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On the set of robust sustainable thresholds

Pedro Gajardo[1] | **Cristopher Hermosilla**[1] | **Athena Picarelli**[2]

[1]Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile

[2]Dipartimento di Scienze Economiche, Università di Verona, Verona, Italy

Correspondence
Athena Picarelli, Dipartimento di Scienze Economiche, Università di Verona, Polo Santa Marta, Via Cantarane 24, Verona 37129, Italy.
Email: athena.picarelli@univr.it

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Abstract

In natural resource management, or more generally in the study of sustainability issues, the objective often consists of maintaining the state of a given system within a desirable configuration, typically established in terms of standards or thresholds. For instance, in fisheries management, the procedure for designing policies may include maintaining the spawning stock biomass over a precautionary threshold and ensuring minimal catches. With the evolution of some natural resources, under the action of controls and uncertainties, being represented by a dynamical system in discrete time, the aim of this paper is to characterize the set of robust sustainable thresholds. That is, the thresholds for which there exists a trajectory satisfying, for all possible uncertainty scenarios, prescribed constraints parametrized by such thresholds. This set provides useful information to users and decision-makers, illustrating the tradeoffs between constraints. Using optimal control, maximin and level-set approaches, we characterize the weak Pareto front of the set of robust sustainable thresholds and derive a numerical method for computing the entire set, as we show with a numerical example relying on renewable resource management.

Recommandation for Resource Managers

- For biological, ecological or social systems, identifying robust sustainable thresholds that it is possible

not to exceed over time regardless of uncertain future scenarios, is a fundamental challenge to operate in a sustainable way.

- The set of robust sustainable thresholds provides useful information to users and decision-makers, illustrating the tradeoffs between constraints.
- The computation of the weak-Pareto front of the set of sustainable thresholds provides a tool for the management and visualization of multiple objectives related to sustainability that can be accomplished in a robust manner, allowing to observe the (in)compatibility of multiple objectives and their tradeoffs.

KEYWORDS

discrete-time systems, dynamic programming, level-set approach, mixed constraints, robust viability, set of robust sustainable thresholds, viability theory

1 | INTRODUCTION

In natural resource management or broadly in the study of sustainability issues, determining biological, ecological or social constraints to satisfy over time emerges as a crucial issue. Mathematically speaking, one of the objectives of decision-makers can be seen as maintaining the state of a given system within a desirable configuration, typically established in terms of constraints parametrized by standards or thresholds. For instance, in fisheries management, the procedure for designing policies may include maintaining the spawning stock biomass over a precautionary threshold and ensuring minimal catches. In this example, the first requirement is associated with the sustainability of the resource and the second with economic, social, or food security issues. The focus on constraints is well adapted to address biodiversity conservation problems or climate change issues. In this framework, reference points not to exceed for biological, ecological, economic, or social indicators represent sustainable management objectives. As examples of this approach, consider the concept of safe minimum standards (SMS) (Margolis & Naevdal, 2008), where tipping thresholds and risky areas are introduced, or the tolerable windows approach (TWA) (Bruckner et al., 1999), based on safe boundaries and feasibility regions. If the constraints induced by thresholds or tipping points have to be satisfied over time, such problems related to sustainability can be formulated into the mathematical framework of viability theory (Aubin, 1990; Aubin et al., 2011; De Lara & Doyen, 2008). Indeed, this approach has been applied by numerous authors to the sustainable management of renewable resources (Bates et al., 2018; Béné & Doyen, 2000; Béné et al., 2001; Doyen et al., 2017; Durand et al., 2017; Krawczyk et al., 2013; Péreau et al., 2012; Pereau et al., 2018), as recently reviewed in Oubraham and Zaccour (2018). Related to the sustainability of fisheries, in Quinn and Collie (2005) (see also Hilborn & Walters, 1992), the authors present a review of modeling approaches focused on single-species population models, remarking that additional work is needed to make definitions of sustainability operational to specify achievable quantitative objectives. In addition, they note that multiple objectives may be incompatible, so tradeoffs in what constitutes sustainability must be made. This challenge also fits very well into the viability theory framework, as presented in Schuhbauer and Sumaila (2016), where the authors provide a review of the application of viability theory to the sustainable management of small-scale fisheries.

A key concept in viability theory and in its application to sustainability issues (see Oubraham & Zaccour, 2018) is the so-called *viability kernel*, consisting of the set of initial states of the system, for instance, the initial endowment of natural resources, from which it is possible to satisfy prescribed constraints over time. Unfortunately, the computation of the viability kernel is not an easy task, and this set may comprise states that are not reachable from the current endowment of natural resources, or more generally from the current state of the system, which makes its complete calculation useless. Taking into account some of the challenges mentioned above, such as developing tools to manage and visualize multiple objectives and their tradeoffs related to sustainability, a different strategy has been proposed from that focused on the viability kernel: to characterize the *set of sustainable thresholds or standards* that can mathematically be regarded as the inverse mapping of the viability kernel. In the context of deterministic systems, this set has been studied recently in Barrios et al. (2018), Gajardo et al. (2018), Martinet (2011), Martinet et al. (2011) and characterized in Doyen and Gajardo (2020) and Gajardo and Hermosilla (2021).

We assume that the evolution of some natural resources, under the action of controls (decisions) and uncertainties, is represented by a dynamical system in discrete time, and given an initial endowment of the resources, our aim is to characterize the *set of robust sustainable thresholds*. This set is composed of the collection of all possible thresholds for which there exists a control strategy (sequence of decisions), along with its corresponding state trajectory, satisfying for all possible uncertainty scenarios prescribed mixed constraints parametrized by such thresholds. In Gajardo and Hermosilla (2021), the starting point of the current work, a characterization of the strong and weak Pareto front for this set is provided. The present paper therefore extends previous results related to the set of sustainable thresholds to uncertain control systems, with a focus on robustness, providing a new tool for the management and visualization of multiple objectives related to sustainability problems, as reviewed in Oubraham and Zaccour (2018), that can be accomplished in a robust manner.

The consideration of uncertain control systems is motivated by practical applications, where limited knowledge about the phenomena that influence the system evolution and the role of uncertainty and its quantification becomes particularly relevant. In environmental management problems, uncertainty typically affects the model as a result of environmental changes that influence natural mechanisms (see, for instance, Lande et al., 2003; Olson & Santanu, 2000). Moreover, uncertainty can also be used to reflect the possibility of measurement errors. In the presence of uncertainty, constraints can be considered in different ways. Typical examples are constraints imposed in probability, expectation and sure (or almost sure) pathwise constraints. In this paper, we consider the latter constraints: given a set of scenarios reflecting the possible future states of the world, the set of robust sustainable thresholds defines the collection of thresholds that are sustainable under any scenario. Accordingly, the set of robust sustainable thresholds provides a good picture of the current state of a system in terms of its guaranteed sustainability under any possible occurrence (Freeman & Kokotović, 1996). In particular, for a given initial state or initial endowment of resources, a small set of robust sustainable thresholds means that there is a limited possibility to operate in a sustainable way. We note that the study of robustness in control theory arises in several frameworks. Among the main motivations for this study, we mention sustainable management problems, bioeconomic modeling, and robust viability (see, for instance, Doyen & Béné, 2003; Doyen et al., 2007; Doyen

& Pereau, 2009; Sepulveda & Lara, 2018; Tichit et al., 2004). Another motivation is to provide quantitative tools for socioeconomic resilience management, in the terms proposed in Grafton et al. (2019), where the authors define social-ecological resilience, with robustness being one of its main characteristics.

In this paper, we first obtain a characterization of the weak Pareto front of the set of robust sustainable thresholds, and then we use such characterization to provide a numerical method for the approximation of the entire set. To achieve these goals, we make use of optimal control tools. In particular, we prove that the weak Pareto front corresponds to the zero-level set of the value function associated with a suitable unconstrained maximin optimal control problem. We then use this characterization and the dynamic programming principle to provide an implementable scheme for approximating the set of robust sustainable thresholds through its weak Pareto front.

We remark that in characterizing the weak Pareto front, we are inspired by the so-called *level-set approach*. Introduced in Osher and Sethian (1988) to describe the propagation of fronts in continuous time, the foundational idea of this approach is to link the set of interest (the set of robust sustainable thresholds in our case) to the level set of a suitable auxiliary function that can be numerically approximated. In the deterministic continuous-time framework, this technique has been successfully applied in Altarovici et al. (2013) and Mitchell et al. (2005) to characterize the set of admissible initial conditions in the presence of (pure) state constraints. The approach was subsequently extended to the stochastic case in Bokanowski et al. (2015), where almost sure pathwise state constraints are taken into account. The use of this technique leads us to work with maximin problems such as those considered in De Doná and Lévine (2013) and Esterhuizen et al. (2020) to describe the boundaries of admissible sets for continuous-time systems in the deterministic and uncontrolled robust frameworks, respectively. However, we stress that in all the aforementioned works, the focus is on the characterization and approximation of the set of sustainable initial conditions for a given threshold, that is, the viability kernel, while we are interested in determining the set of thresholds that are sustainable once the initial condition is fixed.

This manuscript is organized as follows. In Section 2, we present some preliminary concepts on discrete-time systems under constraints introducing the set of robust sustainable thresholds. In this section, we establish the standing assumptions for the rest of the paper. The links between appropriate optimal control problems and the set of robust sustainable thresholds are established in Section 3. Specifically, in Section 3.1, we characterize the weak Pareto front of the set of robust sustainable thresholds, providing a method for computing this front based on the dynamic programming principle. Finally, in Section 4, we illustrate the method introduced in Section 3.1 with an example based on renewable resource management, and in Section 5, we end with some concluding remarks mentioning possible further extensions. Mathematical proofs are relegated to the Appendix.

2 | PRELIMINARIES ON DISCRETE-TIME CONTROL SYSTEMS

Given a finite time horizon $N \in \mathbb{N} \setminus \{0\}$, an initial state $\xi \in \mathbb{R}^d$, for instance, an initial endowment of natural resources, a finite sequence of controls or decisions $\mathbf{u} = (u_k)_{k=0}^N$ and a scenario $\mathbf{w} = (\omega_k)_{k=0}^N$, we consider the uncertain discrete-time control system:

$$x_{k+1} = F(x_k, u_k, \omega_k), k \in [[0 : N]], x_0 = \xi. \quad \left(D_\xi^{\mathbf{u}}(\mathbf{w})\right)$$

The data for the problem include the dynamics $F : \mathbb{R}^d \times \mathbf{U} \times \Omega \to \mathbb{R}^d$, the control space $\mathbf{U}$ and the scenarios' space Ω. Here, we denote by $[r : s]$ the collection of all integers (periods of time) between r and s (inclusive).

The set of possible controls is given by:

$$\mathbb{U} := \left\{ \mathbf{u} = (u_k)_{k=0}^{N} \middle| \ u_0, \ldots, u_{N-1} \in \mathbf{U} \right\} \cong \mathbf{U}^{N+1}.$$

The possible scenarios set Ω is assumed to be constant over time. Consequently, the collection of all possible scenarios is then given by

$$\mathbb{W} := \left\{ \mathbf{w} = (\omega_k)_{k=0}^{N} \middle| \ \omega_k \in \Omega, \forall\ k \in [0 : N] \right\} \cong \Omega^{N+1}.$$

A solution of the uncertain control system ($D_\xi^{\mathbf{u}}(\mathbf{w})$) associated with a control $\mathbf{u} \in \mathbb{U}$ and a scenario $\mathbf{w} \in \mathbb{W}$ is an element of the space

$$\mathbb{X} := \left\{ \mathbf{x} = (x_k)_{k=0}^{N+1} \middle| \ x_0, \ldots, x_{N+1} \in \mathbb{R}^d \right\} \cong \mathbb{R}^{d(N+1)},$$

that satisfies the initial time condition $x_0 = \xi$.

To emphasize its dependence on the initial data of the problem (control, scenario, and state), a solution of ($D_\xi^{\mathbf{u}}(\mathbf{w})$), uniquely determined by the control $\mathbf{u}$, a scenario $\mathbf{w}$ and the initial state ξ, is denoted in the sequel by $\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u})$.

2.1 | Constraints and sustainable thresholds

In many practical applications, the outputs and inputs of dynamical systems such as ($D_\xi^{\mathbf{u}}(\mathbf{w})$) are restricted to prescribed sets, which may reflect biological, physical, economic or social constraints. The uncertain control system considered in this study allows us to consider different probabilistic interpretations of how these constraints are satisfied. In this study, we are mainly interested in a robust approach, which means that the set of restrictions considered must be satisfied by the control sequence $\mathbf{u} \in \mathbb{U}$ together with its corresponding trajectories $\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) = (x_k)_{k=0}^{N+1}$ for any possible scenario $\mathbf{w} \in \mathbb{W}$. To be more precise, we consider the so-called *mixed constraints* that can be represented as the level set of a given constraint mapping $g : \mathbb{R}^d \times \mathbf{U} \to \mathbb{R}^m$

$$g(x_k, u_k) \geq c, \forall \quad k \in [0 : N]. \qquad (\mathrm{I}^c)$$

Here, the parameter $c \in \mathbb{R}^m$ represents a vector of thresholds and determines all the constraints of the problem. In contrast to viability theory (see, for instance, Aubin, 1991), the aim of which is to study admissible initial conditions for a prescribed set of thresholds, the focus of our work is on the characterization of the parameter c once the initial condition is fixed. In particular, for a given initial state, we are interested in finding all the thresholds $c \in \mathbb{R}^m$ for which that initial condition can be robustly sustainable over time. This means that there exists some control that, along with its associated trajectories, satisfies the mixed constraints (I^c) for any possible scenario. The set of all such thresholds is referred to as the *set of robust sustainable thresholds* and, for a given initial condition $\xi \in \mathbb{R}^d$, is defined as follows:

$$\mathbb{S}(\xi) := \left\{ c \in \mathbb{R}^m | \exists\ \mathbf{u} \in \mathbb{U},\ \mathbf{u} \quad \text{and} \quad \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) \quad \text{satisfy} \quad (\mathrm{I}^c), \qquad \text{for any} \quad \mathbf{w} \in \mathbb{W} \right\}. \tag{1}$$

For a given threshold vector $c \in \mathbb{R}^m$, the robust viability kernel (De Lara & Doyen, 2008) associated with the uncertain control system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$ is given by

$$\mathbb{V}(c) := \left\{ \xi \in \mathbb{R}^d | \exists\ \mathbf{u} \in \mathbb{U}, \mathbf{u} \quad \text{and} \quad \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) \quad \text{satisfy} \quad (\mathrm{I}^c) \qquad \text{for any} \quad \mathbf{w} \in \mathbb{W} \right\}.$$

Similar to Doyen and Gajardo (2020) and Gajardo and Hermosilla (2021) for the deterministic case, there is a duality between the robust viability kernel and the set of robust sustainable thresholds. This relation is expressed by the following equivalence: for any $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}^m$, we have

$$\xi \in \mathbb{V}(c) \quad \Leftrightarrow \quad c \in \mathbb{S}(\xi). \tag{2}$$

As in the deterministic case, the relevance of the set of robust sustainable thresholds stays in the tradeoff between the dimension of the state space $\mathbb{R}^d$ and the number of restrictions $m \in \mathbb{N}$. When several state variables are involved, the numerical approximation of the set $\mathbb{V}(c)$ may be too expensive or computationally impractical, even if only a few restrictions are taken into account; this phenomenon is known in the literature as the *curse of dimensionality in dynamic programming*. However, in the same situation (several state variables with few constraints), the computational time required to estimate $\mathbb{S}(\xi)$ can be considerably lower. Indeed, roughly speaking, the complexity of computing $\mathbb{V}(c)$ and $\mathbb{S}(\xi)$ is the same, but the latter belongs to a lower-dimensional Euclidean space, which makes the numerical computation of $\mathbb{S}(\xi)$ somewhat more tractable than that of $\mathbb{V}(c)$, as we describe in this study.

The set of robust sustainable thresholds $\mathbb{S}(\xi)$ is intended to provide a picture of the current state of the system ξ in terms of the thresholds that can be maintained in a sustainable way over time. For instance, having a small set $\mathbb{S}(\xi)$ indicates that the current state ξ is vulnerable in the sense that there are few options for operating on the system in terms of sustainability. Figure 1 shows the set of robust sustainable thresholds for two different initial states ξ and ξ' in the case in which the threshold space is of dimension two, that is, where (I^c) consists of only two constraints. In this picture, having $\mathbb{S}(\xi') \subset \mathbb{S}(\xi)$, one can infer that state ξ' is worse than ξ.

2.2 | Pareto front

Due to the structure of constraint (I^c), one clearly has that if $c^* \geq c$ (componentwise), then for any $\xi \in \mathbb{R}^d$

$$c^* \in \mathbb{S}(\xi) \quad \Rightarrow \quad c \in \mathbb{S}(\xi). \tag{3}$$

In other words, $\mathbb{S}(\xi) + \mathbb{R}_-^m = \mathbb{S}(\xi)$, so that the set of robust sustainable thresholds is fully characterized by its boundary and, in particular, by its weak Pareto front. In this context, we

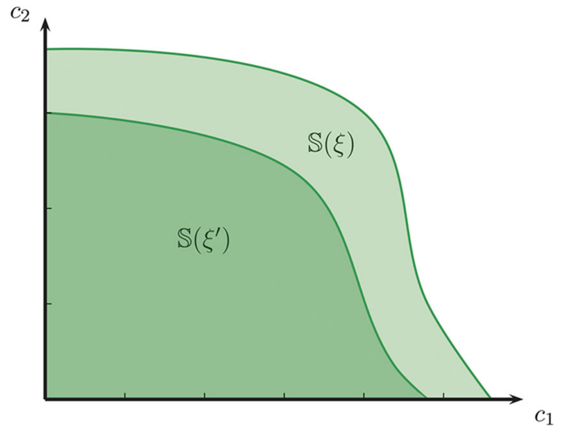

FIGURE 1 Sketch of the set of robust sustainable thresholds for two different given initial states ξ and ξ'

recall that a vector $c^* \in \mathbb{R}^m$ is said to be strongly (Pareto) dominated by c if $c > c^*$ (componentwise). Therefore, given a set $S \subset \mathbb{R}^m$, its weak Pareto front is the collection of all $c^* \in S$, which are not strongly dominated by another element of S:

$$\forall\, c \in S, \exists\, i \in [[1 : m]], c_i^* \geq c_i.$$

We call weak Pareto maxima the elements of the weak Pareto front.

The goal of this paper is to study, for a given initial condition $\xi \in \mathbb{R}^d$, the weak Pareto front of the set of robust sustainable thresholds $\mathbb{S}(\xi)$ obtaining, in this way, a full description of this set. Similar to Gajardo and Hermosilla (2021), to achieve this goal, we make use of optimal control theory, as explained in detail in Section 3.

2.3 | Standing assumptions

We assume, throughout the paper, that the data of the dynamical system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$ and the constraint (I^c) satisfy the following conditions, which we term *standing assumptions*:

(H1) $F(\cdot,\cdot,\omega)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $\omega \in \Omega$.
(H2) For each $i \in [0 : m]$, g_i is upper semicontinuous[1] and bounded below.
(H3) $\mathbf{U}$ is a nonempty compact subset of $\mathbb{R}^l$.

Under these assumptions, given an initial condition $\xi \in \mathbb{R}^d$, a scenario $\mathbf{w} \in \mathbb{W}$ and a threshold vector $c \in \mathbb{R}^m$, the set of feasible solutions to the dynamical system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$ with constraint (I^c) is compact in $\mathbb{X}$ (eventually empty), as shown in Appendix A.1. In our setting, compactness is important because it allows us to ensure the existence of optimal trajectories, which turns out to be a key point when proving the dynamic programming principle (Proposition 3.4).

Standing assumptions (H1) and (H3) are quite usual in the modeling of natural resource problems or, more generally, in the modeling of sustainability issues (De Lara & Doyen, 2008). Assumption (H1) refers to continuous changes of state in terms of the previous state and the control. The compactness of the control space $\mathbf{U}$ simply represents the limited set of decisions that decision makers usually have. Assumption (H2) is also quite natural if these constraints represent biological, ecological, economic, or social indicators that are nonnegative. A consequence of assumption (H2) is that the set of robust sustainable thresholds will never be empty. Indeed, if $\underline{c} = (\underline{c}_1, \ldots, \underline{c}_m)$ is a vector of lower bounds of constraints g_i, then immediately $\underline{c} \in \mathbb{S}(\xi)$ for any initial condition ξ. Of course, if $\mathbb{S}(\xi)$ is reduced to $\{\underline{c}\} + \mathbb{R}^m_-$, then we are in the presence of the worst scenario for operating in a sustainable way.

3 | AN OPTIMAL CONTROL APPROACH FOR STUDYING THE SET OF ROBUST SUSTAINABLE THRESHOLDS

In this part, we show that the set of robust sustainable thresholds can be studied by tools of optimal control and value functions. For this purpose, let us consider a generic maximin optimal control problem:

$$\vartheta_\xi(c) := \sup_{\mathbf{u} \in \mathbb{U}} \left\{ \inf_{\mathbf{w} \in \mathbb{W}} \mathcal{J}\left(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right) \middle| \; \mathbf{u} \text{ and } \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) \text{ satisfy } (\mathrm{I}^c), \; \forall \, \mathbf{w} \in \mathbb{W} \right\}. \tag{4}$$

Here, the function $\mathcal{J} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is an arbitrary payoff. Note that here, we see the optimal value of the problem as a function of the threshold vector.

The link between the set of robust sustainable thresholds and the optimal control problem described above is made explicit in the following statement.

Proposition 3.1. *Assume that $\mathcal{J} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is bounded below and upper semicontinuous. Then, for any $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}^m$, one has*

$$c \in \mathbb{S}(\xi) \quad \Leftrightarrow \quad \vartheta_\xi(c) \in \mathbb{R}.$$

Furthermore, in either of these two cases, there is an optimal control for the optimization problem related to $\vartheta_\xi(c)$.

Proposition 3.1 states that to describe the set of robust sustainable thresholds, one can first attempt to solve an optimal control problem to test the sustainability of a given threshold. A dynamic programming principle can be stated for the value function defined in (4), which in principle could be used for computing the set of robust sustainable thresholds and its Pareto front. From a practical point of view two major issues arise with it: (i) the value function $\vartheta_\xi(c)$ is likely to have infinite values (otherwise any threshold will be sustaintable) and (ii) the mixed constraints need to be verified at each step of computation, requiring additional computational effort. To overcome these issues, we take a different approach (called the *level-set approach*), which we describe next.

3.1 | The weak pareto front

In this section, we analyze the weak Pareto front of the set of robust sustainable thresholds, introducing a method for computing this front using optimal values of unconstrained optimal control problems. As we note in Section 2.2, from the weak Pareto front, we can obtain the entire set $\mathbb{S}(\xi)$. The method consists of identifying points in the front from outside the set, meaning that from a nonrobust sustainable threshold (given an initial state), we will be able to identify a vector of thresholds in the weak Pareto front (see Figure 2).

First, let us introduce the following unconstrained optimal control problem:

$$\mathcal{W}_\xi(c) = \max_{\mathbf{u}\in\mathbb{U}} \inf_{\mathbf{w}\in\mathbb{W}} \left\{ \min_{k=0,\ldots,N} \Phi^c(x_k, u_k) \,\middle|\, \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) = (x_k)_{k=0}^{N+1} \right\}, \tag{5}$$

where $c \in \mathbb{R}^m$ is a given threshold vector and $\Phi^c : \mathbb{R}^d \times \mathbf{U} \to \mathbb{R}$ is given by

$$\Phi^c(x, u) = \min_{i=1,\ldots,m} (g_i(x, u) - c_i).$$

Since the function to be maximized in (5) is upper semicontinuous (it is the infimum of upper semicontinuous functions that depend on each scenario $\mathbf{w}$) and $\mathbb{U}$ is a nonempty compact set, we can write the maximum instead of the supremum in (5) because there exists some optimal control $\mathbf{u} \in \mathbb{U}$.

Furthermore, for any initial state $\xi \in \mathbb{R}^d$, we have

$$c \in \mathbb{S}(\xi) \quad \Leftrightarrow \quad \mathcal{W}_\xi(c) \geq 0. \tag{6}$$

The above equivalence reveals the strong link between the level set of the value function $\mathcal{W}_\xi(c)$ and the set of robust sustainable thresholds $\mathbb{S}(\xi)$. Moreover, we shall see that there is a direct way to determine a point in the weak Pareto front of $\mathbb{S}(\xi)$ from any threshold $c \notin \mathbb{S}(\xi)$ (an *unsustainable threshold*) using the value function $\mathcal{W}_\xi(c)$. This situation is illustrated in Figure 2. If c is unsustainable, then $\mathcal{W}_\xi(c) < 0$, which is equivalent to stating that for any control $\mathbf{u} \in \mathbb{U}$, there is a scenario $\mathbf{w} \in \mathbb{W}$, an index $i \in [1 : m]$ and an instant $k \in [0 : N]$ such that $g_i(x_k, u_k) < c_i$, where $\mathbf{u} = (u_k)_{k=0}^N$ and $\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) = (x_k)_{k=0}^{N+1}$.

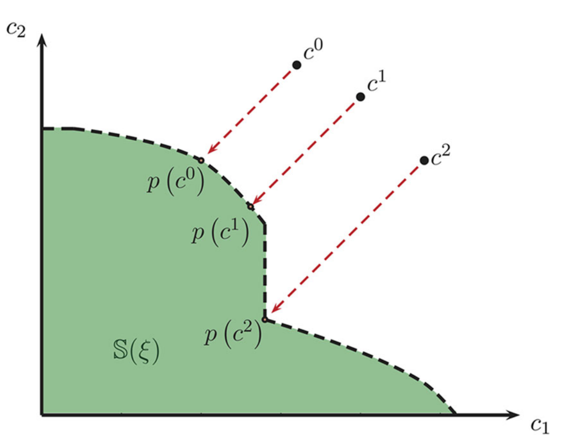

FIGURE 2 Illustration of Theorem 3.2 showing the construction of the weak Pareto front from unsustainable thresholds c^0, c^1, and c^2 when $m = 2$ (i.e., two constraints). The weak Pareto front of the example is depicted by the dashed-black line

Theorem 3.2. *For all $c^*, \bar{c} \in \mathbb{R}^m$, we have the following results:*

(1) *c^* belongs to the weak Pareto front of $\mathbb{S}(\xi)$ if and only if $\mathcal{W}_\xi(c^*) = 0$.*
(2) *If $\mathcal{W}_\xi(\bar{c}) < 0$, then*

$$p(\bar{c}) := \bar{c} + \mathcal{W}_\xi(\bar{c})\mathbf{1}$$

is a weak Pareto maximum of $\mathbb{S}(\xi)$, where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^m$.

Remark 3.3. Observe that the first statement of Theorem 3.2 characterizes the weak Pareto front of $\mathbb{S}(\xi)$. Moreover, we can see that if $\mathcal{W}_\xi(c) > 0$, then c is in the interior of $\mathbb{S}(\xi)$. This property is deduced from the proof of Theorem 3.2 (in Appendix A.3), noting that $c + \frac{1}{2}\mathcal{W}_\xi(c)\mathbf{1} \in \mathbb{S}(\xi)$ implies

$$c + \frac{1}{2}\mathcal{W}_\xi(c)[-1, 1]^m \subset \mathbb{S}(\xi).$$

3.1.1 | Dynamic programming principle

In this part, we use the dynamic programming principle for computing the optimal value $\mathcal{W}_\xi(c)$, thus proposing an implementable method for obtaining the weak Pareto front of the set of robust sustainable thresholds.

For any $n \in [\![0 : N]\!]$, we introduce the following value function:

$$V_n^c(\xi) := \max_{\mathbf{u}\in\mathbb{U}} \inf_{\mathbf{w}\in\mathbb{W}} \left\{ \min_{k=n,\ldots,N} \Phi^c(x_k, u_k) \,\middle|\, x_{k+1} = F(x_k, u_k, \omega_k),\ k \in [\![n : N]\!],\ x_n = \xi \right\}.$$

Note that, for any $\xi \in \mathbb{R}^d$, we have $V_0^c(\xi) = \mathcal{W}_\xi(c)$. Therefore, to compute $\mathcal{W}_\xi(c)$ and then to be able to use Theorem 3.2 to obtain weak Pareto maxima of the set of robust sustainable thresholds $\mathbb{S}(\xi)$, we will apply the dynamic programming principle for obtaining $V_0^c(\xi)$ from the value functions $V_1^c(\cdot), \ldots, V_N^c(\cdot)$.

The dynamic programming principle for maximin problems such as the one that determines the value functions $(V_n^c)_{n=0}^N$ is known (De Lara & Doyen, 2008). We provide a proof for the sake of completeness (see Appendix A.5). In this context, the dynamic programming principle is established as follows.

Proposition 3.4. *For any $n \in [\![0 : N-1]\!]$, $c \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^d$, we have*

$$V_n^c(\xi) = \max_{u\in\mathbf{U}} \min \left\{ \inf_{\omega\in\Omega} V_{n+1}^c(F(\xi, u, \omega)), \Phi^c(\xi, u) \right\}, \tag{7}$$

and $V_N^c(\xi) = \max_{u\in\mathbf{U}} \Phi^c(\xi, u)$.

3.1.2 | A scheme for computing the weak Pareto front

In summary, if we combine Theorem 3.2 and Proposition 3.4, we obtain a method (Algorithm 1) for computing the weak Pareto front of the set of robust sustainable thresholds.

To describe this method, let us define two meshes $\mathbf{X}_h \subset \mathbb{R}^d$ and $S_h \subseteq \mathbb{R}^m$ of size $0 < h \ll 1$ as computational domains[2] (state and thresholds). For each $n \in [\![0:N]\!]$, we compute the function $V_n^c(\cdot)$ in Proposition 3.4 for all $c \in S_h$ and every $\xi' \in \mathbf{X}_h$ reachable in n steps (from ξ). This procedure can be time- and space-consuming, as the method is based on the dynamic programming principle. However, from Theorem 3.2 and Proposition 3.4, the method introduced below in Algorithm 1 will not need a large mesh S_h, as we show in Section 4. In other words, the function $\mathcal{W}_\xi(c)$ does not need to be evaluated in a large set of thresholds c. Nevertheless, this procedure also suffers from the so-called *curse of dimensionality*.

Algorithm 1: Computing the weak Pareto front

```
Input: ξ ∈ ℝ^d, N ∈ ℕ, F : ℝ^d × U × Ω → ℝ^d, g : ℝ^d × U → ℝ^m
Let X_h ⊂ ℝ^d and S_h ⊆ ℝ^m be two meshes of size 0 < h ≪ 1 for
  computational domains (state and thresholds).
For n ∈ ⟦0:N⟧, let X_h^n ⊂ X_h be the set of points in X_h reachable from ξ in
  n steps.
Let 𝕊 and P_w be two empty arrays.
for c_i ∈ S_h do
    for ξ' ∈ X_h^N do
        Compute V_N^{c_i}(ξ') = max_{u∈U} Φ_N^{c_i}(ξ', u).
Set n = N − 1.
while n ≥ 0 do
    for c_i ∈ S_h do
        for ξ' ∈ X_h^n do
            Compute V_n^{c_i}(ξ') = max_{u∈U} min { min_{ω∈Ω} V_{n+1}^{c_i}(F(ξ', u, ω)), Φ_n^{c_i}(ξ', u) }.
    Set n = n − 1.
for c_i ∈ S_h do
    Save c_i + V_0^{c_i}(ξ)1 in P_w
return P_w and 𝕊 = P_w + ℝ^m_−
```

4 | SIMULATIONS

In this section, we illustrate the computation of the set of robust sustainable thresholds $\mathbb{S}(\xi)$ for one example based on renewable resource management, inspired by Clark (1990) and Doyen and Gajardo (2020). In the example, the stock of a renewable resource in period k is represented by $x_k \geq 0$, and its dynamics with harvesting (or catch) u_k are described by

$$x_{k+1} = F(x_k, u_k, \omega_k) = f(x_k, \omega_k) - u_k,$$

where f stands for the renewable function of the stock, depending on the scenario $\omega_k \in \Omega = \{\omega_a, \omega_b\}$, for all $k \in [\![0 : N]\!]$.

For the above control system, suppose that a social planner has the objective of ensuring both minimal resource stocks in nature and minimal harvesting. The first requirement is associated with the sustainability of the resource and the second with economic, social, or food security issues. Reformulated from a viability perspective, the problem relates to sustaining both stock and harvest through the thresholds $x^{\lim}$ and $h^{\lim}$ as follows:

$$\begin{cases} x_{k+1} = f(x_k, \omega_k) - u_k, \\ x_0 = \xi \text{ given (the current state of the resource)} \\ x_k \geq x^{\lim} \\ u_k \geq h^{\lim}. \end{cases} \tag{8}$$

We shall study this very simple example because when the dynamics $f(\cdot,\omega)$ are nondecreasing for every $\omega \in \Omega$, we can analytically compute the set of robust sustainable thresholds when the horizon is infinite, and then we are able to compare this analytical expression with the result given by our method for computing $\mathbb{S}(\xi)$ (Algorithm 1). A first interesting property we can show in this framework is presented in the following lemma.

Lemma 4.1. *Under the assumption that the function $f(\cdot,\omega)$ in (8) is nondecreasing, for all $\omega \in \Omega$, one has*

$$\mathbb{S}(\xi) = \hat{\mathbb{S}}(\xi) := \textstyle\bigcap_{\omega \in \Omega} \mathbb{S}^{\omega}(\xi),$$

where $\mathbb{S}^{\omega}(\xi)$ is the set of sustainable thresholds for the (deterministic) system

$$\begin{cases} x_{k+1} = f(x_k, \omega) - u_k, \\ x_0 = \xi \\ x_k \geq x^{\lim} \\ u_k \geq h^{\lim} \end{cases} \tag{9}$$

associated with the constant scenario $\mathbf{w}_{\omega} = (\omega_k)_{k=0}^{N}$ such that $\omega_k = \omega$ for all $k \in [0 : N]$.

Remark 4.2. Note that, in a general setting, if $\omega \in \Omega$ and we consider the constant scenario $\mathbf{w}_{\omega} = (\omega_k)_{k=0}^{N}$, where $\omega_k = \omega$ for $k = 0, \ldots, N$, we can define the set of sustainable thresholds associated with ω by

$$\mathbb{S}^{\omega}(\xi) := \left\{ c \in \mathbb{R}^m | \exists \ \mathbf{u} \in \mathbb{U}, \ \mathbf{u} \ \text{ and } \ \mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}) \ \text{ satisfy } \ (\mathrm{I}^c) \ \text{ for } \ \mathbf{w} = \mathbf{w}_{\omega} \right\}.$$

This set corresponds to the deterministic set of sustainable thresholds associated with the dynamics $F(\cdot, \cdot, \omega)$ in $(D_{\xi}^{\mathbf{u}}(\mathbf{w}))$ defined in Doyen and Gajardo (2020) and Gajardo and Hermosilla (2021). From the definition of the set of robust sustainable thresholds $\mathbb{S}(\xi)$ in (1), it is straightforward to verify that

$$\mathbb{S}(\xi) \subseteq \hat{\mathbb{S}}(\xi) = \bigcap_{\omega \in \Omega} \mathbb{S}^{\omega}(\xi).$$

Lemma 4.1 links the characterization of the set of robust sustainable thresholds $\mathbb{S}(\xi)$ to one of its deterministic versions $\{\mathbb{S}^{\omega}(\xi), \omega \in \Omega\}$ obtained for constant scenarios. Interestingly, as already noted in Doyen and Gajardo (2020) and Gajardo and Hermosilla (2021), for infinite horizon problems, that is, if $N = \infty$, the monotonicity of f allows for an analytical computation of $\mathbb{S}_{\infty}^{\omega}$ and then, by virtue of Lemma 4.1, of $\mathbb{S}_{\infty}(\xi)$.

In particular, let us consider the Beverton–Holt population dynamics

$$f(x, \omega) = (1 + r(\omega))x\left(1 + \frac{r(\omega)}{K(\omega)}x\right)^{-1} \tag{10}$$

where the intrinsic growth $r(w) \in \{r(\omega_a), r(\omega_b)\}$ and carrying capacity $K(\omega) \in \{K(\omega_a), K(\omega_b)\}$, are positive parameters depending on the scenario $\omega \in \Omega$. For this Beverton-Holt growth function (10) and a fixed scenario ω, the maximal sustainable yield (MSY) level is a tipping point in the determination of the sustainable thresholds of the deterministic system (9), and it is attained at the biomass level $x_{\mathrm{MSY}}^{\omega}$ given by

$$x_{\mathrm{MSY}}^{\omega} = \frac{K(\omega)}{1 + \sqrt{1 + r(\omega)}}.$$

When the horizon is infinite ($N = +\infty$), the viability kernel (associated with 9 for scenario ω fixed) has been calculated analytically in De Lara and Doyen (2008), and as shown in Doyen and Gajardo (2020) and Gajardo and Hermosilla (2021), the (deterministic) set of sustainable thresholds associated with (9) is given by

$$\mathbb{S}_{\infty}^{\omega}(\xi) = \{(x^{\lim}, h^{\lim}) | x^{\lim} \leq \min\{x_0, K(\omega)\};\ h^{\lim} \leq \sigma_{\omega}(x^{\lim})\}, \tag{11}$$

where the function $\sigma_{\omega}(\cdot)$ is defined by

$$\sigma_{\omega}(x) = f(x, \omega) - x, \tag{12}$$

and represents the harvesting (or yield) at equilibrium when the steady-state stock is x for a fixed scenario ω.

Since $\mathbb{S}(\xi)$ coincides with $\hat{\mathbb{S}}(\xi) = \bigcap_{\omega \in \Omega} \mathbb{S}^{\omega}(\xi)$ and $\mathbb{S}^{\omega}(\xi)$ approaches $\mathbb{S}_{\infty}^{\omega}(\xi)$ when $N \to \infty$, the objective of this example is to compare $\mathbb{S}(\xi)$ computed by our method for N large enough with respect to $\hat{\mathbb{S}}_{\infty}(\xi) = \bigcap_{\omega \in \Omega} \mathbb{S}_{\infty}^{\omega}(\xi)$ computed analytically using (11).

In Figure 3, we show the set of robust sustainable thresholds $\mathbb{S}(\xi)$ considering different time horizons $N = 10$ (first row), $N = 25$ (second row), and $N = 50$ (third row) for three initial endowments of the resource ξ (displayed in each column). Additionally, through (11), we analytically compute the sets of sustainable thresholds associated with the deterministic system (9) for the constant scenarios $\omega \in \Omega = \{\omega_a, \omega_b\}$ and infinite horizon ($N = +\infty$). We illustrate these sets by depicting the weak Pareto fronts of $\mathbb{S}_{\infty}^{\omega_a}(\xi)$ (red) and $\mathbb{S}_{\infty}^{\omega_b}(\xi)$ (blue), from which it is easy to identify the set

$$\hat{\mathbb{S}}_{\infty}(\xi) = \bigcap_{\omega \in \Omega} \mathbb{S}_{\infty}^{\omega}(\xi) = \mathbb{S}_{\infty}^{\omega_a}(\xi) \cap \mathbb{S}_{\infty}^{\omega_b}(\xi).$$

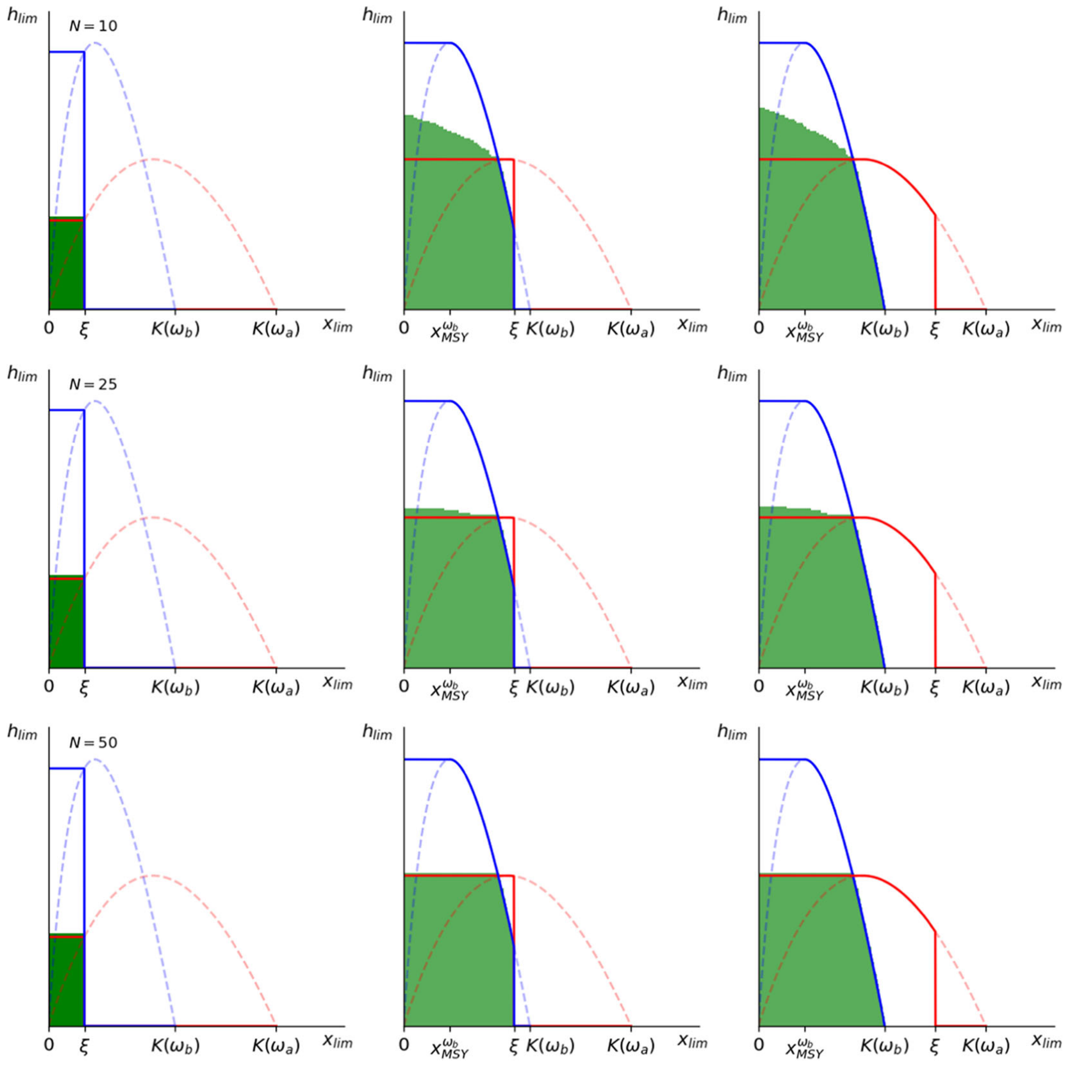

FIGURE 3 Set of robust sustainable thresholds $\mathbb{S}(\xi)$ (green) for different time horizons $N = 10$ (first row), $N = 25$ (second row), and $N = 50$ (third row), and three initial stocks ξ (displayed in each column). The red and blue curves in each case correspond to the weak Pareto fronts of sets $\mathbb{S}_\infty^{\omega_a}(\xi)$ and $\mathbb{S}_\infty^{\omega_b}(\xi)$, respectively, from which one identifies the set $\hat{\mathbb{S}}_\infty(\xi) = \mathbb{S}_\infty^{\omega_a}(\xi) \cap \mathbb{S}_\infty^{\omega_b}(\xi)$. The dashed curves represent functions σ_{ω_a} and σ_{ω_a} defined in (12) used for computing $\mathbb{S}_\infty^{\omega_a}(\xi)$ and $\mathbb{S}_\infty^{\omega_b}(\xi)$ (see 11). The parameters for the resource dynamics were set to $r(\omega_a) = 0.39$, $r(\omega_b) = 2$, $K(\omega_a) = 90$, and $K(\omega_b) = 50$

The procedure for obtaining the set of robust sustainable thresholds $\mathbb{S}(\xi)$ was conducted by computing the weak Pareto front $\mathcal{P}^{\mathcal{W}}(\mathbb{S}(\xi))$ and then using the equality

$$\mathbb{S}(\xi) = \mathcal{P}^{\mathcal{W}}(\mathbb{S}(\xi)) + \mathbb{R}_-^m.$$

The Pareto front $\mathcal{P}^{\mathcal{W}}(\mathbb{S}(\xi))$ is computed using the elements and results presented in Section 3.1, specifically with the method outlined in Algorithm 1.

In more detail, in the positive orthant of $\mathbb{R}^2$ (space of thresholds), we consider the mesh

$$S_d = \{(jd, \bar{h}^{\lim}) | j = 0, 1, ..., N_d\} \cup \{(\bar{x}^{\lim}, jd) | j = 0, 1, ..., N_d\}, \tag{13}$$

with $0 < d \ll 1$ as the size of the mesh, $N_d \in \mathbb{N}$, and $\bar{x}^{\lim}$, $\bar{h}^{\lim} > 0$ large enough. For each vector of thresholds $c = (x^{\lim}, h^{\lim})$ in the mesh S_d, we compute $\mathcal{W}_\xi(c)$ defined in (5). Taking $\bar{x}^{\lim}$ and $\bar{h}^{\lim}$ sufficiently large ensures that vectors c in the mesh are not in $\mathbb{S}(\xi)$. Hence, from Theorem 3.2, we obtain $\mathcal{W}_\xi(c) < 0$, and we find that $p(c) := c + \mathcal{W}_\xi(c)\mathbf{1}$ is in the weak Pareto front for all thresholds c in S_d. Thus, we obtain the weak Pareto front of $\mathbb{S}(\xi)$ and, a fortiori, the entire set $\mathbb{S}(\xi)$.

Since

$$\mathbb{S}(\xi) = \hat{\mathbb{S}}(\xi) = \mathbb{S}^{\omega_a}(\xi) \cap \mathbb{S}^{\omega_b}(\xi)$$

(see Lemma 4.1) and because the set $\hat{\mathbb{S}}(\xi)$ approaches $\hat{\mathbb{S}}_\infty(\xi)$ when the time horizon N increases, for large N, as in the third row of Figure 3, we should obtain

$$\mathbb{S}(\xi) \xrightarrow{N \to \infty} \hat{\mathbb{S}}_\infty(\xi) = \mathbb{S}_\infty^{\omega_a}(\xi) \cap \mathbb{S}_\infty^{\omega_b}(\xi)$$

as the numerical tests reported in Figure 3 confirm.

As we mention in Section 2.3, where standing assumptions are introduced, the set of robust sustainable thresholds $\mathbb{S}(\xi)$ is never empty. In this example, even if we consider negative intrinsic growth parameters $r(\omega) < 0$, since the constraint functions (indicators) $g_1(x, u) = x$ and $g_2(x, u) = u$ are nonnegative, one always has $(0, 0) \in \mathbb{S}(\xi)$. Of course, if $r(\omega) < -1$, from the definitions of the dynamics in (10), we obtain negative levels of the resource, and then that situation would not be modeling a realistic problem.

5 | CONCLUDING REMARKS

In natural resource management, biodiversity conservation problems or climate change, operationalizing definitions of sustainability is a major challenge. Along this line, approaches based on constraints and thresholds have been proposed for biological, ecological, economic, or social indicators as goals for sustainable management objectives. Whenever the constraints induced by thresholds have to be satisfied over time, such problems can be formulated into the mathematical framework of viability theory, as has been proposed by numerous authors in recent decades (Oubraham & Zaccour, 2018). The focus of these approaches has mainly been the computation or estimation of the viability kernel, the core concept in the mentioned theory, and consisting of the set of initial states of the system from which it is possible to satisfy prescribed constraints over time. In this study, our approach is in the inverse, consisting of determining the thresholds for which there exists a trajectory satisfying constraints parametrized by such thresholds. The set of all these thresholds has been studied in the deterministic case, and in the present work, we have extended the definition to uncertain systems, defining the set of robust sustainable thresholds, that is, thresholds that it is possible not to exceed over time regardless of uncertain future scenarios. This new concept and its computation provide a tool for the management and visualization of multiple objectives related to sustainability that can be accomplished in a robust manner, allowing us to observe the (in) compatibility of multiple objectives and their tradeoffs. Additionally, it may contribute to socioeconomic resilience management according to Grafton et al. (2019), where robustness is introduced as one of the main characteristics of socio-ecological resilience.

For the sake of exposition, we presented only autonomous systems without considering target constraints at the final time. However, after mild modifications, all the results we have shown can be stated when the dynamics and the constraints are nonautonomous as well as when a target constraint is taken into account.

To conclude, let us mention some possible extensions of the work we have presented in this paper.

5.1 | Strong Pareto front

Proposition 3.1 states that to describe the set of robust sustainable thresholds, one can first attempt to solve an optimal control problem to test the sustainability of a given threshold. Moreover, in a similar way as done in Gajardo and Hermosilla (2021) for the deterministic case, one can show that if the payoff is constructed taking into account the constraint mapping, then one obtains a procedure for constructing the strong Pareto front of the set of robust sustainable thresholds.

Under suitable assumptions, the strong Pareto front can be proven to be the smallest subset of the set of robust sustainable thresholds. that allows us to recover this set by adding the negative orthant. It follows that the strong Pareto front can be interpreted, to some extent, as the set of extreme points of the set of robust sustainable thresholds whenever this set is convex, which explains why it could also be interesting its computation.

5.2 | Set of stochastic sustainable thresholds

In this paper, we have focused on a worst-case scenario approach. Other cases that may be worth studying in the presence of uncertainty are when the constraints are satisfied only in some scenarios, which can be quantified with a probability measure. For example, one may be interested in studying the *set of stochastic sustainable thresholds* related to a given confidence level. Research involving the set of stochastic sustainable thresholds will be the object of future research.

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AUTHOR CONTRIBUTIONS

Pedro Gajardo: Writing original draft (equal). **Cristopher Hermosilla**: Writing original draft (equal). **Athena Picarelli**: Writing original draft (equal).

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon request.

ORCID

Pedro Gajardo http://orcid.org/0000-0003-3681-8691
Cristopher Hermosilla https://orcid.org/0000-0003-2975-5944
Athena Picarelli https://orcid.org/0000-0003-1103-5324

ENDNOTES

[1]For each $i \in [0:m]$, the function $g_i : \mathbb{R}^d \times \mathbf{U} \to \mathbb{R}$ is upper semicontinuous if for all $(x, u) \in \mathbb{R}^d \times \mathbf{U}$ and for all sequences $(x_n, u_n) \to (x, u)$, one has $\limsup_{n\to\infty} g_i(x_n, u_n) \le g_i(x, u)$.

[2]The computational domain for the state is a discrete and bounded representation of the state space $\mathbb{R}^d$, or more precisely of the set of all trajectories starting from the considered initial condition. To determine the computational domain of the thresholds, it is enough to consider a rectangle (Cartesian product of intervals), where the lower bound corresponds to lower bounds of constraints g_i and the upper bounds are chosen sufficiently large to ensure that the weak Pareto front of the rectangle corresponds to nonrobust sustainable thresholds. Thus, the computational domain can be taken as the weak Pareto front of the rectangle.

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APPENDIX A: MATHEMATICAL PROOFS

A.1 | The set of feasible solutions is compact

Proof. From the standing assumptions in Section 2.3, one has that the map $\mathbf{u} \mapsto \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u})$ is continuous and that the space $\mathbb{U}$ is compact. Then,

$$A_\xi^{\mathbf{w}} := \left\{(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U} | \mathbf{x} = \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u})\right\}$$

is a compact subset of $\mathbb{X} \times \mathbb{U}$. Moreover, the set

$$B_c^{\mathbf{w}} := \left\{(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U} \middle| \, g(x_k, u_k) \geq c, \forall \, k \in [0{:}N]\right\}$$

is closed in $\mathbb{X} \times \mathbb{U}$. Therefore, as the set of admissible trajectories is the projection of $A_\xi^{\mathbf{w}} \cap B_c^{\mathbf{w}}$ over $\mathbb{R}^d$, one can conclude that the set of feasible solutions to the dynamical system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$-$(\mathrm{I}^c)$ is a compact (possibly empty) subset of $\mathbb{X}$. An analogous argument can be used to prove that the set of admissible controls for the dynamical system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$ with constraint (I^c) is compact in $\mathbb{U}$. □

A.2 | Proposition 3.1

Proof. Let us note that the mapping $\mathbf{u} \mapsto \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u})$ is continuous for any $\xi \in \mathbb{R}^d$ and $\mathbf{w} \in \mathbb{W}$ fixed, so $\mathbf{u} \mapsto \mathcal{J}(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u})$ is upper semicontinuous. This implies that the functional $\mathbf{u} \mapsto \inf_{\mathbf{w} \in \mathbb{W}} \mathcal{J}(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u})$ is also upper semicontinuous. Note that this functional is also finite because $\mathcal{J}$ is bounded below and $\mathbb{W}$ is nonempty. Moreover, for any $\xi \in \mathbb{R}^d$ and $\mathbf{w} \in \mathbb{W}$ fixed, let

$$\mathbb{U}_\xi^{\mathbf{w}}(c) := \left\{\mathbf{u} \in \mathbb{U} \middle| \, \mathbf{u} \quad \text{and} \quad \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) \quad \text{satisfy} \quad (\mathrm{I}^c)\right\}$$

be the set of admissible controls for the dynamical system $(D_\xi^{\mathbf{u}}(\mathbf{w}))$ with constraint (I^c). As remarked previously, this set is closed in $\mathbb{U}$ for any given $c \in \mathbb{R}^m$ and therefore compact. In particular, since

$$\left\{\mathbf{u} \in \mathbb{U} | \mathbf{u} \quad \text{and} \quad \mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}) \quad \text{satisfy} \quad (\mathrm{I}^c), \; \forall \, \mathbf{w} \in \mathbb{W}\right\} = \bigcap_{\mathbf{w} \in \mathbb{W}} \mathbb{U}_\xi^{\mathbf{w}}(c),$$

the set on the right-hand side is also compact.

Note that $c \in \mathbb{S}(\xi)$ if and only if $\bigcap_{\mathbf{w}\in\mathbb{W}} \mathbb{U}_{\xi}^{\mathbf{w}}(c)$ is nonempty. Therefore, the maximum in the definition of $\vartheta_{\xi}(c)$ is attained, and so $\vartheta_{\xi}(c) < +\infty$; this is a consequence of maximizing a finite and upper semicontinuous payoff over a nonempty compact set. Furthermore, if $\vartheta_{\xi}(c) < +\infty$, then clearly $\bigcap_{\mathbf{w}\in\mathbb{W}} \mathbb{U}_{\xi}^{\mathbf{w}}(c) \neq \emptyset$, and the conclusion follows. □

A.3 | Theorem 3.2

Proof. Let us set for any $c \in \mathbb{R}^m$

$$R^c(\mathbf{x}, \mathbf{u}) := \min_{i=1,\ldots,m} \left[\min_{k=0,\ldots,N} g_i(x_k, u_k) - c_i \right], \forall\, \mathbf{x} \in \mathbb{X},\ \ \mathbf{u} \in \mathbb{U}. \tag{A.1}$$

Therefore,

$$\mathcal{W}_{\xi}(c) = \sup_{\mathbf{u}\in\mathbb{U}} \inf_{\mathbf{w}\in\mathbb{W}} R^c\left(\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right).$$

(1) First, we assume that $\mathcal{W}_{\xi}(c^*) = 0$. From (6), one has $c^* \in \mathbb{S}(\xi)$. Let us suppose that c^* does not belong to the weak Pareto front. Then, there exists $c \in \mathbb{S}(\xi)$ with $c > c^*$. For δ defined by

$$\delta := \min_{i=1,\ldots,m} \left\{c_i - c_i^*\right\} > 0,$$

since $c \in \mathbb{S}(\xi)$, there exists $\mathbf{u} = (u_k)_{k=0}^{N} \in \mathbb{U}$ such that for any scenario $\mathbf{w} \in \mathbb{W}$, we have

$$g_i(x_k, u_k) \geq c_i, \forall\, i \in [1{:}m],\ \ k \in [0{:}N],$$

where $\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}) = (x_k)_{k=0}^{N+1}$. This implies that

$$0 = \mathcal{W}_{\xi}(c^*) \geq \inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}\left(\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right) \geq \min_{i=1,\ldots,m} c_i - c_i^* = \delta,$$

obtaining thus a contradiction. Hence, we conclude that c^* belongs to the weak Pareto front of $\mathbb{S}(\xi)$.

Now, suppose that c^* belongs to the weak Pareto front of $\mathbb{S}(\xi)$. Therefore, $c^* \in \mathbb{S}(\xi)$, and from (6), we deduce that $\mathcal{W}_{\xi}(c^*) \geq 0$. If $\mathbf{u} \in \mathbb{U}$ is an optimal control for $\mathcal{W}_{\xi}(c^*)$; that is, $\inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}(\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}), \mathbf{u}) = \mathcal{W}_{\xi}(c^*)$. Suppose by contradiction that $\inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}(\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}), \mathbf{u}) > 0$, and define

$$c := c^* + \frac{1}{2} \inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}\left(\mathbf{x}_{\xi}^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right)\mathbf{1} > c^*.$$

Note that

$$\mathcal{W}_\xi(c) \geq \inf_{\mathbf{w}\in\mathbb{W}} R^c\left(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right) = \inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}\left(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right) - \frac{1}{2}\inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}\left(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right).$$

Then, we can deduce that

$$\mathcal{W}_\xi(c) \geq \frac{1}{2}\inf_{\mathbf{w}\in\mathbb{W}} R^{c^*}\left(\mathbf{x}_\xi^{\mathbf{w}}(\mathbf{u}), \mathbf{u}\right) \geq 0.$$

In particular, we must have that $c \in \mathbb{S}(\xi)$. Therefore, we have found some robust sustainable threshold c for which $c > c^*$, which contradicts the fact that c^* is assumed to be a weak Pareto maximum.

(2) The result follows from the equality $\mathcal{W}_\xi(p(\bar{c})) = \mathcal{W}_\xi(\bar{c}) - \mathcal{W}_\xi(\bar{c}) = 0$, which is straightforward. We conclude by applying the first statement of Theorem 3.2.

□

A.4 | The mapping $\xi \mapsto V_n^c(\xi)$ is upper semicontinuous

Proof. Let us prove that the mapping $\xi \mapsto V_n^c(\xi)$ is upper semicontinuous for any $n \in [0{:}N]$. Indeed, if $n \in [0{:}N]$, then for any sequence $\{\xi^j\}_{j\in\mathbb{N}}$ that converges to $\bar{\xi}$, there is a sequence of optimal controls $\left\{\left(u_k^j\right)_{k=0}^N\right\}_{j\in\mathbb{N}} \subseteq \mathbb{U}$ such that

$$V_n^c(\xi^j) = \inf_{\mathbf{w}\in\mathbb{W}}\left\{\min_{k=n,\ldots,N}\Phi^c\left(x_k, u_k^j\right) \middle| \; x_{k+1} = F\left(x_k, u_k^j, \omega_k\right), k \in [n{:}N], x_n = \xi^j\right\} \tag{A.1}$$

By the compactness of $\mathbb{U}$, we can assume that $\left\{\left(u_k^j\right)_{k=0}^N\right\}_{j\in\mathbb{N}}$ converges to some $(\bar{u}_k)_{k=0}^N$. Note further that the mapping

$$(\xi, \mathbf{u}) \mapsto \inf_{\mathbf{w}\in\mathbb{W}}\{\min_{k=n,\ldots,N}\Phi^c(x_k, u_k) | \; x_{k+1} = F(x_k, u_k, \omega_k), k \in [n:N], x_n = \xi\}$$

is upper semicontinuous, and so, by taking limsup in (A.1), we obtain

$$\limsup_{j\to+\infty} V_n^c(\xi^j) \leq \inf_{\mathbf{w}\in\mathbb{W}}\left\{\min_{k=n,\ldots,N}\Phi^c(x_k, \bar{u}_k) \middle| \; x_{k+1} = F(x_k, \bar{u}_k, \omega_k), k \in [n:N], x_n = \bar{\xi}\right\}.$$

Whence, using the definitions of the value funcion $V_n^c(\bar{\xi})$ we conclude. □

A.5 | Proposition 3.4

Proof. We set $W_N^c(\xi) := \max_{u\in\mathbf{U}}\Phi^c(\xi, u)$, and for any $n \in [0{:}N-1]$, let us recursively define the function

$$W_n^c(\xi) := \sup_{u \in \mathbf{U}} \min\left\{ \inf_{\omega \in \Omega} \overset{c}{\underset{n+1}{V}} \left(F(\xi, u, \omega)\right), \Phi^c(\xi, u)\right\}.$$

The value $W_n^c(\xi)$ is consistent with the right-hand side of (7) with the supremum instead of the maximum. Let us check that this maximum is attained. Since V_{n+1}^c and $\Phi^c(\xi, \cdot)$ are upper semicontinuous (see Appendix A.4) and $F(\xi, \cdot, \omega)$ is continuous for any $n \in [\![0{:}N]\!]$, we find that the functional to be maximized in the definition of $W_n^c(\xi)$ is upper semicontinuous. Thus, since $\mathbf{U}$ is compact, this maximum is attained.

Let us prove by (backward) induction that $V_n^c(\xi) = W_n^c(\xi)$ for any $n \in [\![0{:}N]\!]$.

In the sequel, unless otherwise stated, we use the notation $\mathbf{u} = (u_k)_{k=0}^N \in \mathbb{U}$ and $\mathbf{w} = (\omega_k)_{k=0}^N \in \mathbb{W}$. Let us first check that the dynamic programming principle holds for the case of $n = N$. By definition, we have

$$V_N^c(\xi) := \max_{\mathbf{u} \in \mathbb{U}} \inf_{\mathbf{w} \in \mathbb{W}} \Phi^c(\xi, u_N) = \max_{u \in \mathbf{U}} \inf_{\omega \in \Omega} \Phi^c(\xi, u) = \max_{u \in \mathbf{U}} \Phi^c(\xi, u),$$

and we obtain $V_N^c(\xi) = W_N^c(\xi)$.

Let $n \in [\![0 : N-1]\!]$, and assume that $V_k^c(x) = W_k^c(x)$ for any $k \in [\![n+1 : N]\!]$ and any $x \in \mathbb{R}^d$. Let us verify that $V_n^c(\xi) = W_n^c(\xi)$

Let $\bar{\mathbf{u}} = (\bar{u}_k)_{k=0}^N \in \mathbb{U}$ be an optimal control for $V_n^c(\xi)$. Note then that

$$V_n^c(\xi) = \inf_{\mathbf{w} \in \mathbb{W}} \left\{ \Phi^c(\xi, \bar{u}_n), \min_{k=n+1,\ldots,N} \Phi^c(x_k, \bar{u}_k) \;\middle|\; \begin{array}{l} x_{k+1} = F(x_k, \bar{u}_k, \omega_k) \\ k \in [\![n+1 : N]\!], \\ x_{n+1} = F(\xi, \bar{u}_n, \omega_n) \end{array} \right\}$$

$$= \min\left\{ \Phi^c(\xi, \bar{u}_n), \inf_{\mathbf{w} \in \mathbb{W}} \left\{ \min_{k=n+1,\ldots,N} \Phi^c(x_k, \bar{u}_k) \;\middle|\; \begin{array}{l} x_{k+1} = F(x_k, \bar{u}_k, \omega_k) \\ k \in [\![n+1 : N]\!], \\ x_{n+1} = F(\xi, \bar{u}_n, \omega_n) \end{array} \right\} \right\}$$

Furthermore, it also holds that

$$\inf_{\mathbf{w} \in \mathbb{W}} \left\{ \min_{k=n+1,\ldots,N} \Phi^c(x_k, \bar{u}_k) \;\middle|\; \begin{array}{l} x_{k+1} = F(x_k, \bar{u}_k, \omega_k) \\ k \in [\![n+1 : N]\!], \\ x_{n+1} = F(\xi, \bar{u}_n, \omega_n) \end{array} \right\}$$

$$= \inf_{\omega \in \Omega} \inf_{\mathbf{w} \in \mathbb{W}} \left\{ \min_{k=n+1,\ldots,N} \Phi^c(x_k, \bar{u}_k) \;\middle|\; \begin{array}{l} x_{k+1} = F(x_k, \bar{u}_k, \omega_k) \\ k \in [\![n+1 : N]\!], \\ x_{n+1} = F(\xi, \bar{u}_n, \omega) \end{array} \right\}$$

$$\leq \inf_{\omega \in \Omega} \max_{\mathbf{u} \in \mathbb{U}} \inf_{\mathbf{w} \in \mathbb{W}} \left\{ \min_{k=n+1,\ldots,N} \Phi^c(x_k, u_k) \;\middle|\; \begin{array}{l} x_{k+1} = F(x_k, u_k, \omega_k) \\ k \in [\![n+1 : N]\!], \\ x_{n+1} = F(\xi, \bar{u}_n, \omega) \end{array} \right\}$$

$$= \inf_{\omega \in \Omega} V_{n+1}^c(F(\xi, \bar{u}_n, \omega)).$$

Whence we obtain $V_n^c(\xi) \leq W_n^c(\xi)$.

Now, to prove the other inequality, let us note that since the maximum is attained in the definition of any of the functions $W_k^c(\cdot)$, there is an optimal feedback control $\vartheta_k : \mathbb{R}^d \to \mathbf{U}$ such that

$$W_k^c(x) := \min\left\{\inf_{\omega_k \in \Omega} W_{k+1}^c(F(x, \vartheta_k(x), \omega_k), \Phi^c(x, \vartheta_k(x))\right\}, \forall\, x \in \mathbb{R}^d.$$

The use of W_{k+1}^c instead of V_{k+1}^c in the preceding equality is a consequence of the induction hypothesis.

Take $\mathbf{w} \in \mathbb{W}$ arbitrary, and define

$$x_{k+1} = F(x_k, \vartheta_k(x_k), \omega_k), \forall\, k \in [[n : N]], x_n = \xi.$$

It then follows that

$$W_k^c(x_k) \leq \min\left\{W_{k+1}^c(x_{k+1}), \Phi^c(x, \vartheta_k(x_k))\right\}, \forall\, k \in [[n : N]].$$

Therefore, using this inequality repeatedly we obtain

$$W_n^c(\xi) = W_n^c(x_n) \leq \min_{k=n,\,\ldots,\,N} \Phi^c(x_k, \vartheta_k(x_k)).$$

Since this is true for any $\mathbf{w} \in \mathbb{W}$, it yields

$$W_n^c(\xi) \leq \inf_{\mathbf{w} \in \mathbb{W}} \left\{\min_{k=n,\,\ldots,\,N} \Phi^c(x_k, \vartheta_k(x_k)) \,\middle|\, x_{k+1} = F(x_k, \vartheta_k(x_k), \omega_k),\ \ k \in [[n : N]], x_n = \xi\right\}$$

Finally, by taking the supremum over $\mathbf{u} \in \mathbb{U}$, we obtain that $W_n^c(\xi) \leq V_n^c(\xi)$. Thus, by induction, the conclusion follows. □

A.6 | Lemma 4.1

Proof. Thanks to Remark 4.2, to prove Lemma 4.1, we only need to show that $\hat{\mathbb{S}}(\xi) \subseteq \mathbb{S}(\xi)$. For this purpose, let us consider a threshold $c = (x^{\lim}, h^{\lim}) \in \hat{\mathbb{S}}(\xi)$. It is then immediate to verify that the set of admissible controls $\mathbb{U}_\xi^{\mathbf{w}_\omega}(c)$ is nonempty for all $\omega \in \Omega$. Let $\mathbf{u}^\omega = (u_k^\omega)_{k=0}^N$ be an element of $\mathbb{U}_\xi^{\mathbf{w}_\omega}(c)$ for $\omega \in \Omega$ and $\bar{\mathbf{u}} = (\bar{u}_{\ k})_{k=0}^N \in \mathbb{U}$ be defined by $\bar{u}_{\ k} := \inf_{\omega \in \Omega} u_k^\omega$ for $k \in [0 : N]$. With $\mathbf{u}^\omega$ being admissible for all $\omega \in \Omega$, one has

$$\bar{u}_{\ k} \geq h^{\lim}, \forall\, k \in [[0 : N]].$$

Furthermore, thanks to the monotonicity of $f(\cdot,\omega)$ and the definition of $\bar{\mathbf{u}}$, it can be easily shown that for any $\mathbf{w} = (\omega_k)_{k=0}^N \in \mathbb{W}$, the trajectory $\mathbf{x}_\xi^{\mathbf{w}}(\bar{\mathbf{u}}) = (x_k)_{k=0}^{N+1}$ satisfies

$$x_k \geq x_k^{\omega_k}, \forall\, k \in [[0 : N]],$$

where we denoted by $\mathbf{x}_{\xi}^{\mathbf{w}_{\omega}}(\mathbf{u}^{\omega}) = (x_k^{\omega})_{k=0}^{N+1}$ the admissible (deterministic) trajectory associated with $\omega \in \Omega$. From the last inequality and the admissibility of the control $\mathbf{u}^{\omega}$, we obtain

$$x_k \geq x^{\lim}, \forall\, k \in [[0:N]],$$

and then we can conclude that $\bar{\mathbf{u}} \in \mathbb{U}_{\xi}^{\mathbf{w}}(c)$. Thanks to the arbitrariness of $\mathbf{w} \in \mathbb{W}$, the latter implies that $\bar{\mathbf{u}} \in \bigcap_{\mathbf{w} \in \mathbb{W}} \mathbb{U}_{\xi}^{\mathbf{w}}(c)$, that is, $c \in \mathbb{S}(\xi)$. $\square$