



Topological Singular Set of Vector-Valued Maps, II: Γ -convergence for Ginzburg–Landau type functionals

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Abstract

We prove a Γ -convergence result for a class of Ginzburg–Landau type functionals with \mathcal{N} -well potentials, where \mathcal{N} is a closed and $(k - 2)$ -connected submanifold of \mathbb{R}^m , in arbitrary dimension. This class includes, for instance, the Landau-de Gennes free energy for nematic liquid crystals. The energy density of minimisers, subject to Dirichlet boundary conditions, converges to a generalised surface (more precisely, a flat chain with coefficients in $\pi_{k-1}(\mathcal{N})$) which solves the Plateau problem in codimension k . The analysis relies crucially on the set of topological singularities, that is, the operator \mathbf{S} we introduced in the companion paper [17].

1. Introduction

Let $n \geq 0$, $k \geq 2$, $m \geq 2$ be integers, and let $\Omega \subseteq \mathbb{R}^{n+k}$ be a bounded, smooth domain. Let $\varepsilon > 0$ be a small parameter. For $u \in W^{1,k}(\Omega, \mathbb{R}^m)$, we define the functional

$$E_\varepsilon(u) := \int_\Omega \left(\frac{1}{k} |\nabla u|^k + \frac{1}{\varepsilon^k} f(u) \right). \quad (1.1)$$

Here, $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is a non-negative, continuous potential, whose zero-set $\mathcal{N} := f^{-1}(0)$ is assumed to be a smooth, compact, $(k - 2)$ -connected manifold without boundary. The aim of this paper is to understand the asymptotic behaviour of the functionals E_ε in the limit as $\varepsilon \rightarrow 0$, by a Γ -convergence approach. Our analysis builds upon the results obtained in a companion paper, [17].

Functionals of the form (1.1), which describe a kind of penalised k -harmonic map problem (see e.g. [19, 40]), arise naturally in different contexts. A well-known example is the Ginzburg–Landau functional, which corresponds to the case $k = m = 2$ and $f(u) := (|u|^2 - 1)^2$, so that the zero-set of f is the unit circle, $\mathcal{N} = \mathbb{S}^1 \subseteq \mathbb{R}^2$. The Ginzburg–Landau functional was originally introduced as a (simplified) model for superconductivity, but has attracted considerable attention in

the mathematical community since the pioneering work by BETHUEL, BRÉZIS AND HÉLEIN [8]. Another example, arising from materials science, is the Landau-de Gennes model for nematic liquid crystals (in the so-called one-constant approximation, see e.g. [23]). In this case, $k = 2$ and the zero-set of f is a real projective plane $\mathcal{N} = \mathbb{R}P^2$, whose elements can be interpreted as the preferred configurations for the material. Functionals of the form (1.1) have also applications to mesh generation in numerical analysis, via the so-called cross-field algorithms (see e.g. [18]).

Minimisers of (1.1) subject to a boundary condition

$$u|_{\partial\Omega} = v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$$

may not satisfy uniform energy bounds, due to topological obstructions carried by the boundary datum v . When this phenomenon occurs, the energy of minimisers is of order $|\log \varepsilon|$ (see e.g. [8, 11, 49] in case $k = 2$, $\mathcal{N} = \mathbb{S}^1$). A similar phenomenon arises for tangent vector fields on a closed manifold, due to the Poincaré-Hopf theorem (see e.g. [34]). The analysis of the Ginzburg–Landau case shows that the energy of minimisers (and other critical points) concentrates, to leading order, on a n -dimensional surface; see e.g. [8, 9, 41, 51]. From a variational viewpoint, the Ginzburg–Landau functional itself can be considered an approximation of an n -dimensional “weighted area” functional, in a sense that can be made precise by Γ -convergence [2, 3, 39, 51]. Therefore, the Ginzburg–Landau functional and its variants have been proposed as tools to construct “weak minimal surfaces” or, more precisely, stationary varifolds of codimension greater than one [4, 9, 42, 48, 52]. Energy concentration results have also been established for Landau-de Gennes minimisers [5, 15, 16, 22, 28, 35, 36, 43, 46]. To our best knowledge, minimisers of functionals associated with more general manifolds \mathcal{N} , in the logarithmic energy regime, have been studied only in case $n = 0$, $k = 2$ so far [15, 44, 45].

In this paper, we show that the re-scaled functionals $|\log \varepsilon|^{-1} E_\varepsilon$ do converge to an n -dimensional weighted area functional, thus extending the results in [2, 39] to more general potentials f . The key tool is the topological singular set of vector-valued maps, that is, the operator \mathbf{S} we introduced in [17], which identifies the appropriate topology of the Γ -convergence. The operator \mathbf{S} effectively serves as a replacement, or rather a generalisation, of the distributional Jacobian, which is commonly used when the distinguished manifold is a sphere, $\mathcal{N} = \mathbb{S}^{k-1}$. In order to overcome the algebraic issues that make the distributional Jacobian incompatible with the topology of other manifolds \mathcal{N} , we work in the setting of *flat chains* with coefficients in $\pi_{k-1}(\mathcal{N})$ [26]. In the context of manifold-constrained problems, the use of flat chains with coefficients in an Abelian group was proposed by PAKZAD AND RIVIÈRE [47] and traces its roots back in the earlier literature on the subject: the very notion of “minimal connection”, introduced by BRÉZIS ET AL. [13], can be interpreted as the flat norm of the distributional Jacobian.

We state our main Γ -convergence result, Theorem C, in Section 2, after introducing some background and notation. Here, we present an application (Theorem A below) to the asymptotic analysis of minimisers of (1.1) in the limit as $\varepsilon \rightarrow 0$. We make the following assumptions on the potential f :

$$(H_1) \quad f \in C^1(\mathbb{R}^m) \text{ and } f \geq 0.$$

(H₂) The set $\mathcal{N} := f^{-1}(0) \neq \emptyset$ is a smooth, compact manifold without boundary. Moreover, \mathcal{N} is $(k-2)$ -connected, that is $\pi_0(\mathcal{N}) = \pi_1(\mathcal{N}) = \dots = \pi_{k-2}(\mathcal{N}) = 0$, and $\pi_{k-1}(\mathcal{N}) \neq 0$. In case $k = 2$, we also assume that $\pi_1(\mathcal{N})$ is Abelian.

(H₃) There exists a positive constant λ_0 such that $f(y) \geq \lambda_0 \operatorname{dist}^2(y, \mathcal{N})$ for any $y \in \mathbb{R}^m$.

The assumption (H₂) is consistent with the setting of [17] and is satisfied, for instance, when $k = 2$ and $\mathcal{N} = \mathbb{S}^1$ (the Ginzburg–Landau case) or $k = 2$ and $\mathcal{N} = \mathbb{R}P^2$ (the Landau–de Gennes case). The assumption (H₃) is both a non-degeneracy condition around the minimising set \mathcal{N} and a growth condition.

Remark 1. We do not expect the assumption (H₃) to be sharp. In fact, (H₃) may probably be relaxed so as to include potentials that behave as $\operatorname{dist}^s(\cdot, \mathcal{N})$, for some $s > 2$, in a neighbourhood of \mathcal{N} .

We consider minimisers $u_{\varepsilon, \min}$ of (1.1), subject to the boundary condition $u = v$ on $\partial\Omega$. On the boundary datum v , we assume

(H₄) $v \in W^{1-1/k, k}(\partial\Omega, \mathcal{N})$ — that is, $v \in W^{1-1/k, k}(\partial\Omega, \mathbb{R}^m)$ and $v(x) \in \mathcal{N}$ for \mathcal{H}^{n+k-1} -a.e. $x \in \partial\Omega$.

Under the assumptions (H₁)–(H₄), the rescaled energy densities

$$\mu_{\varepsilon, \min} := \left(\frac{1}{k} |\nabla u_{\varepsilon, \min}|^k + \frac{1}{\varepsilon^k} f(u_{\varepsilon, \min}) \right) \frac{\mathrm{d}x \llcorner \Omega}{|\log \varepsilon|}$$

have uniformly bounded mass (see e.g. Remark 9 below; here, $\mathrm{d}x \llcorner \Omega$ denotes the Lebesgue measure restricted to Ω). Up to extraction of a subsequence, we may assume that $\mu_{\varepsilon, \min}$ converges weakly* (as measures in \mathbb{R}^{n+k}) to a non-negative measure μ_{\min} , as $\varepsilon \rightarrow 0$. We provide a variational characterisation of μ_{\min} in terms of flat chains with coefficients in $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$, where $|\cdot|_*$ is a suitable norm, defined in Section 2 below. (For instance, in case $k = 2$ and $\mathcal{N} = \mathbb{S}^1$, $|d|_* = \pi |d|$ for any $d \in \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.) We denote the mass of such a flat chain S by $\mathbb{M}(S)$, and the restriction of S to a set E by $S \llcorner E$. We have

Theorem A. *Under the assumptions (H₁)–(H₄), there exists a finite-mass n -chain S_{\min} , with coefficients in $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$ and support in $\overline{\Omega}$, such that $\mu_{\min}(E) = \mathbb{M}(S_{\min} \llcorner E)$ for any Borel set $E \subseteq \mathbb{R}^{n+k}$. Moreover, S_{\min} minimises the mass in its homology class—that is, for any $(n+1)$ -chain R with coefficients in $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$ and support in $\overline{\Omega}$, we have*

$$\mathbb{M}(S_{\min}) \leq \mathbb{M}(S_{\min} + \partial R).$$

In other words, in the limit as $\varepsilon \rightarrow 0$ the energy of minimisers concentrates, to leading order, on the support of a flat chain S_{\min} that solves a homological Plateau problem. The homology class of S_{\min} is uniquely determined by the domain Ω and the boundary datum v (that is, S_{\min} belongs to the class $\mathcal{C}(\Omega, v)$ defined by (2.6) below). We stress that Theorem A does not require any topological assumption, such as simply connectedness, on the domain Ω . However, the homology class of S_{\min}

does depend on the topology of the domain and it can be described more easily if Ω has a simple topology (see the examples in Section 2 below). On the other hand, the topological assumption (H_2) on the manifold \mathcal{N} is essential. An analogue of Theorem A in case $k = 2$ and the fundamental group of \mathcal{N} is non-Abelian would already be of interest in terms of the applications; manifolds with non-Abelian fundamental group arise quite naturally, for instance, in materials science (e.g., as a model for biaxial liquid crystals). Unfortunately, the very statement of Theorem A does not make sense in the non-Abelian setting, because homology requires the coefficient group to be Abelian. Convergence results in case $n = 0, k = 2$ (see e.g. [15, 44]) suggest that the energy concentration set may inherit some minimality properties, even if $\pi_1(\mathcal{N})$ is non-Abelian. However, a general convergence result in the non-Abelian setting, along the lines of Theorem A, would presumably require some ‘ad-hoc’ tools from Geometric Measure Theory.

Remark 2. Theorem A characterises the asymptotic behaviour of the energy of minimisers, to leading order:

$$E_\varepsilon(u_{\varepsilon, \min}) = \mathbb{M}(S_{\min}) |\log \varepsilon| + o(|\log \varepsilon|) \quad \text{as } \varepsilon \rightarrow 0.$$

In some cases, the next-to-leading order term can be characterised, too. For instance, when $n = 0, k = 2$, the energy concentrates on a finite number of points and the next-to-leading order term in the energy expansion is a ‘renormalised energy’ which describes the interaction among the singular points. The renormalised energy was introduced, in the Ginzburg–Landau setting, by BETHUEL ET AL. [8] and it was extended very recently by MONTEIL ET AL. [44, 45] to more general functionals. This raises the question as to whether a renormalised energy may be derived in case $n = 0, k > 2$. A higher-order energy expansion for the three-dimensional Ginzburg–Landau functional ($n = 1, k = 2, \mathcal{N} = \mathbb{S}^1$) was obtained by CONTRERAS AND JERRARD [21], in a setting where the energy concentrates on a cluster of ‘nearly parallel’ vortex filaments.

We deduce Theorem A from our Γ -convergence result, Theorem C in Section 2. The proof of the Γ -lower bound is based on the same strategy as in [2]. However, the construction of a recovery sequence is rather different from [2]. The main building block, Proposition 4 in Section 3.2, is inspired by the ‘dipole construction’ [6, 7, 13]. Here, dipoles are suitably inserted into a non-constant and, in fact, singular background.

As an auxiliary result, we prove the following lower energy bound, which may be of independent interest.

Proposition B. *Suppose that (H_1) – (H_4) hold. Let $\Omega \subseteq \mathbb{R}^k$ be a bounded, Lipschitz domain that is homeomorphic to a ball. Then, for any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ such that $u = v$ on $\partial\Omega$, it holds that*

$$E_\varepsilon(u) \geq |\sigma|_* |\log \varepsilon| - C,$$

where $\sigma \in \pi_{k-1}(\mathcal{N})$ is the homotopy class of v and C is a positive constant that depends only on Ω, v .

If $\Omega \subseteq \mathbb{R}^k$ is homeomorphic to a ball and $v \in W^{1-1/k, k}(\partial\Omega, \mathcal{N})$, the homotopy class of v can be defined as in [14]. In the Ginzburg–Landau case, this inequality was proved by SANDIER [50] (with $k = 2$) and JERRARD [38]; for the Landau-de Gennes functional, see e.g. [5, 16]. The proof of Proposition B in contained in Appendix C (in fact, a slightly stronger statement is given there).

Remark 3. In case $\sigma = 0$, Proposition B does not provide any information. However, there could be critical points of the functional E_ε whose energy diverges logarithmically even if the boundary datum is homotopically trivial. In other words, energy concentration may happen not only because of global topological constraints, but also for other reasons, such as symmetry. See, for instance, IGNAT ET AL. [37] for an analysis of two-dimensional Landau-de Gennes solutions ($n = 0, k = 2, \mathcal{N} = \mathbb{R}P^2$).

The paper is organised as follows: in Section 2 we recall some notation from [17] and we state the main Γ -convergence result, Theorem C. We prove the Γ -upper bound first, in Section 3, and give the proof of the Γ -lower bound in Section 4. Theorem A is deduced from Theorem C in Section 5. A series of appendices, with proofs of technical results, completes the paper.

2. Setting of the Problem and Statement of the Γ -convergence Result

Throughout the paper, we will write $A \lesssim B$ as a shorthand for $A \leq CB$, where C is a positive constant that only depends on n, k, f, \mathcal{N} , and Ω . If $F \subseteq \mathbb{R}^{n+k}$ is a rectifiable set of dimension d and $u \in W_{\text{loc}}^{1, k}(\mathbb{R}^{n+k}, \mathbb{R}^m)$ we will write

$$E_\varepsilon(u, F) := \int_F \left(\frac{1}{k} |\nabla u|^k + \frac{1}{\varepsilon^k} f(u) \right) d\mathcal{H}^d.$$

Additional notation will be set later on. Throughout the paper, we assume that (H_1) – (H_4) are satisfied.

2.1. Choice of the Norm on $\pi_{k-1}(\mathcal{N})$

Under the assumption (H_2) , the group $\pi_{k-1}(\mathcal{N})$ is Abelian (and we use additive notation for the group operation). We recall that a function $|\cdot|: \pi_{k-1}(\mathcal{N}) \rightarrow [0, +\infty)$ is called a norm if it satisfies the following properties:

- (i) $|\sigma| = 0$ if and only if $\sigma = 0$
- (ii) $|\!-\sigma| = |\sigma|$ for any $\sigma \in \pi_{k-1}(\mathcal{N})$
- (iii) $|\sigma_1 + \sigma_2| \leq |\sigma_1| + |\sigma_2|$ for any $\sigma_1, \sigma_2 \in \pi_{k-1}(\mathcal{N})$.

As in [17], we assume that the norm satisfies

$$\inf_{\sigma \in \pi_{k-1}(\mathcal{N}) \setminus \{0\}} |\sigma| > 0, \tag{2.1}$$

that is, $|\cdot|$ induces the discrete topology on $\pi_{k-1}(\mathcal{N})$.

Remark 4. We do *not* require that $|n\sigma| = n|\sigma|$ for any $n \in \mathbb{N}$, $\sigma \in \pi_{k-1}(\mathcal{N})$; this is consistent with the theory of flat chains as developed in [26, 55].

While the results of [17] hold for any norm on $\pi_{k-1}(\mathcal{N})$ that satisfies (2.1), Theorem A only holds for a specific choice of the norm. Let us define such a norm, following the approach in [20, Chapter 6]. A natural attempt, motivated by the analogy with the functional (1.1), is to define

$$E_{\min}(\sigma) := \inf \left\{ \frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} v|^k : v \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma \right\} \quad (2.2)$$

for any $\sigma \in \pi_{k-1}(\mathcal{N})$. Here ∇_{\top} denotes the tangential gradient on \mathbb{S}^{k-1} , that is, the restriction of the Euclidean gradient ∇ to the tangent plane to the sphere. Due to the compact embedding $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \hookrightarrow C(\mathbb{S}^{k-1}, \mathcal{N})$, the set $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma$ is sequentially $W^{1,k}$ -weakly closed and hence, the infimum in (2.2) is achieved. However, the function E_{\min} fails to be a norm, in general, because it may not satisfy the triangle inequality (iii). To overcome this issue, for any $\sigma \in \pi_{k-1}(\mathcal{N})$ we define

$$|\sigma|_* := \inf \left\{ \sum_{i=1}^q E_{\min}(\sigma_i) : q \in \mathbb{N}, (\sigma_i)_{i=1}^q \in \pi_{k-1}(\mathcal{N})^q, \sum_{i=1}^q \sigma_i = \sigma \right\}. \quad (2.3)$$

Proposition 1. *The function $|\cdot|_*$ is a norm on $\pi_{k-1}(\mathcal{N})$ that satisfies (2.1) and $|\sigma|_* \leq E_{\min}(\sigma)$ for any $\sigma \in \pi_{k-1}(\mathcal{N})$. The infimum in (2.3) is achieved, for any $\sigma \in \pi_{k-1}(\mathcal{N})$. Moreover, the set*

$$\mathfrak{S} := \{\sigma \in \pi_{k-1}(\mathcal{N}) : |\sigma|_* = E_{\min}(\sigma)\} \quad (2.4)$$

is finite, and for any $\sigma \in \pi_{k-1}(\mathcal{N})$ there exists a decomposition $\sigma = \sum_{i=1}^q \sigma_i$ such that $|\sigma|_ = \sum_{i=1}^q |\sigma_i|_*$ and $\sigma_i \in \mathfrak{S}$ for any i .*

The proof of this result will be given in Appendix A. In case $\mathcal{N} = \mathbb{S}^{k-1}$, the group $\pi_{k-1}(\mathbb{S}^{k-1})$ is isomorphic to \mathbb{Z} , $\mathfrak{S} = \{-1, 0, 1\}$, and for any $d \in \mathbb{Z}$ we have

$$|d|_* = (k-1)^{k/2} \mathcal{L}^k(B_1^k) |d|,$$

where $\mathcal{L}^k(B_1^k)$ is the Lebesgue measure of the unit ball in \mathbb{R}^k and $|d|$ is the standard absolute value of d (see Example A.1).

Remark 5. When $k = 2$, the infimum in (2.2) is achieved by a minimising geodesic in the homotopy class σ , parametrised by multiples of arc-length. As a consequence, $E_{\min}(\sigma)$ is — up to a multiplicative constant — the length squared of a minimising geodesic in the class σ , and $E_{\min}^{1/2}$ is a norm on $\pi_1(\mathcal{N})$. However, $E_{\min}^{1/2}$ may not coincide with $|\cdot|_*$, not even up to a multiplicative constant. For instance, when \mathcal{N} is the flat torus, $\mathcal{N} = \mathbb{R}^2/(2\pi\mathbb{Z})^2 = \mathbb{S}^1 \times \mathbb{S}^1$, we have $\pi_1(\mathcal{N}) \simeq \mathbb{Z} \times \mathbb{Z}$,

$$E_{\min}^{1/2}(d_1, d_2) = \pi^{1/2} (d_1^2 + d_2^2)^{1/2} \quad \text{and} \quad |(d_1, d_2)|_* = \pi (|d_1| + |d_2|)$$

for any $(d_1, d_2) \in \mathbb{Z} \times \mathbb{Z}$. We did not investigate whether, for arbitrary $k > 2$ and \mathcal{N} , $E_{\min}^{1/k}$ is a norm on $\pi_{k-1}(\mathcal{N})$.

2.2. Notation for Flat Chains

We follow the notation adopted in [17, Section 2]. In particular, we denote by $\mathbb{F}_q(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$ the space of flat q -dimensional chains in \mathbb{R}^{n+k} with coefficients in the normed group $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$. We denote the flat norm by \mathbb{F} , and the mass by \mathbb{M} . The support of a flat chain S is denoted by $\text{spt } S$. The restriction of S to a Borel set $E \subseteq \mathbb{R}^{n+k}$ is denoted $S \llcorner E$. Given $f \in C^1(\mathbb{R}^{n+k}, \mathbb{R}^{n+k})$, we write f_*S for the push-forward of S through f . (The reader is referred e.g. to [26, 55] for the definitions of these objects.)

Given a domain $\Omega \subseteq \mathbb{R}^{n+k}$, we define $\mathbb{F}_q(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ as the set of flat chains such that $\text{spt } S \subseteq \overline{\Omega}$. We also define $\mathbb{M}_q(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ as the set of flat chains $S \in \mathbb{F}_q(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that $\mathbb{M}(S) < +\infty$. We will say that two chains $S_1, S_2 \in \mathbb{M}_q(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ are *cobordant in $\overline{\Omega}$* if and only if there exists a finite-mass chain $R \in \mathbb{M}_{q+1}(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that

$$S_2 - S_1 = \partial R.$$

In this case, we write $S_1 \sim_{\overline{\Omega}} S_2$. The cobordism in $\overline{\Omega}$ defines an equivalence relation on the space of finite-mass chains, $\mathbb{M}_q(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$. Moreover, due to the isoperimetric inequality (see e.g. [25, 7.6]), cobordism classes are closed with respect to the \mathbb{F} -norm.

The group of flat q -chains relative to a domain $\Omega \subseteq \mathbb{R}^{n+k}$ is defined as the quotient

$$\mathbb{F}_q(\Omega; \pi_{k-1}(\mathcal{N})) := \frac{\mathbb{F}_q(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))}{\{S \in \mathbb{F}_q(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N})) : \text{spt } S \subseteq \mathbb{R}^{n+k} \setminus \Omega\}}$$

To avoid notation, the equivalence class of a chain $S \in \mathbb{F}_q(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$ will still be denoted by S . The quotient norm may equivalently be rewritten as

$$\mathbb{F}_\Omega(S) = \inf \left\{ \mathbb{M}(P \llcorner \Omega) + \mathbb{M}(Q \llcorner \Omega) : P \in \mathbb{F}_{q+1}(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N})), \right. \\ \left. Q \in \mathbb{F}_q(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N})), \text{spt}(S - \partial P - Q) \subseteq \mathbb{R}^{n+k} \setminus \Omega \right\} \quad (2.5)$$

(see [17, Section 2.1]).

For any $S \in \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$ and $R \in \mathbb{F}_k(\mathbb{R}^{n+k}; \mathbb{Z})$ such that $\mathbb{M}(R) + \mathbb{M}(\partial R) < +\infty$, $\text{spt } R \subseteq \Omega$, and $\text{spt}(\partial S) \cap \text{spt } R = \text{spt } S \cap \text{spt}(\partial R) = \emptyset$, we denote the intersection index of S and R (as defined in [17, Section 2.1]) by $\mathbb{I}(S, R) \in \pi_{k-1}(\mathcal{N})$. For instance, if S is carried by a n -polyhedron with constant multiplicity $\sigma \in \pi_{k-1}(\mathcal{N})$, R is carried by a k -polyhedron with unit multiplicity and (the supports of) S, R intersect transversally, then $\mathbb{I}(S, R) = \pm\sigma$, where the sign depends on the relative orientation of S and R . The intersection index \mathbb{I} is a bilinear pairing and satisfies suitable continuity properties (see e.g. [17, Lemma 8]).

2.3. The Topological Singular Set

In [17], we constructed the topological singular set, $\mathbf{S}_y(u)$, for $u \in (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m)$ and $y \in \mathbb{R}^m$. Here, we introduce a variant of that construction and define $\mathbf{S}_y(u)$ in case $u \in W^{1,k}(\Omega, \mathbb{R}^m)$, without assuming that $u \in L^\infty(\Omega, \mathbb{R}^m)$. In both cases, the operator $\mathbf{S}_y(u)$ generalises the Jacobian determinant of u — and indeed, the Jacobian of $u: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is well-defined in a distributional sense if $u \in (L^\infty \cap W^{1,k-1})(\mathbb{R}^k, \mathbb{R}^k)$, and in a pointwise sense if $u \in W^{1,k}(\mathbb{R}^k, \mathbb{R}^k)$. The starting point of the construction is the following topological property:

Proposition 2. ([30]). *Under the assumption (H_2) , there exist a compact, polyhedral complex $\mathcal{X} \subseteq \mathbb{R}^m$ of dimension $m - k$ and a smooth map $\varrho: \mathbb{R}^m \setminus \mathcal{X} \rightarrow \mathcal{N}$ such that $\varrho(z) = z$ for any $z \in \mathcal{N}$, and*

$$|\nabla \varrho(z)| \leq \frac{C}{\text{dist}(z, \mathcal{X})}$$

for any $z \in \mathbb{R}^m \setminus \mathcal{X}$ and some constant $C = C(\mathcal{N}, m, \mathcal{X}) > 0$.

This result, or variants thereof, was proved in [30, Lemma 6.1], [12, Proposition 2.1], [33, Lemma 4.5]. While in our previous paper [17] we required \mathcal{X} to be a smooth complex, in this paper we require \mathcal{X} to be polyhedral, because this will simplify some technical points in the proofs.

Let us fix once and for all a polyhedral complex \mathcal{X} and a map ϱ , as in Proposition 2. Let $\delta^* \in (0, \text{dist}(\mathcal{N}, \mathcal{X}))$ be fixed, and let $B^* := B^m(0, \delta^*) \subseteq \mathbb{R}^m$. Let*

$$Y := L^1(B^*, \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))), \quad \bar{Y} := L^1(B^*, \mathbb{F}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N})))$$

be the set of Lebesgue-measurable maps $S: B^* \rightarrow \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$, respectively $S: B^* \rightarrow \mathbb{F}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$ (we use the notation $y \in B^* \mapsto S_y$ in both cases), such that

$$\|S\|_Y := \int_{B^*} \mathbb{F}_\Omega(S_y) \, dy < +\infty, \quad \text{respectively} \quad \|S\|_{\bar{Y}} := \int_{B^*} \mathbb{F}(S_y) \, dy < +\infty.$$

The sets Y, \bar{Y} are complete normed moduli, with the norms $\|\cdot\|_Y, \|\cdot\|_{\bar{Y}}$ respectively. The space $\mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$, respectively $\mathbb{F}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$, embeds canonically into Y , respectively \bar{Y} . If need be, we will identify a chain $S \in \mathbb{F}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$ with an element of \bar{Y} , i.e. the constant map $y \mapsto S$.

By [17, Theorem 3.1], there exists a unique operator

$$\mathbf{S}: (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m) \rightarrow Y$$

that is continuous (if $u_j \rightarrow u$ strongly in $W^{1,k-1}(\Omega)$ and $\sup_j \|u_j\|_{L^\infty(\Omega)} < +\infty$, then $\mathbf{S}(u_j) \rightarrow \mathbf{S}(u)$ in Y) and satisfies

(P₀) for any smooth u , a.e. $y \in B^*$ and any $R \in \mathbb{F}_k(\mathbb{R}^{n+k}; \mathbb{Z})$ such that $\mathbb{M}(R) + \mathbb{M}(\partial R) < +\infty$, $\text{spt}(R) \subseteq \Omega$, $\text{spt}(\partial R) \subseteq \Omega \setminus \text{spt} \mathbf{S}_y(u)$, there holds

$$\mathbb{I}(\mathbf{S}_y(u), R) = \text{homotopy class of } \varrho \circ (u - y) \text{ on } \partial R.$$

We recall that \mathbb{I} denotes the intersection index, defined as in [17, Section 2.1].

Proposition 3. *There exists a (unique) continuous operator*

$$\bar{\mathbf{S}}: W^{1,k}(\Omega, \mathbb{R}^m) \rightarrow \bar{Y}$$

that satisfies (P_0) and the following properties:

(P_1) For any $u \in (L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$ and a.e. $y \in B^*$, $\bar{\mathbf{S}}_y(u) = \mathbf{S}_y(u)$ — more precisely, the chain $\bar{\mathbf{S}}_y(u)$ belongs to the equivalence class $\mathbf{S}_y(u) \in \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$.

(P_2) For any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ and any Borel subset $E \subseteq \bar{\Omega}$, there holds

$$\int_{B^*} \mathbb{M}(\bar{\mathbf{S}}_y(u) \llcorner E) \, dy \lesssim \int_E |\nabla u|^k.$$

(P_3) If $u_0, u_1 \in W^{1,k}(\Omega, \mathbb{R}^m)$ are such that $u_0|_{\partial\Omega} = u_1|_{\partial\Omega} \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ (in the sense of traces), then $\bar{\mathbf{S}}_{y_0}(u_0) \sim_{\bar{\Omega}} \bar{\mathbf{S}}_{y_1}(u_1)$ for a.e. $y_0, y_1 \in B^*$.

The proof of Proposition 3 will be given in Appendix B. Taking account of (P_1) , we abuse of notation and write \mathbf{S} instead of $\bar{\mathbf{S}}$ from now on. As a consequence of (P_3) , for any boundary datum $v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ there exists a unique cobordism class $\mathcal{C}(\Omega, v) \subseteq \mathbb{M}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that

$$\mathbf{S}_y(u) \in \mathcal{C}(\Omega, v) \tag{2.6}$$

for any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ with trace v on $\partial\Omega$ and for a.e. $y \in B^*$.

2.4. The Γ -convergence Result

The main result of this paper is a generalisation of [2, Theorem 5.5]. We let $W_v^{1,k}(\Omega, \mathbb{R}^m)$ denote the set of maps $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ such that $u = v$ on $\partial\Omega$ (in the sense of traces).

Theorem C. *Suppose that the assumptions (H_1) – (H_4) are satisfied. Then, the following properties hold:*

(i) Compactness and lower bound. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $W_v^{1,k}(\Omega, \mathbb{R}^m)$ that satisfies $\sup_{\varepsilon>0} |\log \varepsilon|^{-1} E_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists a (non re-labelled) countable subsequence and a finite-mass chain $S \in \mathcal{C}(\Omega, v)$ such that $\mathbf{S}(u_\varepsilon) \rightarrow S$ in \bar{Y} and, for any open subset $A \subseteq \mathbb{R}^{n+k}$,*

$$\mathbb{M}(S \llcorner A) \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log \varepsilon|}.$$

(ii) Upper bound. *For any finite-mass chain $S \in \mathcal{C}(\Omega, v)$, there exists a sequence (u_ε) in $W_v^{1,k}(\Omega, \mathbb{R}^m)$ such that $\mathbf{S}(u_\varepsilon) \rightarrow S$ in \bar{Y} and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leq \mathbb{M}(S).$$

Theorem A follows almost immediately from Theorem C, combined with general properties of the Γ -convergence and standard facts in measure theory. There is a variant of Theorem C for the problem with no boundary conditions, which is analogous to [2, Theorem 1.1]. We will say that a chain S is a finite-mass, n -dimensional relative boundary if it has form $S = (\partial R)\llcorner\Omega$, where $R \in \mathbb{M}_{n+1}(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$ is such that $\mathbb{M}(\partial R) < +\infty$.

Proposition D. *Suppose that the assumptions (H₁)–(H₃) are satisfied. Then, the following properties hold:*

- (i) *Compactness and lower bound. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $W^{1,k}(\Omega, \mathbb{R}^m)$ that satisfies $\sup_{\varepsilon>0} |\log \varepsilon|^{-1} E_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists a (non re-labelled) countable subsequence and a finite-mass, n -dimensional relative boundary S such that $\mathbf{S}(u_\varepsilon) \rightarrow S$ in Y and, for any open subset $A \subseteq \Omega$,*

$$\mathbb{M}(S \llcorner A) \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log \varepsilon|}.$$

- (ii) *Upper bound. For any finite-mass, n -dimensional relative boundary S , there exists a sequence (u_ε) in $W^{1,k}(\Omega, \mathbb{R}^m)$ such that $\mathbf{S}(u_\varepsilon) \rightarrow S$ in Y and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leq \mathbb{M}(S).$$

Proposition D is not quite informative as it stands, because minimisers of the functional (1.1) under no boundary conditions are constant. However, since Γ -convergence is stable with respect to continuous perturbations, Proposition D can be extended to non-trivial minimisation problems with lower-order terms or under integral constraints, as long as these are compatible with the topology of Γ -convergence.

2.5. A Few Examples

We illustrate our results by means of a few simple examples. If $A \subseteq \mathbb{R}^{n+k}$ is an n -dimensional polyhedral (or smooth) set, with a given orientation, the unit-multiplicity chain carried by A will be denoted $\llbracket A \rrbracket \in \mathbb{M}_n(\mathbb{R}^{n+k}; \mathbb{Z})$.

Example 2.1. First, we suppose the domain is the unit ball in the critical dimension, i.e. $n = 0$ and $\Omega = B^k$, and consider the target $\mathcal{N} = \mathbb{S}^{k-1} \subseteq \mathbb{R}^k$. We need to identify the class $\mathcal{C}(\Omega, v)$ defined by (2.6). For simplicity, suppose that the boundary datum $v: \partial B^k \rightarrow \mathbb{S}^{k-1}$ is smooth, of degree d . (General data $v \in W^{1-1/k, k}(\partial B^k, \mathbb{S}^{k-1})$ could also be considered, by appealing to BREZIS AND NIRENBERG's theory of the degree in VMO, [14]). Let $u: B^k \rightarrow \mathbb{R}^k$ be any smooth extension of v . Let $y \in \mathbb{R}^k$ be a regular value for u (i.e., $\det \nabla u(x) \neq 0$ for any $x \in u^{-1}(y)$) such that $|y| < 1$. Then, the inverse image $u^{-1}(y)$ consists of a finite number points. Let $r > 0$ be a sufficiently small radius. By definition of \mathbf{S} , we have

$$\mathbf{S}_y(u) = \sum_{x \in u^{-1}(y)} d(x) \llbracket x \rrbracket \in \mathbb{M}_0(\bar{B}^k; \mathbb{Z}),$$

where $d(x)$ is the degree of the map $(u - y)/|u - y|: \partial B_r(x) \rightarrow \mathbb{S}^{k-1}$. The class $\mathcal{C}(\Omega, v)$ consists of all and only the chains that differ from $\mathbf{S}_y(u)$ by a boundary. It is not difficult to characterise $\mathcal{C}(\Omega, v)$ using the following topological property, which holds true for any (normed, Abelian) coefficient group \mathbf{G} and any connected, open set $D \subseteq \mathbb{R}^d$.

Fact. *Let T be a 0-chain of the form $T = \sum_{i=1}^q \sigma_j \llbracket z_i \rrbracket$, for $z_j \in \bar{D}$, $\sigma_j \in \mathbf{G}$. Then, there exists $R \in \mathbb{M}_1(\bar{D}; \mathbf{G})$ such that $\partial R = T$ if and only if $\sum_{j=1}^q \sigma_j = 0$.*

For a proof of this fact, see e.g. [31, Proposition 2.7]. Now, Brouwer's theory of the degree (or Property (P₀) above) implies that

$$\sum_{x \in u^{-1}(y)} d(x) = \sum_{x \in u^{-1}(y)} \text{sign}(\det \nabla u(x)) = d,$$

therefore

$$\mathcal{C}(\Omega, v) = \left\{ \sum_{j=1}^q \sigma_j \llbracket z_j \rrbracket : q \in \mathbb{N}, (\sigma_j)_{j=1}^q \in \mathbb{Z}^q, (z_j)_{j=1}^q \in (\bar{B}^k)^q, \sum_{j=1}^q \sigma_j = d \right\}.$$

In agreement with the Ginzburg–Landau theory, mass-minimising chains in $\mathcal{C}(\Omega, v)$ consist of exactly $|d|$ points, with multiplicities equal to 1 or -1 according to the sign of d . This argument extends to more general manifolds \mathcal{N} , with no essential change; we obtain

$$\mathcal{C}(\Omega, v) = \left\{ \sum_{j=1}^q \sigma_j \llbracket z_j \rrbracket : q \in \mathbb{N}, \sigma_j \in \pi_{k-1}(\mathcal{N}), z_j \in \bar{B}^k, \sum_{j=1}^q \sigma_j = \sigma \right\},$$

where $\sigma \in \pi_{k-1}(\mathcal{N})$ is the homotopy class of the boundary datum $v: \partial B^k \rightarrow \mathcal{N}$. Mass-minimising chains in $\mathcal{C}(\Omega, v)$ have the form $\sum_{j=1}^q \sigma_j \llbracket z_j \rrbracket$, where the multiplicities σ_j belong to the set \mathfrak{S} defined in (2.4) and satisfy $\sum_{j=1}^q E_{\min}(\sigma_j) = |\sigma|_*$.

Example 2.2. Next, we discuss the case $n = 1$, $\Omega = B^{k+1}$. Suppose that the boundary datum $v: \partial B^{k+1} \rightarrow \mathcal{N}$ is smooth, except for finitely many isolated singularities at the points x_1, \dots, x_p . Let D_1, \dots, D_p be pairwise-disjoint closed geodesic disks in ∂B^{k+1} , centred at the points x_1, \dots, x_p . Each D_i is given the orientation induced by the outward-pointing unit normal to B^{k+1} . Using orientation-preserving coordinate charts, we may identify $v|_{\partial D_i}: \partial D_i \rightarrow \mathcal{N}$ with a map $\mathbb{S}^{k-1} \rightarrow \mathcal{N}$; the homotopy class of the latter is an element of $\pi_{k-1}(\mathcal{N})$, which we denote σ_i . The coefficients σ_i must satisfy the topological constraint

$$\sum_{i=1}^p \sigma_i = 0. \quad (2.7)$$

Indeed, let $D^+ \subseteq \partial B^{k+1}$ be a small geodesic disk that does not contain any singular point x_i , and let $D^- := \partial B^{k+1} \setminus D^+$. Topologically, D^- is a disk which contains

all the singular points of v ; therefore, the homotopy class of v restricted to ∂D^- is the sum of all the σ_i 's above. However, the homotopy class of v on ∂D^+ must be trivial, because v is smooth in D^+ . Thus, (2.7) follows.

We consider the chain

$$\mathbf{S}^{\text{bd}}(v) := \sum_{i=1}^p \sigma_i \llbracket x_i \rrbracket \in \mathbb{M}_0(\partial\Omega; \pi_{k-1}(\mathcal{N})).$$

Thanks to (2.7), $\mathbf{S}^{\text{bd}}(v)$ is the boundary of some 1-chain supported in \bar{B}^{k+1} . More precisely, let $u \in W^{1,k}(B^{k+1}, \mathbb{R}^m)$ be any extension of v . The results of [17] (see, in particular, Proposition 1, Proposition 3 and Lemma 18) imply that

$$\partial \mathbf{S}_y(u) = \mathbf{S}^{\text{bd}}(v)$$

for a.e. $y \in \mathbb{R}^m$ of norm small enough. Chains in the same homology class have the same boundary; therefore, for any chain $T \in \mathcal{C}(\Omega, v)$, there holds $\partial T = \mathbf{S}^{\text{bd}}(v)$. Conversely, two chains in \bar{B}^{k+1} that have the same boundary belong to same homology class (relative to \bar{B}^{k+1}), because the domain \bar{B}^{k+1} is contractible. As a consequence, we have

$$\mathcal{C}(\Omega, v) = \left\{ T \in \mathbb{M}_1(\bar{\Omega}; \pi_{k-1}(\mathcal{N})) : \partial T = \mathbf{S}^{\text{bd}}(v) \right\}. \quad (2.8)$$

In particular, mass-minimising chains in $\mathcal{C}(\Omega, v)$ will be carried by a finite union of segments, connecting the singularities of the boundary datum according to their multiplicities. In case $\mathcal{N} = \mathbb{S}^{k-1}$, such union of segments realises a ‘minimising connection’, in the sense of BREZIS ET AL. [13]. For $k = 2$ and $\mathcal{N} = \mathbb{R}\mathbb{P}^2$, the condition (2.7) implies that v has an *even* number of non-orientable singularities; mass-minimising chains connect the non-orientable singularities in pairs.

The characterisation (2.8) extends to general data $v \in W^{1-1/k,k}(\partial B^{k+1}, \mathcal{N})$, provided that we define $\mathbf{S}^{\text{bd}}(v)$ in a suitable way (see [17, Section 3]). It also extend to more general domains $\Omega \subseteq \mathbb{R}^{n+k}$, so long as the n -th homology group $H_n(\Omega; \pi_{k-1}(\mathcal{N}))$ is trivial.

Example 2.3. If the domain has a non-trivial topology, then $\mathcal{C}(\Omega, v)$ may contain non-trivial chains even if the boundary datum is smooth. For instance, take $n = 1$, $k = 2$, $\mathcal{N} = \mathbb{S}^1$. Let $\Omega \subseteq \mathbb{R}^3$ be a solid torus of revolution, defined as the image of the map $\Psi : B^2 \times \mathbb{R} \rightarrow \mathbb{R}^3$,

$$\Psi(x, \theta) := ((x_1 + 2) \cos \theta, (x_1 + 2) \sin \theta, x_2) \quad \text{for } x = (x_1, x_2) \in B^2, \theta \in \mathbb{R}.$$

We consider the smooth map $u : \Omega \rightarrow \mathbb{R}^2$ given by $u(\Psi(x, \theta)) := x$ for $(x, \theta) \in B^2 \times \mathbb{R}$. The trace of u at the boudary, v , takes its values in \mathbb{S}^1 and its restriction on each meridian curve of the torus $\partial\Omega$ has degree 1. Therefore, $\mathcal{C}(\Omega, v)$ is the homology class of $\llbracket u^{-1}(0) \rrbracket \in \mathbb{M}_1(\bar{\Omega}; \mathbb{Z})$, where $u^{-1}(0)$ is the zero-set of u (i.e. the circle $\Psi(\{(0, 0)\} \times \mathbb{R})$) with the orientation induced by Ψ . The elements of $\mathcal{C}(\Omega, v)$ can be characterised by means of the intersection index \mathbb{I} . More precisely, let D be the closure of $\Psi(B^2 \times \{0\})$. D is a 2-disk in the plane orthogonal to $(0, 1, 0)$;

we give D the orientation induced by $(0, 1, 0)$. By the Poincaré-Lefschetz duality (see e.g. [27, Theorem 3, p. 631]), for any $T \in \mathbb{M}_1(\overline{\Omega}; \mathbb{Z})$ we have

$$T \in \mathcal{C}(\Omega, \nu) \quad \text{if and only if} \quad \partial T = 0 \quad \text{and} \quad \mathbb{I}(T, \llbracket D \rrbracket) = 1.$$

By a slicing argument, we deduce that the (unique) mass-minimising chain S_{\min} in $\mathcal{C}(\Omega, \nu)$ is carried by an equator of $\partial\Omega$:

$$S_{\min} := \llbracket \Psi(\{(-1, 0)\} \times \mathbb{R}) \rrbracket,$$

with the orientation induced by Ψ . (See, e.g., [16, Section 5.4] for a similar example, in case $\mathcal{N} = \mathbb{R}\mathbb{P}^2$.)

3. Upper Bounds

3.1. Notations and Sketch of the Construction

We say that a map $u: \Omega \rightarrow \mathbb{R}^m$ is *locally piecewise affine* if u is continuous in Ω and, for any polyhedral set $K \subset\subset \Omega$, the restriction $u|_K$ is piecewise affine. A set $P \subseteq \Omega$ is called *locally n -polyhedral* if, for any compact set $K \subseteq \Omega$, there exists a finite union Q of convex, compact, n -dimensional polyhedra such that $P \cap K = Q \cap K$. In a similar way, we say that a finite-mass chain $S \in \mathbb{M}_n(\overline{\Omega}; \pi_{k-1}(\mathcal{N}))$ is *locally polyhedral* if, for any compact set $K \subseteq \Omega$, there exists a polyhedral chain T such that $(S - T) \llcorner K = 0$. If M is a polyhedral complex and $j \geq 0$ is an integer, we denote by M_j the j -skeleton of M , i.e. the union of all its faces of dimension less than or equal to j . We set $M_{-1} := \emptyset$.

Maps with nice and η -minimal singularities. To construct a recovery sequence, we will work with \mathcal{N} -valued maps with well-behaved singularities, in a sense that is made precise by the definition below. Let M, S be polyhedral sets in \mathbb{R}^{n+k} of dimension $n, n-1$ respectively, and let $u: \Omega \subseteq \mathbb{R}^{n+k} \rightarrow \mathbb{R}^m$.

Definition 3.1. ([1, 2]) We say that u has a *nice singularity at M* if u is locally Lipschitz on $\overline{\Omega} \setminus M$ and there exists a constant C such that

$$|\nabla u(x)| \leq C \operatorname{dist}^{-1}(x, M) \quad \text{for a.e. } x \in \Omega \setminus M.$$

We say that u has a *nice singularity at (M, S)* if u is locally Lipschitz on $\overline{\Omega} \setminus (M \cup S)$ and, for any $p > 1$, there is a constant C_p such that

$$|\nabla u(x)| \leq C_p \left(\operatorname{dist}^{-1}(x, M) + \operatorname{dist}^{-p}(x, S) \right) \quad \text{for a.e. } x \in \Omega \setminus (M \cup S).$$

We say that u has a *locally nice singularity at M* (respectively, at (M, S)) if, for any open subset $W \subset\subset \Omega$, the restriction $u|_W$ has a nice singularity at M (respectively, at (M, S)).

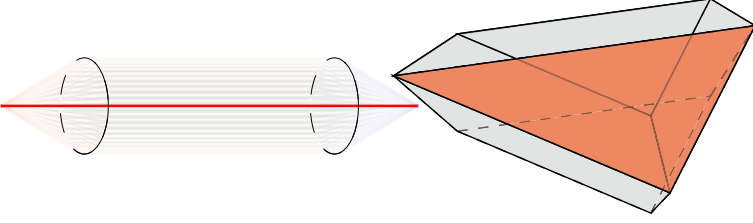


Fig. 1. The set $U(K, \delta, \gamma)$, in case $n = 1, k = 2$ (left) and $n = 2, k = 1$ (right). In both cases, the polyhedron K is in red

Remark 6. If u has a nice singularity at (M, S) then $u \in W^{1,k-1}(\Omega, \mathbb{R}^m)$, since both M and S have codimension strictly larger than $k - 1$ (see e.g. [2, Lemma 8.3] for more details). In particular, if $u: \Omega \rightarrow \mathcal{N}$ has a nice singularity at (M, S) , then $\mathbf{S}_y(u) \in \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$ is well-defined for a.e. $y \in B^*$. Actually, $\mathbf{S}_{y_1}(u) = \mathbf{S}_{y_2}(u)$ for a.e. $y_1, y_2 \in B^*$ [17, Proposition 3], and we will write $\mathbf{S}(u) := \mathbf{S}_{y_1}(u) = \mathbf{S}_{y_2}(u)$. The chain $\mathbf{S}(u)$ is supported on M , and its multiplicities coincide with the homotopy class of u around each n -face of M (see [17, Lemma 18]).

Throughout Section 3, we will work with maps with nice (or locally nice) singularities. However, in order to obtain sharp energy estimates, we will need to impose a further restriction on the behaviour of our maps near the singularities. Let $u: \Omega \rightarrow \mathcal{N}$ be a map with nice singularity at (M, S) , where M, S are polyhedral sets of dimension $n, n - 1$ respectively. We triangulate M , i.e. we write M as a finite union of closed simplices such that, if K', K are simplices with $K \neq K'$, $K \cap K' \neq \emptyset$, then $K \cap K'$ is a boundary face of both K and K' . Let $K \subseteq M$ be a n -dimensional simplex of the triangulation, and let K^\perp be the k -plane orthogonal to K through the origin. Given positive parameters δ, γ , we define the set

$$U(K, \delta, \gamma) := \left\{ x' + x'' : x' \in K, x'' \in K^\perp, |x''| \leq \min(\delta, \gamma \operatorname{dist}(x', \partial K)) \right\} \quad (3.1)$$

(see Figure 1). We will identify each $x \in U(K, \delta, \gamma)$ with a pair $x = (x', x'')$, where x', x'' are as in (3.1). By choosing δ, γ small enough (uniformly in K), we can make sure that the sets $U(K, \delta, \gamma)$ have pairwise disjoint interiors.

Definition 3.2. Let $u: \Omega \rightarrow \mathcal{N}$ be a map with nice singularity at (M, S) , and let $\eta > 0$. We say that u is η -minimal if there exist positive numbers δ, γ , a triangulation of M and, for any n -simplex K of the triangulation, a Lipschitz map $\phi_K: \mathbb{S}^{k-1} \rightarrow \mathcal{N}$ that satisfy the following properties.

- (i) If $K \subseteq M, K' \subseteq M$ are n -simplices with $K \neq K'$, then $U(K, \delta, \gamma)$ and $U(K', \delta, \gamma)$ have disjoint interiors.
- (ii) For any n -dimensional simplex $K \subseteq M$ and a.e. $x = (x', x'') \in U(K, \delta, \gamma)$, we have $u(x) = \phi_K(x''/|x''|)$.
- (iii) For any n -dimensional simplex $K \subseteq M$ and any map $\zeta \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N})$ that is homotopic to ϕ_K , we have

$$\int_{\mathbb{S}^{k-1}} |\nabla_{\top} \phi_K|^k d\mathcal{H}^{k-1} \leq \int_{\mathbb{S}^{k-1}} |\nabla_{\top} \zeta|^k d\mathcal{H}^{k-1} + \eta.$$

The operator ∇_{\top} is the tangential gradient on \mathbb{S}^{k-1} , i.e. the restriction of the Euclidean gradient ∇ to the tangent plane to the sphere.

Remark 7. Thanks to the Sobolev embedding $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \hookrightarrow C(\mathbb{S}^{k-1}, \mathcal{N})$, smooth maps are dense in $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N})$. Therefore, for any $\eta > 0$ and any homotopy class $\sigma \in \pi_{k-1}(\mathcal{N})$, there exists a smooth map $\phi: \mathbb{S}^{k-1} \rightarrow \mathcal{N}$ in the homotopy class σ that satisfies

$$\int_{\mathbb{S}^{k-1}} |\nabla_{\top}\phi|^k d\mathcal{H}^{k-1} \leq \int_{\mathbb{S}^{k-1}} |\nabla_{\top}\zeta|^k d\mathcal{H}^{k-1} + \eta \quad (3.2)$$

for any $\zeta \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma$.

Remark 8. It is possible to find C^1 -maps that satisfy a stronger version of (3.2), with $\eta = 0$. Indeed, the compact Sobolev embedding $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \hookrightarrow C(\mathbb{S}^{k-1}, \mathcal{N})$ implies that homotopy classes of maps $\mathbb{S}^{k-1} \rightarrow \mathcal{N}$ are sequentially closed with respect to the weak $W^{1,k}$ -convergence. Then, for each homotopy class $\sigma \in \pi_{k-1}(\mathcal{N})$, there exists a map ϕ_{σ} that minimises the L^k -norm of the gradient in σ . The map ϕ_{σ} solves the k -harmonic map equation and, by Sobolev embedding, is continuous. Then, regularity results for k -harmonic maps (e.g. [24, Proposition 5.4]) imply that $\phi_{\sigma} \in C^{1,\alpha}(\mathbb{S}^{k-1}, \mathcal{N})$. However, the weaker condition (3.2) is enough for our purposes.

Construction of a recovery sequence: a sketch. In most of this section, we focus on the proof of Theorem C.(ii), i.e. we study the problem in the presence of boundary conditions; only at the end of section, we present the proof of Proposition D.(ii). As in [2], in order to define a recovery sequence, we first construct a map $w: \Omega \rightarrow \mathcal{N}$ with (locally) nice singularity and prescribed singular set $\mathbf{S}(w) = S$. However, w must also satisfy the boundary condition, $w = v$ on $\partial\Omega$, where $v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ is a datum. This boundary condition makes the construction of w substantially harder. For such a w to exist, we need a topological assumption on S , namely, that S belongs to the homology class (2.6) determined by Ω and v . Our approach is rather different from that of [2, Theorem 5.3]. In [2], the authors first construct w inside Ω , then interpolate near $\partial\Omega$, using the symmetries of the target \mathbb{S}^{k-1} , so as to match the boundary datum. On the contrary, we start from a map that satisfies the boundary conditions and we modify it inside Ω so to obtain $\mathbf{S}(w) = S$. Before giving the details, we sketch the main steps of our construction.

First, we consider a locally piecewise affine extension $u_* \in (L^{\infty} \cap W^{1,k})(\Omega, \mathbb{R}^m)$ of v . Since we have assumed that \mathcal{X} is polyhedral, the singular set $\mathbf{S}_y(u_*)$ will be locally polyhedral, for a.e. y . By projecting u_* onto \mathcal{N} (using HARDT ET AL. [29], see Section 3.3), we define a map $w_*: \Omega \rightarrow \mathcal{N}$ such that $w_* = v$ on $\partial\Omega$, $\mathbf{S}(w_*) = \mathbf{S}_y(u_*)$ (for a well-chosen y) is locally polyhedral, and w_* has a locally nice singularity at $\text{spt } \mathbf{S}(w_*)$. We cannot make sure that the singularity is nice up to the boundary of Ω , because the boundary datum is not regular enough.

Let S be a finite-mass n -chain in the homology class $\mathcal{C}(\Omega, v)$ defined by (2.6). Thanks to (P₃), we know that $\mathbf{S}(w_*) = \mathbf{S}_y(u_*) \in \mathcal{C}(\Omega, v)$ and hence, $\mathbf{S}(w_*)$ and S

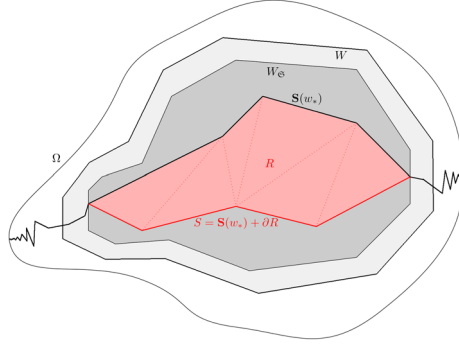


Fig. 2. Sketch of the construction of a recovery sequence. Inside $W_{\mathfrak{G}}$, the chain S (in red) takes multiplicities in the set $\mathfrak{G} \subseteq \pi_{k-1}(\mathcal{N})$. Outside W , the original map w_* and the modified map w coincide

differ by a boundary. By approximation (see Section 3.4.2), we reduce to the case

$$S = \mathbf{S}(w_*) + \partial R,$$

where R is a polyhedral $(n+1)$ -chain with compact support in Ω . Actually, we can make a further assumption on S . Let $W_{\mathfrak{G}} \subset\subset \Omega$ be an open set, with polyhedral boundary, whose closure contains the support of R (see Figure 2). Up to a density argument (Proposition 6), we can assume that $S \llcorner W_{\mathfrak{G}}$ takes its multiplicities in the set $\mathfrak{G} \subseteq \pi_{k-1}(\mathcal{N})$ defined by (2.4). Roughly speaking, we replace each polyhedron K of $S \llcorner W_{\mathfrak{G}}$ with a finite number of polyhedra, very close to each other, whose multiplicities add up to the multiplicity of K . This is possible, because \mathfrak{G} generates $\pi_{k-1}(\mathcal{N})$ by Proposition 1. The assumption on the multiplicity of $S \llcorner W_{\mathfrak{G}}$ turns out to be essential to obtain sharp energy bounds for our recovery sequence.

Let W be another open set, with polyhedral boundary, such that $W_{\mathfrak{G}} \subset\subset W \subset\subset \Omega$ (see Figure 2). In particular, W contains the support of R . We aim to modify w_* inside W , so to obtain a new map $w: \Omega \rightarrow \mathcal{N}$ with locally nice singularities and $\mathbf{S}(w) = \mathbf{S}(w_*) + \partial R = S$. In other words, we need to “move” the singularities of w_* along the boundary of R . This is the key step in the construction. We achieve this goal by a suitable generalisation of the so-called “insertion of dipoles”, Proposition 4 in Section 3.2. For any $(n+1)$ -polyhedron T of R , we modify w_* in a neighbourhood of T by inserting an \mathcal{N} -valued map that depends only on the $k-1$ coordinates in the orthogonal directions to T . To define w near ∂T , we use radial projections repeatedly, first onto the n -skeleton of T , then onto its $(n-1)$ -skeleton, and so on. Eventually, we obtain a map $w: \Omega \rightarrow \mathcal{N}$ that agrees with w_* out of a neighbourhood of $\text{spt } R$ (in particular, it matches the boundary datum), has locally nice singularities at S and satisfies $\mathbf{S}(w) = S$. By local surgery ([2, Lemma 9.3], stated below as Lemma 6), we can also make sure that $w|_W$ is η -minimal.

The map w does not belong to the energy space $W^{1,k}(\Omega, \mathbb{R}^m)$, unless $S = 0$, because it has a singularity of codimension k . Therefore, we must regularise w to

construct a recovery sequence. For $x \in W$, we define

$$u_\varepsilon(x) := \min \left(\frac{\text{dist}(x, \text{spt } S)}{\varepsilon}, 1 \right) w(x).$$

Since w is η -minimal in W , a fairly explicit computation allows us to estimate the energy of u_ε on W , in terms of the area of $\text{spt } S$ and the maps ϕ_K given by Definition 3.2. Moreover, for any simplex K of $S \llcorner W_\mathfrak{S}$, the multiplicity σ_K of S at K belongs to \mathfrak{S} and hence,

$$\frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} \phi_K|^k \, d\mathcal{H}^{k-1} \leq |\sigma_K|_* + \eta,$$

because of Definition 3.2 and (2.4). Thanks to this inequality, we can indeed estimate $E_\varepsilon(u_\varepsilon, W)$ in terms of the mass of S , up to remainder terms that can be made arbitrarily small. However, this approach is not viable near the boundary of Ω , because the regularity of w degenerates near $\partial\Omega$. Instead, we define u_ε on $\Omega \setminus W$ by adapting [49, Proposition 2.1], see Section 3.3. The two pieces—inside and outside W —are glued together by linear interpolation.

3.2. Insertion of Dipoles Along a Simplex

Our next result, Proposition 4, is the main building block in the construction of the recovery sequence.

Proposition 4. *Let $D \subseteq \mathbb{R}^{n+k}$ be a bounded domain. Let $\Sigma \subseteq D$ be a polyhedral set of dimension n , and $u \in W^{1,k-1}(D, \mathcal{N})$ a map with nice singularity at Σ . Let $T \subset\subset D$ be an oriented simplex of dimension $n+1$ and $\sigma \in \pi_{k-1}(\mathcal{N})$. Then, there exists a map $\tilde{u} \in W^{1,k-1}(D, \mathcal{N})$, with nice singularity at a polyhedral set of dimension n , such that $\tilde{u} = u$ in a neighbourhood of ∂D and $\mathbf{S}(\tilde{u}) = \mathbf{S}(u) + \sigma \partial \llbracket T \rrbracket$.*

Perhaps it is worth commenting on the assumptions of Proposition 4. In terms of regularity of \mathcal{N} , we do not need to work with smooth manifolds: a compact, connected Lipschitz neighbourhood retract would do. The assumption that \mathcal{N} is $(k-2)$ -connected could also be relaxed. $(k-2)$ -connectedness is used in [17, 47] to construct $\mathbf{S}(u)$ for arbitrary $u \in W^{1,k-1}(\Omega, \mathcal{N})$; however, if u has nice singularities and $\pi_{k-1}(\mathcal{N})$ is Abelian, then $\mathbf{S}(u)$ can be defined in a straightforward way. On the other hand, we must assume that \mathcal{N} is $(k-1)$ -free (that is, the fundamental group of \mathcal{N} acts trivially on $\pi_{k-1}(\mathcal{N})$). Should \mathcal{N} not be $(k-1)$ -free, we could not identify free homotopy classes of maps $\mathbb{S}^{k-1} \rightarrow \mathcal{N}$ with elements of $\pi_{k-1}(\mathcal{N})$. In this case, the product of free homotopy classes $\mathbb{S}^{k-1} \rightarrow \mathcal{N}$ is multi-valued and hence, the equality $\mathbf{S}(\tilde{u}) = \mathbf{S}(u) + \sigma \partial \llbracket T \rrbracket$ may fail.

The proof of Proposition 4 (see Figure 3) is based on a construction known as “insertion of dipoles”. Several variants of this construction are available in the literature (see e.g. [6, 7, 13, 27, 47]), but all of them rely on the following fact: a map $B^{k-1} \rightarrow \mathcal{N}$ that takes a constant value on ∂B^{k-1} may be identified with a map $\mathbb{S}^{k-1} \rightarrow \mathcal{N}$, by collapsing the boundary of the disk to a point. As a consequence, if a continuous map $\phi: B^{k-1} \rightarrow \mathcal{N}$ is constant on ∂B^{k-1} , then we

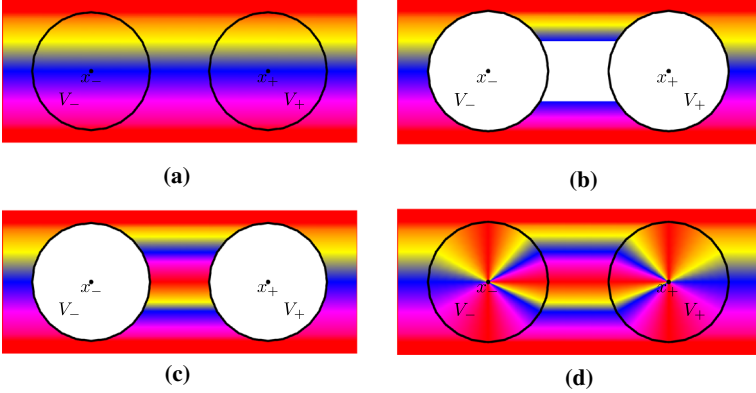


Fig. 3. Idea of the proof of Proposition 4: an example with $k = 2$, $n = 0$ and $\mathcal{N} = \mathbb{S}^1$. The initial map u is plotted in (a); the values of u are represented by the colour code. We aim to insert singularities of degrees 1, -1 at the points x_+ , x_- . First, we reparametrise u , creating a ‘slit’ along the segment of endpoints x_+ and x_- (b). Then, we fill the slit by inserting a map that winds around the circle exactly once, as we move in the direction orthogonal to the segment of endpoints x_+ , x_- (c). Finally, we define \tilde{u} in the disks V_+ , V_- in such a way that \tilde{u} is homogeneous inside each disk (d). The new map \tilde{u} behaves as required. For instance, there are exactly three yellow points on ∂V_+ ; as we move anticlockwise around ∂V_+ , two of them carry the orientation ‘from red to blue’ and the other one carries the opposite orientation ‘from blue to red’. If we orient the target \mathbb{S}^1 ‘from red to yellow to blue’, then the degree of \tilde{u} on ∂V_+ is 1

may define the homotopy class of ϕ as an element of $\pi_{k-1}(\mathcal{N})$. (In principle, we should distinguish between free or based homotopy, according to whether the boundary value of ϕ is allowed to vary during the homotopy or not; however, the assumption (H₂) guarantees that these two notions are equivalent.)

Lemma 1. *Let K be a convex polyhedron, let $h : K \rightarrow \mathcal{N}$ be a Lipschitz map, and let $\sigma \in \pi_{k-1}(\mathcal{N})$. Then, there exists a Lipschitz map $u : K \times B^{k-1} \rightarrow \mathcal{N}$ such that*

$$u(x', x'') = h(x') \quad \text{for any } (x', x'') \in K \times \partial B^{k-1} \quad (3.3)$$

and, for any $\sigma \in \pi_{k-1}(\mathcal{N})$, the homotopy class of $u(x', \cdot)$ is σ .

The proof of Lemma 1 is completely standard, but we provide it for the sake of convenience.

Proof of Lemma 1. We choose a point $x'_0 \in K$ and consider the map $\psi : [0, 1] \times K \rightarrow K$ as $\psi(t, x') := tx' + (1-t)x'_0$. We define $u : K \times (B^{k-1} \setminus B_{1/2}^{k-1}) \rightarrow \mathcal{N}$ as

$$u(x', x'') := (h \circ \psi)(2|x''| - 1, x') \quad \text{for } x' \in K, 1/2 \leq |x''| \leq 1.$$

The map u is Lipschitz and satisfies (3.3); moreover, for $|x''| = 1/2$ we have $u(x', x'') = h(x'_0)$. Now, we take a smooth map $\phi : B^{k-1} \rightarrow \mathcal{N}$ that is constant on ∂B^{k-1} — say, $\phi = z_0 \in \mathcal{N}$ on ∂B^{k-1} — and has homotopy class σ .

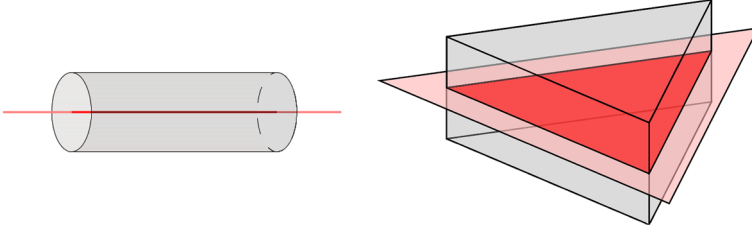


Fig. 4. The set $V(K, \delta, \gamma)$, in case $n = 1, k = 2$ (left) and $n = 2, k = 1$ (right). In both cases, the polyhedron K is in pink and \tilde{K} is in red

Let $\zeta : [0, 1] \rightarrow \mathcal{N}$ be a Lipschitz curve with $\zeta(0) = z_0, \zeta(1) = h(x'_0)$. We define $u : K \times B_{1/2}^{k-1} \rightarrow \mathcal{N}$ as

$$u(x', x'') := \begin{cases} \zeta(4|x''| - 1) & \text{if } 1/4 \leq |x''| < 1/2 \\ \phi(4x'') & \text{if } |x''| < 1/4. \end{cases}$$

For any $x' \in K$, the map $u(x', \cdot)$ is (freely) homotopic to σ , via a reparametrisation and a change of base-point. Therefore, the homotopy class of $u(x', \cdot)$ is σ . \square

Proof of Proposition 4. We triangulate $\Sigma \cup T$, that is, we write $\Sigma \cup T$ as a finite union of closed simplices in such a way that, for any simplices K, K' with $K \neq K'$, $K \cap K'$ is either empty or a boundary face of both K and K' . We denote by T_n the n -skeleton of this triangulation (i.e., the union of all simplices of dimension n or less). We will construct a sequence of maps $u^{n+1}, u^n, \dots, u^1, u^0$ by modifying the given map u first along the simplices of dimension $n+1$ that are contained in T , then along those of dimension n , and so on. In order to do so, we first need to construct a suitable covering of T .

Step 1. (Construction of a covering of T) Let $K \subseteq T$ be a simplex of dimension $j > 0$. Let K^\perp be the orthogonal $(n+k-j)$ -plane to K through the origin. We fix positive numbers δ_K, γ_K and define

$$\tilde{K} := \{x' \in K : \text{dist}(x', \partial K) > \gamma_K\}, \quad (3.4)$$

$$V_K := \{x' + x'' : x' \in \tilde{K}, x'' \in K^\perp, |x''| < \delta_K\}, \quad (3.5)$$

$$\Gamma_K := \{x' + x'' : x' \in \tilde{K}, x'' \in K^\perp, |x''| = \delta_K\} \quad (3.6)$$

(see Figure 4). If K is a 0-dimensional simplex, i.e. a point, we define $V_K := B^{n+k}(K, \delta_K)$ and $\Gamma_K := \partial V_K$. By choosing δ_K, γ_K in a suitable way, we can make sure that the following properties are satisfied:

- (a) $V_K \subset\subset D$ for any simplex $K \subseteq T$.
- (b) For any j -dimensional simplex $K \subseteq T$, we have

$$\partial V_K \setminus \Gamma_K \subseteq \bigcup_{K' \subseteq T : \dim K' < j} V_{K'}$$

(in case $j = 0$, both sides of the inclusion are empty).

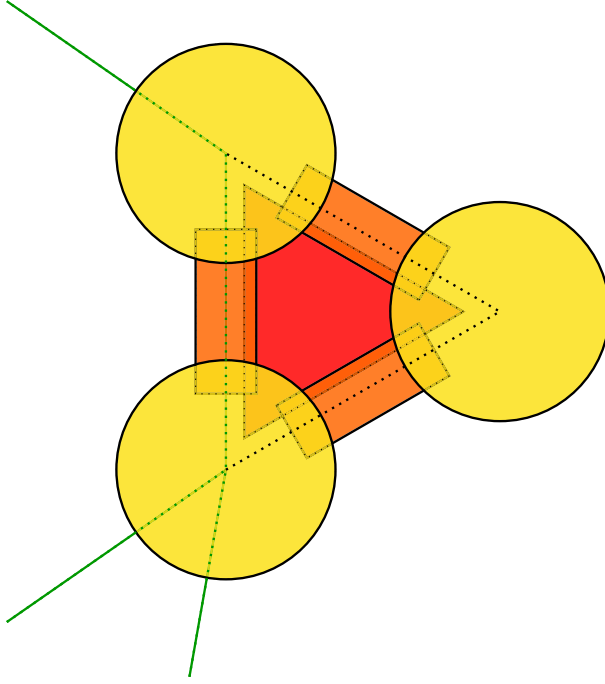


Fig. 5. The covering of T , in case $n = 1$ and $k = 2$ (view from the top). The set Σ is in green

- (c) For any simplices $K \subseteq T$, $K' \subseteq T$ with $K \neq K'$, $\dim K = \dim K'$, we have $\overline{V_K} \cap \overline{V_{K'}} = \emptyset$.
- (d) For any simplices $K \subseteq T$, $K' \subseteq \Sigma \cup T$ with $K \not\subseteq K'$, we have $\overline{V_K} \cap K' = \emptyset$.
- (e) No simplex $K \subseteq T$ is entirely contained in $\cup\{\overline{V_{K'}} : \dim K' < \dim K\}$.

Property (b) implies that the V_K 's do cover T . To construct a covering that satisfies (a)–(e), we first cover the 0-skeleton of T by pairwise disjoint balls that are compactly contained in D . Then, we cover each 1-dimensional simplex in T by a “thin cylinder”, whose bases are contained in the balls we have chosen before. Next, we cover each 2-dimensional simplex by a “thin shell”, and so on, as illustrated in Figure 5. At each step, we can make sure that the properties (a)–(e) are satisfied, because the simplices have pairwise disjoint interiors and only intersect along their boundaries. As a consequence of (d), for any simplex $K \subseteq T$ it holds that

$$\begin{cases} \overline{V_K} \cap (\Sigma \cup T_n) = \emptyset & \text{if } \dim K = n + 1 \\ \Gamma_K \cap (\Sigma \cup T_n) = \emptyset & \text{if } \dim K = n. \end{cases} \quad (3.7)$$

For any integer $j \in \{0, 1, \dots, n + 1\}$, we define

$$V^{=j} := \bigcup_{K \subseteq T : \dim K = j} V_K, \quad V^{<j} := \bigcup_{i=0}^{j-1} V^{=i}, \quad V^{\geq j} := \bigcup_{i=j}^{n+1} V^{=i}$$

and $V^{<0} := \emptyset$.

Step 2. (Construction of u^{n+1}) Let $K \subseteq T$ be a $(n+1)$ -simplex of the triangulation, with the orientation induced by T . We identify V_K with $\tilde{K} \times B^{k-1}(0, \delta_K)$, where \tilde{K} is given by (3.4). We construct a Lipschitz map $u_K^{n+1}: V_K \rightarrow \mathcal{N}$ as follows. First, we let

$$u_K^{n+1}(x', x'') := u\left(x', 2x'' - \frac{\delta_K x''}{|x''|}\right) \quad \text{for } x' \in \tilde{K}, \delta_K/2 \leq |x''| \leq \delta_K. \quad (3.8)$$

Thus, $u_K^{n+1} = u$ on Γ_K , while $u_K^{n+1}(x', x'') = u(x', 0)$ for $|x''| = \delta_K/2$. Since the trace of u_K^{n+1} on $\tilde{K} \times \partial B^{k-1}(0, \delta_K/2)$ only depends on the variable x' , we may apply Lemma 1 and define u_K^{n+1} in $\tilde{K} \times B^{k-1}(0, \delta_K/2)$ in such a way that, for any $x' \in \tilde{K}$,

$$\text{the homotopy class of } u_K^{n+1}(x', \cdot)|_{B^{k-1}(0, \delta_K/2)} \text{ is } (-1)^{n+1} \sigma. \quad (3.9)$$

The sign $(-1)^{n+1}$ will be useful to compensate for orientation effects, later on in the proof.

We define a map

$$u^{n+1}: (D \setminus V^{<n+1}) \cup V^{=n+1} \rightarrow \mathcal{N}$$

as follows: $u^{n+1}(x) := u_K^{n+1}(x)$ if $x \in V_K$ for some $(n+1)$ -simplex K , and $u^{n+1}(x) := u(x)$ otherwise. This definition is consistent. Indeed, the sets $\overline{V_K}$ are pairwise disjoint, due to (c). Moreover, if a point x belongs both to $\overline{V_K}$ and to $D \setminus V^{<n+1}$, then $x \in \Gamma_V$ because of (b), so $u_K^{n+1}(x) = u(x)$ by (i). Therefore, the map u^{n+1} is well-defined and locally Lipschitz out of Σ , with nice singularity at Σ .

Step 3. (Construction of u^n) Let $K \subseteq T$ be a n -simplex. We identify V_K with $\tilde{K} \times B^k(0, \delta_K)$. The map u^{n+1} is Lipschitz continuous on Γ_K , due to (3.7). Let $\sigma_K \in \pi_{k-1}(\mathcal{N})$ be the homotopy class of u^{n+1} on an arbitrary slice of Γ_K , of the form $\{x'\} \times \partial B^k(0, \delta_K)$. If $\sigma_K = 0$ then, by adapting the arguments of Lemma 1, we can construct a Lipschitz continuous map $u_K^n: V_K \rightarrow \mathcal{N}$ such that $u_K^n = u^{n+1}$ on Γ_K . If $\sigma_K \neq 0$, we define $u_K^n: V_K \rightarrow \mathcal{N}$ as

$$u_K^n(x', x'') := u^{n+1}\left(x', \frac{\delta_K x''}{|x''|}\right) \quad \text{for } (x', x'') \in \tilde{K} \times B^k(0, \delta_K).$$

In both cases, by a straightforward computation, we obtain

$$|\nabla u_K^n(x', x'')| \lesssim |x''|^{-1} \quad \text{for a.e. } (x', x'') \in \tilde{K} \times B^k(0, \delta_K), \quad (3.10)$$

where the proportionality constant at the right-hand side depends on δ_K and u^{n+1} . We define

$$u^n: (D \setminus V^{<n}) \cup V^{\geq n} \rightarrow \mathcal{N}$$

as follows: $u^n(x) := u_K^n(x)$ if $x \in V_K$ for some n -simplex K , and $u^n(x) := u^{n+1}(x)$ otherwise. Thanks to (b), (c) and (3.10), we can argue as in Step 2 and check that u^n is locally Lipschitz out of $\Sigma \cup T_n$, with nice singularity at $\Sigma \cup T_n$.

Step 4. (Construction of u^j for $j < n$) We proceed by induction. Let $j \in \{0, 1, \dots, n-1\}$. Suppose we have constructed a map

$$u^{j+1}: \left(D \setminus V^{<j+1}\right) \cup V^{\geq j+1} \rightarrow \mathcal{N}$$

that is locally Lipschitz out of $\Sigma \cup T_n$ and has a nice singularity at $\Sigma \cup T_n$. Let $K \subseteq T$ be a j -simplex. By identifying V_K with $\tilde{K} \times B^{n+k-j}(0, \delta_K)$, we define $u_K^j: V_K \rightarrow \mathcal{N}$,

$$u_K^j(x', x'') := u^{j+1}\left(x', \frac{\delta_K x''}{|x''|}\right) \quad \text{for } (x', x'') \in \tilde{K} \times B^{n+k-j}(0, \delta_K).$$

The map u_K^j is locally Lipschitz out of the set

$$A := \left\{ (x', x'') \in \tilde{K} \times B^{n+k-j}(0, \delta_K) : \left(x', \frac{\delta_K x''}{|x''|}\right) \in \Sigma \cup T_n \right\}.$$

By Property (d), the only simplices of $\Sigma \cup T_n$ that intersect \overline{V}_K are those that contain K . Therefore, if H_1, H_2, \dots, H_p denote the n -dimensional (closed) simplices of $\Sigma \cup T_n$ that contain K , then

$$(\Sigma \cup T_n) \cap \overline{V}_K = \bigcup_{i=1}^p (H_i \cap \overline{V}_K) \quad (3.11)$$

Moreover, Property (d) and the convexity of H_i imply that

$$H_i \cap \overline{V}_K = \tilde{K} \times \left(\tilde{H}_i \cap \bar{B}^{n+k-j}(0, \delta_K)\right), \quad (3.12)$$

where $\tilde{H}_i \subseteq \mathbb{R}^{n+k-j}$ is a cone (i.e., $\lambda x \in \tilde{H}_i$ for any $x \in \tilde{H}_i$ and any $\lambda \geq 0$). As a consequence,

$$\begin{aligned} A &\stackrel{(3.11), (3.12)}{=} \bigcup_{i=1}^p \left(\tilde{K} \times (\tilde{H}_i \cap \bar{B}^{n+k-j}(0, \delta_K))\right) \stackrel{(3.12)}{=} \bigcup_{i=1}^p (H_i \cap V_K) \\ &\stackrel{(3.11)}{=} (\Sigma \cup T_n) \cap V_K, \end{aligned}$$

that is, u_K^j is locally Lipschitz out of $\Sigma \cup T_n$. We claim that

$$|\nabla u_K^j(x)| \lesssim \text{dist}^{-1}(x, \Sigma \cup T_n) \quad \text{for a.e. } x \in V_K, \quad (3.13)$$

where the proportionality constant at the right-hand side may depend on δ_K . Given $x = (x', x'') \in V_K$, let $y(x) := (x', \delta_K x'' / |x''|)$. By the induction hypothesis, u^{j+1} has a nice singularity at $\Sigma \cup T_n$. Therefore, an explicit computation gives

$$|\nabla u_K^j(x)| \lesssim |x''|^{-1} \text{dist}^{-1}(y(x), \Sigma \cup T_n) \quad (3.14)$$

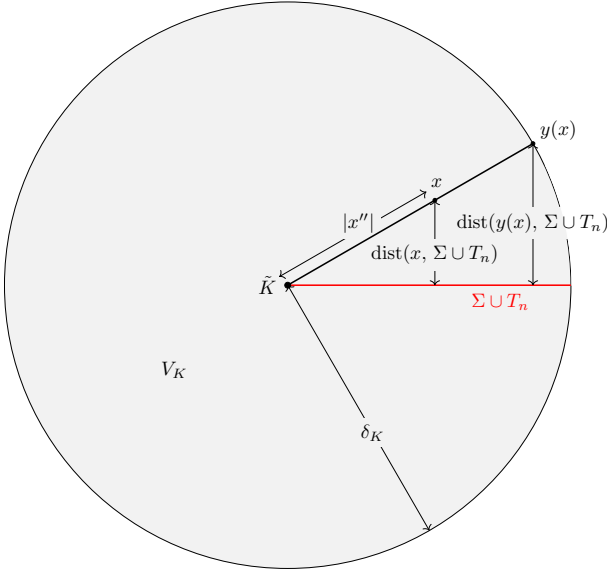


Fig. 6. Proof of (3.15). The picture represents a slice of V_K , of the form $\{x'\} \times B^{n+k-j}(0, \delta_K)$

for a.e. $x \in V_K$. By (3.11) and (3.12), the set $\Sigma \cup T_n$ agrees with $\tilde{K} \times \cup_i \tilde{H}_i$ in $\overline{V_K}$, and $\cup_i \tilde{H}_i$ is a cone. Then, by a geometric argument (see Figure 6), we have

$$\frac{|x''|}{\text{dist}(x, \Sigma \cup T_n)} = \frac{\delta_K}{\text{dist}(y(x), \Sigma \cup T_n)} \quad (3.15)$$

By combining (3.14) and (3.15), (3.13) follows. Finally, we define

$$u^j: \left(D \setminus V^{<j} \right) \cup V^{\geq j} \rightarrow \mathcal{N}$$

as follows: $u^j(x) := u_K^j(x)$ if $x \in V_K$ for some j -simplex $K \subseteq T$, and $u^j(x) := u^{j+1}(x)$ otherwise. Thanks to (b), (c) and (3.13), the map u^j is well-defined, locally Lipschitz out of $\Sigma \cup T_n$ and has a nice singularity at $\Sigma \cup T_n$.

Step 5. (Conclusion) By induction, we have constructed a sequence of maps $u^{n+1}, u^n, \dots, u^1, u^0$. Let $\tilde{u} := u^0: D \rightarrow \mathcal{N}$. By construction, the map \tilde{u} has a nice singularity at $\Sigma \cup T_n$ and agrees with u out of $V^{<n+1} \cup V^{=n+1}$. In particular, $\tilde{u} = u$ in a neighbourhood of ∂D , because of (a).

It only remains to compute $\mathbf{S}(\tilde{u})$. Let K be an n -simplex of T . By Property (e), K is not entirely contained in $\overline{V^{<n}}$; we take a point $x \in K \setminus \overline{V^{<n}}$. Let K^\perp be the orthogonal k -plane to K at x , and let $F := \overline{V_K} \cap K^\perp$. By Property (d), the only $(n+1)$ -simplices that intersect F are those that contain K ; we call them H_1, \dots, H_p . We consider the restriction of \tilde{u} to the $(k-1)$ -sphere ∂F . By construction (see (3.8) and (3.9) in Step 2), $\tilde{u}|_{\partial F}$ consists (up to homotopy) of a reparametrisation of $u|_{\partial F}$, with the insertion of ‘bubbles’ around the points $\partial F \cap H_i$. Each bubble carries the

homotopy class σ or $-\sigma$, depending on the orientation of H_i (which, we recall, is the one induced by T). The net topological contribution of all the bubbles may vanish or not, depending on whether the point x belongs to the boundary of T or not. As a result, we have

$$\begin{aligned} & \left(\text{homotopy class of } \tilde{u}|_{\partial F} : \partial F \simeq \mathbb{S}^{k-1} \rightarrow \mathcal{N} \right) \\ &= \left(\text{homotopy class of } u|_{\partial F} : \partial F \simeq \mathbb{S}^{k-1} \rightarrow \mathcal{N} \right) \\ & \quad + \sigma \left(\text{multiplicity of } \partial[[T]] \text{ at } x \right). \end{aligned}$$

The sign of the second term in the right-hand side depends on the choice of the sign we made in Equation (3.9) (see, for instance, Property (iv) in Lemma 8 of [17]). Then, by Remark 6, $\mathbf{S}(\tilde{u}) = \mathbf{S}(u) + \sigma \partial[[T]]$. \square

3.3. Projection of a $W^{1,k}$ -map onto \mathcal{N}

Before we pass to the construction of a recovery sequence, we gather some useful results, based on earlier work by HARDT ET AL. [29, Lemma 2.3], [30], and RIVIÈRE [49, Proposition 2.1]; see also [2, Proposition 6.4] for similar statements in case $\mathcal{N} = \mathbb{S}^{k-1}$.

For any $y \in \mathbb{R}^m$, we consider the map $\tilde{\varrho}_y : z \mapsto \varrho(z - y)$ which is well defined for $z \in \mathbb{R}^m \setminus (\mathcal{X} + y)$. This is not a retraction onto \mathcal{N} , in general, because it does not restrict to the identity on \mathcal{N} . However, for sufficiently small $|y|$ — say, $y \in B_\sigma^m$ with $\sigma > 0$ small enough — the restriction $\tilde{\varrho}_y|_{\mathcal{N}}$ is a small perturbation of the identity and, in particular, it is a diffeomorphism. For $y \in B_\sigma^m$ and $z \in \mathbb{R}^m \setminus (\mathcal{X} + y)$, let us define

$$\varrho_y(z) := \left((\tilde{\varrho}_y|_{\mathcal{N}})^{-1} \circ \varrho \right) (z - y). \quad (3.16)$$

This map is indeed a smooth retraction of $\mathbb{R}^m \setminus (\mathcal{X} + y)$ onto \mathcal{N} . We also define a function $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$\psi(z) := \min \left\{ \frac{\text{dist}(z, \mathcal{X})}{\text{dist}(\mathcal{N}, \mathcal{X})}, 1 \right\} \quad \text{for } z \in \mathbb{R}^m. \quad (3.17)$$

The function ψ is Lipschitz and $\psi = 1$ on \mathcal{N} . By Proposition 2 and (3.17), we have

$$|\nabla \varrho_y(z)| \lesssim \frac{1}{\text{dist}(z - y, \mathcal{X})} \lesssim \frac{1}{\psi(z - y)} \quad (3.18)$$

for any $y \in B_\sigma^m$ and $z \in \mathbb{R}^m \setminus (\mathcal{X} + y)$. The proportionality constants here depend on σ , but $\sigma = \sigma(\mathcal{N}, \mathcal{X}, \varrho)$ is fixed once and for all. Finally, let $\xi_\varepsilon(t) := \min(t/\varepsilon, 1)$ for $t \geq 0$.

Lemma 2. *Let Λ be a positive number, and let $u \in (L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$ be such that $\|u\|_{L^\infty(\Omega)} \leq \Lambda$. For $y \in B_\sigma^m$, $\varepsilon > 0$ and $x \in \Omega$, define*

$$w_y(x) := (\varrho_y \circ u)(x), \quad w_{\varepsilon,y}(x) := (\xi_\varepsilon \circ \psi)(u(x) - y) w_y(x).$$

Then, the following properties hold:

- (i) For a.e. $y \in B_\sigma^m$, $w_y \in W^{1,k-1}(\Omega, \mathcal{N})$ and $\mathbf{S}(w_y) = \mathbf{S}_y(u)$.
 (i) For a.e. $y \in B_\sigma^m$ and sufficiently small ε , $w_{\varepsilon,y} \in (L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$ and $\|w_{\varepsilon,y}\|_{L^\infty(\Omega)} \leq \max\{|z| : z \in \mathcal{N}\}$.
 (iii) For any open set $D \subseteq \Omega$, it holds that

$$\begin{aligned} & \int_{B_\sigma^m} \left(E_\varepsilon(w_{\varepsilon,y}, D) + \varepsilon^{-k} \mathcal{L}^{n+k}\{x \in D : w_{\varepsilon,y}(x) \neq w_y(x)\} \right) dy \\ & \leq C_\Lambda \left(|\log \varepsilon| \|\nabla u\|_{L^k(D)}^k + \mathcal{L}^{n+k}(D) \right), \end{aligned}$$

where C_Λ is a positive constant that only depends on \mathcal{N} , k , \mathcal{X} and Λ .

- (iv) For a.e. $y \in B_\sigma^m$ there exists a (non-relabelled) subsequence $\varepsilon \rightarrow 0$ such that $w_{\varepsilon,y} \rightarrow w_y$ strongly in $W^{1,k-1}(\Omega, \mathbb{R}^m)$.

Remark 9. Statement (iii) of Lemma 2 implies, via an averaging argument, that

$$\inf \left\{ E_\varepsilon(u) : u \in W_v^{1,k}(\Omega, \mathbb{R}^m) \right\} \lesssim |\log \varepsilon|$$

for any $v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ and any $\varepsilon > 0$.

Proof of Lemma 2. Throughout the proof, we denote by C_Λ a generic positive constant that only depends on \mathcal{N} , k , \mathcal{X} and Λ (and may change from one occurrence to the other).

Step 1. (Proof of (i)) For a.e. y , we have $\varrho \circ (u - y) \in W^{1,k-1}(\Omega, \mathcal{N})$ (see e.g. [17, Lemma 14] for a proof of this claim). Moreover, by CANEVARI AND ORLANDI [17, Lemma 17] we know that

$$\mathbf{S}_{y'}(\varrho \circ (u - y)) = \mathbf{S}_y(u) \quad \text{for a.e. } y, y' \in B_\sigma^m.$$

Now, w_y is obtained from $\varrho \circ (u - y)$ by composition with a map, $(\tilde{\varrho}_y|_{\mathcal{N}})^{-1}$, which is homotopic to the identity on \mathcal{N} . Therefore, from the identity above we obtain

$$\mathbf{S}_{y'}(w_y) = \mathbf{S}_y(u) \quad \text{for a.e. } y, y' \in B^*. \quad (3.19)$$

This can be first checked when u is smooth, using [17, Lemma 18], and remains true for a general u by a density argument, using the continuity of \mathbf{S} and e.g. [17, Lemma 14].

Step 2. (Proof of (ii), (iii)) It is immediate to see that $\|w_{\varepsilon,y}\|_{L^\infty(\Omega)} \leq \max\{|z| : z \in \mathcal{N}\}$. By (i), $w_{\varepsilon,y} \in (L^\infty \cap W^{1,k-1})(\Omega, \mathbb{R}^m)$ for a.e. y , and by the chain rule, we have the pointwise bound

$$|\nabla w_{\varepsilon,y}(x)| \leq C_\Lambda \left((\xi'_\varepsilon \circ \psi)(u(x) - y) |\nabla u(x)| + (\xi_\varepsilon \circ \psi)(u(x) - y) |\nabla w_y(x)| \right)$$

for a.e. $x \in \Omega$. Thanks to (3.18), we deduce that

$$\begin{aligned} |\nabla w_{\varepsilon,y}(x)| & \leq C_\Lambda \left((\xi'_\varepsilon \circ \psi)(u(x) - y) + \frac{(\xi_\varepsilon \circ \psi)(u(x) - y)}{\psi(u(x) - y)} \right) |\nabla u(x)| \\ & \leq C_\Lambda \left(\frac{\mathbb{1}_{\{\psi(u(x)-y) \leq \varepsilon\}}}{\varepsilon} + \frac{\mathbb{1}_{\{\psi(u(x)-y) \geq \varepsilon\}}}{\psi(u(x) - y)} \right) |\nabla u(x)| \end{aligned} \quad (3.20)$$

(where, as usual, $\mathbb{1}_A$ denotes the characteristic function of a set A). On the other hand, the L^∞ -norm of $w_{\varepsilon,y}$ is uniformly bounded in terms of \mathcal{N} only, and hence it holds that

$$f(w_{\varepsilon,y}) \lesssim \mathbb{1}_{\{w_{\varepsilon,y} \neq w_y\}} = \mathbb{1}_{\{\psi(u-y) < \varepsilon\}}. \quad (3.21)$$

Together, (3.20) and (3.21) imply that

$$\begin{aligned} & E_\varepsilon(w_{\varepsilon,y}, D) + \varepsilon^{-k} \mathcal{L}^{n+k} \{x \in D: w_{\varepsilon,y}(x) \neq w_y(x)\} \\ & \leq C_\Lambda \int_\Omega \frac{\mathbb{1}_{\{\psi(u(x)-y) \leq \varepsilon\}}}{\varepsilon^k} dx \\ & \quad + C_\Lambda \int_D \left(\frac{\mathbb{1}_{\{\psi(u(x)-y) \leq \varepsilon\}}}{\varepsilon^k} + \frac{\mathbb{1}_{\{\psi(u(x)-y) \geq \varepsilon\}}}{\psi(u(x)-y)^k} \right) |\nabla u(x)|^k dx \end{aligned}$$

We integrate the previous inequality for $y \in B_\sigma^m$, apply Fubini theorem and make the change of variable $z = u(x) - y$:

$$\begin{aligned} & \int_{B_\sigma^m} \left(E_\varepsilon(w_{\varepsilon,y}, D) + \varepsilon^{-k} \mathcal{L}^{n+k} \{x \in D: w_{\varepsilon,y}(x) \neq w_y(x)\} \right) dy \\ & \leq C_\Lambda \int_D \int_{B_{\sigma+\Lambda}^m} \left\{ \left(\frac{\mathbb{1}_{\{\psi(z) \leq \varepsilon\}}}{\varepsilon^k} + \frac{\mathbb{1}_{\{\psi(z) \geq \varepsilon\}}}{\psi(z)^k} \right) |\nabla u(x)|^k + \frac{\mathbb{1}_{\{\psi(z) \leq \varepsilon\}}}{\varepsilon^k} \right\} dz dx. \end{aligned}$$

Since \mathcal{X} is a finite union of simplices of codimension k or higher, for ε sufficiently small it holds that

$$\int_{B_{\sigma+\Lambda}^m} \mathbb{1}_{\{\psi(z) \leq \varepsilon\}} dz \leq C_\Lambda \varepsilon^k, \quad \int_{B_{\sigma+\Lambda}^m} \frac{\mathbb{1}_{\{\psi(z) \geq \varepsilon\}}}{\psi(z)^k} dz \leq C_\Lambda |\log \varepsilon|$$

(see e.g. [2, Lemma 8.3]). As a consequence, we obtain (iii).

Step 3. (Proof of (iv)) For a.e. $y \in B_\sigma^m$, the set $\{\psi(u-y) = 0\} = (u-y)^{-1}(\mathcal{X})$ has Lebesgue measure equal to zero (see e.g. [17, proof of Lemma 14]). Then, since $\xi_\varepsilon \rightarrow 1$ pointwise on $(0, +\infty)$ as $\varepsilon \rightarrow 0$, we have $w_{\varepsilon,y} \rightarrow w_y$ a.e. as $\varepsilon \rightarrow 0$, for a.e. y . Using the chain rule, (3.18) and (3.20), we obtain that

$$|\nabla w_{\varepsilon,y}(x) - \nabla w_y(x)| \leq C_\Lambda \left(\frac{1}{\varepsilon} + \frac{1}{\psi(u(x)-y)} \right) \mathbb{1}_{\{\psi(u(x)-y) \leq \varepsilon\}} |\nabla u(x)|.$$

for a.e. $x \in \Omega$. We raise both sides of this inequality to the $(k-1)$ -th power, integrate over $(x, y) \in \Omega \times B_\sigma^m$, apply Fubini theorem and make the change of variable $z = u(x) - y$:

$$\begin{aligned} & \int_{B^*} \|\nabla w_{\varepsilon,y} - \nabla w_y\|_{L^{k-1}(\Omega)}^{k-1} dy \leq C_\Lambda \int_\Omega \int_{B_{\sigma+\Lambda}^m} \left(\frac{1}{\varepsilon^{k-1}} \right. \\ & \quad \left. + \frac{1}{\psi(z)^{k-1}} \right) \mathbb{1}_{\{\psi(z) \leq \varepsilon\}} |\nabla u(x)|^{k-1} dz dx. \end{aligned}$$

We apply [2, Lemma 8.3] to estimate the integral with respect to z : since \mathcal{X} has codimension k , we obtain

$$\int_{B_{\sigma+\Lambda}^m} \left(\frac{1}{\varepsilon^{k-1}} + \frac{1}{\psi(z)^{k-1}} \right) \mathbb{1}_{\{\psi(z) \leq \varepsilon\}} dz \leq C_\Lambda \varepsilon,$$

so

$$\int_{B_\sigma^m} \|\nabla w_{\varepsilon,y} - \nabla w_y\|_{L^{k-1}(\Omega)}^{k-1} dy \leq C_\Lambda \varepsilon \|\nabla u\|_{L^{k-1}(\Omega)}^{k-1}.$$

By Fatou lemma, we deduce

$$\int_{B_\sigma^m} \liminf_{\varepsilon \rightarrow 0} \|\nabla w_{\varepsilon,y} - \nabla w_y\|_{L^{k-1}(\Omega)}^{k-1} dy = 0,$$

so (iv) follows. \square

3.4. Construction of a Recovery Sequence

3.4.1. Construction of an \mathcal{N} -valued Map with Nice Singularity at a Locally Polyhedral Set In this section, we give the construction of a recovery sequence. We first construct a map $\Omega \rightarrow \mathcal{N}$ that matches the Dirichlet boundary datum and has nice singularities along a locally polyhedral set.

Lemma 3. *Any boundary datum $v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ can be extended to a map $u_* \in (L^\infty \cap W_v^{1,k})(\Omega, \mathbb{R}^m)$ that satisfies the following properties, for a.e. $y \in \mathbb{R}^m$:*

- (a) $\mathbb{M}(\mathbf{S}_y(u_*)) < +\infty$ and $\mathbf{S}_y(u_*) \perp \partial\Omega = 0$;
- (b) the chain $\mathbf{S}_y(u_*)$ is locally polyhedral;
- (c) the chain $\mathbf{S}_y(u_*)$ takes its multiplicities in a finite subset of $\pi_{k-1}(\mathcal{N})$, which depends only on \mathcal{N} , ϱ , \mathcal{X} ;
- (d) there exists a locally $(n-1)$ -polyhedral set P_y such that $\varrho \circ (u_* - y)$ has a locally nice singularity at $\text{spt } \mathbf{S}_y(u_*) \cup P_y$.

The proof of Lemma 3 relies on the following fact.

Lemma 4. *Any boundary datum $v \in W^{1-1/k,k}(\partial\Omega, \mathbb{R}^m)$ has a locally piecewise affine extension $u_* \in (L^\infty \cap W_v^{1,k})(\Omega, \mathbb{R}^m)$.*

We give a proof of Lemma 4, for the convenience of the reader only.

Proof of Lemma 4. Arguing component-wise, we reduce to the case $m = 1$. Let $u \in W_v^{1,k}(\Omega)$ be an extension of v . By a truncation argument, we can make sure that $v \in L^\infty(\Omega)$. Let $\Gamma_1 := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/2\}$ and, for any integer $j \geq 2$, let $\Gamma_j := \{x \in \Omega : (j+1)^{-1} < \text{dist}(x, \partial\Omega) < (j-1)^{-1}\}$. Using a partition of unity, we construct a sequence of smooth functions $\varphi_j \in C_c^\infty(\Gamma_j)$ such that $\sum_{j \geq 1} \varphi_j = 1$. Thanks to e.g. [53, Theorem 1], for any j there exists a triangulation \mathcal{T}_j of \mathbb{R}^{n+k} such that the piecewise affine interpolant u_j of $\varphi_j u$ along \mathcal{T}_j is well-defined (that is, all the vertices of \mathcal{T}_j are Lebesgue points of $\varphi_j u$) and there holds

$$\|\nabla u_j - \nabla(\varphi_j u)\|_{L^k(\mathbb{R}^{n+k})} \leq 2^{-j}. \quad (3.22)$$

Moreover, the proof of [53, Theorem 1] shows that for any $r > 0$, we can choose \mathcal{T}_j such that all the simplices of \mathcal{T}_j have diameter $\leq r$. In particular, we can make sure that u_j is still supported in Γ_j . Now, we define $u_* := \sum_{j \geq 1} u_j$. Since the support of u_j intersects the support of u_i only for finitely many i , the function u_* is locally piecewise affine. Moreover, $u_* \in L^\infty(\Omega)$ because, by construction, $\|u_j\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ for any j , and $u \in W^{1,k}(\Omega)$ due to (3.22). Finally, for any $N \geq 1$ the function $\sum_{j=1}^N (u_j - \varphi_j u)$ is compactly supported in Ω , and hence $\sum_{j=1}^N (u_j - \varphi_j u) \in W_0^{1,k}(\Omega)$. Passing to the limit as $N \rightarrow +\infty$, we conclude that $u_* - u \in W_0^{1,k}(\Omega)$, and the lemma follows. \square

Proof of Lemma 3. Let u_* be the locally piecewise extension of v given by Lemma 4. Statement (a) follows from (P₂) in Proposition 3, because $u_* \in W^{1,k}(\Omega, \mathbb{R}^m)$. Let $K \subseteq \Omega$ be a (closed) $(n+k)$ -simplex such that $u_*|_K$ is affine. Since we have assumed that \mathcal{X} is polyhedral, for any $y \in \mathbb{R}^m$ the inverse image $(u_* - y)^{-1}(\mathcal{X}) \cap K$ is polyhedral too. Take $y \in \mathbb{R}^m$ such that $(u_* - y)|_K$ is transverse to each cell of \mathcal{X} . By [17, Corollary 1], we have $\mathbf{S}_y(u_*) \llcorner K = \mathbf{S}_y(u_*|_K)$ and by definition (see [17, Section 3.2] and Section B below), the latter is a polyhedral chain supported on $(u_* - y)^{-1}(\mathcal{X}) \cap K$. Thus, $\mathbf{S}_y(u_*)$ is locally polyhedral. Moreover, $\mathbf{S}_y(u_*)$ take its multiplicities in the set

$$\{\pm (\text{homotopy class of } \varrho \text{ around } H) : H \text{ is a } (m-k)\text{-polyhedron of } \mathcal{X}\},$$

which is a finite subset of $\pi_{k-1}(\mathcal{N})$, because \mathcal{X} is a finite union of polyhedra. Finally, let us prove Statement (d). Take an open set $W \subset\subset \Omega$, and take $y \in \mathbb{R}^m$ such that $u_*|_W$ is transverse to each cell of \mathcal{X} . Let K be a $(n+k)$ -simplex such that $K \cap W \neq \emptyset$ and $u_*|_K$ is affine. By transversality, we see that

$$\text{dist}(u_*(x) - y, \mathcal{X}) \geq C_{K,y} \text{dist}\left(x, (u_* - y)^{-1}(\mathcal{X})\right) \quad \text{for any } x \in K,$$

where $C_K > 0$ is a constant that depends on the (constant) gradient of u_* on K and on y . Since W is covered by finitely many simplices, we have

$$\text{dist}(u_*(x) - y, \mathcal{X}) \geq C_{W,y} \text{dist}\left(x, (u_* - y)^{-1}(\mathcal{X})\right) \quad \text{for any } x \in W,$$

where $C_{W,y} := \min_{K: K \cap W \neq \emptyset} C_{K,y} > 0$. Then, by applying the chain rule and Proposition 2, we conclude that $\varrho \circ (u_* - y)|_W$ has a nice singularity at $(\text{spt } \mathbf{S}_y(u_*) \cup P_y) \cap W$, where $P_y := (u_* - y)^{-1}(\mathcal{X}_{m-k-1})$. \square

3.4.2. Reduction of the Problem Throughout the rest of Section 3, we fix the boundary datum $v \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ and let u_* be the map given by Lemma 3. We also fix $y^* \in \mathbb{R}^m$, with $|y^*|$ sufficiently small, in such a way that Statements (a)–(d) in Lemma 3 are satisfied. Let $w_* := \varrho_{y^*} \circ u_*$, where ϱ_{y^*} is defined by (3.16). By Lemma 3, the map w_* has a locally nice singularity at $\text{spt } \mathbf{S}_{y^*}(u_*) \cup P_{y^*}$, where P_{y^*} is a locally polyhedral set of dimension $n-1$. By Lemma 2, we can choose y^* so to have $w_* \in W^{1,k-1}(\Omega, \mathcal{N})$ and $\mathbf{S}(w_*) = \mathbf{S}_{y^*}(u_*)$ as well.

Remark 10. For a generic map $w \in W^{1,k-1}(\Omega, \mathbb{R}^m)$, $\mathbf{S}(w)$ is only well-defined as a *relative flat chain*, $\mathbf{S}(w) \in \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$ (see [17, Section 3]). However, $\mathbf{S}_{y_*}(u_*)$ is well-defined as an element of $\mathbb{F}_n(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$, because $u_* \in W^{1,k}(\Omega, \mathbb{R}^m)$ (see Proposition 3). With a slight abuse of notation, we will regard $\mathbf{S}(w_*)$ as an element of $\mathbb{F}_n(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$, too.

Let S be a finite-mass n -chain, supported in $\overline{\Omega}$, that is cobordant to $\mathbf{S}(w_*)$. By definition of $\mathcal{C}(\Omega, v)$, Equation (2.6), S and $\mathbf{S}(w_*)$ differ by a boundary. By an approximation argument, we will reduce to the case S has a special form.

Proposition 5. *Let $S \in \mathcal{C}(\Omega, v)$ be a finite mass chain. Then, there exists a sequence of polyhedral $(n+1)$ -chains R_j , with compact support in Ω , such that $\mathbf{S}(w_*) + \partial R_j \rightarrow S$ (with respect to the \mathbb{F} -norm) and $\mathbb{M}(\mathbf{S}(w_*) + \partial R_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$.*

The proof of Proposition 5 is left to Appendix D.1. Thanks to Proposition 5, and a diagonal argument, we can assume with no loss of generality that S has the form

$$S = \mathbf{S}(w_*) + \partial R, \quad (3.23)$$

where R is a polyhedral $(n+1)$ -chain, compactly supported in Ω . There is one further assumption we can make. Let $W_{\mathfrak{S}} \subset\subset \Omega$ be an open set, with polyhedral boundary, such that $\partial W_{\mathfrak{S}}$ is transverse to $\text{spt } S$ (more precisely, there exist triangulations of $\partial W_{\mathfrak{S}}$ and $\text{spt } S$ such that any simplex of the triangulation of $\partial W_{\mathfrak{S}}$ is transverse to any simplex of the triangulation of $\text{spt } S$) and

$$\text{spt } R \subseteq \overline{W_{\mathfrak{S}}}, \quad S \llcorner \partial W_{\mathfrak{S}} = 0. \quad (3.24)$$

The condition $S \llcorner \partial W_{\mathfrak{S}} = 0$ is satisfied because, by transversality, $\text{spt } S \cap \partial W_{\mathfrak{S}}$ has dimension $(n-1)$ or less and hence, it cannot support a non-trivial polyhedral n -chain.

Proposition 6. *There exists a sequence of polyhedral $(n+1)$ -chains R_j , supported in $\overline{W_{\mathfrak{S}}}$, such that the following hold:*

- (i) $S + \partial R_j \rightarrow S$, with respect to the \mathbb{F} -norm, as $j \rightarrow +\infty$;
- (ii) $\mathbb{M}(S + \partial R_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$;
- (iii) for any j , $(S + \partial R_j) \llcorner \partial W_{\mathfrak{S}} = 0$;
- (iv) for any j , the chain $(S + \partial R_j) \llcorner W_{\mathfrak{S}}$ takes multiplicities in the set $\mathfrak{S} \subseteq \pi_{k-1}(\mathcal{N})$ defined by (2.4).

The proof of Proposition 6 will be given in Appendix D.1. Thanks to Proposition 6, it is not restrictive to assume that

$$S \llcorner W_{\mathfrak{S}} \text{ takes its multiplicities in } \mathfrak{S}, \quad (3.25)$$

in addition to (3.23), (3.24). Indeed, if (3.24) does not hold, we replace S with a chain of the form $S + \partial R_j$ as given by Proposition 6, we replace R with $R + R_j$, then we use a diagonal argument to pass to the limit as $j \rightarrow +\infty$.

3.4.3. Construction of an \mathcal{N} -valued Map with Prescribed Singular Set Our next task is to construct a map $w: \Omega \rightarrow \mathcal{N}$, with locally nice singularities, in such a way that $\mathbf{S}(w) = S$. To do so, we fix an open set $W \subset \subset \Omega$ such that $W_{\mathfrak{S}} \subset \subset W$ and ∂W is transverse to $\text{spt } S$ (i.e., there exist triangulations of $\partial W_{\mathfrak{S}}$ and $\text{spt } S$ such that any simplex of the triangulation of $\partial W_{\mathfrak{S}}$ is transverse to any simplex of the triangulation of $\text{spt } S$). We also fix a small parameter $\eta > 0$.

Lemma 5. *For any W as above and any $\eta > 0$, there exists a map $w \in W^{1,k-1}(\Omega, \mathcal{N})$ that satisfies the following properties:*

- (i) $w = w_*$ a.e. in $\Omega \setminus W$;
- (ii) w has a locally nice singularity at $(\text{spt } S, Q_*)$, where $Q_* \supseteq (\text{spt } S)_{n-1}$ is a locally $(n-1)$ -polyhedral set;
- (iii) $\mathbf{S}(w) = S$;
- (iv) $w|_W$ is η -minimal.

Lemma 5 follows from Proposition 4, combined with the following lemma from [2]:

(Lemma 9.3, [2])

Lemma 6. *Let $K \subseteq \mathbb{R}^{n+k}$ be a n -simplex, and let δ, γ be positive parameters.*

Let $u: U(K, \delta, \gamma) \rightarrow \mathcal{N}$ be a map with nice singularity at K , and let $\sigma \in \pi_{k-1}(\mathcal{N})$ the homotopy class of u around K . Let $\phi: \mathbb{S}^{k-1} \rightarrow \mathcal{N}$ be a Lipschitz map in the homotopy class σ . Then, there exists a map $\tilde{u}: U(K, \delta, \gamma) \rightarrow \mathcal{N}$ that satisfies the following properties:

- (i) $\tilde{u} = u$ on $\partial U(K, \delta, \gamma)$;
- (ii) \tilde{u} has a nice singularity at $(K, \partial K)$;
- (iii) $\mathbf{S}(\tilde{u}) = \mathbf{S}(u)$;
- (iv) $\tilde{u}(x) = \phi(x''/|x''|)$ for any $x = (x', x'') \in U(K, \delta/4, \gamma/4)$.

In [2], this result is proved in the particular case $\mathcal{N} = \mathbb{S}^{k-1}$. However, the same proof applies to a general target \mathcal{N} : the map \tilde{u} is constructed by a suitable reparametrisation of the domain $U(K, \delta, \gamma)$, and the arguments do not rely on properties of the target \mathcal{N} other than (Lipschitz) path-connectedness. Property (iii) follows from Remark 6 and (ii), (iv).

Proof of Lemma 5. By (3.24), we have $\text{spt } R \subseteq \overline{W}_{\mathfrak{S}} \subseteq W$. By triangulating, we can write R in the form

$$R = \sum_{i=1}^q \sigma_i \llbracket T_i \rrbracket,$$

where the coefficients σ_i belong to $\pi_{k-1}(\mathcal{N})$ and each $T_i \subset \subset W$ is a convex $(n+1)$ -simplex. We apply Proposition 4, so to modify w_* in a neighbourhood of T_1 . We obtain a new map $w_1 \in W^{1,k}(\Omega, \mathcal{N})$ that has a locally nice singularity at $\text{spt } \mathbf{S}(w_*) \cup (T_1)_n \cup P_{y_*}$ (with $(T_1)_n$ is the n -skeleton of a suitable triangulation of T_1), satisfies $w_1 = w_*$ on $\Omega \setminus W$ and $\mathbf{S}(w_1) = \mathbf{S}(w_*) + \sigma_1 \partial \llbracket T_1 \rrbracket$. Now, we use Proposition 4 to modify w_1 in a neighbourhood of T_2 , and so on. By applying iteratively Proposition 4, we construct a sequence of maps w_1, w_2, \dots, w_q . The

map w_q has a locally nice singularity at $\text{spt } \mathbf{S}(w_*) \cup (\text{spt } R)_n \cup P_{y_*}$, satisfies $w_q = w_*$ on $\Omega \setminus W$ and $\mathbf{S}(w_q) = \mathbf{S}(w_*) + \partial R = S$.

To complete the proof, it only remains to modify w_q so as to satisfy (iv). Since $W \subset\subset \Omega$ has polyhedral boundary, the restriction $S \llcorner W$ is a polyhedral chain. Let K be a n -face of $\text{spt } \mathbf{S}(w_*) \cup (\text{spt } R)_n$. The interior of K is contained in W and hence, for sufficiently small parameters $\delta > 0$, $\gamma > 0$, the interior of $U(K, \delta, \gamma)$ is contained in W . Let $\sigma_K \in \pi_{k-1}(\mathcal{N})$ be the homotopy class of w_q around K . By Remark 7, there exists a smooth map $\phi_K: \mathbb{S}^{k-1} \rightarrow \mathcal{N}$ that satisfies

$$\int_{\mathbb{S}^{k-1}} |\nabla_{\top} \phi_K|^k d\mathcal{H}^{k-1} \leq \int_{\mathbb{S}^{k-1}} |\nabla_{\top} \psi|^k d\mathcal{H}^{k-1} + \eta$$

for any $\psi \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma_K$. If $\sigma_K = 0$, we choose ϕ_K to be constant. We apply Lemma 6 to $u = w_q$ and $\phi = \phi_K$. By doing so for each K , we obtain a map $w: \Omega \rightarrow \mathcal{N}$ that agrees with w_* on $\Omega \setminus \overline{W}$ and is η -minimal on W . By Remark 6, $\mathbf{S}(w) = \mathbf{S}(w_q) = S$. Moreover, since ϕ_K is constant if $\sigma_K = 0$, w has a locally nice singularity at $(\text{spt } S, Q_*)$ where $Q_* := (\text{spt } \mathbf{S}(w_*))_{n-1} \cup (\text{spt } R)_{n-1} \cup P_{y_*}$. Therefore, w has all the desired properties. \square

3.4.4. ε -regularisation The map $w: \Omega \rightarrow \mathcal{N}$ given by Lemma 5 has a singularity of codimension k at $\text{spt } S$, so $w \notin W^{1,k}(\Omega, \mathcal{N})$ unless $S = 0$. Therefore, in order to define a recovery sequence, we need to regularise w around $\text{spt } S$. We do so by defining the maps

$$w_\varepsilon(x) := \min \left\{ \frac{\text{dist}(x, \text{spt } S)}{\varepsilon}, 1 \right\} w(x) \quad \text{for any } x \in \Omega. \quad (3.26)$$

Lemma 7. *For sufficiently small ε , the map w_ε defined by (ii) belongs to $(L^\infty \cap W_{\text{loc}}^{1,k})(\Omega, \mathbb{R}^m)$. Moreover, the following properties holds:*

- (i) $w_\varepsilon \rightarrow w$ strongly in $W_{\text{loc}}^{1,k-1}(\Omega)$ as $\varepsilon \rightarrow 0$.
- (ii) For any open set $D \subset\subset \Omega$ with polyhedral boundary, it holds that

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(w_\varepsilon, D)}{|\log \varepsilon|} \leq C_{w,D} \mathbb{M}(S \llcorner \overline{D}),$$

where the constant $C_{w,D}$ depends on the map w and on $\text{dist}(D, \partial\Omega)$.

(iii) We have

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(w_\varepsilon, W)}{|\log \varepsilon|} \leq (1 + C\eta) \mathbb{M}(S \llcorner \overline{W}) + C \mathbb{M}(\overline{W} \setminus W_\ominus),$$

where C is a constant that depends only on \mathcal{N} , \mathcal{X} , q and k .

Proof. Let $Z_\varepsilon := \{x \in \mathbb{R}^{n+k} : \text{dist}(x, \text{spt } S) < \varepsilon\}$, and let ζ_ε be the characteristic function of Z_ε (i.e. $\zeta_\varepsilon := 1$ on Z_ε , $\zeta_\varepsilon := 0$ elsewhere).

Step 1. (Proof of (i)) Let $D \subset\subset \Omega$ be an open set. We choose a number p , with $1 < p < (k+1)/(k-1)$. Since w has a locally nice singularity at $(\text{spt } S, Q_*)$, at a.e. point of D we have

$$\begin{aligned} |\nabla w_\varepsilon| &\lesssim \left(\frac{\text{dist}(\cdot, \text{spt } S)\zeta_\varepsilon}{\varepsilon} + 1 - \zeta_\varepsilon \right) |\nabla w| + \frac{\zeta_\varepsilon}{\varepsilon} \\ &\leq C_{w,D} \left(\left(\frac{\text{dist}(\cdot, \text{spt } S)\zeta_\varepsilon}{\varepsilon} + 1 - \zeta_\varepsilon \right) (\text{dist}^{-1}(\cdot, \text{spt } S) + \text{dist}^{-p}(\cdot, Q_*)) + \frac{\zeta_\varepsilon}{\varepsilon} \right) \\ &\leq C_{w,D} \left((1 - \zeta_\varepsilon) \text{dist}^{-1}(\cdot, \text{spt } S) + \text{dist}^{-p}(\cdot, Q_*) + \frac{\zeta_\varepsilon}{\varepsilon} \right), \end{aligned} \tag{3.27}$$

where $C_{w,D}$ is a constant that depends on w , $\text{dist}(D, \partial\Omega)$ and p , but not on ε . Therefore,

$$\int_{D \cap Z_\varepsilon} |\nabla w_\varepsilon|^{k-1} \leq C_{w,D} \left(\int_{D \cap Z_\varepsilon} \text{dist}^{p-kp}(x, Q_*) \, dx + \frac{\mathcal{L}^{n+k}(D \cap Z_\varepsilon)}{\varepsilon^{k-1}} \right).$$

By our choice of p , we have $p - kp > -(k+1)$. Since Q_* has codimension $k+1$, [2, Lemma 8.3] implies that the function $\text{dist}^{p-kp}(\cdot, Q_*)$ is integrable and that

$$\mathcal{L}^{n+k}(Z_\varepsilon) \lesssim \varepsilon^k. \tag{3.28}$$

As a consequence, we have

$$\int_D |\nabla(w - w_\varepsilon)|^{k-1} \lesssim \int_{D \cap Z_\varepsilon} |\nabla w|^{k-1} + \int_{D \cap Z_\varepsilon} |\nabla w_\varepsilon|^{k-1} \rightarrow 0$$

as $\varepsilon \rightarrow 0$, and (i) follows.

Step 2. (Proof of (ii)) Let $D \subset\subset \Omega$ and $1 < p < 1 + 1/k$. From (3.27), we deduce

$$\begin{aligned} E_\varepsilon(w_\varepsilon, D) &\leq C_{w,D} \left(\int_{D \setminus Z_\varepsilon} \text{dist}^{-k}(x, \text{spt } S) \, dx + \int_D \text{dist}^{-kp}(x, Q_*) \, dx \right. \\ &\quad \left. + \frac{\mathcal{L}^{n+k}(D \cap Z_\varepsilon)}{\varepsilon^k} \right) \end{aligned}$$

The second and third term at the right-hand side are uniformly bounded with respect to $\varepsilon \rightarrow 0$, due to [2, Lemma 8.3] and (3.28). Since $\text{spt } S \cap D$ is contained in a finite union of polyhedra of codimension k or higher and D has polyhedral boundary, a computation based on Fubini theorem gives

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{D \setminus Z_\varepsilon} \text{dist}^{-k}(\cdot, \text{spt } S) \lesssim \mathcal{H}^n(\text{spt } S \cap \overline{D}).$$

On the other hand, $\mathcal{H}^n(\text{spt } S \cap \overline{D}) \lesssim \mathbb{M}(S \llcorner \overline{D})$ because the coefficient group $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$ is discrete (Proposition 1). Thus, (ii) follows (and in particular, $w_\varepsilon \in W_{\text{loc}}^{1,k}(\Omega, \mathbb{R}^m)$).

Step 3. (Proof of (iii)) The inequality (3.28) implies

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon| \varepsilon^k} \int_W f(w_\varepsilon) \lesssim \limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{L}^{n+k}(Z_\varepsilon)}{|\log \varepsilon| \varepsilon^k} = 0, \quad (3.29)$$

so we only need to estimate the gradient terms. By Lemma 5, $w|_W$ is η -minimal, with nice singularity at $((\text{spt } S) \cap W, Q_* \cap W)$. Therefore, there exist positive numbers δ, γ , a triangulation of $(\text{spt } S) \cap W$ and, for any n -simplex K of the triangulation, a Lipschitz map $\phi_K: \mathbb{S}^{k-1} \rightarrow \mathcal{N}$ that satisfy the conditions (i)–(iii) in Definition 3.2. By taking smaller δ, γ if necessary, we can also assume that the interior of $U(K, \delta, \gamma)$ is contained in W , for any n -simplex K of the triangulation. Let $F := W \setminus \cup_K U(K, \delta, \gamma)$, where the union is taken over all n -simplices K of the triangulation. We estimate separately the energy on F and on each $U(K, \delta, \gamma)$.

Let us estimate the energy on F first. Since $Q_* \supseteq (\text{spt } S)_{n-1}$, the definition (3.1) of $U(K, \delta, \gamma)$ implies that

$$\text{dist}(x, Q^*) \gtrsim \text{dist}(x, \text{spt } S) \quad \text{for any } x \in F. \quad (3.30)$$

(The proportionality constant at the right-hand side depends on δ, γ .) Let us choose a number p with $1 < p < 1 + 1/k$. Since w has a locally nice singularity at $(\text{spt } S, Q_*)$, we obtain

$$\begin{aligned} |\nabla w(x)| &\leq C_{w,W} \left(\text{dist}^{-1}(x, \text{spt } S) + \text{dist}^{-p}(x, Q_*) \right) \\ (3.30) \quad &\leq C_{w,W} \text{dist}^{-p}(x, Q_*) \end{aligned}$$

for a.e. $x \in F$ and some constant $C_{w,W}$ that depends on w, W, p, δ and γ . This implies

$$\frac{1}{k} \int_F |\nabla w_\varepsilon|^k \leq C_{w,W} \int_F \text{dist}^{-kp}(x, Q_*) \, dx + \frac{C_{w,W}}{\varepsilon^k} \mathcal{L}^{n+k}(Z_\varepsilon).$$

The right-hand side is uniformly bounded with respect to ε , due to [2, Lemma 8.3] and (3.28), so

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{k |\log \varepsilon|} \int_F |\nabla w_\varepsilon|^k = 0. \quad (3.31)$$

Next, we estimate the energy on $U(K, \delta, \gamma)$, with K an n -dimensional simplex in the triangulation of $(\text{spt } S) \cap W$. We write $U := U(K, \delta, \gamma)$ for brevity, and let $x = (x', x'')$ denote the variable in U , as in (3.1). Using Condition (ii) in Definition (3.2), we can compute explicitly the gradient of w_ε , and we obtain

$$\begin{aligned} |\nabla w_\varepsilon(x)| &\leq \left(\frac{C \zeta_\varepsilon(x)}{\varepsilon} + \frac{1 - \zeta_\varepsilon(x)}{|x''|} \right) \left| (\nabla_\top \phi_K) \left(\frac{x''}{|x''|} \right) \right| + \frac{C \zeta_\varepsilon(x)}{\varepsilon} \\ &\leq \frac{1 - \zeta_\varepsilon(x)}{|x''|} \left| (\nabla_\top \phi_K) \left(\frac{x''}{|x''|} \right) \right| + \frac{C_{w,W} \zeta_\varepsilon(x)}{\varepsilon} \end{aligned}$$

for a.e. $x \in U$, where ∇_\top denotes the tangential gradient on \mathbb{S}^{k-1} . (In the second inequality, we use that ϕ_K is Lipschitz.) We raise to the power k both sides of this

inequality, integrate over U , apply Fubini theorem and pass to polar coordinates for the integral with respect to x'' :

$$\begin{aligned} \frac{1}{k} \int_U |\nabla w_\varepsilon|^k &\leq \left(\frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} \phi_K|^k d\mathcal{H}^{k-1} \right) \left(\int_\varepsilon^\delta \frac{d\rho}{\rho} \right) \mathcal{H}^n(K) \\ &\quad + \frac{C_{w,W} \mathcal{L}^{n+k}(Z_\varepsilon)}{\varepsilon^k} \\ &\stackrel{(3.28)}{\leq} \left(\frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} \phi_K|^k d\mathcal{H}^{k-1} \right) \left(\log \frac{\delta}{\varepsilon} \right) \mathcal{H}^n(K) + C_{w,W}. \end{aligned}$$

Using Condition (iii) in Definition 3.2, we deduce

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{k |\log \varepsilon|} \int_U |\nabla w_\varepsilon|^k \leq (E_{\min}(\sigma_K) + \eta) \mathcal{H}^n(K), \quad (3.32)$$

where $\sigma_K \in \pi_{k-1}(\mathcal{N})$ is the homotopy class of ϕ_K and $E_{\min}(\sigma_K)$ is defined by (2.2). We need to distinguish two cases, depending on whether the interior of K is contained $W_\mathfrak{S}$ or not. If the interior of K is contained in $W_\mathfrak{S}$, then $\sigma_K \in \mathfrak{S}$ because of (3.25), and (3.32) becomes

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{k |\log \varepsilon|} \int_U |\nabla w_\varepsilon|^k \leq (|\sigma_K|_* + \eta) \mathcal{H}^n(K) \leq (1 + C\eta) \mathbb{M}(S \llcorner K) \quad (3.33)$$

for some constant C that depends only on \mathcal{N} . (Here again, we have used that $\mathbb{M}(S \llcorner K) \gtrsim \mathcal{H}^n(K)$, due to Proposition (1).) Suppose now that the interior of K is not contained in $W_\mathfrak{S}$. The intersection between the interior of K and $\partial W_\mathfrak{S}$ has dimension $n - 1$ at most, because we have taken $\partial W_\mathfrak{S}$ to be transverse to $\text{spt } S$. Therefore, up to refining the triangulation, we may assume that the interior of K is contained in $W \setminus \bar{W}_\mathfrak{S}$. Then, thanks to (3.23) and (3.24), S agrees with $\mathbf{S}(w_*)$ in the interior of K . The chain $\mathbf{S}(w_*)$ takes its multiplicity in a finite set that depends only on \mathcal{N} , \mathcal{X} , ϱ (by Lemma 3) and hence, $E_{\min}(\sigma_K) \leq C$. Thus, (3.32) becomes

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{k |\log \varepsilon|} \int_U |\nabla w_\varepsilon|^k \lesssim \mathcal{H}^n(K) \lesssim \mathbb{M}(S \llcorner K). \quad (3.34)$$

Combining (3.29), (3.31), (3.33) and (3.34), the inequality (iii) follows. \square

3.4.5. Proof of Theorem C.(ii) and Proposition D.(ii)

Proof of Theorem C.(ii). Let $S \in \mathcal{C}(\Omega, \nu)$ be a finite-mass chain, and let $\eta > 0$ be a small number. Given a countable sequence $\varepsilon \rightarrow 0$, we aim to construct $u_\varepsilon \in (L^\infty \cap W_v^{1,k})(\Omega, \mathbb{R}^m)$, where ε ranges in a non-relabelled subsequence, in such a way that

$$\lim_{\varepsilon \rightarrow 0} \int_{B^m(0, \text{dist}(\cdot, \mathcal{N}, \mathcal{X}))} \mathbb{F}(\mathbf{S}_y(u_\varepsilon) - S) dy = 0, \quad (3.35)$$

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leq (1 + C\eta) \mathbb{M}(S) + C\eta, \quad (3.36)$$

where C is a constant that does not depend on η . If we do so, the theorem will follow, by a diagonal argument. As we have seen, thanks to Proposition 5, Proposition 6 and a diagonal argument, it is not restrictive to assume that S satisfies (3.23), (3.24), (3.25). Moreover, we have

$$S \perp \partial\Omega \stackrel{(3.23),(3.24)}{=} \mathbf{S}(w_*) \perp \partial\Omega = \mathbf{S}_{y_*}(u_*) \perp \partial\Omega = 0 \quad (3.37)$$

by Lemma 3 and hence, by taking a larger $W_{\mathfrak{G}}$ if necessary, we can assume without loss of generality that

$$\mathbb{M}(S \perp (\overline{\Omega} \setminus W_{\mathfrak{G}})) + \int_{\Omega \setminus W_{\mathfrak{G}}} |\nabla u_*|^k \leq \eta. \quad (3.38)$$

Step 1. (Definition of u_ε) To define the recovery sequence near the boundary of Ω , we apply Lemma 2 to u_* and y_* , and consider the map

$$w_{\varepsilon, y_*} := (\xi_\varepsilon \circ \psi) \circ (u_* - y_*) \cdot w_* = (\xi_\varepsilon \circ \psi) \circ (u_* - y_*) \cdot (\varrho_{y_*} \circ u_*)$$

(with ξ_ε, ψ as in Lemma 2). Thanks to Lemma 2 and an averaging argument, by possibly modifying the value of y_* we have

$$\begin{aligned} E_\varepsilon(w_{\varepsilon, y_*}, \Omega \setminus W_{\mathfrak{G}}) + \varepsilon^{-k} \mathcal{L}^{n+k} \{x \in \Omega \setminus W_{\mathfrak{G}} : w_{\varepsilon, y_*}(x) \neq w_*(x)\} \\ \lesssim |\log \varepsilon| \int_{\Omega \setminus W_{\mathfrak{G}}} |\nabla u_*|^k + 1 \stackrel{(3.38)}{\lesssim} \eta |\log \varepsilon| + 1. \end{aligned} \quad (3.39)$$

Our recovery sequence will coincide with w_ε given by (ii) in W , where $W_{\mathfrak{G}} \subset \subset W \subset \subset \Omega$ is the open set introduced in Section 3.4.3. We need to interpolate between w_ε and w_{ε, y_*} near W . To this end, we take a small parameter $\theta > 0$, and we let $D_\theta := \{x \in \Omega \setminus \overline{W} : \text{dist}(x, W) < \theta\}$. For $x \in D_\theta$, let $t_\theta(x) := \theta^{-1} \text{dist}(x, W)$. We define

$$u_\varepsilon(x) := \begin{cases} w_\varepsilon(x) & \text{if } x \in \overline{W} \\ (1 - t_\theta(x)) w_\varepsilon(x) + t_\theta(x) w_{\varepsilon, y_*}(x) & \text{if } x \in D_\theta \\ w_{\varepsilon, y_*}(x) & \text{if } x \in \Omega \setminus (\overline{W} \cup D_\theta). \end{cases}$$

We have $u_\varepsilon \in (L^\infty \cap W_v^{1,k})(\Omega, \mathbb{R}^m)$ and $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty$.

Step 2. (Bounds on $E_\varepsilon(u_\varepsilon)$) The energy of u_ε on $\Omega \setminus (\overline{W} \cup D_\theta)$ is bounded from above by (3.39). The energy of u_ε is bounded from above by Lemma 7:

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, W)}{|\log \varepsilon|} &\leq (1 + C\eta) \mathbb{M}(S) + C \mathbb{M}(S \perp (\overline{W} \setminus W_{\mathfrak{G}})) \\ &\stackrel{(3.38)}{\leq} (1 + C\eta) \mathbb{M}(S) + C\eta. \end{aligned} \quad (3.40)$$

It remains to estimate the energy of u_ε on D_θ . We first note that $|\nabla t_\theta| = \theta^{-1}$ and hence,

$$|\nabla u_\varepsilon| \leq |\nabla w_\varepsilon| + |\nabla w_{\varepsilon, y_*}| + \theta^{-1} |w_\varepsilon - w_{\varepsilon, y_*}| \leq |\nabla w_\varepsilon| + |\nabla w_{\varepsilon, y_*}| + C\theta^{-1}. \quad (3.41)$$

By Lemma 5, $w = w_*$ a.e. in $\Omega \setminus W$ and in particular, $w = w_*$ a.e. in D_θ . Therefore, for a.e. $x \in D_\theta$ such that $w_\varepsilon(x) = w(x)$ and $w_{\varepsilon, y_*}(x) = w_*(x)$, we have $u_\varepsilon(x) = w_*(x) \in \mathcal{N}$. Since the maps u_ε are uniformly bounded, we deduce that

$$f(u_\varepsilon) \lesssim \mathbb{1}_{\{w_\varepsilon \neq w\}} + \mathbb{1}_{\{w_{\varepsilon, y_*} \neq w_*\}}. \quad (3.42)$$

From (3.41) and (3.42), we obtain

$$\begin{aligned} E_\varepsilon(u_\varepsilon, D_\theta) &\lesssim E_\varepsilon(w_\varepsilon, D_\theta) + E_\varepsilon(w_{\varepsilon, y_*}, D_\theta) + \theta^{-k} \mathcal{L}^{n+k}(D_\theta) \\ &\quad + \varepsilon^{-k} \mathcal{L}^{n+k}\{w_\varepsilon \neq w\} + \varepsilon^{-k} \mathcal{L}^{n+k}(D_\theta \cap \{w_{\varepsilon, y_*} \neq w_*\}). \end{aligned}$$

The set $\{w_\varepsilon \neq w\}$ is the ε -neighbourhood of $\text{spt } S$, which is a locally polyhedral set of codimension k , so

$$\mathcal{L}^{n+k}\{w_\varepsilon \neq w\} \lesssim \varepsilon^k \quad (3.43)$$

(see Lemma [2, Lemma 8.3] and (3.28)). Moreover, $\mathcal{L}^{n+k}(D_\theta) \lesssim \theta$. Then,

$$\begin{aligned} E_\varepsilon(u_\varepsilon, D_\theta) &\lesssim E_\varepsilon(w_\varepsilon, D_\theta) + E_\varepsilon(w_{\varepsilon, y_*}, D_\theta) \\ &\quad + \varepsilon^{-k} \mathcal{L}^{n+k}(D_\theta \cap \{w_{\varepsilon, y_*} \neq w_*\}) + \theta^{1-k} + 1. \end{aligned} \quad (3.44)$$

We choose $\theta = \theta(\varepsilon)$ in such a way that $\theta(\varepsilon) \rightarrow 0$ and $\theta(\varepsilon)^{1-k} |\log \varepsilon|^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$; for instance, we take $\theta(\varepsilon) := |\log \varepsilon|^{-1/(2k-2)}$. With this choice of θ , from (3.44), Lemma 7 and (3.39) we deduce

$$\frac{E_\varepsilon(u_\varepsilon, D_{\theta(\varepsilon)})}{|\log \varepsilon|} \leq C_{w, W} \mathbb{M}(S \llcorner \overline{D_{\theta(\varepsilon)}}) + C\eta + o_{\varepsilon \rightarrow 0}(1)$$

where $C_{w, W}$ is a constant that depends on w and $\text{dist}(W, \partial\Omega)$, but not on ε . By taking the limit as $\varepsilon \rightarrow 0$, and recalling that ∂W is transverse to $\text{spt } S$, we conclude that

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, D_{\theta(\varepsilon)})}{|\log \varepsilon|} = C_{w, W} \mathbb{M}(S \llcorner \partial W) + C\eta = C\eta. \quad (3.45)$$

Combining (3.39), (3.40) and (3.45), the inequality (3.36) follows.

Step 3. ($u_\varepsilon \rightarrow w$ in $W^{1, k-1}(\Omega)$) To complete the proof, it only remains to check (3.35). As an intermediate step, we prove that $u_\varepsilon \rightarrow w$ strongly in $W^{1, k-1}(\Omega)$. Up to extraction of a subsequence, we have $w_\varepsilon \rightarrow w$ in $W^{1, k-1}(W)$ and $w_{\varepsilon, y_*} \rightarrow w_* = w$ in $W^{1, k-1}(\Omega \setminus W)$ by Lemma 7 and Lemma 2, respectively. Thus, we only need to check that

$$\int_{D_{\theta(\varepsilon)}} |\nabla u_\varepsilon|^{k-1} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.46)$$

From (3.41), using that $w = w_*$ a.e. on $D_{\theta(\varepsilon)}$, we deduce

$$\int_{D_{\theta(\varepsilon)}} |\nabla u_\varepsilon|^{k-1} \lesssim \int_{D_{\theta(\varepsilon)}} \left(|\nabla w_\varepsilon|^{k-1} + |\nabla w_{\varepsilon, y_*}|^{k-1} + \frac{|w_\varepsilon - w|^{k-1}}{\theta(\varepsilon)^{k-1}} + \frac{|w_{\varepsilon, y_*} - w_*|^{k-1}}{\theta(\varepsilon)^{k-1}} \right).$$

The sequences w_ε and w_{ε, y_*} are strongly compact in $W_{\text{loc}}^{1,k-1}(\Omega)$, $W^{1,k-1}(\Omega)$ respectively. Since $\mathcal{L}^{n+k}(D_{\theta(\varepsilon)}) \rightarrow 0$, we have

$$\int_{D_{\theta(\varepsilon)}} \left(|\nabla w_\varepsilon|^{k-1} + |\nabla w_{\varepsilon, y_*}|^{k-1} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Then, keeping in mind that w_{ε, y_*} , w_ε are uniformly bounded, and using (3.39), (3.43), we obtain

$$\int_{D_{\theta(\varepsilon)}} |\nabla u_\varepsilon|^{k-1} \lesssim \varepsilon^k |\log \varepsilon| \theta(\varepsilon)^{1-k} + o_{\varepsilon \rightarrow 0}(1).$$

Now (3.46) follows, because we have chosen $\theta(\varepsilon)$ in such a way that $\theta(\varepsilon)^{1-k} |\log \varepsilon|^{-1} \rightarrow 0$.

Step 4. (Proof of (3.35)) Let us take a larger, bounded domain $\Omega' \supset \supset \Omega$ and a map $V \in (L^\infty \cap W^{1,k})(\Omega' \setminus \overline{\Omega}, \mathbb{R}^m)$ with trace v on $\partial\Omega$. We define

$$\tilde{u}_\varepsilon := \begin{cases} u_\varepsilon & \text{on } \Omega \\ V & \text{on } \Omega' \setminus \Omega, \end{cases} \quad \tilde{w} := \begin{cases} w & \text{on } \Omega \\ V & \text{on } \Omega' \setminus \Omega. \end{cases}$$

Since the traces of u_ε , w agree with that of V on $\partial\Omega$, we have $\tilde{u}_\varepsilon \in (L^\infty \cap W^{1,k})(\Omega', \mathbb{R}^m)$, $\tilde{w} \in (L^\infty \cap W^{1,k-1})(\Omega', \mathbb{R}^m)$, $\sup_\varepsilon \|\tilde{u}_\varepsilon\|_{L^\infty(\Omega')} < +\infty$ and $\tilde{u}_\varepsilon \rightarrow \tilde{w}$ strongly in $W^{1,k-1}(\Omega')$. By continuity of \mathbf{S} [17, Theorem 3.1], this implies

$$\int_{B^m(0, \text{dist}(\mathcal{N}, \mathcal{X}))} \mathbb{F}_{\Omega'}(\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w})) \, dy \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Since $\tilde{u}_\varepsilon = \tilde{w}$ a.e. on $\Omega' \setminus \overline{\Omega}$ and the operator \mathbf{S} is local [17, Corollary 1], we have $\mathbf{S}_y(\tilde{u}_\varepsilon) \llcorner (\Omega' \setminus \overline{\Omega}) = \mathbf{S}_y(\tilde{w}) \llcorner (\Omega' \setminus \overline{\Omega})$ for a.e. y , and hence $\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w})$ is supported in $\overline{\Omega}$ for a.e. y . For chains supported in a compact subset of Ω' , the relative flat norm $\mathbb{F}_{\Omega'}$ is equivalent to \mathbb{F} (see e.g. [17, Remark 2.2]). Therefore, we have

$$\int_{B^m(0, \text{dist}(\mathcal{N}, \mathcal{X}))} \mathbb{F}(\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w})) \, dy \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.47)$$

By [17, Eq. (3.25)] we have $\mathbf{S}_y(\tilde{u}_\varepsilon) \llcorner \partial\Omega = 0$ and $\mathbf{S}_y(\tilde{w}) \llcorner \partial\Omega = \mathbf{S}_y(V) \llcorner \partial\Omega = 0$ for a.e. y , so

$$\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w}) = (\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w})) \llcorner \Omega.$$

Since $\tilde{u}_\varepsilon = u_\varepsilon$ and $\tilde{w} = w$ a.e. on Ω , [17, Corollary 1] implies

$$\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w}) = (\mathbf{S}_y(u_\varepsilon) - \mathbf{S}(w)) \llcorner \Omega = \mathbf{S}_y(u_\varepsilon) - S \llcorner \Omega$$

and finally, recalling (3.37), we obtain

$$\mathbf{S}_y(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{w}) = \mathbf{S}_y(u_\varepsilon) - S. \quad (3.48)$$

From (3.47) and (3.48) we deduce (3.35), and the proof is complete. \square

The proof of Proposition D.(ii) follows along the same lines, and in fact, is even simpler.

Proof of Proposition D. Let S be an n -dimensional relative boundary of finite mass — that is, S has the form $S = (\partial R) \llcorner \Omega$, where R is an $(n+1)$ -chain of finite mass such that $\mathbb{M}(\partial R) < +\infty$. By a density argument, we can assume without loss of generality that R is polyhedral. By Proposition 6, we can also assume that ∂R takes its multiplicities in the set $\mathfrak{S} \subseteq \pi_{k-1}(\mathcal{N})$ defined by (2.4). Finally, by translating the support of R and applying Thom's transversality theorem, we can assume that

$$(\partial R) \llcorner \partial \Omega = 0. \quad (3.49)$$

Let $w_* \in \mathcal{N}$ be a constant, and let $\eta > 0$ be a small parameter. We repeat the same arguments of Lemma 5 and modify the constant map w_* in a neighbourhood of $\text{spt } R$. We obtain a new map $w: \mathbb{R}^{n+k} \rightarrow \mathcal{N}$ that

- (i) has a nice singularity at $(\text{spt}(\partial R), (\text{spt}(\partial R))_{n-1})$;
- (ii) satisfies $\mathbf{S}(w) = \mathbf{S}(w_*) + \partial R = \partial R$;
- (iii) is η -minimal.

Let

$$u_\varepsilon(x) := \left\{ \frac{\text{dist}(x, \text{spt}(\partial R))}{\varepsilon}, 1 \right\} w(x) \quad \text{for } x \in \mathbb{R}^{n+k}.$$

By the same computations as in Lemma 7, we obtain that $w_\varepsilon \rightarrow w$ strongly in $W^{1,k-1}(\mathbb{R}^{n+k})$ and that

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, \Omega)}{|\log \varepsilon|} \leq (1 + C\eta) \mathbb{M}((\partial R) \llcorner \Omega'), \quad (3.50)$$

where $\Omega' \supset \supset \Omega$ is any open set, with polyhedral boundary, such that $\partial \Omega$ is transverse to $\text{spt}(\partial R)$. (The latter condition is generic, by Thom's transversality theorem.) The continuity of \mathbf{S} [17, Theorem 3.1], together with the fact that the operator \mathbf{S} is local [17, Corollary 1], implies $\mathbf{S}(u_{\varepsilon|\Omega}) \rightarrow \mathbf{S}(w) \llcorner \Omega = (\partial R) \llcorner \Omega = S$ in Y . We let $\Omega' \searrow \Omega$ in (3.50), and we deduce

$$\limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, \Omega)}{|\log \varepsilon|} \leq (1 + C\eta) \mathbb{M}((\partial R) \llcorner \overline{\Omega}) \stackrel{(3.49)}{=} (1 + C\eta) \mathbb{M}(S).$$

Since η may be taken arbitrarily small, Proposition D.(ii) follows, by a diagonal argument. \square

4. Compactness and Lower Bounds

4.1. A Local Compactness Result

The aim of this section is to prove Statement (i) of Theorem C. As an intermediate step, we will prove the following result, which is a local version of Theorem C.(i). We recall that we have fixed a number $\delta^* \in (0, \text{dist}(\mathcal{N}, \mathcal{X}^c))$ and that $B^* := B^m(0, \delta^*) \subseteq \mathbb{R}^m$.

Proposition 7. *Let $U \subset\subset U'$ be bounded domains in \mathbb{R}^{n+k} . Let $(u_\varepsilon)_\varepsilon$ be a countable sequence of maps in $W^{1,k}(U', \mathbb{R}^m)$ such that*

$$\sup_{\varepsilon>0} \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|} < +\infty. \quad (4.1)$$

Then, there exist a (non-relabelled) subsequence and a finite-mass chain $S \in \mathbb{M}_n(\bar{U}'; \pi_{k-1}(\mathcal{N}))$ such that

$$\lim_{\varepsilon \rightarrow 0} \int_{B^*} \mathbb{F}_U(\mathbf{S}_y(u_\varepsilon) - S) \, dy = 0 \quad (4.2)$$

$$\mathbb{M}(S \llcorner U) \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|} \quad (4.3)$$

(\mathbb{F}_U is the relative flat norm, see (2.5)).

Throughout this section, we fix bounded domains $U \subset\subset U' \subseteq \mathbb{R}^{n+k}$ and a countable sequence (u_ε) in $W^{1,k}(U', \mathbb{R}^m)$ that satisfies (4.1). By an approximation argument, using the continuity of \mathbf{S} (Proposition 3 and [17, Theorem 3.1]), we can assume without loss of generality that the maps u_ε are *smooth and bounded*. For any $\varepsilon > 0$ and $y \in B^*$, we define the measure

$$\mu_{\varepsilon,y}(A) := \mathbb{M}(\mathbf{S}_y(u_\varepsilon) \llcorner (A \cap U')) \quad \text{for any Borel set } A \subseteq \mathbb{R}^{n+k}. \quad (4.4)$$

Thanks to (P₂) (Proposition 3), $\mu_{\varepsilon,y}$ is a bounded Radon measure for a.e. y .

4.1.1. Choice of a Grid As in [2], we define a grid \mathcal{G} of size $h > 0$ as a collection of closed cubes of the form

$$\mathcal{G} = \mathcal{G}(a, h) := \left\{ a + hz + [0, h]^{n+k} : z \in \mathbb{Z}^{n+k} \right\}, \quad (4.5)$$

for some $a \in \mathbb{R}^{n+k}$. For $j \in \mathbb{N}$, $0 \leq j \leq n+k$, we denote by \mathcal{G}_j the collection of the (closed) j -cells of \mathcal{G} , and we define the j -skeleton of \mathcal{G} , $R_j := \cup_{K \in \mathcal{G}_j} K$. We let \tilde{R}_k be the union of all the cells $K \in \mathcal{G}_k$ that are parallel to the k -plane spanned by $\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+k}\}$. Given an open set $V \subseteq U'$, we denote by $R_k(V)$ the union of the k -cells $K \in \mathcal{G}$ such that $K \cap V \neq \emptyset$ (so $R_k(V) \supseteq R_k \cap V$). Given $\mathcal{G} = \mathcal{G}(a, h)$, the grid

$$\mathcal{G}' := \mathcal{G}(a + (h/2, h/2, \dots, h/2), h)$$

will be called the dual grid of \mathcal{G} . We will denote by \mathcal{G}'_k the collections of k -cells of \mathcal{G}' and by R'_k its k -skeleton. For each $K \in \mathcal{G}_k$ there exists a unique $K' \in \mathcal{G}'_n$, called the dual cell of K , such that $K \cap K' \neq \emptyset$.

We are now going to construct a sequence of grids \mathcal{G}^ε with suitable properties. The construction is analogous to [2, Lemma 3.11]. Let us take a function $h : (0, 1) \rightarrow \mathbb{R}^+$ such that

$$\varepsilon^\alpha \ll h(\varepsilon) \ll |\log \varepsilon|^{-1} \quad \text{for any } \alpha > 0, \text{ as } \varepsilon \rightarrow 0. \quad (4.6)$$

For instance, we may take $h(\varepsilon) := |\log \varepsilon|^{-2}$.

Lemma 8. *For any fixed parameter $\delta > 0$ and any $\varepsilon < 1$ there exists a grid \mathcal{G}^ε of size $h(\varepsilon)$ that satisfies the following properties:*

$$h(\varepsilon)^n E_\varepsilon \left(u_\varepsilon, \tilde{R}_k^\varepsilon \cap U' \right) \leq (1 + \delta) E_\varepsilon \left(u_\varepsilon, U' \right) \quad (4.7)$$

$$h(\varepsilon)^n E_\varepsilon \left(u, R_k^\varepsilon \cap U' \right) \lesssim \delta^{-1} E_\varepsilon \left(u_\varepsilon, U' \right) \quad (4.8)$$

$$h(\varepsilon)^{n+1} E_\varepsilon \left(u, R_{k-1}^\varepsilon \cap U' \right) \lesssim \delta^{-1} E_\varepsilon \left(u_\varepsilon, U' \right) \quad (4.9)$$

$$h(\varepsilon)^n \int_{B^*} \int_{U'} \frac{d\mu_{\varepsilon,y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon)} dy \lesssim \delta^{-1} E_\varepsilon \left(u_\varepsilon, U' \right). \quad (4.10)$$

Here $\mu_{\varepsilon,y}$ is the measure defined by (4.4).

Proof. We take a grid $\mathcal{G}^\varepsilon := \mathcal{G}(a, h(\varepsilon))$ of the form (4.5). We claim that it is possible to choose $a \in (0, h(\varepsilon))^{n+k}$ in such a way that (4.7)–(4.10) are satisfied. For (4.7)–(4.9), we can repeat verbatim the arguments in [2]. As for (4.10), let us call $R_{k-1}^\varepsilon(a)$ the $(k-1)$ -skeleton of the grid $\mathcal{G}(a, h(\varepsilon))$. Thanks to (P₂) in Section 2, $\mu_{\varepsilon,y}$ is a finite, non-negative Radon measure for a.e. $y \in B^*$. By applying [25, Lemma 5.2], together with a scaling argument, we obtain

$$\begin{aligned} h(\varepsilon)^n \int_{(0, h(\varepsilon))^{n+k}} \left(\int_{U'} \frac{d\mu_{\varepsilon,y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon(a))} \right) d\mathcal{L}^{n+k}(a) \\ \lesssim \mu_{\varepsilon,y}(\mathbb{R}^{n+k}) = \mathbb{M}(\mathbf{S}_y(u_\varepsilon) \llcorner U') \end{aligned}$$

for a.e. $y \in B^*$. By integrating the previous inequality with respect to y and applying (P₂), we obtain

$$\begin{aligned} h(\varepsilon)^n \int_{(0, h(\varepsilon))^{n+k}} \left(\int_{B^*} \int_{U'} \frac{d\mu_{\varepsilon,y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon(a))} dy \right) d\mathcal{L}^{n+k}(a) \\ \lesssim \|\nabla u_\varepsilon\|_{L^k(U')}^k \lesssim E_\varepsilon(u_\varepsilon, U'). \end{aligned}$$

Now the lemma follows by an averaging argument, see e.g. [2, Lemma 8.4]. \square

Throughout the rest of this section, we suppose that (4.6) is satisfied, we fix $\delta \in (0, 1)$ and we consider the sequence of grids \mathcal{G}^ε given by Lemma 8. Without loss of generality, we will also assume that

$$R_{n+k}^\varepsilon(\bar{U}) \subset\subset U' \quad (4.11)$$

($R_{n+k}^\varepsilon(\bar{U})$ is the union of the closed cubes $K \in \mathcal{G}^\varepsilon$ such that $K \cap \bar{U} \neq \emptyset$).

Lemma 9. *For any $\alpha \in (0, k/(k^2 - k + 2))$, there holds*

$$\sup_{x \in R_{k-1}^\varepsilon(U)} \text{dist}(u_\varepsilon(x), \mathcal{N}) \leq \frac{C(\delta, \alpha) \varepsilon^\alpha}{h(\varepsilon)^{(n+k)/2}} (E_\varepsilon(u_\varepsilon, U') + 1)^{1/2},$$

where $C(\delta, \alpha)$ is a positive constant that only depends on \mathcal{N} , k , f , n , δ and α .

Proof. We repeat the arguments of [2, Lemma 3.4]. Let $d_\varepsilon := \text{dist}(u_\varepsilon, \mathcal{N})$, let $\lambda \in (0, 1/k)$ be a parameter, and let $G: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined by $G(t) := t^{2\lambda/(k-k\lambda)+1}$. Thanks to (H₃), we have $d_\varepsilon^2 \lesssim f(u_\varepsilon)$. Therefore, by (4.9) and (4.11), we obtain

$$\begin{aligned} h(\varepsilon)^{n+1} \int_{R_{k-1}^\varepsilon(U)} \left(\frac{1}{k} |\nabla d_\varepsilon|^k + \varepsilon^{-k} d_\varepsilon^2 \right) d\mathcal{H}^{k-1} &\lesssim h(\varepsilon)^{n+1} E_\varepsilon(u_\varepsilon, R_{k-1}^\varepsilon(U)) \\ &\lesssim \delta^{-1} E_\varepsilon(u_\varepsilon, U'). \end{aligned} \quad (4.12)$$

The Young inequality and the chain rule imply that

$$\begin{aligned} \delta^{-1} E_\varepsilon(u_\varepsilon, U') &\gtrsim h(\varepsilon)^{n+1} \int_{R_{k-1}^\varepsilon(U)} \left(\frac{1}{k} |\nabla d_\varepsilon|^k + \varepsilon^{-k} d_\varepsilon^2 \right) d\mathcal{H}^{k-1} \\ &\geq C(\lambda) \varepsilon^{-k\lambda} h(\varepsilon)^{n+1} \int_{R_{k-1}^\varepsilon(U)} |\nabla d_\varepsilon|^{k-k\lambda} d_\varepsilon^{2\lambda} d\mathcal{H}^{k-1} \\ &\geq C(\lambda) \varepsilon^{-k\lambda} h(\varepsilon)^{n+1} \int_{R_{k-1}^\varepsilon(U)} |\nabla(G \circ d_\varepsilon)|^{k-k\lambda} d\mathcal{H}^{k-1} \end{aligned} \quad (4.13)$$

Since we have assumed that $\lambda < 1/k$, we have $k - k\lambda > k - 1$ and hence, for any $(k-1)$ -cell $K \subseteq R_{k-1}^\varepsilon(U)$, we can bound the oscillation of $G \circ d_\varepsilon$ on K by Sobolev embedding:

$$\begin{aligned} (\text{osc}(G \circ d_\varepsilon, K))^{k-k\lambda} &\leq C(\delta, \lambda) h(\varepsilon)^{1-k\lambda} \int_{R_{k-1}^\varepsilon(U)} |\nabla(G \circ d_\varepsilon)|^{k-k\lambda} d\mathcal{H}^{k-1} \\ &\stackrel{(4.13)}{\leq} C(\delta, \lambda) \varepsilon^{k\lambda} h(\varepsilon)^{-n-k\lambda} E_\varepsilon(u_\varepsilon, U'). \end{aligned}$$

The inverse G^{-1} of G is well-defined and Hölder continuous of exponent $(k - k\lambda)/(2\lambda + k - k\lambda)$, so

$$\begin{aligned} \text{osc}(d_\varepsilon, K) &\leq C(\delta, \lambda) \left(\varepsilon^{k\lambda} h(\varepsilon)^{-n-k\lambda} E_\varepsilon(u_\varepsilon, U') \right)^{1/(2\lambda+k-k\lambda)} \\ &\leq C(\delta, \lambda) \varepsilon^{k\lambda/(2\lambda+k-k\lambda)} h(\varepsilon)^{-(n+k\lambda)/(2\lambda+k-k\lambda)} E_\varepsilon(u_\varepsilon, U')^{1/(2\lambda+k-k\lambda)} \end{aligned} \quad (4.14)$$

On the other hand, we can bound the integral average of d_ε on K thanks to (4.12):

$$\int_K d_\varepsilon d\mathcal{H}^{k-1} \leq \left(\int_K d_\varepsilon^2 d\mathcal{H}^{k-1} \right)^{1/2} \lesssim \delta^{-1/2} \varepsilon^{k/2} h(\varepsilon)^{-(n+k)/2} E_\varepsilon(u_\varepsilon, U')^{1/2}. \quad (4.15)$$

Combining (4.14) with (4.15), and letting $\lambda \nearrow 1/k$, the lemma follows. \square

4.1.2. A Polyhedral Approximation of $\mathbf{S}_y(u_\varepsilon)$ Let $y \in B^*$ be fixed in such a way that $\mathbf{S}_y(u_\varepsilon)$ has finite mass for any ε . (Thanks to (P₂), the set of y such that this property is not satisfied is negligible, because the sequence (u_ε) is assumed to be countable.) We are going to construct a polyhedral approximation of $\mathbf{S}_y(u_\varepsilon)$, supported on the dual n -skeleton $(R^\varepsilon)'_n$ of the grid.

Thanks to Lemma 9, there exists $\varepsilon_0 > 0$ (depending on δ_* , but not on y) such that

$$\text{dist}(u_\varepsilon(x), \mathcal{N}) < \text{dist}(\mathcal{N}, \mathcal{X}) - \delta_* < \text{dist}(\mathcal{N}, \mathcal{X}) - |y| \quad (4.16)$$

for any $x \in R_{k-1}^\varepsilon(U)$ and any $\varepsilon \in (0, \varepsilon_0]$. As a consequence, the projection $\varrho(u_\varepsilon - y)$ is well-defined and smooth on $R_{k-1}^\varepsilon(U)$ for $\varepsilon \in (0, \varepsilon_0]$. For any $K \in \mathcal{G}_k^\varepsilon$, let $\gamma^\varepsilon(K) \in \pi_{k-1}(\mathcal{N})$ be the homotopy class of $\varrho(u_\varepsilon - y)$ on ∂K . The quantity $\gamma^\varepsilon(K)$ does not depend on the choice of $y \in B^*$, because $\varrho(u_\varepsilon - y)|_{\partial K}$ and $\varrho(u_\varepsilon)|_{\partial K}$ are homotopic to each other, due to (4.16); a homotopy is defined by $(x, t) \in \partial K \times [0, 1] \mapsto \varrho(u_\varepsilon(x) - ty)$. We define the polyhedral chain

$$T^\varepsilon := \sum_{K \in \mathcal{G}_k^\varepsilon, K \cap U \neq \emptyset} \gamma^\varepsilon(K) \llbracket K' \rrbracket \in \mathbb{M}_n(\overline{U}; \pi_{k-1}(\mathcal{N})), \quad (4.17)$$

where $K' \in (\mathcal{G}^\varepsilon)'_n$ is the dual cell to K . The chain T^ε depends on the choice of the grid, but not on y .

Lemma 10. *For any $\varepsilon \in (0, \varepsilon_0]$ and any $y \in B^*$ such that $\mathbf{S}_y(u_\varepsilon)$ has finite mass, there holds*

$$\mathbb{F}_U(\mathbf{S}_y(u_\varepsilon) \llcorner U - T^\varepsilon) \lesssim h(\varepsilon)^{n+1} \int_{U'} \frac{d\mu_{\varepsilon,y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon)}. \quad (4.18)$$

Moreover, $\partial T^\varepsilon \llcorner U = 0$.

Proof. Essentially, this lemma is a particular instance of the Deformation Theorem for flat chains [26, Theorem 7.3] (see also [2, Lemma 3.8] for a statement which is specifically tailored for application to Ginzburg–Landau functionals). Nevertheless, we provide details for the convenience of the reader.

Let $\varepsilon \in (0, \varepsilon_0]$ be fixed. By [2, Lemma 3.8.(i)] there exists a locally Lipschitz retraction $\zeta^\varepsilon: \mathbb{R}^{n+k} \setminus R_{k-1}^\varepsilon \rightarrow (R^\varepsilon)'_n$, which maps each cube of \mathcal{G}^ε into itself and satisfies

$$|\nabla \zeta^\varepsilon(x)| \lesssim h(\varepsilon) \text{dist}(x, R_{k-1}^\varepsilon)^{-1} \quad \text{for a.e. } x \in \mathbb{R}^{n+k} \setminus R_{k-1}^\varepsilon. \quad (4.19)$$

By (4.16), we have $u_\varepsilon(x) - y \notin \mathcal{X}$ for any $x \in R_{k-1}^\varepsilon(U)$. By construction (see [17, Section 3]), this implies $\text{spt}(\mathbf{S}_y(u_\varepsilon)) \cap R_{k-1}^\varepsilon(U) = \emptyset$, so the push-forward $\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) \llcorner U$ is well-defined. Let $\tau^\varepsilon: [0, 1] \times (\mathbb{R}^{n+k} \setminus R_{k-1}^\varepsilon) \rightarrow \mathbb{R}^{n+k}$ be given by

$$\tau^\varepsilon(t, x) := (1-t)x + t\zeta^\varepsilon(x)$$

and let I be the 1-chain, with integer multiplicity, carried by the interval $[0, 1]$ with positive orientation. We remark that

$$\tau_*^\varepsilon(I \times \partial \mathbf{S}_y(u_\varepsilon)) \llcorner U = 0. \quad (4.20)$$

Indeed, since ζ^ε maps each cell K of \mathcal{G}^ε into itself, we have $(\tau^\varepsilon)^{-1}(U) \subseteq [0, 1] \times R_{n+k}^\varepsilon(\bar{U}) \subset\subset [0, 1] \times U'$ by (4.11). This implies

$$\begin{aligned} \tau_*^\varepsilon(I \times \partial \mathbf{S}_y(u_\varepsilon)) \llcorner U &= \tau_*^\varepsilon((I \times \partial \mathbf{S}_y(u_\varepsilon)) \llcorner ([0, 1] \times R_{n+k}^\varepsilon(\bar{U}))) \llcorner U \\ &= \tau_*^\varepsilon(I \times (\partial \mathbf{S}_y(u_\varepsilon) \llcorner R_{n+k}^\varepsilon(\bar{U}))) \llcorner U = 0 \end{aligned}$$

because $\partial \mathbf{S}_y(u_\varepsilon) \llcorner U' = 0$ [17, Theorem 3.1]. This proves (4.20). As a consequence, by applying the homotopy formula (see e.g. [26, Eq. (6.3) p. 172]), we deduce that

$$(\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) - \mathbf{S}_y(u_\varepsilon)) \llcorner U = \partial \tau_*^\varepsilon(I \times \mathbf{S}_y(u_\varepsilon)) \llcorner U. \quad (4.21)$$

From [26, Eq. (6.5) p. 172] and (4.19), we obtain

$$\begin{aligned} \mathbb{M}(\tau_*^\varepsilon(I \times \mathbf{S}_y(u_\varepsilon))) &\lesssim h(\varepsilon)^n \int_{U'} \frac{|\zeta^\varepsilon(x) - x|}{\text{dist}^n(x, R_{k-1}^\varepsilon)} d\mu_{\varepsilon, y}(x) \\ &\lesssim h(\varepsilon)^{n+1} \int_{U'} \frac{d\mu_{\varepsilon, y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon)}. \end{aligned}$$

Then, by the properties of the relative flat norm (see e.g. [17, Lemma 2]) and (4.21), we deduce

$$\mathbb{F}_U(\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) - \mathbf{S}_y(u_\varepsilon)) \lesssim h(\varepsilon)^{n+1} \int_{U'} \frac{d\mu_{\varepsilon, y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon)}. \quad (4.22)$$

To conclude the proof of (4.18), it suffices to show that $\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon))$ agrees with T^ε inside U . By [26, Lemma 7.2], $\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) \llcorner U$ is a n -polyhedral chain of the grid $(\mathcal{G}^\varepsilon)'$; in particular, its multiplicity is constant on every n -cell of $(\mathcal{G}^\varepsilon)'$. We want to compute such multiplicities. Let us take $K \in \mathcal{G}_k^\varepsilon$ and its dual cell $K' \in (\mathcal{G}^\varepsilon)'_n$, and let x be the unique element of $K \cap K'$. By construction of ζ^ε (see [2, Lemma 3.8 and Figure 3.2]), we have $(\zeta^\varepsilon)^{-1}(x) = K \setminus \partial K$. By Thom's parametric transversality theorem, we can assume with no loss of generality that K intersects transversally the support of $\mathbf{S}_y(u_\varepsilon)$. Then, by definition of push-forward, we have

$$\begin{aligned} \text{multiplicity of } \zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) \text{ at } x &= \mathbb{I}(\mathbf{S}_y(u_\varepsilon), \llbracket (\zeta^\varepsilon)^{-1}(x) \rrbracket) \\ &= \mathbb{I}(\mathbf{S}_y(u_\varepsilon), \llbracket K \rrbracket) \stackrel{(P_0)}{=} \gamma^\varepsilon(K) \end{aligned}$$

and hence

$$(\zeta_*^\varepsilon(\mathbf{S}_y(u_\varepsilon)) - T^\varepsilon) \llcorner U = 0. \quad (4.23)$$

Now, (4.18) follows from (4.22) and (4.23). Moreover, (4.23) implies

$$\partial T^\varepsilon \llcorner U = \zeta_*^\varepsilon(\partial \mathbf{S}_y(u_\varepsilon)) \llcorner U = 0,$$

because $\mathbf{S}_y(u_\varepsilon)$ has no boundary inside U' [17, Theorem 3.1, (S₃)]. \square

To bound the mass of T^ε , we will use the following result.

Lemma 11. *There exist positive numbers $\delta_1 = \delta_1(\mathcal{N}, f)$, $C_0 = C_0(\mathcal{N}, f)$ and, for $r > 0$, $\varepsilon_r = \varepsilon_r(r, \mathcal{N}, f)$, $C_r = C_r(r, \mathcal{N}, f)$ such that the following statement holds. Let $Q_h^k := [-h/2, h/2]^k$ be a cube of edge length $h > 0$. Suppose that $u \in W^{1,k}(Q_h^k, \mathbb{R}^m)$ satisfies*

$$u|_{\partial Q_h^k} \in W^{1,k}(\partial Q_h^k, \mathbb{R}^m) \quad \text{and} \quad \text{dist}(u(x), \mathcal{N}) \leq \delta_1 \quad \text{for a.e. } x \in \partial \Omega.$$

Let $\gamma \in \pi_{k-1}(\mathcal{N})$ be the homotopy class of u on ∂Q_h^k . Let $0 < \varepsilon < h^{k/2}/2$ be such that

$$\frac{\varepsilon}{h^{k/2}} \left| \log \frac{\varepsilon}{h^{k/2}} \right| |\gamma|_* \leq \varepsilon_r.$$

Then,

$$E_\varepsilon(u, Q_h) + C_0 h r E_\varepsilon(u, \partial Q_h) \geq |\gamma|_* \left| \log \frac{\varepsilon}{h^{k/2}} \right| - C_r |\gamma|_* (1 + \log |\gamma|_*).$$

The proof of Lemma 11 will be given in Appendix C.

Lemma 12. *For any r, δ and for sufficiently small ε , there holds*

$$(1 - c_{r,\delta}(\varepsilon)) \mathbb{M}(T^\varepsilon \llcorner U) \lesssim \delta^{-1} (1+r) \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|}, \quad (4.24)$$

where $c_{r,\delta}(\varepsilon) > 0$ is such that $c_{r,\delta}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, if L is the k -plane spanned by $\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+k}\}$, then there holds

$$(1 - c_{r,\delta}(\varepsilon)) \mathbb{M}(\pi_{L,*}(T^\varepsilon \llcorner U)) \leq (1 + \delta + C_r \delta^{-1}) \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|}. \quad (4.25)$$

Proof. We first remark that

$$\mathbb{M}(T^\varepsilon \llcorner U) \leq h(\varepsilon)^n \sum_{K \in \mathcal{G}_k^\varepsilon, K \cap U \neq \emptyset} |\gamma^\varepsilon(K)|_*. \quad (4.26)$$

Let $K \in \mathcal{G}_k^\varepsilon$ be a k -cell such that $K \cap U \neq \emptyset$. We claim that

$$|\gamma^\varepsilon(K)|_* \lesssim \delta^{-1} h(\varepsilon)^{-n} E_\varepsilon(u_\varepsilon, U') \quad (4.27)$$

Indeed, thanks to (P₀) and the definition of \mathbb{I} (see e.g. [17, Section 2.1]), we have

$$|\gamma^\varepsilon(K)|_* = |\mathbb{I}(\mathbf{S}_y(u_\varepsilon), \llbracket K \rrbracket)|_* \leq \mathbb{M}(\mathbf{S}_y(u_{\varepsilon|K}))$$

for any $y \in B^*$. By averaging both sides with respect to $y \in B^*$, and by applying (P₂) from Proposition 3, we obtain

$$|\gamma^\varepsilon(K)|_* \leq \int_{B^*} \mathbb{M}(\mathbf{S}_y(u_{\varepsilon|K})) \, dy \lesssim \|\nabla u_\varepsilon\|_{L^k(K)}^k \lesssim E_\varepsilon(u_\varepsilon, R_k^\varepsilon \cap U').$$

We can bound the right-hand side from above with the help of (4.8), so the claim (4.27) follows.

From (4.1), (4.6) and (4.27), we deduce

$$\sup_{K \in \mathcal{G}_k^\varepsilon, K \cap U \neq \emptyset} \frac{\varepsilon}{h(\varepsilon)^{k/2}} \left| \log \left(\frac{\varepsilon}{h(\varepsilon)^{k/2}} \right) \right| |\gamma^\varepsilon(K)|_* \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and this fact, together with (4.16), shows that the assumptions of Lemma 11 are satisfied for ε small enough. By applying Lemma 11, (4.6) and (4.27), we obtain the following bound:

$$|\gamma^\varepsilon(K)|_* |\log \varepsilon| (1 + o_{\varepsilon \rightarrow 0}(1)) \leq E_\varepsilon(u_\varepsilon, K) + Cr h(\varepsilon) E_\varepsilon(u_\varepsilon, \partial K).$$

We multiply both sides by $h(\varepsilon)^n |\log \varepsilon|^{-1}$ and sum over K . Thanks to (4.26), we obtain

$$(1 + o_{\varepsilon \rightarrow 0}(1)) \mathbb{M}(T^\varepsilon \llcorner U) \leq h(\varepsilon)^n \frac{E_\varepsilon(u_\varepsilon, R_k^\varepsilon \cap U')}{|\log \varepsilon|} + Cr h(\varepsilon)^{n+1} \frac{E_\varepsilon(u_\varepsilon, R_{k-1}^\varepsilon \cap U')}{|\log \varepsilon|}.$$

The right-hand side can now be bounded from above with the help of Lemma 8, so (4.24) follows. The proof of (4.25) is analogous; in this case, we sum over the cells K that are parallel to the k -plane spanned by $\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+k}\}$ and use (4.7). \square

4.1.3. Proof of Proposition 7 By combining the results in the previous section, we prove the following lemma, which is analogous to [2, Proposition 3.1]. For any n -plane $L \subseteq \mathbb{R}^{n+k}$, we denote by $\pi_L: \mathbb{R}^{n+k} \rightarrow L$ the orthogonal projection onto L .

Lemma 13. *Let $U \subset\subset U'$ be bounded domains in \mathbb{R}^{n+k} . Let $(u_\varepsilon)_\varepsilon$ be a countable sequence of smooth, bounded maps that satisfy (4.1). Let $L \subseteq \mathbb{R}^{n+k}$ be a n -plane. Then, there exist a (non-relabelled) subsequence and a finite-mass chain $S \in \mathbb{M}_n(\bar{U}'; \pi_{k-1}(\mathcal{N}))$ such that*

$$\int_{B^*} \mathbb{F}_U(\mathbf{S}_y(u_\varepsilon) - S) \, dy \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (4.28)$$

$$\mathbb{M}(\pi_{L,*}(S \llcorner U)) \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|}. \quad (4.29)$$

Proof. Up to rotations we can assume without loss of generality that L is the k -plane spanned by $\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{n+k}\}$. By Lemma 10 and Lemma 12, we know that $\partial T^\varepsilon \llcorner U = 0$ and $\mathbb{M}(T^\varepsilon \llcorner U)$ is uniformly bounded with respect to ε . Then, by applying compactness results for the flat norm (see e.g. [17, Lemmas 5 and 6] for a statement in terms of the relative flat norm), we find a (non-relabelled) subsequence and a finite-mass chain $S \in \mathbb{M}_n(\bar{U}; \pi_{k-1}(\mathcal{N}))$ such that

$$\mathbb{F}_U(T^\varepsilon - S) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (4.30)$$

$$\mathbb{M}(\pi_{L,*}(S \llcorner U)) \leq \left(1 + \delta + C\delta^{-1}r\right) \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|}. \quad (4.31)$$

The triangle inequality and Lemma 10 imply

$$\begin{aligned} & \int_{B^*} \mathbb{F}_U(\mathbf{S}_y(u_\varepsilon) - S) \, dy \\ & \leq \int_{B^*} \mathbb{F}_U(\mathbf{S}_y(u_\varepsilon) - T^\varepsilon) \, dy + \mathcal{L}^n(B^*) \mathbb{F}_U(T^\varepsilon - S) \\ & \leq h(\varepsilon)^{n+1} \int_{B^*} \int_{U'} \frac{d\mu_{\varepsilon,y}(x)}{\text{dist}^n(x, R_{k-1}^\varepsilon)} \, dy + \mathcal{L}^n(B^*) \mathbb{F}_U(T^\varepsilon - S) \\ & \stackrel{(4.10)}{\lesssim} \delta^{-1} h(\varepsilon) \frac{E_\varepsilon(u_\varepsilon, U')}{|\log \varepsilon|} + \mathcal{L}^n(B^*) \mathbb{F}_U(T^\varepsilon - S) \end{aligned}$$

and the right-hand side tends to zero as $\varepsilon \rightarrow 0$, due to (4.1) and (4.30). Thus, (4.28) follows. By passing to the limit in (4.31) first as $r \rightarrow 0$, then as $\delta \rightarrow 0$, we obtain (4.29). \square

Now, Proposition 7 can be deduced from Lemma 13 by a localisation argument, with the help of the following lemma.

Lemma 14. *Let $S \in \mathbb{M}_n(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$ be a chain of finite mass. Then, there holds*

$$\mathbb{M}(S) = \sup_{(U_i, L_i)_{i \in \mathbb{N}}} \sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i,*}(S \llcorner U_i)),$$

where the supremum is taken over all sequences of pairwise disjoint open sets U_i and n -planes $L_i \subseteq \mathbb{R}^{n+k}$.

The proof will be given in Appendix D.2. Once Lemma 14 is proved, Proposition 7 follows by repeating verbatim the arguments of [2, Theorem 1.1.(i)], so we skip the proof of Proposition 7.

4.2. Compactness and Lower Bounds for the Boundary Value Problem

The aim of this section is to complete the proof of Theorem C.(i). We will deduce Theorem C.(i) from its local counterpart, i.e. Proposition 7, with the help of the extension result, Lemma 2.

Proof of Theorem C.(i). Let $(u_\varepsilon)_\varepsilon \subseteq W_v^{1,k}(\Omega, \mathbb{R}^m)$ be such that $\sup_\varepsilon |\log \varepsilon|^{-1} E_\varepsilon(u_\varepsilon) < +\infty$. Let $\tilde{u} \in (L^\infty \cap W^{1,k})(\mathbb{R}^{n+k}, \mathbb{R}^m)$ be such that $\tilde{u} = v$ on $\partial\Omega$. Let Ω', Ω'' be bounded domains in \mathbb{R}^{n+k} , such that $\Omega \subset\subset \Omega' \subset\subset \Omega''$. By applying Lemma 2, we find $y \in B^*$, a subsequence $\varepsilon \rightarrow 0$ and maps $w_{\varepsilon,y} \in (L^\infty \cap W^{1,k})(\Omega'' \setminus \overline{\Omega}, \mathbb{R}^m)$ that agree with v on $\partial\Omega$ and satisfy

$$\sup_\varepsilon \left(\|w_{\varepsilon,y}\|_{L^\infty(\Omega'' \setminus \overline{\Omega})} + \frac{E_\varepsilon(w_{\varepsilon,y}, \Omega'' \setminus \overline{\Omega})}{|\log \varepsilon|} \right) < +\infty. \quad (4.32)$$

Lemma 2 also implies that the sequence $(w_{\varepsilon,y})$ converges $W^{1,k-1}(\Omega'' \setminus \overline{\Omega})$ -strongly to a limit w_y , and that $\mathbf{S}(w_y) = \mathbf{S}_y(\tilde{u})$. Then, the continuity of \mathbf{S} [17, Theorem 3.1] implies

$$\int_{B^*} \mathbb{F}_{\Omega'' \setminus \overline{\Omega}}(\mathbf{S}_{y'}(w_{\varepsilon,y}) - \mathbf{S}_y(\tilde{u})) \, dy' \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.33)$$

We define the map $\tilde{u}_\varepsilon \in (L^\infty \cap W^{1,k})(\Omega'', \mathbb{R}^m)$ by setting $\tilde{u}_\varepsilon := u_\varepsilon$ on Ω and $\tilde{u}_\varepsilon := w_{\varepsilon,y}$ on $\Omega'' \setminus \overline{\Omega}$. Since the operator \mathbf{S} is local [17, Corollary 1], we have

$$\mathbf{S}_{y'}(w_{\varepsilon,y}) \llcorner (\Omega'' \setminus \overline{\Omega}) = \mathbf{S}_{y'}(\tilde{u}_\varepsilon) \llcorner (\Omega'' \setminus \overline{\Omega})$$

for a.e. $y' \in B^*$. Therefore, from (4.33) and [17, Lemma 3] we obtain

$$\int_{B^*} \mathbb{F}_{\Omega'' \setminus \overline{\Omega}}(\mathbf{S}_{y'}(\tilde{u}_\varepsilon) - \mathbf{S}_y(\tilde{u})) \, dy' \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.34)$$

We are now in the position to apply our local result, Proposition 7, to the sequence \tilde{u}_ε and the open sets $\Omega' \subset\subset \Omega''$. As a result, we obtain a finite-mass chain \tilde{S} such that, up to subsequences,

$$\int_{B^*} \mathbb{F}_{\Omega'}(\mathbf{S}_{y'}(\tilde{u}_\varepsilon) - \tilde{S}) \, dy' \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.35)$$

By [17, Lemma 3],

$$\int_{B^*} \mathbb{F}_{\Omega' \setminus \overline{\Omega}}(\mathbf{S}_{y'}(\tilde{u}_\varepsilon) - \tilde{S}) \, dy' \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

This condition, combined with (4.34), implies that $\mathbf{S}_y(\tilde{u}) \llcorner (\Omega' \setminus \overline{\Omega}) = \tilde{S} \llcorner (\Omega' \setminus \overline{\Omega})$ and hence, the chain

$$S := \tilde{S} - \mathbf{S}_y(\tilde{u}) \llcorner (\Omega' \setminus \overline{\Omega}) = \tilde{S} \llcorner \overline{\Omega}$$

is supported in $\overline{\Omega}$. At the same time, we have $\mathbf{S}_{y'}(u_\varepsilon) = \mathbf{S}_{y'}(\tilde{u}_\varepsilon) \llcorner \overline{\Omega}$ for a.e. $y' \in B^*$. For chains supported in a compact subset of Ω' , the relative flat norm $\mathbb{F}_{\Omega'}$ is equivalent to \mathbb{F} (see e.g. [17, Remark 2.2]) and hence, (4.35) implies

$$\int_{B^*} \mathbb{F}(\mathbf{S}_{y'}(u_\varepsilon) - S) \, dy' \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By (P₃), $\mathbf{S}_{y'}(u_\varepsilon) \in \mathcal{C}(\Omega, \nu)$ for any ε and a.e. $y' \in B^*$. The set $\mathcal{C}(\Omega, \nu)$ is closed with respect to the \mathbb{F} -norm (this follows from the isoperimetric inequality, see e.g. [26, Statement (7.6)]). Therefore, $S \in \mathcal{C}(\Omega, \nu)$.

It only remain to prove the upper bound on the mass of S . Let $A \subseteq \mathbb{R}^{n+k}$ be an open set. We extract a (non-relabelled) subsequence, in such a way that $\liminf_{\varepsilon \rightarrow 0} |\log \varepsilon|^{-1} E_\varepsilon(u_\varepsilon, A \cap \Omega)$ is achieved as a limit. For any integer $j \geq 1$, let $A_j := \{x \in A : \text{dist}(x, \partial A) \geq 1/j\}$. By applying Proposition 7 and a diagonal argument, we find a subsequence such that

$$\mathbb{M}(S \llcorner (A_j \cap \Omega')) \leq \limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(\tilde{u}_\varepsilon, A \cap \Omega'')}{|\log \varepsilon|} \quad \text{for any } j \geq 1.$$

By construction, S is supported in $\overline{\Omega}$, so $S \llcorner (A_j \cap \Omega') = S \llcorner A_j$. Then, by applying Lemma 2, we obtain

$$\mathbb{M}(S \llcorner A_j) \leq \lim_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log \varepsilon|} + C \int_{\Omega'' \setminus \Omega} |\nabla \tilde{u}|^k \quad \text{for any } j \geq 1,$$

for some constant C that does not depend on ε, j, Ω'' . Letting $j \rightarrow +\infty, \Omega'' \searrow \Omega$, we conclude that

$$\mathbb{M}(S \llcorner A) \leq \lim_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log \varepsilon|}$$

and the proof is complete. \square

Statement (i) in Proposition D also follows by Proposition 7, in a similar way.

5. Proof of Theorem A

Let $u_{\varepsilon, \min}$ be a minimiser of the functional E_ε subject to the boundary condition $u = v$ on $\partial\Omega$, and let

$$\mu_{\varepsilon, \min} := \left(\frac{1}{k} |\nabla u_{\varepsilon, \min}|^k + \frac{1}{\varepsilon^k} f(u_{\varepsilon, \min}) \right) \frac{dx \llcorner \Omega}{|\log \varepsilon|}.$$

We have $\sup_\varepsilon \mu_{\varepsilon, \min}(\mathbb{R}^{n+k}) < +\infty$ by Remark 9 and hence, up to a subsequence, $\mu_{\varepsilon, \min}$ converges weakly* to a limit μ_{\min} , in the sense of measures on \mathbb{R}^{n+k} . By applying Theorem C.(i), we find a chain $S_{\min} \in \mathcal{C}(\Omega, v)$ such that

$$\mathbb{M}(S_{\min} \llcorner A) \leq \liminf_{\varepsilon \rightarrow 0} \mu_{\varepsilon, \min}(A) \quad \text{for any open set } A \subseteq \mathbb{R}^{n+k}. \quad (5.1)$$

Theorem C.(ii) implies that S_{\min} is mass-minimising in $\mathcal{C}(\Omega, v)$. Moreover, by the properties of weak* convergence, from (5.1) we obtain

$$\mathbb{M}(S_{\min} \llcorner A) \leq \mu_{\min}(A) \quad \text{for any open set } A \subseteq \mathbb{R}^{n+k} \text{ such that } \mu_{\min}(\partial A) = 0. \quad (5.2)$$

Let $E \subseteq \mathbb{R}^{n+k}$ be a Borel set, let $U \subseteq \mathbb{R}^{n+k}$ be an open set and let $K \subseteq \mathbb{R}^{n+k}$ be a compact set such that $K \subseteq E \subseteq U$. For any $t \in (0, \text{dist}(K, \partial U))$, let $U_t := \{x \in U : \text{dist}(x, \partial U) > t\} \supseteq K$. Since μ_{\min} is a finite measure, we have $\mu_{\min}(\partial U_t) = 0$ for all but countably many $t \in (0, \text{dist}(K, \partial U))$. Therefore, there holds

$$\mathbb{M}(S_{\min} \llcorner K) \leq \mathbb{M}(S_{\min} \llcorner U_t) \stackrel{(5.2)}{\leq} \mu_{\min}(U_t) \leq \mu_{\min}(U) \quad \text{for a.e. } t.$$

Letting $U \searrow K, K \nearrow E$, we conclude that $\mathbb{M}(S_{\min} \llcorner E) \leq \mu_{\min}(E)$. (The measure $\mathbb{M}(S_{\min} \llcorner \cdot)$ is Radon, because by construction, it is the weak* limit of a sequence of Radon measures, associated with polyhedral approximations of S_{\min} ; see [26, Section 4].) As a consequence, $\mu_{\min} - \mathbb{M}(S_{\min} \llcorner \cdot)$ is a non-negative measure. However, Theorem C.(ii) implies that $\mu_{\min}(\mathbb{R}^{n+k}) = \lim_{\varepsilon \rightarrow 0} \mu_{\varepsilon, \min}(\mathbb{R}^{n+k}) \leq \mathbb{M}(S_{\min})$, so $\mu_{\min} = \mathbb{M}(S_{\min} \llcorner \cdot)$.

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A. The Norm on $\pi_{k-1}(\mathcal{N})$: Proof of Proposition 1

The aim of this section is to prove Proposition 1. In Section 2, we have defined

$$E_{\min}(\sigma) := \inf \left\{ \frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} v|^k : v \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma \right\} \quad (\text{A.1})$$

for any $\sigma \in \pi_{k-1}(\mathcal{N})$, with ∇_{\top} the tangential gradient on \mathbb{S}^{k-1} (i.e. the restriction of ∇ to the tangent plane to \mathbb{S}^{k-1}). The compact Sobolev embedding $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \hookrightarrow C(\mathbb{S}^{k-1}, \mathcal{N})$ implies that $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma$ is sequentially $W^{1,k}$ -weakly closed, so the infimum at the right-hand side is achieved. We must have

$$\inf_{\sigma \in \pi_{k-1}(\mathcal{N}) \setminus \{0\}} E_{\min}(\sigma) > 0, \quad (\text{A.2})$$

for otherwise there would exist a sequence of non-null-homotopic maps $v_j \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N})$ that converge $W^{1,k}$ -strongly, and hence uniformly, to a constant. Moreover, there holds

$$E_{\min}(\sigma) = E_{\min}(-\sigma) \quad \text{for any } \sigma \in \pi_{k-1}(\mathcal{N}). \quad (\text{A.3})$$

Indeed, for any $v \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma$ and any $x \in \mathbb{S}^{k-1}$, define $\bar{v}(x) := v(-x_1, x_2, \dots, x_k)$. The map that sends $v \mapsto \bar{v}$ is a bijection $W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma \rightarrow W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap (-\sigma)$ that preserves the L^k -norm of the gradient, and hence (A.3) follows.

Our candidate norm $|\cdot|_*$ on $\pi_{k-1}(\mathcal{N})$, which was also introduced in Section 2, is defined for any $\sigma \in \pi_{k-1}(\mathcal{N})$ by

$$|\sigma|_* := \inf \left\{ \sum_{i=1}^q E_{\min}(\sigma_i) : q \in \mathbb{N}, (\sigma_i)_{i=1}^q \in \pi_{k-1}(\mathcal{N})^q, \sum_{i=1}^q \sigma_i = \sigma \right\}. \quad (\text{A.4})$$

Proposition 8. *The function $|\cdot|_*$ is a norm on $\pi_{k-1}(\mathcal{N})$ that satisfies*

$$\inf_{\sigma \in \pi_{k-1}(\mathcal{N}) \setminus \{0\}} |\sigma|_* > 0 \quad (\text{A.5})$$

and

$$|\sigma|_* \leq E_{\min}(\sigma) \quad \text{for any } \sigma \in \pi_{k-1}(\mathcal{N}). \quad (\text{A.6})$$

The infimum in Equation (A.4) is achieved, for any $\sigma \in \pi_{k-1}(\mathcal{N})$. Moreover, the set

$$\mathfrak{S} := \{\sigma \in \pi_{k-1}(\mathcal{N}) : |\sigma|_* = E_{\min}(\sigma)\} \quad (\text{A.7})$$

is finite, and for any $\sigma \in \pi_{k-1}(\mathcal{N})$ there exists a decomposition $\sigma = \sum_{i=1}^q \sigma_i$ such that $|\sigma|_* = \sum_{i=1}^q |\sigma_i|_*$ and $\sigma_i \in \mathfrak{S}$ for any i .

Proof. The function $|\cdot|_*$ is certainly non-negative, and its definition (A.4) immediately implies the triangle inequality, $|\sigma_1 + \sigma_2|_* \leq |\sigma_1|_* + |\sigma_2|_*$. The property $|\sigma|_* = |-\sigma|_*$ follows by (A.3), while (A.2) yields (A.5) (in particular, $|\sigma|_* = 0$ only if $\sigma = 0$). The property (A.6) is immediate from the definition of $|\cdot|_*$.

We check now that the set \mathfrak{S} is finite. Under the assumption (H₂), Hurewicz theorem (see e.g. [31, Theorem 4.37 p. 371]) implies that $\pi_{k-1}(\mathcal{N})$ is isomorphic to the homology group $H_{k-1}(\mathcal{N})$. The latter is Abelian and finitely generated, because the manifold \mathcal{N} is compact and hence, homotopically equivalent to a finite cell complex. Therefore, we have

$$\pi_{k-1}(\mathcal{N}) \simeq H_{k-1}(\mathcal{N}) \simeq \mathbb{Z}^p \oplus T,$$

where $p \geq 0$ is an integer and T is a finite group. Let $(g_i)_{i=1}^p$ be a basis for the torsion-free part of $H_{k-1}(\mathcal{N})$ (i.e., the quotient $H_{k-1}(\mathcal{N})/T \simeq \mathbb{Z}^p$). By de Rham theorem, there exist closed, smooth $(k-1)$ -forms $\omega_1, \dots, \omega_p$ that satisfy

$$\int_{c_i} \omega_j = \delta_{ij} \quad \text{for any } i, j,$$

where c_i is a smooth $(k-1)$ -cycle in the homology class g_i . Let $\sigma \in \pi_{k-1}(\mathcal{N})$. By abusing of notation, and identifying g_i with its image under the Hurewicz isomorphism, we can write uniquely

$$\sigma = \sum_{i=1}^p d_i g_i + \sigma_T,$$

where $d_i \in \mathbb{Z}$ and $\sigma_T \in T$. Then, for any $v \in W^{1,k}(\mathbb{S}^{k-1}, \mathcal{N}) \cap \sigma$, we have

$$\begin{aligned} |d_i| &= \left| \int_{\mathbb{S}^{k-1}} v^* \omega_i \right| \leq \|\omega_i\|_{L^\infty(\mathcal{N}, \Lambda^{k-1} T^* \mathcal{N})} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} v|^{k-1} \\ g &\leq C_{k, \mathcal{N}} \left(\frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} v|^k \right)^{\frac{k-1}{k}} \end{aligned}$$

where $C_{k, \mathcal{N}} > 0$ is a constant depending only on k and the ω_i 's. This implies

$$E_{\min}(\sigma) \geq C'_{k, \mathcal{N}} \left(\sum_{i=1}^p |d_i| \right)^{\frac{k}{k-1}} \quad (\text{A.8})$$

for a different constant $C'_{k,\mathcal{N}}$. On the other hand, the definition of $|\cdot|_*$ immediately gives the upper bound

$$|\sigma|_* \leq \left(\max_{i=1,\dots,p} E_{\min}(g_i) \right) \sum_{i=1}^p |d_i| + \max_{\sigma_T \in T} E_{\min}(\sigma_T). \quad (\text{A.9})$$

If $\sigma \in \mathfrak{S}$ then, by comparing (A.8) and (A.9), we obtain $\sum_i |d_i| \leq M$ for some constant $M > 0$ depending only on k, \mathcal{N} . Therefore, \mathfrak{S} is a finite set.

For any $\sigma \in \pi_{k-1}(\mathcal{N})$ there exists a finite decomposition $\sigma = \sum_{i=1}^q \sigma_i$ which achieves the infimum in the right-hand side of (A.4). Indeed, it suffices to minimise among the decompositions with $q \leq (\inf_{g \in \pi_{k-1}(\mathcal{N}) \setminus \{0\}} E_{\min}(g))^{-1} E_{\min}(\sigma)$ and $E_{\min}(\sigma_i) \leq E_{\min}(\sigma)$ for any i , and there are only finitely many such decompositions because of (A.2), (A.8). Let $\sigma = \sum_{i=1}^q \sigma_i$ be a decomposition that achieves the minimum in (A.4). Then, the triangle inequality implies

$$\sum_{i=1}^q E_{\min}(\sigma_i) = |\sigma|_* \leq \sum_{i=1}^q |\sigma_i|_*$$

and, since $|\sigma_i|_* \leq E_{\min}(\sigma_i)$ for any i , we must have $|\sigma_i|_* = E_{\min}(\sigma_i)$, i.e. $\sigma_i \in \mathfrak{S}$, for any i . \square

Example A.1. Let $k = 2$, $\mathcal{N} = \mathbb{S}^1$. Then, $\pi_{k-1}(\mathcal{N}) \simeq \mathbb{Z}$ and $E_{\min}(d) = \pi d^2$ for any $d \in \mathbb{Z}$, since the infimum in (A.1) is achieved by a curve that parametrises the unit circle $|d|$ times, with constant speed and orientation depending on the sign of d . Therefore, $\mathfrak{S} = \{-1, 0, 1\}$ and $|d|_* = \pi |d|$ for any $d \in \mathbb{Z}$.

More generally, when $\mathcal{N} = \mathbb{S}^{k-1}$ the constant that appears in the lower bound (A.8) can be computed explicitly, and we have

$$E_{\min}(d) \geq \beta_k |d|^{\frac{k}{k-1}} \quad \text{for any } d \in \pi_{k-1}(\mathbb{S}^{k-1}) \simeq \mathbb{Z}, \quad (\text{A.10})$$

where $\beta_k := (k-1)^{k/2} \mathcal{L}^k(B_1^k)$. On the other hand, by using the identity as a comparison map for (A.1), we see that $E_{\min}(1) \leq (k-1)^{k/2} \text{vol}(\mathbb{S}^{k-1})/k = \beta_k$, hence $E_{\min}(1) = E_{\min}(-1) = \beta_k$. It follows that

$$E_{\min}(d) \stackrel{(\text{A.10})}{>} \beta_k |d| \geq |d|_* \quad \text{if } |d| > 1.$$

Therefore, also in case $\mathcal{N} = \mathbb{S}^{k-1}$ we have $\mathfrak{S} = \{-1, 0, 1\}$. By Proposition 1, we conclude that $|d|_* = \beta_k |d|$ for any $d \in \pi_{k-1}(\mathbb{S}^{k-1}) \simeq \mathbb{Z}$.

B. The Operator $\bar{\mathfrak{S}}$: Proof of Proposition 3

The aim of this section is to prove Proposition 3, which we recall here for the convenience of the reader. We recall that $\delta^* \in (0, \text{dist}(\mathcal{N}, \mathcal{X}))$ is a fixed constant, $B^* := B^m(0, \delta^*) \subseteq \mathbb{R}^m$, and $\bar{Y} := L^1(B^*, \mathbb{F}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N})))$ is equipped with the norm

$$\|S\|_{\bar{Y}} := \int_{B^*} \mathbb{F}(S_y) \, dy < +\infty.$$

Proposition 9. *There exists a continuous operator $\bar{\mathfrak{S}}: W^{1,k}(\Omega, \mathbb{R}^*) \rightarrow \bar{Y}$ that satisfies the following properties:*

(P₀) For any smooth u , a.e. $y \in B^*$ and any $R \in \mathbb{F}_k(\mathbb{R}^{n+k}; \mathbb{Z})$ such that $\mathbb{M}(R) + \mathbb{M}(\partial R) < +\infty$, $\text{spt}(R) \subseteq \Omega$, $\text{spt}(\partial R) \subseteq \Omega \setminus \text{spt} \mathbf{S}_y(u)$, there holds

$$\mathbb{I}(\mathbf{S}_y(u), R) = \text{homotopy class of } \varrho \circ (u - y) \text{ on } \partial R.$$

(P₁) For any $u \in (L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$ and a.e. $y \in B^*$, $\bar{\mathbf{S}}_y(u) = \mathbf{S}_y(u)$ (more precisely, the chain $\bar{\mathbf{S}}_y(u)$ belongs to the equivalence class $\mathbf{S}_y(u) \in \mathbb{F}_n(\Omega; \pi_{k-1}(\mathcal{N}))$).
 (P₂) For any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ and any Borel subset $E \subseteq \bar{\Omega}$, there holds

$$\int_{B^*} \mathbb{M}(\bar{\mathbf{S}}_y(u) \llcorner E) \, dy \lesssim \int_E |\nabla u|^k.$$

(P₃) Let $u_0, u_1 \in W^{1,k}(\Omega, \mathbb{R}^m)$ be such that $u_0|_{\partial\Omega} = u_1|_{\partial\Omega} \in W^{1-1/k,k}(\partial\Omega, \mathcal{N})$ (in the sense of traces). Then, for a.e. $y_0, y_1 \in B^*$ there exists $R \in \mathbb{M}_{n+1}(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$ such that $\bar{\mathbf{S}}_{y_1}(u_1) - \bar{\mathbf{S}}_{y_0}(u_0) = \partial R$.

In the proof, we will use the following

Lemma 15. Let $\rho > 0$, and let $\Gamma_\rho := \{x \in \mathbb{R}^{n+k} \setminus \bar{\Omega} : \text{dist}(x, \Omega) < \rho\}$. Then, for any finite-mass chain $T \in \mathbb{M}_n(\mathbb{R}^{n+k}; \pi_{k-1}(\mathcal{N}))$, there holds

$$\mathbb{F}(T \llcorner \bar{\Omega}) \leq (1 + \rho^{-1}) \mathbb{F}(T) + \mathbb{M}(T \llcorner \Gamma_\rho).$$

Proof. For any $t > 0$, let $\Omega_t := \{x \in \mathbb{R}^{n+k} : \text{dist}(x, \Omega) < t\}$. There holds

$$\int_0^\rho \mathbb{F}(T \llcorner \Omega_t) \, dt \leq (1 + \rho) \mathbb{F}(T)$$

(see e.g. [17, Lemma 4, Eq. (2.8)]). By an averaging argument, we can find $t \in (0, \rho)$ such that

$$\mathbb{F}(T \llcorner \Omega_t) \leq (1 + \rho^{-1}) \mathbb{F}(T). \quad (\text{B.1})$$

Now, there holds $T \llcorner \bar{\Omega}_t = T \llcorner \bar{\Omega} + T \llcorner \Gamma_t$ and hence,

$$\mathbb{F}(T \llcorner \bar{\Omega}) \leq \mathbb{F}(T \llcorner \Omega_t) + \mathbb{F}(T \llcorner \Gamma_t) \stackrel{(\text{B.1})}{\leq} (1 + \rho^{-1}) \mathbb{F}(T) + \mathbb{M}(T \llcorner \Gamma_t),$$

so the lemma follows. \square

Proof of Proposition 9. For the sake of clarity, we split the proof into steps.

Step 1. (Construction of $\bar{\mathbf{S}}$) First, we consider a smooth map $u \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R}^m)$ and the topological singular operator, $\mathbf{S}_y(u)$, as defined in [17, Section 3.2, Eq. (3.4)]. By definition, we can write

$$\mathbf{S}_y(u) = \sum_K \gamma(K) \llbracket (u - y)^{-1}(K) \rrbracket \quad \text{for a.e. } y \in B^*. \quad (\text{B.2})$$

Here, the sum is taken over all $(m - k)$ -dimensional polyhedra K in \mathcal{X} . The coefficient $\gamma(K) \in \pi_{k-1}(\mathcal{N})$ is the homotopy class of ϱ restricted to a small $(k - 1)$ -sphere Σ around K , $\varrho|_\Sigma : \Sigma \simeq \mathbb{S}^{k-1} \rightarrow \mathcal{N}$. For a.e. $y \in B^*$, the set $(u - y)^{-1}(K)$ is a smooth, compact n -dimensional manifold (as a consequence of Thom's transversality theorem, see e.g. [32, Theorem 2.7 p. 79]) and $\llbracket (u - y)^{-1}(K) \rrbracket$ denotes the smooth chain carried by $(u - y)^{-1}(K)$, with unit multiplicity and a suitable orientation (see [17, Section 3.2] for more details).

We claim that, for any $u, u_0, u_1 \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R}^m)$ and any open set $U \subseteq \mathbb{R}^{n+k}$, there holds

$$\int_{B^*} \mathbb{M}(\mathbf{S}_y(u) \llcorner U) \lesssim \int_U |\nabla u|^k \quad (\text{B.3})$$

$$\begin{aligned} & \int_{B^*} \mathbb{F}(\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0)) \\ & \lesssim \|u_1 - u_0\|_{L^k(\mathbb{R}^{n+k})} \left(\|\nabla u_0\|_{L^k(\mathbb{R}^{n+k})}^{k-1} + \|\nabla u_1\|_{L^k(\mathbb{R}^{n+k})}^{k-1} \right). \end{aligned} \quad (\text{B.4})$$

These inequalities differ from the corresponding ones in [17, Theorem 3.1] because the multiplicative constants in front of the right-hand sides do not depend on the L^∞ -norm of u, u_0, u_1 . We postpone the proof of (B.3)–(B.4). As a consequence of (B.4), by a density argument we can extend \mathbf{S} to a continuous operator $W^{1,k}(\mathbb{R}^{n+k}, \mathbb{R}^m) \rightarrow L^1(B^*, \mathbb{F}(\mathbb{R}^{n+k}, \pi_{k-1}(\mathcal{N})))$, still denoted \mathbf{S} for simplicity. The property (B.3) is preserved for any $u \in W^{1,k}(\mathbb{R}^{n+k}, \mathbb{R}^m)$, by the lower semi-continuity of \mathbb{M} (see e.g. [17, Lemma 3 and Lemma 5]).

Since the domain $\Omega \subseteq \mathbb{R}^{n+k}$ is bounded and smooth, by reflection about $\partial\Omega$ and multiplication with a cut-off function we can define a linear extension operator $T: W^{1,k}(\Omega, \mathbb{R}^m) \rightarrow W^{1,k}(\mathbb{R}^{n+k}, \mathbb{R}^m)$, such that

$$\|Tu\|_{L^k(\mathbb{R}^{n+k})} \lesssim \|u\|_{L^k(\Omega)}, \quad \|\nabla(Tu)\|_{L^k(\mathbb{R}^{n+k})} \lesssim \|\nabla u\|_{L^k(\Omega)} + \|u\|_{L^k(\Omega)}. \quad (\text{B.5})$$

For any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ and a.e. $y \in B^*$, the chain $\mathbf{S}_y(Tu)$ has finite mass, due to (B.3). Therefore, the restriction

$$\bar{\mathbf{S}}_y(u) := \mathbf{S}_y(Tu) \llcorner \bar{\Omega}$$

is well-defined and belongs to $\mathbb{M}_n(\bar{\Omega}; \pi_{k-1}(\mathcal{N}))$.

Step 2. ($\bar{\mathbf{S}}$ is continuous) Let $(u_j)_{j \in \mathbb{N}}$ be a sequence in $W^{1,k}(\Omega, \mathbb{R}^m)$ such that $u \rightarrow u$ in $W^{1,k}(\Omega)$. From (B.4) and (B.5), we deduce

$$\begin{aligned} & \int_{B^*} \mathbb{F}(\mathbf{S}_y(Tu_j) - \mathbf{S}_y(Tu)) \, dy \\ & \lesssim \|u_j - u\|_{L^k(\Omega)} \left(\|\nabla u_j\|_{L^k(\Omega)}^{k-1} + \|\nabla u\|_{L^k(\Omega)}^{k-1} \right) + \|u_j - u\|_{L^k(\Omega)}^k \end{aligned} \quad (\text{B.6})$$

Let $\rho > 0$ and $\Gamma_\rho := \{x \in \mathbb{R}^{n+k} \setminus \bar{\Omega} : \text{dist}(x, \Omega) < \rho\}$. By applying Lemma 15 and (B.3), (B.6), we obtain

$$\begin{aligned} \|\bar{\mathbf{S}}(u_j) - \bar{\mathbf{S}}(u)\|_{\bar{Y}} & \lesssim \left(1 + \rho^{-1}\right) \int_{B^*} \mathbb{F}(\mathbf{S}_y(Tu_j) - \mathbf{S}_y(Tu)) \, dy \\ & \quad + \int_{B^*} \mathbb{M}(\mathbf{S}_y(Tu_j) \llcorner \Gamma_\rho) \, dy + \int_{B^*} \mathbb{M}(\mathbf{S}_y(Tu) \llcorner \Gamma_\rho) \, dy \\ & \lesssim \left(1 + \rho^{-1}\right) \|u_j - u\|_{L^k(\Omega)} \left(\|\nabla u_j\|_{L^k(\Omega)}^{k-1} + \|\nabla u\|_{L^k(\Omega)}^{k-1} \right) \\ & \quad + \left(1 + \rho^{-1}\right) \|u_j - u\|_{L^k(\Omega)}^k + \|\nabla(Tu_j)\|_{L^k(\Gamma_\rho)}^k \\ & \quad + \|\nabla(Tu)\|_{L^k(\Gamma_\rho)}^k. \end{aligned}$$

By taking the limit in the inequality above first as $j \rightarrow +\infty$, then as $\rho \rightarrow 0$, we conclude that $\bar{\mathbf{S}}(u_j) \rightarrow \bar{\mathbf{S}}(u)$ in \bar{Y} .

Step 3. (Proof of (P₁)) By construction, $\bar{\mathbf{S}}_y(u) = \mathbf{S}_y(u)$ for any $u \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R}^m)$ and a.e. $y \in B^*$. By continuity of both $\bar{\mathbf{S}}$ and \mathbf{S} [17, Theorem 3.1], we deduce that $\bar{\mathbf{S}} = \mathbf{S}$ on $(L^\infty \cap W^{1,k})(\Omega, \mathbb{R}^m)$.

Step 4. (Proof of (P₂)) Let $E \subseteq \bar{\Omega}$ be a Borel set and $U \supseteq E$ be open. By (B.3), we have

$$\int_{B^*} \mathbb{M}(\bar{\mathbf{S}}_y(u) \llcorner E) \, dy \leq \int_{B^*} \mathbb{M}(\mathbf{S}_y(u) \llcorner U) \, dy \lesssim \int_U |\nabla u|^k$$

and (P₂) follows by letting $U \searrow E$.

Step 5. (Proof of (P₃)) Take $u_0, u_1 \in W^{1,k}(\Omega, \mathbb{R}^m)$. For $i \in \{0, 1\}$ and $M > 0$, we define

$$u_{i,M} := \begin{cases} u_i & \text{if } |u_i| \leq M \\ \frac{Mu_i}{|u_i|} & \text{otherwise.} \end{cases}$$

Since $u_{i,M} \rightarrow u_i$ strongly in $W^{1,k}(\Omega)$ as $M \rightarrow 0$, the continuity of $\bar{\mathbf{S}}$ gives, upon extraction of a (non-relabelled) subsequence,

$$\mathbb{F}(\mathbf{S}_y(u_{i,M}) - \mathbf{S}_y(u_i)) \rightarrow 0 \quad \text{as } M \rightarrow +\infty, \text{ for a.e. } y \in B^* \text{ and } i \in \{0, 1\}. \quad (\text{B.7})$$

Let $\mathbb{B} := \{\partial R : R \in \mathbb{M}_{n+1}(\bar{\Omega}; \mathbb{R}^{n+k})\}$. By [17, Proposition 2], we have $\bar{\mathbf{S}}_{y_0}(u_{1,M}) - \bar{\mathbf{S}}_{y_1}(u_{0,M}) \in \mathbb{B}$ for any $M > 0$ and a.e. $y_0, y_1 \in B^*$. On the other hand, the set \mathbb{B} is closed with respect to the \mathbb{F} -norm, as a consequence of the isoperimetric inequality (see e.g. [25, 7.6]). Therefore, (P₃) follows by (B.7).

Step 6. (Proof of (B.3)) Let $u \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R}^m)$ and let $E \subseteq \mathbb{R}^{n+k}$ be a Borel set. Since \mathcal{X} contains finitely many $(m-k)$ -cells K , there exists a constant C such that $|\gamma(K)|_* \leq C$ for any K . Then, using the definition (B.2) of $\mathbf{S}_y(u)$, we deduce

$$\mathbb{M}(\mathbf{S}_y(u) \llcorner E) \lesssim \sum_K \mathcal{H}^n((u-y)^{-1}(K) \cap E), \quad (\text{B.8})$$

where the sum is taken over all the $(m-k)$ -dimensional polyhedra K in \mathcal{X} . We fix K and assume, without loss of generality, that $K \subseteq \{y \in \mathbb{R}^m : y_1 = \dots = y_k = 0\} \simeq \mathbb{R}^{m-k}$. Let ζ^\perp be the orthogonal projection $\mathbb{R}^m \rightarrow \{y \in \mathbb{R}^m : y_{m-k+1} = \dots = y_m = 0\} \simeq \mathbb{R}^k$. Then,

$$(u-y)^{-1}(K) \subseteq (\zeta^\perp \circ u)^{-1}(\zeta^\perp(y))$$

If we integrate this inequality over $y \in B^*$, and use the variable $y = (z, z^\perp) \in \mathbb{R}^{m-k} \times \mathbb{R}^m$, we obtain

$$\begin{aligned} & \int_{B^*} \mathcal{H}^n((u-y)^{-1}(K) \cap E) \\ & \leq \int_{[-\delta^*, \delta^*]^{m-k} \times [-\delta^*, \delta^*]^k} \mathcal{H}^n\left(\left(\zeta^\perp \circ u\right)^{-1}(z^\perp) \cap E\right) \, d(z, z^\perp) \\ & \leq (2\delta^*)^{m-k} \int_{\mathbb{R}^k} \mathcal{H}^n\left(\left(\zeta^\perp \circ u\right)^{-1}(z^\perp) \cap E\right) \, dz^\perp. \end{aligned}$$

The right-hand side can be estimated by applying the coarea formula:

$$\int_{B^*} \mathcal{H}^n((u-y)^{-1}(K) \cap E) \lesssim \int_E \left| \nabla(\zeta^\perp \circ u) \right|^k \lesssim \int_E |\nabla u|^k. \quad (\text{B.9})$$

Combining (B.8) and (B.9), (P₂) follows.

Step 7. (Proof of (B.4)) Let $u_0, u_1 \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R}^m)$, and let $u: [0, 1] \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^m$ be defined by $u(t, x) := (1-t)u_0(x) + tu_1(x)$. Let $\pi: [0, 1] \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ be the canonical projection, $\pi(t, x) := x$. By [17, Proposition 4], we have

$$\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0) = \partial(\pi_* \mathbf{S}_y(u)).$$

Therefore, using (B.2), we obtain

$$\mathbb{F}(\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0)) \leq \mathbb{M}(\pi_* \mathbf{S}_y(u)) \leq \sum_K \mathbb{M}(\pi_* \llbracket (u-y)^{-1}(K) \rrbracket), \quad (\text{B.10})$$

where the sum is taken over all the $(m-k)$ -polyhedra K in \mathcal{X} . Fix such a K . As above, we assume that that $K \subseteq \{y \in \mathbb{R}^m : y_1 = \dots = y_k = 0\}$. Let ζ, ζ^\perp be the orthogonal projections of \mathbb{R}^m onto $\{y \in \mathbb{R}^m : y_1 = \dots = y_{m-k} = 0\} \simeq \mathbb{R}^{m-k}$, $\{y \in \mathbb{R}^m : y_{m-k+1} = \dots = y_m = 0\} \simeq \mathbb{R}^k$, respectively. We write $z := \zeta(y)$, $z^\perp := \zeta^\perp(y)$ and identify $y = (z, z^\perp)$. Then, for a suitable choice of orientation of K , we obtain

$$\llbracket (u-y)^{-1}(K) \rrbracket = \llbracket (\zeta^\perp \circ u)^{-1}(z^\perp) \rrbracket \llcorner \left((\zeta \circ u - z)^{-1}(K) \right). \quad (\text{B.11})$$

where $\llbracket (\zeta^\perp \circ u)^{-1}(z^\perp) \rrbracket$ is the chain carried by the set $(\zeta^\perp \circ u)^{-1}(z^\perp)$, with unit multiplicity, oriented by the Jacobian of $\zeta^\perp \circ u$ (see e.g. [17, p. 72]). Let us define $v := \zeta^\perp \circ u$, $K_z := (\zeta \circ u - z)^{-1}(K)$. By integrating (B.11) with respect to $y \in B^*$, we obtain

$$\begin{aligned} & \int_{B^*} \mathbb{M}(\pi_* \llbracket (u-y)^{-1}(K) \rrbracket) dy \\ & \leq \int_{[-\delta^*, \delta^*]^{m-k} \times \mathbb{R}^k} \mathbb{M}(\pi_* (v^{-1}(z^\perp) \llcorner K_z)) d(z, z^\perp). \end{aligned}$$

We may write $v(t, x) = (1-t)v_0(x) + tv_1(x)$, where $v_0 := \zeta^\perp \circ u_0$, $v_1 := \zeta^\perp \circ u_1$. By applying [17, Lemma 15], we deduce

$$\begin{aligned} & \int_{B^*} \mathbb{M}(\pi_* \llbracket (u-y)^{-1}(K) \rrbracket) dy \\ & \lesssim (2\delta^*)^{m-k} \int_{\mathbb{R}^{n+k}} |v_0 - v_1| (|\nabla v_0|^{k-1} + |\nabla v_1|^{k-1}). \end{aligned} \quad (\text{B.12})$$

Combining (B.10) and (B.12), using that the function ζ is 1-Lipschitz, and applying the Hölder inequality, (B.4) follows. \square

C. Energy Lower Bounds When $n = 0$

The aim of this section is to prove energy lower bounds in the critical dimension, i.e. when $n = 0$. In the contest of the Ginzburg–Landau theory, i.e. when $\mathcal{N} = \mathbb{S}^{k-1}$, energy bounds of this type were proved by JERRARD [38] and, in case $k = 2$, by SANDIER [50].

Let $\delta_0 > 0$, $r > 0$ be small numbers. Suppose that a map $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ satisfies

$$\text{dist}(u(x), \mathcal{N}) \leq \delta_0 \quad \text{for a.e. } x \in \Omega \text{ such that } \text{dist}(x, \partial\Omega) < r. \quad (\text{C.1})$$

Then, we can define the homotopy class of u (or, more precisely, of $\varrho \circ u$) on $\partial\Omega$ as an element of $\pi_{k-1}(\mathcal{N})$. This is immediate in case u is continuous on $\partial\Omega$ and $\overline{\Omega}$ is homeomorphic to a disk. If Ω has not the topology of a disk, this is still possible due to the Hurewicz isomorphism $\pi_{k-1}(\mathcal{N}) \simeq H_{k-1}(\mathcal{N})$, which holds true thanks to (H₂) (see e.g. [31, Theorem 4.37 p. 371] and (C.10) below). If u is not continuous we can define its homotopy class by approximating $\varrho \circ u$ with smooth functions $\Omega \rightarrow \mathcal{N}$, as in [14] (see also (C.10) below for more details).

Proposition 10. *Let $\Omega \subseteq \mathbb{R}^k$ be a bounded, Lipschitz domain and let $r > 0$. There exist a number $\delta_0 > 0$, depending only on \mathcal{N} , and positive constants ε_0, M , depending only on $\Omega, r, \mathcal{N}, k$ and f , such that the following statement holds. Suppose that $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ satisfies (C.1), and let $\sigma \in \pi_{k-1}(\mathcal{N})$ be the homotopy class of u on $\partial\Omega$. Let $\varepsilon \in (0, 1/2)$ be such that $\varepsilon |\log \varepsilon| |\sigma|_* \leq \varepsilon_0$. Then,*

$$E_\varepsilon(u) \geq |\sigma|_* |\log \varepsilon| - M |\sigma|_* (1 + \log |\sigma|_*).$$

The aim of this section is to prove Proposition 10. Once Proposition 10 is proved, Proposition B follows by an extension argument in a neighbourhood of $\partial\Omega$ (see e.g. [10, Theorem 2]). Lemma 11 also follows from Proposition 10, by exactly the same arguments as in [2, Lemma 3.10].

C.1. Reduction to the Cone-Valued Case

For the purposes of this section, it will be convenient to consider the nearest-point projection onto \mathcal{N} . If $z \in \mathbb{R}^m$ is sufficiently close to \mathcal{N} , there exists a unique $\pi(z) \in \mathcal{N}$ such that $|z - \pi(z)| \leq |z - w|$ for any $w \in \mathcal{N}$. Moreover, the map $z \mapsto \pi(z)$ is a smooth in a neighbourhood of \mathcal{N} . Throughout the rest of the section, we fix a small parameter θ_0 and assume that π is well-defined and smooth in a θ_0 -neighbourhood of \mathcal{N} .

Lemma 16. *If $u: \Omega \rightarrow \mathbb{R}^m$ is a smooth map that satisfies $\text{dist}(u(x), \mathcal{N}) < \theta_0$ for any $x \in \Omega$ and if $d := \text{dist}(u, \mathcal{N})$, then there holds*

$$|\nabla u|^2 \geq C_1 |\nabla d|^2 + (1 - C_2 d) |\nabla(\pi \circ u)|^2 \quad \text{on } \Omega,$$

where C_1, C_2 are positive constants that only depend on \mathcal{N} .

Proof. Let $x_0 \in \Omega$ be arbitrarily fixed. Let v_1, \dots, v_p be a smooth orthonormal frame for the normal space to \mathcal{N} , locally defined in a neighbourhood of $(\pi \circ u)(x_0)$. Then, for each x in a neighbourhood of x_0 there exist numbers $\alpha_1(x), \dots, \alpha_p(x)$ such that

$$u(x) = (\pi \circ u)(x) + \sum_{i=1}^p \alpha_i(x) (v_i \circ \pi \circ u)(x).$$

The functions α_i are as regular as u . By differentiating this equation, raising both sides to the square, using the fact that $\partial_j(\pi \circ u)$ is tangent to \mathcal{N} and $v \cdot \partial_j v = 0$, we obtain

$$\begin{aligned} & |\nabla u|^2 - |\nabla(\pi \circ u)|^2 \\ &= \sum_{i=1}^p \left(|\nabla \alpha_i|^2 + \alpha_i^2 |\nabla(v_i \circ \pi \circ u)|^2 + 2\alpha_i \nabla(\pi \circ u) : \nabla(v_i \circ \pi \circ u) \right). \end{aligned}$$

Since \mathcal{N} is smooth and compact, we have $|\nabla v_i| \leq C$ for some constant C that only depends on \mathcal{N} and not on u . Therefore, setting $d := \text{dist}(u, \mathcal{N}) = (\sum_i \alpha_i^2)^{1/2}$, we obtain

$$|\nabla u|^2 - |\nabla(\pi \circ u)|^2 \geq \sum_{i=1}^p |\nabla \alpha_i|^2 - Cd |\nabla(\pi \circ u)|^2. \quad (\text{C.2})$$

On the other hand, by differentiating the identity $d = (\sum_i \alpha_i^2)^{1/2}$, we see that

$$|\nabla d|^2 \leq \left(\sum_{i=1}^p |\nabla \alpha_i| \right)^2 \lesssim \sum_{i=1}^p |\nabla \alpha_i|^2. \quad (\text{C.3})$$

By combining (C.2) and (C.3), the lemma follows. \square

Lemma 17. *Suppose that $f: \mathbb{R}^m \rightarrow \mathbb{R}$ satisfies the assumptions (H₁)–(H₃). Then, there exist positive constants α , β and a smooth function $\phi: \mathbb{R}^m \rightarrow [0, 1]$ such that the following holds:*

- (i) $\phi(y) = 1$ for any $y \in \mathcal{N}$;
- (ii) $\phi(y) = 0$ if $\text{dist}(y, \mathcal{N}) \geq \theta_0$, and in particular $\pi(y)$ is well-defined for any $y \in \mathbb{R}^m$ such that $\phi(y) > 0$;
- (iii) for any $u \in W^{1,k}(\Omega, \mathbb{R}^m)$, there holds

$$\frac{1}{k} |\nabla u|^k + \frac{1}{\varepsilon^k} f(u) \geq \alpha |\nabla(\phi \circ u)|^k + \frac{1}{k} (\phi \circ u)^k |\nabla(\pi \circ u)|^k + \frac{\beta}{\varepsilon^k} (1 - \phi \circ u)^2$$

pointwise a.e. on Ω .

Proof. Let $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ be given. By a density argument, we can assume without loss of generality that u is smooth. Let $d := \text{dist}(u, \mathcal{N})$, and let $x_0 \in \Omega$ be such that $d(x_0) < \theta_0$. By applying Lemma 16, and using the convexity of the function $t \mapsto t^{k/2}$, we see that the inequality

$$|\nabla u|^k \geq C_1 |\nabla d|^k + (1 - C_2 d) |\nabla(\pi \circ u)|^k \quad (\text{C.4})$$

holds pointwise in a neighbourhood of x_0 (though we may need to re-define the constants C_1, C_2 .) Let $\xi \in C_c^\infty[0, +\infty)$ be a non-increasing function, such that $\xi = 1$ in a neighbourhood of 0 and $\xi(\min\{\theta_0/2, 1/(2C_2)\}) = 0$. We set

$$\phi(y) := (1 - C_2 \text{dist}(y, \mathcal{N}))^{1/k} \xi(\text{dist}(y, \mathcal{N}))$$

for any $y \in \mathbb{R}^m$. This defines a smooth function $\phi: \mathbb{R}^m \rightarrow [0, 1]$ which satisfies (i) and (ii). Since $(\phi \circ u)^k \leq 1 - C_2 d$ and $|\nabla(\phi \circ u)| \lesssim |\nabla d|$, from (C.4) we deduce that

$$\frac{1}{k} |\nabla u|^k \geq \alpha |\nabla(\phi \circ u)|^k + \frac{1}{k} (\phi \circ u)^k |\nabla(\pi \circ u)|^k \quad (\text{C.5})$$

pointwise in the open set $\{d < \theta_0\}$. Here α is a positive constant that only depends on \mathcal{N} , k and ξ . Because the function $\phi \circ u$ is identically equal to zero on the open set $\{d > \theta_0/2\}$, the inequality (C.5) actually holds in the whole of Ω .

We consider now the potential term $f(u)$. Due to the assumption (H₃), $f(u) \gtrsim d^2$ and hence, there exists a positive number $\beta > 0$ such that

$$f(u) \geq \beta \left(1 - (\phi \circ u)^k\right)^2 \quad \text{in } \Omega. \quad (\text{C.6})$$

By combining (C.5) and (C.6), and using the elementary inequality $1 - x^k \geq 1 - x$ for $0 \leq x \leq 1$, the lemma follows. \square

C.2. Proof of Proposition 10

Throughout this section, we fix a bounded, smooth map $u: \Omega \rightarrow \mathbb{R}^m$ and we let $s := \phi \circ u$, $v := \pi \circ u$, where ϕ is the function given by Lemma 17 and π is the nearest-point projection onto \mathcal{N} . Thanks to Lemma 17, in order to provide lower bounds for $E_\varepsilon(u)$ it suffices to bound from below the functional

$$G_\varepsilon(s, v) = G_\varepsilon(s, v; \Omega) := \int_\Omega \left(\alpha |\nabla s|^k + \frac{s^k}{k} |\nabla v|^k + \frac{\beta}{\varepsilon^k} (1 - s)^2 \right). \quad (\text{C.7})$$

To this end, we adapt Jerrard's approach in [38]. We explain here the main steps of the construction and point out the differences, referring the reader to [38] for more details.

Let us fix a small number η_0 , such that

$$0 < \eta_0 < \text{dist}(\mathcal{N}, \mathcal{X}) - \theta_0. \quad (\text{C.8})$$

Let $V \subseteq \Omega$ be an open set such that $s > 0$ on ∂V . By Lemma 17, we have $\text{dist}(u(x), \mathcal{N}) < \theta_0$ for any $x \in \partial V$. Therefore, by (C.8), we have $\text{spt}(\mathbf{S}_y(u)) \cap \partial V = \emptyset$ for a.e. $y \in \mathbb{R}^m$ such that $|y| \leq \eta_0$. In fact, $\mathbf{S}_y(u)$ is a 0-chain of finite mass, and hence we can write

$$\mathbf{S}_y(u) \llcorner \bar{V} := \sum_{i=1}^q \sigma_i \llbracket x_i \rrbracket \quad (\text{C.9})$$

where $\sigma_i \in \pi_{k-1}(\mathcal{N})$ and $x_i \in V$. The quantity $\mathbb{I}(\mathbf{S}_y(u), \llbracket V \rrbracket) := \sum_{i=1}^q \sigma_i \in \pi_{k-1}(\mathcal{N})$ plays the rôle of the topological degree and indeed, it coincides with the homotopy class of $\pi \circ u$ on ∂V because of Proposition 9.(P₀) and (C.8) (see [17, Section 2.4 and Theorem 3.1] for the details on the case u is not smooth). In particular, $\mathbb{I}(\mathbf{S}_y(u), \llbracket V \rrbracket)$ is independent of the choice of y .

As in [38], we define an “approximate homotopy class”, which allows us to disregard sets where s is small but u does not carry topological obstruction. We let $S := \{x \in \Omega : s(x) \leq 1/2\}$. The components \tilde{S} of S are closed sets and it is possible to define $\mathbb{I}(\mathbf{S}_y(u), \llbracket \tilde{S} \rrbracket)$ as above. We define the “essential part” of S :

$$S_E := \cup \left\{ \tilde{S} \text{ component of } S : \mathbb{I}(\mathbf{S}_y(u), \llbracket \tilde{S} \rrbracket) \neq 0 \right\}.$$

For $V \subseteq \Omega$, we define the “approximate homotopy class” of u on ∂V as

$$\text{hc}(u, \partial V) := \sum_{\tilde{S}} \mathbb{I}(\mathbf{S}_y(u), \llbracket \tilde{S} \rrbracket) \in \pi_{k-1}(\mathcal{N}), \quad (\text{C.10})$$

where the sum is taken over all the components \tilde{S} of S_E such that $\tilde{S} \subset\subset V$. If $V \subseteq \Omega$ is an open disk and $s > 1/2$ on ∂V , then $\text{hc}(u, \partial V)$ is the homotopy class of $v : \partial V \simeq \mathbb{S}^{k-1} \rightarrow \mathcal{N}$.

For any $\rho > 0$, we define

$$\lambda_\varepsilon(\rho) := \min_{0 \leq \mu \leq 1} \left\{ \frac{\mu^k}{\rho} + \frac{C_0}{\varepsilon} (1 - \mu)^N \right\}$$

and

$$\Lambda_\varepsilon(\rho) := \int_0^\rho \min \left\{ \lambda_\varepsilon(s), \frac{C_1}{\varepsilon} \right\} d\rho,$$

where $C_0 > 0$, $C_1 > 0$ and $N > 1$ are parameters that we will need to choose, depending on k , α and β . It can be shown (see [38, Theorem 2.1, proof of (2.2)]) that

$$\lambda_\varepsilon(\rho) \geq \frac{1}{\rho} \left(1 - C \frac{\varepsilon^v}{\rho^v} \right),$$

where $C > 0$ only depends on k , C_0 and $v := 1/(N - 1) > 0$. As a consequence, $\lambda_\varepsilon(\rho) \geq C_1/\varepsilon$ if $\rho \geq C_2\varepsilon$, for some constant $C_2 > 0$ depending on k , C_0 , C_1 and N . Therefore, after integration we obtain that

$$\Lambda_\varepsilon(\rho) \geq \log \frac{\rho}{\varepsilon} - C \quad (\text{C.11})$$

for any $\rho \geq 0$, where the constant C only depends on k , α , β . We have the following analogue of [38, Proposition 3.2].

Lemma 18. *Let $\varepsilon \leq \rho_1 \leq \rho_2$ and let $u \in W^{1,k}(B_{\rho_2}^k \setminus B_{\rho_1}^k, \mathbb{R}^m)$ be smooth. Suppose that $\text{hc}(u, \partial B_\rho^k) = \sigma$ for any $\rho \in (\rho_1, \rho_2)$. Then, there holds*

$$E_\varepsilon(u, B_{\rho_2}^k \setminus B_{\rho_1}^k) \geq |\sigma|_* \left(\Lambda_\varepsilon \left(\frac{\rho_2}{|\sigma|_*} \right) - \Lambda_\varepsilon \left(\frac{\rho_1}{|\sigma|_*} \right) \right).$$

Proof. First of all, given $\rho > 0$ and a map $v \in W^{1,k}(\partial B_\rho^k, \mathcal{N})$ in the homotopy class $\sigma \in \pi_{k-1}(\mathcal{N})$, there holds

$$\frac{1}{k} \int_{\partial B_\rho^k} |\nabla v|^k \geq \frac{E_{\min}(\sigma)}{\rho} \quad (\text{C.12})$$

where $E_{\min}(\sigma)$ is defined by (A.1). This inequality follows immediately from the definition of $E_{\min}(\sigma)$, combined with a scaling argument.

Now, for any $\rho \geq \varepsilon > 0$ and any smooth $u: \partial B_\rho^k \rightarrow \mathbb{R}^m$ such that $\mu := \min_{\partial B_\rho^k} \phi \circ u > 1/2$, it holds that

$$G_\varepsilon(s, v; \partial B_\rho^k) \geq \frac{E_{\min}(\sigma)}{\rho} \mu^k + \frac{C_0}{\varepsilon} (1 - \mu)^N.$$

Here G_ε is defined by (C.7), $s := \phi \circ u$, $v := \pi \circ u$, and σ denotes the homotopy class of v on ∂B_ρ^k . The constants C_0, N are suitably chosen at this stage. The proof of this claim follow by repeating, almost word by word, the arguments in [38, Theorem 2.1]; the only difference is that we need to apply (C.12) instead of [38, Lemma 2.4]. Due to (A.6), we obtain

$$G_\varepsilon(s, v; \partial B_\rho^k) \geq \frac{\mu^k}{\rho/|\sigma|_*} + \frac{C_0}{\varepsilon} (1 - \mu)^N \geq \lambda_\varepsilon \left(\frac{\rho}{|\sigma|_*} \right). \quad (\text{C.13})$$

On the other hand, in case $0 \leq \mu \leq 1/2$, [38, Lemma 2.3] implies that

$$G_\varepsilon(s, v; \partial B_\rho^k) \geq \frac{C_1}{\varepsilon} \quad (\text{C.14})$$

for some $C_1 > 0$ that depends on k, C_0 and N . Therefore, by integrating the inequalities (C.13)–(C.14) with respect to ρ , we deduce that

$$\begin{aligned} G_\varepsilon(s, v; B_{\rho_2}^k \setminus B_{\rho_1}^k) &\geq \int_{\rho_1}^{\rho_2} \min \left(\lambda_\varepsilon \left(\frac{\rho}{|\sigma|_*} \right), \frac{C_1}{\varepsilon} \right) \\ d\rho &= |\sigma|_* \int_{\rho_1/|\sigma|_*}^{\rho_2/|\sigma|_*} \min \left(\lambda_\varepsilon(s), \frac{C_1}{\varepsilon} \right) ds \end{aligned}$$

and, thanks to Lemma 17, the lemma follows. \square

We also have an analogue of [38, Proposition 3.3].

Lemma 19. *Suppose that $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ is smooth and that $S_E \subset\subset \Omega$. Then, there exists a finite collection of closed, pairwise disjoint balls $(B_i)_{i=1}^p$, of radii $\rho_i \geq \varepsilon$, such that $S_E \subseteq \cup_{i=1}^p B_i$, $B_i \cap S_E \neq \emptyset$ for any i , and*

$$E_\varepsilon(u; B_{\rho_i} \cap \Omega) \geq \frac{C_1}{\varepsilon} \rho_i.$$

Proof. We claim that, if \tilde{S} is a connected component of S_E such that $\tilde{S} \subset\subset \Omega$, then

$$\int_{\tilde{S}} |\nabla u|^k \gtrsim \left| \text{hc}(u, \partial \tilde{S}) \right|_* \quad (\text{C.15})$$

This inequality parallels [38, Lemma 3.2]; once (C.15) is established, the rest of the proof follows exactly as in [38]. The definition (C.10) of hc and (C.9) imply that

$$\left| \text{hc} \left(u, \partial \tilde{S} \right) \Big|_* = \left| \mathbb{I}(\mathbf{S}_y(u), \llbracket \tilde{S} \rrbracket) \Big|_* \leq \mathbb{M} \left(\mathbf{S}_y(u) \llcorner \tilde{S} \right) \quad (\text{C.16})$$

for a.e. $y \in \mathbb{R}^m$ such that $|y| \leq \eta_0$. On the other hand, (P₂) gives that

$$\int_{\mathbb{R}^m} \mathbb{M} \left(\mathbf{S}_y(u) \llcorner \tilde{S} \right) dy \lesssim \int_{\tilde{S}} |\nabla u|^k,$$

so there exists (a set of positive measure of) y such that $|y| \leq \eta_0$ and $\mathbb{M}(\mathbf{S}_y(u) \llcorner \tilde{S}) \lesssim \int_{\tilde{S}} |\nabla u|^k$. Then, (C.15) follows from (C.16). \square

Lemma 19 and the definition of Λ_ε imply that

$$E_\varepsilon(u; B_i) \geq |\text{hc}(u, \partial B_i)|_* \Lambda_\varepsilon \left(\frac{\rho_i}{|\text{hc}(u, \partial B_i)|_*} \right) \quad \text{for any } i. \quad (\text{C.17})$$

The last step in the proof of Proposition 10 is the so-called ‘‘ball construction’’ [38, Proposition 4.1]. If u satisfies (C.1) for some $r > 0$ then, by choosing $\delta_0 = \delta_0(\mathcal{N}) < \theta_0$ sufficiently small, we obtain as a consequence

$$|s(x)| \geq \frac{1}{2} \quad \text{for any } x \in \Omega \text{ such that } \text{dist}(x, \partial \Omega) < r. \quad (\text{C.18})$$

Moreover, we can assume without loss of generality that u satisfies

$$E_\varepsilon(u) \leq |\text{hc}(u, \partial \Omega)|_* |\log \varepsilon| + C \quad (\text{C.19})$$

for some ε -independent constant C , for otherwise Proposition 10 holds trivially.

Lemma 20. *There exists a constant $\varepsilon_0 > 0$ such that the following statement holds. Let $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ be a smooth function that satisfies (C.18) for some $r > 0$ and (C.19). For any $\tau > 0$ and any $\varepsilon \in (0, 1/2)$ such that*

$$4\tau |\text{hc}(u, \partial \Omega)|_* < r, \quad \varepsilon |\log \varepsilon| |\text{hc}(u, \partial \Omega)|_* \leq \varepsilon_0, \quad (\text{C.20})$$

there exists a finite collection of closed ball $(\tilde{B}_i)_{i=1}^q$, of radii r_i , that satisfy the following properties:

- (i) *the interiors of the balls are pairwise disjoint;*
- (ii) *$S_E \subset \cup_{i=1}^q \tilde{B}_i$ and $\tilde{B}_i \cap S_E \neq \emptyset$ for any i ;*
- (iii) *letting $s := \min_i r_i / |\text{hc}(u, \partial \tilde{B}_i)|_*$, we have*

$$E_\varepsilon(u, \tilde{B}_i \cap \Omega) \geq \frac{r_i}{s} \Lambda_\varepsilon(s);$$

- (iv) $\tau/2 \leq s \leq \tau$;

$$(v) |\text{hc}(u, \partial \Omega)|_* = \sum_{i=1}^q |\text{hc}(u, \partial \tilde{B}_i)|_*.$$

Lemma 20 follows by repeating the arguments of [38, Proposition 4.1] (see also [2, Remark at p. 22]), and using Lemmas 18, 19 and (C.17).

Proof of Proposition 10. We assume that u is smooth, satisfies (C.18) (as a consequence of our assumption (C.1)) and (C.19). We apply Lemma 20 and use the fact that, by definition of s , $|\mathrm{hc}(u, \partial \tilde{B}_i)|_* \leq r_i/s$ for any i :

$$\begin{aligned} E_\varepsilon(u, \Omega) &\stackrel{(i)-(iii)}{\geq} \sum_{i=i}^P \frac{r_i}{s} \Lambda_\varepsilon(s) \geq \sum_{i=i}^P \left| \mathrm{hc}(u, \partial \tilde{B}_i) \right|_* \Lambda_\varepsilon(s) \stackrel{(v)}{=} |\mathrm{hc}(u, \partial \Omega)|_* \Lambda_\varepsilon(s) \\ &\stackrel{(iv)}{\geq} |\mathrm{hc}(u, \partial \Omega)|_* \Lambda_\varepsilon\left(\frac{\tau}{2}\right) \stackrel{(C.11)}{\geq} |\mathrm{hc}(u, \partial \Omega)|_* \log \frac{\tau}{2\varepsilon} - C. \end{aligned}$$

The constant C here only depends on k, α, β . Now, we choose $\tau := r/(8|\mathrm{hc}(u, \partial \Omega)|_*)$ (which is admissible in view of (C.20)). Taking (A.5) into account, we obtain the desired estimate in case u is smooth. Now the proposition follows by a density argument. \square

D. Technical Results about Flat Chains

Throughout this appendix, we consider chains with coefficients in a normed Abelian group $(\mathbf{G}, |\cdot|)$ such that

$$\inf_{g \in \mathbf{G} \setminus \{0\}} |g| > 0. \quad (\text{D.1})$$

This assumption is satisfied by $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$, due to Proposition 1.

D.1. Approximation Results for Flat Chains

We give the proof of the approximation results, Proposition 5 and 6, we have used in Section 3.4.2. For convenience, we recall the statements here. Let $\mathfrak{S} \subseteq \mathbf{G}$ be a set of generators for \mathbf{G} . We assume that, for any $g \in \mathbf{G}$, there exist $g_1, \dots, g_p \in \mathfrak{S}$ such that

$$g = \sum_{i=1}^p g_i \quad \text{and} \quad |g| = \sum_{i=1}^p |g_i|. \quad (\text{D.2})$$

The set defined by (A.7) satisfies this assumption, by Proposition 1.

Proposition 11. *Let $S \in \mathbb{M}_n(\mathbb{R}^{n+k}; \mathbf{G})$ be a polyhedral chain. Let $W_\mathfrak{S} \subseteq \mathbb{R}^{n+k}$ be an open set, with polyhedral boundary, such that $\partial W_\mathfrak{S}$ is transverse to $\mathrm{spt} S$ (i.e., there exist triangulations of $\partial W_\mathfrak{S}$ and $\mathrm{spt} S$ such that any simplex of the triangulation of $\partial W_\mathfrak{S}$ is transverse to any simplex of the triangulation of $\mathrm{spt} S$). Then, there exists a sequence of polyhedral $(n+1)$ -chains R_j , supported in $\overline{W_\mathfrak{S}}$, such that the following hold:*

- (i) $S + \partial R_j \rightarrow S$, with respect to the \mathbb{F} -norm, as $j \rightarrow +\infty$;
- (ii) $\mathbb{M}(S + \partial R_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$;
- (iii) for any j , $(S + \partial R_j) \llcorner \partial W_\mathfrak{S} = 0$;
- (iv) for any j , the chain $(S + \partial R_j) \llcorner W_\mathfrak{S}$ takes multiplicities in the set $\mathfrak{S} \subseteq \pi_{k-1}(\mathcal{N})$ defined by (2.4).

Proposition 11 implies Proposition 6.

Proof. Since $\partial W_\mathfrak{S}$ is transverse to $\mathrm{spt} S$, the intersection $\mathrm{spt} S \cap \partial W_\mathfrak{S}$ has dimension $n-1$ at most and hence, $S \llcorner \partial W_\mathfrak{S} = 0$. By triangulating S , we can write $S \llcorner W_\mathfrak{S}$ as a finite sum

$$S \llcorner W_\mathfrak{S} = \sum_K \sigma_K \llbracket K \rrbracket,$$

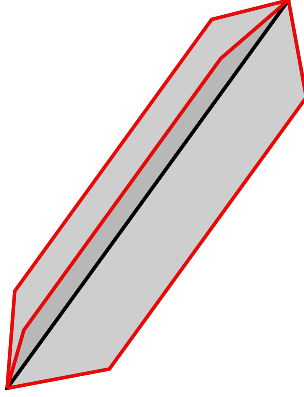


Fig. 7. The chain R_j , in case $n = 1, k = 2$ and S consists of a segment only, $S = \sigma_K \llbracket K \rrbracket$. The chain S is in black, R_j is in gray, and $S + \partial R_j$ is in red

where $\sigma_K \in \pi_{k-1}(\mathcal{N})$ and the K 's are closed n -simplices, whose interiors are contained in $W_{\mathfrak{G}}$ and pairwise disjoint. We fix positive parameters δ, γ and, for any n -simplex K of $S \llcorner W_{\mathfrak{G}}$, we consider the set $U(K, \delta, \gamma)$ defined by (3.1). We choose δ, γ small enough, so that the interiors of the $U(K, \delta, \gamma)$'s are pairwise disjoint and contained in $W_{\mathfrak{G}}$. By assumption (D.2), we can write $\sigma_K = \sum_{i=1}^p \sigma_{K,i}$ where $\sigma_{K,i} \in \mathfrak{G}$ and

$$|\sigma_K| = \sum_{i=1}^p |\sigma_{K,i}|. \quad (\text{D.3})$$

Take distinct vectors $y^{K,1}, \dots, y^{K,p} \in \mathbb{R}^{n+k}$ that are orthogonal to K and satisfy $|y^{K,1}| = \dots = |y^{K,p}| = 1$. For each $i \in \{1, \dots, p\}$, we define $h^{K,i}: [0, 1] \times K \rightarrow \mathbb{R}^{n+k}$ by

$$h^{K,i}(t, x') := x' + t \min\{\delta, \gamma \operatorname{dist}(x', \partial K)\} y^{K,i}$$

for any $(t, x') \in [0, 1] \times K$.

For any integer $j \geq 1$, we define

$$R_j := \sum_K \sum_{i=1}^p \sigma_{K,i} h_*^{K,i} (\llbracket [0, 1/j] \rrbracket \times \llbracket K \rrbracket)$$

(see Figure 7).

The chain R_j is polyhedral, because the $h^{K,i}$'s are piecewise affine, and supported in $\overline{W_{\mathfrak{G}}}$. The support of R_j may intersect $\partial W_{\mathfrak{G}}$ only along its $(n-1)$ -skeleton, so $(\partial R_j) \llcorner \partial W_{\mathfrak{G}} = 0$. We compute the mass of R_j . Since the maps $h^{K,i}$ are Lipschitz, and their Lipschitz constant only depends on γ , which is fixed, the area formula implies

$$\mathbb{M}(R_j) \lesssim \sum_K \sum_{i=1}^p |\sigma_{K,i}|_* \mathcal{H}^{n+1}([0, 1/j] \times K) \stackrel{(\text{D.3})}{\leq} j^{-1} \mathbb{M}(S \llcorner W_{\mathfrak{G}}) \rightarrow 0$$

as $j \rightarrow +\infty$.

Thus, (i) follows. Now, we compute the boundary of R_j . For each simplex K and each i , we have $h^{K,i}(t, x') = x'$ if $x' \in \partial K$ and $h^{K,i}(0, x') = x'$ for any $x' \in K$. As a consequence,

$$\begin{aligned} \partial h_*^{K,i} (\llbracket [0, 1/j] \rrbracket \times \llbracket K \rrbracket) \\ &= h_*^{K,i} (\llbracket [1/j] \rrbracket \times \llbracket K \rrbracket - \llbracket 0 \rrbracket \times \llbracket K \rrbracket) - h_*^{K,i} (\llbracket [0, 1/j] \rrbracket \times \partial \llbracket K \rrbracket) \\ &= h^{K,i}(j^{-1}, \cdot)_* \llbracket K \rrbracket - \llbracket K \rrbracket. \end{aligned}$$

By multiplying this identity by $\sigma_{K,i}$, and taking the sum over i, K , we obtain

$$\partial R_j = \sum_K \sum_{i=1}^p \sigma_{K,i} h^{K,i}(j^{-1}, \cdot)_* \llbracket K \rrbracket - S \llcorner W_{\mathfrak{S}}. \quad (\text{D.4})$$

In particular, $S \llcorner W_{\mathfrak{S}} + \partial R_j = (S + \partial R_j) \llcorner W_{\mathfrak{S}}$ takes multiplicities in \mathfrak{S} . Finally, by applying the area formula to (D.4), we deduce

$$\begin{aligned} \mathbb{M}(S \llcorner W_{\mathfrak{S}} + \partial R_j) &\rightarrow \sum_K \sum_{i=1}^p |\sigma_{K,i}|_* \mathcal{H}^n(K) \stackrel{(\text{D.3})}{=} \mathbb{M}(S \llcorner W_{\mathfrak{S}}) \\ &\text{as } j \rightarrow +\infty, \end{aligned}$$

and (ii) follows. \square

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be a domain and let $S \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$. Recall that S is called locally polyhedral if, for any compact set $K \subseteq \Omega$, there exists a polyhedral chain T such that $(S - T) \llcorner K = 0$. We write $S_0 \sim_{\overline{\Omega}} S_1$ if there exists $R \in \mathbb{M}_{n+1}(\overline{\Omega}; \mathbf{G})$ such that $S_1 = S_0 + \partial R$.

Proposition 12. *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be a bounded, Lipschitz domain. Let $S_0 \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ be a locally polyhedral chain such that $S_0 \llcorner \partial \Omega = 0$. Let $S \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ be such that $S \sim_{\overline{\Omega}} S_0$. Then, there exists a sequence of polyhedral $(n+1)$ -chains R_j , with compact support in Ω , such that $S_0 + \partial R_j \rightarrow S$ (with respect to the \mathbb{F} -norm) and $\mathbb{M}(S_0 + \partial R_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$.*

Proposition 5 follows from Proposition 12, with the help of (P₃) and Lemma 3. We split the proof of Proposition 12 into several lemmas. The first one is a straightforward consequence of the deformation theorem for flat chains; we provide a proof for completeness.

Lemma 21. *Let $q \in \{0, \dots, n+k-1\}$, $T \in \mathbb{M}_q(\mathbb{R}^{n+k}; \mathbf{G})$ and $\eta > 0$ be given. Suppose that T is compactly supported and ∂T is polyhedral. Then, there exist a polyhedral q -chain P and a finite-mass chain $C \in \mathbb{M}_{q+1}(\mathbb{R}^{n+k}; \mathbf{G})$, supported in the η -neighbourhood of $\text{spt } T$, that satisfy*

$$\begin{aligned} T &= P + \partial C, \\ \mathbb{M}(P) &\lesssim \mathbb{M}(T) + \eta \mathbb{M}(\partial T), \quad \mathbb{M}(C) \lesssim \eta \mathbb{M}(T). \end{aligned}$$

Proof. We apply the deformation theorem (see e.g. [26, Theorem 7.3] or [54, Theorem 1.1]) to T . We find a polyhedral q -chain A , a finite-mass q -chain B and a finite-mass $(q+1)$ -chain C that satisfy the following properties:

- (a) $T = A + B + \partial C$;
- (b) $\mathbb{M}(A) \lesssim \mathbb{M}(T) + \eta \mathbb{M}(\partial T)$, $\mathbb{M}(B) \lesssim \eta \mathbb{M}(\partial T)$ and $\mathbb{M}(C) \lesssim \eta \mathbb{M}(T)$;
- (c) A, B, C are supported in the η -neighbourhood of $\text{spt } T$.

Since we have assumed that ∂T is polyhedral, we can take B to be polyhedral, too (see e.g. [54, Theorem 1.1.(7)]). Then, the chains $P := A + B$ and C have all the required properties.

\square

Lemma 22. *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be a bounded, Lipschitz domain. Let $S_0 \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ be such that $S_0 \llcorner \partial\Omega = 0$. Let $R \in \mathbb{M}_{n+1}(\overline{\Omega}; \mathbf{G})$ be such that $\mathbb{M}(\partial R) < +\infty$. Then, there exists a sequence of chains $R_j \in \mathbb{M}_{n+1}(\overline{\Omega}; \mathbf{G})$, compactly supported in Ω , such that $\partial R_j \rightarrow \partial R$ (with respect to the \mathbb{F} -norm) and $\mathbb{M}(S_0 + \partial R_j) \rightarrow \mathbb{M}(S_0 + \partial R)$ as $j \rightarrow +\infty$.*

The proof of Lemma 22 is identical to that of [2, Proposition 8.6]. In [2], the authors work in the setting of currents; however, the arguments used in the proof of Proposition 8.6 carry over to the setting of flat chains, thanks to the results in [26, Sections 5 and 6].

Lemma 23. *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be a bounded, Lipschitz domain. Let $S_0 \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ be a locally polyhedral chain such that $S_0 \llcorner \partial\Omega = 0$, and let $S \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ be such that $S \sim_{\overline{\Omega}} S_0$. Then, there exists a sequence of locally polyhedral chains $S_j \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ with the following properties:*

- (i) $\mathbb{F}(S_j - S) \rightarrow 0$ as $j \rightarrow +\infty$;
- (ii) $\mathbb{M}(S_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$;
- (iii) for any j , we can write $S_j = S_0 + \partial R_j$ for some finite-mass $(n+1)$ -chain R_j with compact support in Ω .

Proof. By assumption, there exists $R \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ such that $S = S_0 + \partial R$. Thanks to Lemma 22 and a diagonal argument, we can assume without loss of generality that R is compactly supported in Ω . For any positive t , let $\Omega_t := \{x \in \Omega : \text{dist}(x, \partial\Omega) > t\}$. We take a positive number t_0 such that $\text{spt } R \subseteq \Omega_{2t_0}$, and an open set U , with polyhedral boundary, such that $\Omega_{2t_0} \subset\subset U \subset\subset \Omega_{t_0}$. Because S and S_0 differ by a boundary, we have

$$\begin{aligned} \partial(S \llcorner U) + \partial\left(S \llcorner \left(\mathbb{R}^{n+k} \setminus U\right)\right) \\ = \partial S = \partial S_0 = \partial(S_0 \llcorner U) + \partial\left(S_0 \llcorner \left(\mathbb{R}^{n+k} \setminus U\right)\right). \end{aligned}$$

However, S and S_0 agree out of U , so $\partial(S \llcorner U) = \partial(S_0 \llcorner U)$. In particular, since S_0 is locally polyhedral in Ω and U is polyhedral, $\partial(S \llcorner U)$ is a polyhedral chain. Thanks to, e.g., [26, Theorem 5.6 and 7.7], there exists a sequence of polyhedral n -chains T_j that \mathbb{F} -converges to $S \llcorner U$, satisfies $\text{spt } T_j \subseteq \Omega_{t_0}$ for any j and

$$\partial T_j = \partial(S \llcorner U) \quad \text{for any } j \in \mathbb{N}, \quad \mathbb{M}(T_j) \rightarrow \mathbb{M}(S \llcorner U) \quad \text{as } j \rightarrow +\infty. \quad (\text{D.5})$$

By definition of the \mathbb{F} -norm, there exist sequences $P_j \in \mathbb{M}_{n+1}(\mathbb{R}^{n+k}; \mathbf{G})$ and $Q_j \in \mathbb{M}_n(\mathbb{R}^{n+k}; \mathbf{G})$ such that

$$S \llcorner U - T_j = \partial P_j + Q_j \quad \text{for any } j \quad (\text{D.6})$$

$$\mathbb{M}(P_j) \rightarrow 0, \quad \mathbb{M}(Q_j) \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (\text{D.7})$$

We do not know a priori whether the chains P_j, Q_j are supported in Ω , so we perform a truncation argument. Define

$$P_{j,t} := (\partial P_j) \llcorner \Omega_t - \partial(P_j \llcorner \Omega_t)$$

for $t \in (0, t_0)$ and $j \in \mathbb{N}$. By applying Fatou's lemma and [26, Theorem 5.7], we obtain that

$$\int_0^{t_0} \liminf_{j \rightarrow +\infty} \mathbb{M}(P_{j,t}) \, dt \leq \liminf_{j \rightarrow +\infty} \int_0^{t_0} \mathbb{M}(P_{j,t}) \, dt \leq \liminf_{j \rightarrow +\infty} \mathbb{M}(P_j) \stackrel{(\text{D.7})}{=} 0.$$

Therefore, for a.e. $t \in (0, t_0)$ there exists a (non-relabelled) subsequence $j \rightarrow +\infty$ such that $\mathbb{M}(P_{j,t}) \rightarrow 0$. By taking the restriction of (D.6) to Ω_t , we obtain

$$S \llcorner U - T_j = \partial \left(\underbrace{P_j \llcorner \Omega_t}_{=: P'_j} \right) + \underbrace{P_{j,t} + Q_t \llcorner \Omega_t}_{=: Q'_j}. \quad (\text{D.8})$$

By construction, P'_j and Q'_j are supported in $\overline{\Omega_t} \subseteq \Omega$, and there holds

$$\mathbb{M}(P'_j) \rightarrow 0, \quad \mathbb{M}(Q'_j) \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (\text{D.9})$$

Moreover, by taking the boundary of both sides of (D.8), we deduce that

$$\partial Q'_j = \partial(S \llcorner U) - \partial T_j \stackrel{(\text{D.5})}{=} 0.$$

By applying Lemma 21 to Q'_j , we find a decomposition

$$Q'_j = Q''_j + \partial C_j, \quad (\text{D.10})$$

where

- (a) Q''_j is a polyhedral n -chain such that $\mathbb{M}(Q''_j) \lesssim \mathbb{M}(Q'_j)$;
- (b) C_j is a $(n+1)$ -chain of finite mass and $\mathbb{M}(C_j) \lesssim j^{-1} \mathbb{M}(Q'_j)$;
- (c) Q''_j and C_j are supported in $\overline{\Omega_{t_0-1/j}}$.

From (a), (b) and (D.9), we deduce that

$$\mathbb{M}(C_j) \rightarrow 0, \quad \mathbb{M}(Q''_j) \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (\text{D.11})$$

Now, we define

$$S_j := T_j + Q''_j + S \llcorner (\mathbb{R}^{n+k} \setminus U). \quad (\text{D.12})$$

By construction, S_j is locally polyhedral. We have

$$S_j - S = T_j + Q''_j - S \llcorner U \stackrel{(\text{D.8})}{=} Q''_j - \partial P'_j - Q'_j \stackrel{(\text{D.10})}{=} -\partial(P'_j + C_j) \quad (\text{D.13})$$

and hence, $\mathbb{F}(S_j - S) \rightarrow 0$ due to (D.9) and (D.11). By the lower semi-continuity of the mass, we deduce that $\mathbb{M}(S) \leq \liminf_{j \rightarrow +\infty} \mathbb{M}(S_j)$. On the other hand, if we apply the triangle inequality to (D.12) and use the identity $\mathbb{M}(S) = \mathbb{M}(S \llcorner U) + \mathbb{M}(S \llcorner (\mathbb{R}^{n+k} \setminus U))$, we obtain

$$\mathbb{M}(S_j) - \mathbb{M}(S) \leq \mathbb{M}(T_j) + \mathbb{M}(Q''_j) - \mathbb{M}(S \llcorner U).$$

The right hand side converges to zero as $j \rightarrow +\infty$, due to (D.5) and (D.11). Thus, we deduce that $\limsup_{j \rightarrow +\infty} \mathbb{M}(S_j) \leq \mathbb{M}(S)$, and hence $\mathbb{M}(S_j) \rightarrow \mathbb{M}(S)$ as $j \rightarrow +\infty$. Finally, we define $R_j := S_j - P'_j - C_j$. Then, (D.13) gives $S_j - S_0 = \partial R_j$ and the lemma follows. \square

Proof of Proposition 12. Let S_0, S be given, as in the statement. By applying Lemma 23, we find a sequence of locally polyhedral chains $S_j \in \mathbb{M}_n(\overline{\Omega}; \mathbf{G})$ and a sequence of finite-mass $(n+1)$ -chains \tilde{R}_j , compactly supported in Ω , such that $S_j \rightarrow S$ in the \mathbb{F} -norm, $\mathbb{M}(S_j) \rightarrow \mathbb{M}(S)$ and $S_j = S_0 + \partial \tilde{R}_j$ for any j . Since S_0, S_j are locally polyhedral in Ω , $\partial \tilde{R}_j$ is polyhedral. We apply Lemma 21 to each \tilde{R}_j . We find polyhedral $(n+1)$ -chains R_j , compactly supported in Ω , and $(n+2)$ -chains C_j of finite mass, such that

$$\tilde{R}_j = R_j + \partial C_j.$$

Then, $S_j = S_0 + \partial(R_j + \partial C_j) = S_0 + \partial R_j$, and the proposition follows. \square

D.2. A Characterisation of the Mass of a Rectifiable Chain

For any linear subspace $L \subseteq \mathbb{R}^{n+k}$, we let $\pi_L : \mathbb{R}^{n+k} \rightarrow L$ be the orthogonal projection onto L . A n -chain of class C^1 is a chain S that can be written in the form $S = f_*P$, with f a map of class C^1 and P a polyhedral chain. The set of rectifiable n -chains is defined as the closure of n -chains of class C^1 with respect to the \mathbb{M} -norm.

Lemma 24. *Let $S \in \mathbb{M}_n(\mathbb{R}^{n+k}; \mathbf{G})$ be a rectifiable n -chain. Then,*

$$\mathbb{M}(S) = \sup_{(U_i, L_i)_{i \in \mathbb{N}}} \sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i, *}(S \llcorner U_i)),$$

where the supremum is taken over all sequences of pairwise disjoint open sets U_i and n -planes $L_i \subseteq \mathbb{R}^{n+k}$.

If the coefficient group satisfies (D.1), as is the case for $\mathbf{G} = \pi_{k-1}(\mathcal{N})$, then any chain of finite mass is rectifiable, by White's Rectifiability Theorem [55, Theorem 7.1]. Therefore, Lemma 24 implies Lemma 14.

Proof. Let $(U_i)_{i \in \mathbb{N}}$ be a sequence of pairwise disjoint open sets, and let $(L_i)_{i \in \mathbb{N}}$ be a sequence of n -planes in \mathbb{R}^{n+k} . For any i , the projection π_{L_i} is a 1-Lipschitz map and hence $\mathbb{M}(\pi_{L_i, *}(S \llcorner U_i)) \leq \mathbb{M}(S \llcorner U_i)$ (see e.g. [26, Eq. (5.1)]). Since the U_i 's are assumed to be pairwise disjoint, we obtain

$$\sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i, *}(S \llcorner U_i)) \leq \sum_{i=0}^{+\infty} \mathbb{M}(S \llcorner U_i) \leq \mathbb{M}(S). \quad (\text{D.14})$$

This proves one of the inequalities. To prove the opposite inequality, we first suppose that S is a C^1 -polyhedron, then a C^1 -chain, and finally we extend the result to an arbitrary rectifiable chain. We denote by $\text{int } A$ the interior of a set $A \subseteq \mathbb{R}^{n+k}$, and by $\text{diam } A$ its diameter.

Step 1. (S is a C^1 -polyhedron) We suppose that $S = f_*(\sigma \llcorner K)$, where $\sigma \in \mathbf{G}$, K is a convex, compact n -polyhedron, and $f : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ is a C^1 -diffeomorphism. Let $\eta > 0$ be arbitrarily fixed. Since f is C^1 and K is compact, there exists $\rho > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq \eta \quad \text{if } (x, y) \in K \times K \text{ and } |x - y| \leq \rho, \quad (\text{D.15})$$

where $\|\cdot\|$ denotes the operator norm on the space of real $(n+k) \times (n+k)$ -matrices. Let $(T_i)_{i=1}^q$ be a collection of n -simplices that triangulate K , such that

$$\max_{1 \leq i \leq q} \text{diam } T_i \leq \rho. \quad (\text{D.16})$$

Let $V_i := \text{int } U(T_i, \rho/2, \rho/2)$ where $U(T_i, \rho/2, \rho/2)$ is defined as in (3.1), and $U_i := f(V_i)$. The U_i 's are pairwise disjoint open sets, because f is a diffeomorphism. Let L be the n -plane passing through the origin that is parallel to K . For any i , we choose a point $x_i \in \text{int } T_i$ and we define $L_i := \nabla f(x_i)(L)$. The L_i 's are indeed n -planes, because $\nabla f(x_i)$ is an invertible linear map. For any $x \in K \cap T_i$ and any $y \in L$, we have

$$\begin{aligned} & |(\pi_{L_i} \circ \nabla f)(x)y - \nabla f(x)y| \\ & \leq |(\pi_{L_i} \circ \nabla f)(x_i)y - \nabla f(x_i)y| + 2\|\nabla f(x_i) - \nabla f(x)\| |y| \\ & \stackrel{(\text{D.15})-(\text{D.16})}{\leq} 2\eta |y|. \end{aligned}$$

Therefore, by applying the area formula we obtain

$$\begin{aligned} & |\mathbb{M}(\pi_{L_i, *}(S \llcorner U_i)) - \mathbb{M}(S \llcorner U_i)| \\ & \leq |\sigma|_* \mathcal{H}^n((\pi_{L_i, *} \circ f)(K \cap V_i)) - \mathcal{H}^n(f(K \cap V_i))| \\ & \leq C\eta |\sigma|_* \mathcal{H}^n(K \cap V_i) \end{aligned}$$

for some constant C depending only on n, k . This implies

$$\begin{aligned} \sum_{i=1}^q \mathbb{M}(\pi_{L_i, *}(S \llcorner U_i)) & \geq \sum_{i=1}^q \mathbb{M}(S \llcorner U_i) - C\eta |\sigma|_* \sum_{i=1}^q \mathcal{H}^n(K \cap V_i) \\ & \geq \mathbb{M}(S \llcorner \cup_i U_i) - C\eta |\sigma|_* \mathcal{H}^n(K), \end{aligned}$$

where η is arbitrarily small. To complete the proof in this case, it only remains to notice that $\mathbb{M}(S) = \mathbb{M}(S \llcorner \cup_i U_i)$, because $\mathcal{H}^n(K \setminus \cup_i V_i) = 0$ and $\mathcal{H}^n(S \setminus \cup_i U_i) = 0$ by the area formula.

Step 2. (S is a C^1 -chain) We suppose that $S = f_*P$, where P is a polyhedral n -chain and $f: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ is a C^1 -diffeomorphism. This case follows easily from the previous one, by additivity. Indeed, let us write $S = \sum_{j=1}^p \sigma_j \llcorner [K_j]$ with $\sigma_j \in \mathbf{G}$, K_j a convex, compact n -polyhedra. Given positive parameters δ, γ , let $W^j := f(\text{int } U(K_j, \delta, \gamma))$. For δ, γ small enough, the W^j have pairwise disjoint interiors. Let $\eta > 0$ be fixed. By applying Step 1, for any j we find a sequence $(V_i^j)_{i \in \mathbb{N}}$ of pairwise disjoint open sets and a sequence $(L_i^j)_{i \in \mathbb{N}}$ of n -planes such that

$$\sum_{i=0}^{+\infty} \mathbb{M}\left(\pi_{L_i^j, *}\left(f_*(\sigma_j \llcorner [K_j]) \llcorner U_i^j\right)\right) \geq \mathbb{M}(f_*(\sigma_j \llcorner [K_j])) - \eta. \quad (\text{D.17})$$

We define $U_i^j := V_i^j \cap W^j$. The U_i^j 's are pairwise disjoint open sets. We have $\mathcal{H}^n(K_j \setminus \text{int } U(K_j, \delta, \gamma)) = 0$ and hence, by the area formula, $\mathbb{M}(f_*(\sigma_j \llcorner [K_j]) \llcorner (\mathbb{R}^{n+k} \setminus W^j)) = 0$. Therefore, we obtain

$$\begin{aligned} & \sum_{i,j} \mathbb{M}\left(\pi_{L_i^j, *}\left(f_*(\sigma_j \llcorner [K_j]) \llcorner U_i^j\right)\right) \\ & = \sum_{i,j} \mathbb{M}\left(\pi_{L_i^j, *}\left(f_*(\sigma_j \llcorner [K_j]) \llcorner V_i^j\right)\right) \stackrel{(\text{D.17})}{\geq} \mathbb{M}(S) - p\eta \end{aligned}$$

and the lemma is proved also in this case, because η may be taken arbitrarily small.

Step 3. (S is a rectifiable chain) Since S is rectifiable, for any $\eta > 0$ there exist a polyhedral n -chain P and a C^1 -diffeomorphism $f: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ such that

$$\mathbb{M}(S - f_*P) \leq \eta \quad (\text{D.18})$$

(see e.g. [55, 1.2 at p. 169] and the references therein). By Step 2, there exists a sequence $(U_i, L_i)_{i \in \mathbb{N}}$ such that

$$\sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i, *}(f_*P \llcorner U_i)) \geq \mathbb{M}(f_*P) - \eta. \quad (\text{D.19})$$

Let us set $Q := S - f_*P$. Then, using the linearity of $\pi_{L_i,*} \cdot \lfloor U_i$, and the triangle inequality for \mathbb{M} , we obtain

$$\begin{aligned} \sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i,*}(S \lfloor U_i)) &\geq \sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i,*}(f_*P \lfloor U_i)) - \sum_{i=0}^{+\infty} \mathbb{M}(\pi_{L_i,*}(Q \lfloor U_i)) \\ &\stackrel{(D.14), (D.19)}{\geq} \mathbb{M}(f_*P) - \eta - \mathbb{M}(Q) \stackrel{(D.18)}{\geq} \mathbb{M}(S) - 3\eta \end{aligned}$$

so the lemma follows. \square

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Topological Singular Set of Vector-Valued Maps, II

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