta $\mathcal{J}_1$ and $\mathcal{J}_2$, showing that the first integrals obtained in [17,27,38,50,56] can be obtained from the symmetry of the system as horizontal gauge momenta.

The computation of the 2 horizontal gauge momenta. We now characterize the coordinate functions of the horizontal gauge symmetries written in the basis $\mathfrak{B}_{\mathfrak{g}_S}$ on $\widetilde{Q}$. That is, let us denote

$$J_1 := \mathbf{i}_{Y_1}\Theta = -yp_x + xp_y \qquad \text{and} \qquad J_2 := \mathbf{i}_{Y_2}\Theta = p_n.$$

Using the orbit projection $\rho_{\widetilde{Q}} : \widetilde{Q} \to \widetilde{Q}/G$, a G-invariant function f on Q can be thought as depending on the variable $\tau = x^2 + y^2$, i.e., $f = f(\tau)$. Following Theorem 3.14(ii), a function $\mathcal{J} = f_1 J_1 + f_2 J_2$ for $f_1, f_2 \in C^\infty(Q)^G$ is a horizontal gauge momenta if and only if (f_1, f_2) is a solution of the linear system of ordinary differential equations on $\widetilde{Q}/G$,

$$R\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \bar{X}_0(f_1) \\ \bar{X}_0(f_2) \end{pmatrix} \quad \text{where} \quad R = 2\tau\begin{pmatrix} 0 & -2\frac{rI}{E}n_3^2(2(\phi')^3 - \phi'') \\ \frac{A}{r}n_3^2 & 0 \end{pmatrix} \tag{5.46}$$

for $A = \phi' + 2\tau\phi''$ and $\bar{X}_0 = T\rho_{\widetilde{Q}}(X_0) = 2\tau\frac{\partial}{\partial\tau}$. The matrix R is computed using that $R = [\kappa|_S]^{-1}[N]$, where

$$[N] = \frac{2I}{r}\tau\begin{pmatrix} 0 & -2\tau n_3^2(2(\phi')^3 - \phi'') \\ An_3^2 & 0 \end{pmatrix} \quad \text{and} \quad [\kappa|_S] = \begin{pmatrix} \frac{E}{r^2}\tau & 0 \\ 0 & I \end{pmatrix}.$$

Since this system admits two independent solutions $f_1 = (f_1^1, f_1^2)$ and $f_2 = (f_2^1, f_2^2)$ on $\widetilde{Q}/G$, then the nonholonomic system admits two G-invariant horizontal gauge momenta $\mathcal{J}_1, \mathcal{J}_2$ defined on $\widetilde{\mathcal{M}}$ of the form

$$\mathcal{J}_1 = f_1^1 J_1 + f_1^2 J_2 \qquad \text{and} \qquad \mathcal{J}_2 = f_2^1 J_1 + f_2^2 J_2, \tag{5.47}$$

recalling that $J_1 = -yp_x + xp_y$ and $J_2 = p_n$

Observations 5.5. Let us denote by $\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2$ the functions on $\widetilde{\mathcal{M}}/G$ associated to (5.47).

(i) The (reduced) first integrals $\bar{\mathcal{J}}_1, \bar{\mathcal{J}}_2$ can be extended by continuity to the *differential space* $\mathcal{M}/G$ and thus $\mathcal{J}_1, \mathcal{J}_2$ are G-invariant functions on $\mathcal{M}$ (see [27] for details) and in this case we say that the system admits $2 = \text{rank}(\mathfrak{g}_S)$ horizontal gauge momenta.

(ii) The system of differential equations (5.46) can be written as

$$R_1 f^2 = (f^1)' \qquad \text{and} \qquad R_2 f^1 = (f^2)',$$

where $R_1 = R_1(\tau) = -2\frac{rI}{E}n_3^2(2(\phi')^3 - \phi'')$ and $R_2 = R_2(\tau) = \frac{A}{r}n_3^2$. Hence $\bar{\mathcal{J}}_1$, $\bar{\mathcal{J}}_2$ are first integrals of Routh type found in [38] (see also [17,23,51,56]) and shown to be horizontal gauge momenta in [27,29].

□

Integrability and reconstruction. The reduced integrability of this system was established in [50] and its complete broad integrability has been extensively studied in [17,26,27,38,56], using the existence of first integrals $\mathcal{J}_1$ and $\mathcal{J}_2$, without relating their existence to the symmetry group. The symmetry origin of $\mathcal{J}_1$ and $\mathcal{J}_2$ was announced in [9], and then proved in [29,51]. Here we want to stress how Theorem 3.14 can be applied and therefore the reduced integrability of the system is ensured. That is, $\bar{\mathcal{J}}_1$, $\bar{\mathcal{J}}_2$, H_{red} are first integrals of the reduced dynamics X_{red} defined on the manifold $\widetilde{\mathcal{M}}/G$ of dimension 4. Moreover, as proved in [38,56] the reduced dynamics is made of periodic motions or of equilibria, and hence, since the symmetry group is compact, the complete dynamics is generically quasi-periodic on tori of dimension 3 (see Theorem 4.12 and [26,38]). Indeed one could can say more on the geometric structure of the phase space $\widetilde{\mathcal{M}}$ of the complete system, it is endowed with the structure of a fibration on tori of dimension at most 3 (see [26] for a detailed study of the geometry of the complete system on $\widetilde{\mathcal{M}}$).

Hamiltonization. Even though the hamiltonization of this example has been studied in [8], in this section we see the hamiltonization as a consequence of Theorem 3.14 and how the resulting Poisson bracket on $\widetilde{\mathcal{M}}/G$ depends on the linear system of ordinary differential equations (5.46).

By Theorem 4.5, the nonholonomic system is *hamiltonizable by a gauge transformation*; that is, on $\widetilde{\mathcal{M}}/G$ the reduced nonholonomic system is described by a Poisson bracket with 2-dimensional leaves given by the common level sets of the horizontal gauge momenta $\bar{\mathcal{J}}_1$, $\bar{\mathcal{J}}_2$, induced by (5.47), (recall that $\bar{\mathcal{J}}_i$ are the functions on $\widetilde{\mathcal{M}}/G$, such that $\rho^*(\bar{\mathcal{J}}_i) = \mathcal{J}_i$).

Following Theorem 4.10, we compute the 2-form B_{HGS}, defining the dynamical gauge transformation, using the momentum equation (5.46). Since $dY^1|_D = 0$, then

$$B_{\text{HGS}} := \langle J, \mathcal{K}_{\mathcal{W}}\rangle - p_1 R_2^1 \mathcal{X}^0 \wedge \mathcal{Y}^2 + p_2 R_1^2 X^0 \wedge \mathcal{Y}^1 + p_2 d\mathcal{Y}^2, \tag{5.48}$$

where $\mathcal{X}^0 = \tau_{\mathcal{M}}^* X^0$ and $\mathcal{Y}^i = \tau_{\mathcal{M}}^* Y^i$. That is,

$$\langle J, \mathcal{K}_{\mathcal{W}}\rangle|_{\mathcal{C}} = M_1\, d\epsilon^1|_{\mathcal{C}} + M_2\, d\epsilon^2|_{\mathcal{C}} = -\frac{Ir}{E(x^2+y^2)}$$
$$\left(p_1(\tfrac{1}{rn_3^2} + 2n_3^2 A)\mathcal{X}^0 \wedge \mathcal{Y}^2 + p_0 n_3(2\phi' n_3 + \tfrac{1}{r}\mathcal{Y}^1 \wedge \mathcal{Y}^2\right)|_{\mathcal{C}},$$

and using that $d\mathcal{Y}^2|_{\mathcal{C}} = \frac{(x^2+y^2)}{n_3} p_2 X^0 \wedge \mathcal{Y}^1|_{\mathcal{C}}$, we obtain

$$B_{\text{HGS}} = (x^2+y^2)p_2(\tfrac{1}{n_3} + 2\tfrac{A}{r}n_3^2)\mathcal{X}^0 \wedge \mathcal{Y}^1 + \tfrac{rI}{E}(\tfrac{1}{rn_3} + 2\phi')(p_1\mathcal{X}^0 \wedge \mathcal{Y}^2 - p_0 n_3^2 \mathcal{Y}^1 \wedge \mathcal{Y}^2). \quad (5.49)$$

Observations 5.6. (i) Since the action is not free, $\mathcal{M}/G$ is a semialgebraic variety that consists in two strata: a singular 1-dimensional stratum corresponding to the points in which the action is not free; and the four dimensional regular stratum $\widetilde{\mathcal{M}}/G$ (where the action is free). Moreover, analyzing the change of coordinates between $\mathbf{B}_{T^*Q}$ and $\mathfrak{B}_{T^*Q}$ we get

$$\tau = x^2 + y^2, \;\; p_0 = xp_x + yp_y, \;\; p_1 = -yp_x + xp_y, \;\; p_2 = p_n,$$

and adding $p_3 = p_x^2 + p_y^2$, we recover the coordinates used in [27,38] on $\widetilde{\mathcal{M}}/G$.

(ii) Since the convexity of the function ϕ that parametrizes the surface Σ is not strictly used, this example also describes the geometry and dynamics of a homogeneous ball rolling on surface of revolution such that its normal vector fields has $n_3 \neq 0$. □

5.4. Comments on the hypotheses of the nonholonomic Noether theorem: examples and counterexamples

Theorem 3.14 shows that a nonholonomic system with symmetries satisfying certain hypotheses admits the existence of k functionally independent G-invariant horizontal gauge momenta. Next, assuming Conditions ($\mathcal{A}$1)-($\mathcal{A}$3), we study what may happen if the other hypotheses of Theorem 3.14 are not satisfied. In particular we study three cases: when the metric is not strong invariant, when $\kappa(X_0, [X_0, Y])$ is different from zero, and finally when Condition ($\mathcal{A}$4) is not verified (i.e., $\dim(Q/G) \neq 1$). For each case we give examples and counterexamples to illustrate our conclusions.

Analyzing the strong invariance condition and $\kappa(X_0, [X_0, Y]) = 0$ Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{C}}, H_{\mathcal{M}})$ with a G-symmetry satisfying Conditions ($\mathcal{A}$1)-($\mathcal{A}$4). Suppose that $(f_1, ..., f_k)$ is a solution of the system of differential equations (3.21), then, from (3.22), we observe that $\mathcal{J} = f_j J_i$ is a horizontal gauge momentum if and only if

$$f^i \kappa(X_0, [Y_i, X_0]) = 0 \quad \text{and} \quad f^i(\kappa(Y_j, [Y_i, Y_l]) + \kappa(Y_l, [Y_i, Y_j])) = 0, \text{ for each } j, l.$$

for a S-orthogonal horizontal space H. That is, in some cases, even if $\kappa(X_0, [X_0, Y_{i_0}]) \neq 0$ for some $Y_{i_0} \in \Gamma(S)$ or the metric is not strong invariant, we may still have a horizontal gauge momentum.

We now present two examples that show the main features of these phenomenon.

The metric is not strong invariant on S. The following is a mathematical example, that has the property that the metric is not strong invariant, and it admits only 1 horizontal gauge momenta even though the rank of the distribution S is 3.

Precisely, consider the nonholonomic system on the manifold $Q = \mathbb{R}^3 \times SE(2)$ with coordinates $(u, v, x) \in \mathbb{R}^3$ and $(y, z, \theta) \in SE(2)$ with Lagrangian given by

$$L(q, \dot{q}) = \frac{1}{2}\left(u^2 + v^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \dot{\theta}^2 + 4(\sin\theta\, \dot{z} + \cos\theta\, \dot{y})\dot{\theta}\right),$$

and constraints 1-forms given by

$$\epsilon^u = du - (1 + \cos x)d\theta \qquad \text{and} \qquad \epsilon^v = dv - \sin x d\theta.$$

The symmetry is given by the action of the Lie group $G = \mathbb{R}^2 \times SE(2)$ defined, at each $(a, b; c, d, \beta) \in G$, by

$$\Psi((a, b; c, d, \beta), (u, v, x, y, z, \theta)) = (u + a, v + b, x, h_\beta \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}, \theta + \beta),$$

where h_β is the 2×2 rotational matrix of angle β. The distribution $S = D \cap V$ is generated by the G-invariant vector fields $\{Y_\theta, Y_1, Y_2\}$ given by

$$Y_\theta := \partial_\theta + (1 + \cos x)\partial_u + \sin x \partial_v,\ Y_1 := \cos\theta\partial_y + \sin\theta\partial_z,$$
$$Y_2 := -\sin\theta\partial_y + \cos\theta\partial_z,$$

and $X_0 = \partial_x$ generates $H = S^\perp \cap D$. It is straightforward to check that Conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ are satisfied and that $\kappa(X_0, [X_0, Y]) = 0$ for all $Y \in \Gamma(S)$. However, the metric is not strong invariant on S: $\kappa(Y_2, [Y_\theta, Y_1]) = 1$ and $\kappa(Y_\theta, [Y_1, Y_2]) = 0$. From (3.22), we can observe that $\mathcal{J} = 2p_1 + p_\theta$ is the only horizontal gauge momentum of the system in spite of the rank of S being 3 (where, as usual, $p_1 = \mathbf{i}_{Y_1}\Theta_{\mathcal{M}}$ and $p_\theta = \mathbf{i}_{Y_\theta}\Theta_{\mathcal{M}}$).

Dropping condition $\kappa(X_0, [X_0, Y]) = 0$. We illustrate with a multidimensional nonholonomic particle the different scenarios obtained when $\kappa(X_0, [X_0, Y]) \neq 0$ for a section $Y \in \Gamma(S)$ (see Table 5.4).

Consider the nonholonomic system on $\mathbb{R}^5$ with Lagrangian $L(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot \kappa\, \dot{q} - V(x_1)$, where κ is the kinetic energy metric

$$\kappa = \begin{pmatrix} 1&0&1&0&1 \\ 0&1&0&0&0 \\ 1&0&1&0&0 \\ 0&0&0&1&1 \\ 1&0&0&1&1 \end{pmatrix},$$

and with the nonintegrable distribution D given, at each $q = (x_1, \ldots, x_5) \in \mathbb{R}^5$, by

$$D_q = \text{span}\{D_1 = f(x_1)\,\partial_{x_1} + b(x_1)\,\partial_{x_3} + c(x_1)\,\partial_{x_4}, D_2 = h(x_1)\,\partial_{x_1} + g(x_1)\,\partial_{x_2},$$
$$D_3 = d(x_1)\,\partial_{x_1} + j(x_1)\,\partial_{x_4} + l(x_1)\,\partial_{x_5}\},$$

where $b(x_1), c(x_1), d(x_1), f(x_1), g(x_1), h(x_1), j(x_1), l(x_1)$ are functions on $\mathbb{R}^5$ depending only on the coordinate x_1. The group $\mathbb{R}^4$ of translations along the x_2, x_3, x_4 and x_5 directions acts on the system and leaves both the Lagrangian and the nonholonomic constraints invariant. It is straightforward to see that this G-symmetry satisfies Conditions ($\mathcal{A}1$)-($\mathcal{A}4$). The fiber of the distribution S over $q \in Q$ is $S_q = \text{span}\{Y_1 := f(x_1)D_2 - h(x_1)D_1, Y_2 := h(x_1)D_3 - d(x_1)D_2\}$. Since the translational Lie group $\mathbb{R}^4$ is abelian then the kinetic energy is strong invariant on V (see Example 3.11). The distribution $H = S^\perp \cap D$ is generated by the vector field $X_0 = \beta_1(x_1)\, D_1 + \beta_2(x_1)\, D_2 + \beta_3(x_1)\, D_3$, for β_1, β_2 and β_3 suitable functions (defined on $\mathbb{R}^5$ but depending only on the coordinate x_1).

For particular choices of the functions $b(x_1), c(x_1), d(x_1), f(x_1), g(x_1), h(x_1), j(x_1), l(x_1)$ the two terms $\kappa(X_0, [Y_1, X_0])$ and $\kappa(X_0, [Y_2, X_0])$ may not vanish. The computations and their expression are rather long and were implemented with Mathematica. The next table shows different situations that we obtain:

multidimensional nonholonomic particle (rank$(S) = 2$)	
behaviour of $\kappa(X_0, [X_0, Y])$	**♯ horizontal gauge momenta**
$\kappa(X_0, [Y_1, X_0]) = 0$ and $\kappa(X_0, [Y_2, X_0]) \neq 0$	0
$\kappa(X_0, [Y_1, X_0]) = 0$ and $\kappa(X_0, [Y_2, X_0]) \neq 0$	1
$\kappa(X_0, [Y_1, X_0]) \neq 0$ and $\kappa(X_0, [Y_2, X_0]) \neq 0$	0

Cases when Condition ($\mathcal{A}4$) is not satisfied (or rank$(H) \neq 1$) When Condition ($\mathcal{A}4$) is not verified, or more precisely when rank$(H) > 1$, it is still possible to work with the *momentum equation* stated in Proposition 3.3. The problem that appears when we want to apply the coordinate momentum equation (3.22) is that we cannot assert the existence of a global basis of H. However, in some examples the horizontal space H may admit a global basis which we denoted by $\{X_1, \ldots, X_n\}$ for $n = \text{rank}(H)$. In this case, we observe that the second summand of the momentum equation (3.22) gives the condition

$$\kappa(X_\alpha, [Y_i, X_\beta]) - \kappa(X_\beta, [Y_i, X_\alpha]) = 0 \quad \text{for all } \alpha, \beta = 1, \ldots, n,$$

and the third summand gives a system of *partial* differential equations whose solutions induce the horizontal gauge momenta. As an illustrative example, we can work out the Chaplygin ball [19,24]; this example has a G-symmetry so that rank$(S) = 1$ and rank$(H) = 2$ with a global basis (see e.g. [4,36]). However, working with the momentum equation (3.17), it is possible to show that the system admits 1 horizontal gauge momentum, recovering the known result in [16,19,24].

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A Appendix: Almost Poisson brackets and gauge transformations

Almost Poisson brackets. An *almost Poisson bracket* on a manifold M is a bilinear bracket $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ that is skew-symmetric and satisfies Leibniz identity (but does not necessarily satisfy Jacobi identity). Due to the bilinear property, an almost Poisson bracket induces a bivector field π on M defined, for each $f, g \in C^\infty(M)$ by

$$\pi(df, dg) = \{f, g\}.$$

The vector field $X_f := \{\cdot, f\}$ is the *hamiltonian vector field of* f. Equivalently, $X_f = -\pi^\sharp(df)$, where $\pi^\sharp : T^*M \to TM$ is the map such that for $\alpha, \beta \in T^*M$, $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$. The *characteristic distribution* of the bracket $\{\cdot,\cdot\}$ is the distribution on M generated by the hamitonian vector fields.
An almost Poisson bracket $\{\cdot,\cdot\}$ is *Poisson* when the Jacobi identity is satisfied, i.e.,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \qquad \text{for } f, g, h \in C^\infty(M).$$

Equivalently, a bivector field π is Poisson if and only if $[\pi, \pi] = 0$ where $[\cdot,\cdot]$ is the Schouten bracket, see e.g. [47]. The characteristic distribution of a Poisson bracket is integrable and foliated by symplectic leaves.

Definition A.1. [53] An almost Poisson bracket $\{\cdot,\cdot\}$ on M is *twisted Poisson* if there exists a closed 3-form Φ on M such that, for each $f, g, h \in C^\infty(M)$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \Phi(X_f, X_g, X_h),$$

where X_f, X_g, X_h are the hamiltonian vector fields of f, g, h, with respect to $\{\cdot,\cdot\}$. In other words, a bivector field π on M is twisted Poisson if $[\pi, \pi] = \frac{1}{2}\pi^\sharp(\Phi)$.

Observation A.2. The characteristic distribution of a twisted Poisson bracket is integrable and it is foliated by almost symplectic leaves. Conversely, it was shown in [6], that any regular almost Poisson bracket with integrable characteristic distribution is a twisted Poisson bracket. □

A regular almost Poisson bracket $\{\cdot,\cdot\}$ on M is determined by a 2-form Ω and a distribution F defined on M so that $\Omega|_F$ is nondegenerate. In fact, for $f \in C^\infty(M)$,

$$X_f = \{\cdot, f\} \quad \text{if and only if} \quad \mathbf{i}_{X_f}\Omega|_F = df|_F, \tag{A.50}$$

(actually, the bracket is determined by the nondegenerate 2-section $\Omega|_F$ on M). The distribution F is the characteristic distribution of the bracket. If F is integrable, then $\{\cdot,\cdot\}$ is a (regular) twisted Poisson bracket by the 3-form $\Phi = d\Omega$ (Ω is not necessarily closed). A Poisson bracket has F integrable and Ω closed.

Gauge transformations of a (regular) bracket by a 2-form.

Definition A.3. [53] Consider a (regular) bracket $\{\cdot,\cdot\}$ on the manifold M as in (A.50) and a 2-form B satisfying that $(\Omega + B)|_F$ is nondegenerate. A *gauge transformation of* $\{\cdot,\cdot\}$ *by the 2-form* B defines a bracket $\{\cdot,\cdot\}_B$ on M given, at each $f \in C^\infty(M)$, by

$$\mathbf{i}_{X_f}(\Omega + B)|_F = df|_F \quad \text{if and only if} \quad X_f = \{\cdot, f\}_B.$$

In this case, we say that the brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_B$ are *gauge related* .

Observations A.4. (i) The brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_B$ have the same characteristic distribution F. Therefore, if an almost Poisson bracket has a nonintegrable characteristic distribution, all gauge related brackets will be almost Poisson with a nonintegrable characteristic distribution.
(ii) If the bracket $\{\cdot,\cdot\}$ is twisted Poisson by a 3-form Φ, then the gauge related bracket $\{\cdot,\cdot\}_B$ is twisted Poisson by the 3-form $(\Phi + dB)$. Moreover, they share the characteristic foliation but the 2-form on each leaf F_μ changes by the term $B_\mu = \iota_\mu B$ for $\iota_\mu : F_\mu \to M$ the inclusion.
(iii) The original definition of a *gauge transformation* in [53] was given on Dirac structures and then the 2-form B does not need to satisfy the nondegenerate condition $(\Omega + B)|_F$. □

Definition A.5. Let $\tau : M \to P$ be a vector bundle and let α be a k-form on the manifold M. We say that α is *semi-basic* with respect to the bundle $M \to P$ if

$$\mathbf{i}_X \alpha = 0 \quad \text{for all } X \in TM \text{ such that } T\tau(X) = 0.$$

The k-form α is *basic* if there exists a k-form $\bar{\alpha}$ on P such that $\tau^*\bar{\alpha} = \alpha$.

Observation A.6. Consider the canonical symplectic 2-form Ω_Q on T^*Q. If B is a semi-basic 2-form with respect to the bundle $T^*Q \to Q$, then $\Omega_Q + B$ is a nondegenerate 2-form on T^*Q. □

Symmetries. Let us consider an almost Poisson manifold $(M, \{\cdot,\cdot\})$ given as in (A.50) and a Lie group G acting on M and leaving $\{\cdot,\cdot\}$ invariant. Then on the reduced manifold M/G there is an almost Poisson bracket $\{\cdot,\cdot\}_{\text{red}}$ defined, at each $f, g \in C^\infty(M/G)$ by

$$\{f, g\}_{\text{red}} \circ \rho = \{\rho^* f, \rho^* g\},$$

where $\rho : M \to M/G$ is the orbit projection.
If a G-invariant 2-form B satisfies that $(\Omega + B)|_F$ is nondegenerate, then the gauge related bracket $\{\cdot,\cdot\}_B$ is G-invariant as well. Both brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_B$ can be reduced to obtain the corresponding reduced brackets $\{\cdot,\cdot\}_{\text{red}}$ and $\{\cdot,\cdot\}^B_{\text{red}}$ on the quotient manifold M/G as the diagram shows that

$$\begin{array}{ccc} (M, \{\cdot,\cdot\}) & \xrightarrow{\text{gauge transf. by} B} & (M, \{\cdot,\cdot\}_B) \\ {\scriptstyle\text{reduction}}\downarrow & & \downarrow \\ (M/G, \{\cdot,\cdot\}_{\text{red}}) & & (M/G, \{\cdot,\cdot\}^B_{\text{red}}) \end{array} \tag{A.51}$$

As was observed in [6,36], the brackets $\{\cdot,\cdot\}_{\text{red}}$ and $\{\cdot,\cdot\}^B_{\text{red}}$ can have different properties. More precisely, they are not necessarily gauge related and hence one can be Poisson while the other not. In fact, $\{\cdot,\cdot\}_{\text{red}}$ and $\{\cdot,\cdot\}^B_{\text{red}}$ are gauge related if and only if the 2-form B is basic with respect to the principal bundle $M \to M/G$.

B Appendix: Some facts on reconstruction theory

The reconstruction of the dynamics from reduced equilibria and reduced periodic orbits has been well studied in [35,43], when the symmetry group is compact and in [2] in the non-compact case. In this subsection we shortly review the basic results of reconstruction theory in the simplest framework, of free and proper group actions. We consider a Lie group G that acts freely and properly on a manifold M. The freeness and properness of the action guarantee that the quotient space M/G has the structure of a manifold and $\tau : M \longrightarrow M/G$ is a principal bundle with structural group G. Let X be a G-invariant vector field on M, then there exists a vector field $\bar{X}$ on M/G, which is τ-related to X. We recall that a G-orbit $\mathcal{O}_{m_0} = G \cdot m_0$, with $m_0 \in M$, is a *relative equilibrium* for X, if it is invariant with respect to the flow of X and its projection to the reduced space M/G is an equilibrium of the reduced dynamics $\bar{X}$. Moreover a G-invariant subset $\mathcal{P}$ of M is called a *relative periodic orbit* for X, if it is invariant by the flow and its projection to the quotient manifold M/G is a periodic orbit of $\bar{X}$.
Let $\mathcal{P}$ be a relative periodic orbit and γ a curve in $\mathcal{P}$. By the periodicity of the reduced dynamics, the integral curves of the complete system, that pass through $\gamma(0)$, returns periodically, with period $T > 0$, to the G-orbit through $\gamma(0)$. The freeness of the action of G on M guarantees that $\forall \gamma$ in $\mathcal{P}$ there exists a unique $p(\hat{\gamma})$ in G such that

$$\phi_T^X(\gamma) = \psi_{p(\hat{\gamma})}(\gamma),$$

where ϕ_T^X is the flow of X at time T, ψ_g is the action of G on M, $\hat{\gamma}$ is the projection of γ on M/G with respect to τ, and the map $p : \mathcal{P} \to G$, $\gamma \mapsto p = p(\hat{\gamma})$ is the so-called *phase* [26]. The phase p is a piecewise smooth map, constant along the orbits of X (i.e. $p \circ \phi_t^X = p$, $\forall t$) and it is equivariant with respect to conjugation, that is $\quad p(h \cdot \gamma) = h\, p(\hat{\gamma}) h^{-1}, \quad \forall h \in G, \forall \gamma \in \mathcal{P}$. Then the following Theorem holds:

Proposition B.1. [2,35,43] *Let $\mathcal{P}$ be a relative periodic orbit of X on M. Then*

(i) if the group G is compact, the flow of X in $\mathcal{P}$ is quasi-periodic with at most $rank\, G + 1$ frequencies;
(ii) if G is non-compact, the flow of X in $\mathcal{P}$ is either quasi-periodic, or a drift.

The non-compact case is the most frequent and also the most interesting, for example one could say more on which of the two behaviours of the dynamics, namely quasi-periodicity or a drift, is "generic" by studying the group G (but this goes beyond our scopes, for more details see [2]).

Observation B.2. In [2,35,43] reconstructions results are given from the point of view of Lie Algebras, while [33] develops a theory in terms of groups. Moreover [33] investigates the structure of the copies of $\mathbb{R}$ and shows that one can define an intrinsic notion of a certain number of frequencies that gives rise to the idea that, in this case, the reconstructed dynamics 'spirals' toward a certain direction. □

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