Advances in Mathematics 319 (2017) 653–698

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The Ziegler spectrum for derived-discrete algebras

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ARTICLE INFO

Article history:
Received 29 March 2016
Received in revised form 28 June 2017
Accepted 17 July 2017
Communicated by Henning Krause

Dedicated to the memory of Gena Puninski

MSC:
16G10
18E30
18E35

Keywords:
Ziegler spectrum
Pure-injective object
Derived-discrete algebra
Compactly generated triangulated category
Homotopy category

ABSTRACT

Let Λ be a derived-discrete algebra. We show that the Krull–Gabriel dimension of the homotopy category of projective Λ-modules, and therefore the Cantor–Bendixson rank of its Ziegler spectrum, is 2, thus extending a result of Bobiński and Krause [8]. We also describe all the indecomposable pure-injective complexes and hence the Ziegler spectrum for derived-discrete algebras, extending a result of Z. Han [17]. Using this, we are able to prove that all indecomposable complexes in the homotopy category of projective Λ-modules are pure-injective, so obtaining a class of algebras for which every indecomposable complex is pure-injective but which are not derived pure-semisimple.

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http://dx.doi.org/10.1016/j.aim.2017.07.016

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Introduction

We prove that if Λ is a derived-discrete algebra then the Krull–Gabriel dimension of the homotopy category, $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$, of projective modules is 2, which is therefore also the Cantor–Bendixson rank of the Ziegler spectrum of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$. This extends a result of Bobiński and Krause who computed this dimension for the derived category. We also show that every indecomposable complex of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ is a homotopy string complex, in particular is pure-injective and hence a point of the Ziegler spectrum, which we also describe. Our proof that there are no further indecomposable objects relies heavily on that description.

The axioms of triangulated categories are an abstraction of the structural properties enjoyed by derived categories in representation theory and algebraic geometry, and stable homotopy categories in algebraic topology. Thus, triangulated categories provide a common language for a broad part of modern mathematics. In representation theory triangulated categories play a central role in tilting theory, which is a technique to enable effective comparisons between the representation theory of different algebras.

Whilst useful, triangulated categories often have complicated structure. One can gain some understanding by studying various aspects. For example, one may try to classify the thick or localising subcategories. In this article we focus on another method of simplification: identifying the indecomposable pure-injective objects and the space, the Ziegler spectrum, originally arising from the model theory of modules, that they form. Methods involving purity and the Ziegler spectrum were extended to compactly generated triangulated categories by Beligiannis, Krause and others; see for example [4,5,25,28] and the references therein.

Let T be a compactly generated triangulated category (see Section 1). We shall focus on three essentially equivalent approaches to studying the structure of T; namely understanding the objects in the following diagram.

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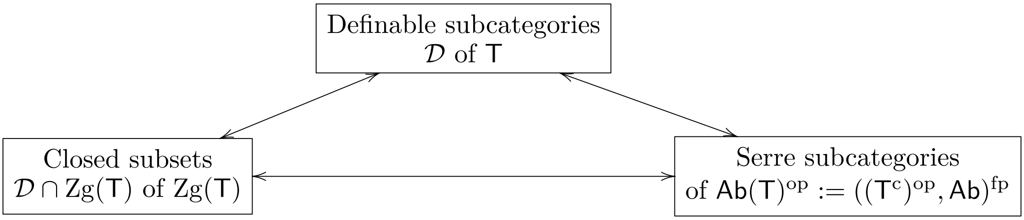


Here $\mathrm{Zg}(\mathsf{T})$, the *Ziegler spectrum* of T, is a topological space whose points are the (isomorphism classes of) indecomposable pure-injective objects of T, and $\mathsf{Ab}(\mathsf{T})^{\mathrm{op}} := ((\mathsf{T}^{\mathrm{c}})^{\mathrm{op}}, \mathsf{Ab})^{\mathrm{fp}}$ is the category of finitely presented contravariant functors from the subcategory T^{c} of compact objects, to abelian groups; this may be regarded as an abelian approximation of T. We refer to Section 1 for precise definitions of the concepts involved in the diagram above.

Associated to a topological space is its *Cantor–Bendixson rank* and associated to an abelian category is its *Krull–Gabriel dimension*. Applied to $\mathrm{Zg}(\mathsf{T})$ and $\mathsf{Ab}(\mathsf{T})^{\mathrm{op}}$, each of these may be regarded as a measure of the complexity of the triangulated category T; the second we denote $\mathrm{KGdim}(\mathsf{T})$. Under suitable conditions, the process of computing the Cantor–Bendixson rank of $\mathrm{Zg}(\mathsf{T})$ and the Krull–Gabriel dimension of $\mathsf{Ab}(\mathsf{T})^{\mathrm{op}}$ proceed in tandem. Indeed, there is a strong link between the two computations. In this article, we shall use this link to study the homotopy category of projective left Λ-modules $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ for Λ a derived-discrete algebra.

Derived-discrete algebras were introduced by Vossieck in [36] and occupy a position between finite and tame representation type: Λ is derived-discrete if given a cohomological dimension vector $\mathbf{v}$, i.e. a $\mathbb{Z}$-tuple of elements of $K_0(\mathsf{mod}\text{-}\Lambda)$ in which almost all entries are zero, there are only finitely many objects X in $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)$ with $(\underline{\dim} H^i(X))_{i\in\mathbb{Z}} = \mathbf{v}$; see [36, §1.1]. It was shown in [36] that Λ is derived-discrete if and only if it is piecewise hereditary of Dynkin type or a gentle one-cycle algebra satisfying a certain condition on its quiver. From now on when we refer to Λ as a derived-discrete algebra we shall exclude the case that Λ is piecewise hereditary of Dynkin type since these categories are very well understood. The philosophy behind studying derived-discrete algebras is that they provide a natural laboratory of computationally feasible but nontrivial derived categories in which to explore derived representation theory, and they provide a template for further study of derived categories of gentle algebras, these being important in cluster-tilting theory. As such, they have been the focus of much recent interest [6–10,16,17,34,36].

From now on Λ will be a derived-discrete algebra over an algebraically closed field $\mathbf{k}$. Our first main result refers to the bottom vertices of the diagram above:

Theorem A. *Let Λ be a derived-discrete algebra. Then* $\mathrm{KGdim}(\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)) = 2$*, which is also the Cantor–Bendixson rank of the Ziegler spectrum of* $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

This extends the result of Bobiński and Krause in [8], who compute the Krull–Gabriel dimension of the abelianisation of the perfect complexes $\mathsf{Ab}(\mathsf{D}(\mathsf{Mod}\text{-}\Lambda))^{\mathrm{op}}$, getting

$$\mathrm{KGdim}(\mathsf{D}(\mathsf{Mod}\text{-}\Lambda)) = 1 \text{ if } \mathrm{gldim}\,\Lambda = \infty \text{ and } \mathrm{KGdim}(\mathsf{D}(\mathsf{Mod}\text{-}\Lambda)) = 2 \text{ if } \mathrm{gldim}\,\Lambda < \infty.$$

We then use the correspondences in the diagram to obtain a description of the points of the Ziegler spectrum and its topology.

The indecomposable objects of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)$ were classified in [7] and the morphisms studied in [10]. This fits inside a more general classification, due to Bekkert and Merklen [3], of indecomposable complexes in the bounded derived category of a gentle algebra and a more general description, in [1], of morphisms between the indecomposable complexes. We build on these results, describing all the indecomposable complexes of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$. We show that every indecomposable is pure-injective, hence a point of the Ziegler spectrum of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

Theorem B. *Let Λ be a derived-discrete algebra. Then each indecomposable complex in $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ is both:*

(1) *a (possibly infinite) homotopy string complex;*
(2) *pure-injective.*

Moreover, the homotopy string complexes, of which we have a complete list, are all the indecomposable pure-injective objects of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

We also describe the Cantor–Bendixson stratification of the Ziegler spectrum, identifying the simple functors and the corresponding points relatively isolated by them. In the case of the derived-discrete algebras $\Lambda(n, n, 0)$, $n \geq 1$ the Ziegler spectrum had already been described by Z. Han [17] using covering theory. Our methods are very different, relying heavily on the description of morphisms in [1] and techniques around the Ziegler spectrum and the functor category. We will require ideas and results coming from various directions; some we summarise and for all we give references to where the details can be found.

In Section 1 we describe purity in a compactly generated triangulated category T. We then go on to define the Ziegler spectrum, $\mathrm{Zg}(\mathsf{T})$, of T, both internally and through the restricted Yoneda functor to the functor category on the compact objects of T. We also describe the corresponding two, anti-equivalent, categories of coherent functors. Localisation of abelian categories, Krull–Gabriel dimension, Cantor–Bendixson rank and the relations between these are briefly recalled in Section 2.

Section 3 reviews what we need on (homotopy) string complexes and the morphisms between them. In particular, we describe the structure of the Auslander-Reiten quiver of the string complexes and determine the Hom-hammocks of each string complex.

In Section 4 we consider the triangulated category $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ where Λ is a derived-discrete algebra. We first show that the Krull–Gabriel dimension is defined, then determine the simple functors in each quotient category obtained from the Krull–Gabriel filtration, and then identify the indecomposable complex relatively isolated by each such functor.

By that point we have essentially complete information on the topology of the Ziegler spectrum of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ and the remainder of the paper is devoted to showing that there are no more indecomposable objects in $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

Our proof of that is structured around the Cantor–Bendixson analysis of the Ziegler spectrum. After some preliminary results in Section 5, we treat the infinite global dimension case in Section 6, this being simplified by there being unbounded, CB-rank 1, compact objects. We have to work harder in the finite global dimension case, in Section 7, where an analysis of the structure of the indecomposables of CB-rank 1 allows us to complete the proof.

Convention. Throughout, Λ will be a derived-discrete algebra over an algebraically closed field $\mathbf{k}$ which is not piecewise hereditary of Dynkin type. The homotopy category of (not necessarily bounded) complexes of projective left Λ-modules will be denoted by the shorthand $\mathsf{K} := \mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

1. The Ziegler spectrum of a compactly generated triangulated category

Throughout this section T will be a compactly generated triangulated category with set-indexed coproducts; we shall denote the suspension functor by $\Sigma\colon \mathsf{T} \to \mathsf{T}$. For the convenience of the reader we briefly recall the definition here. A standard reference for this material is [29].

An object $C \in \mathsf{T}$ is *compact* if, for every set $\{D_i \mid i \in I\}$ of objects in T, the canonical morphism

$$\mathrm{Hom}_{\mathsf{T}}(C, \bigoplus_{i\in I} D_i) \to \bigoplus_{i\in I} \mathrm{Hom}_{\mathsf{T}}(C, D_i)$$

is an isomorphism. We will denote the full (triangulated) subcategory of compact objects in T by T^{c}. We say that T is *compactly generated* if T^{c} is skeletally small and, for every object $0 \neq D$ in T, there exists a nonzero morphism $C \to D$ for some $C \in \mathsf{T}^{\mathrm{c}}$.

1.1. Purity in a compactly generated triangulated category

Notions of purity and the Ziegler spectrum were first developed in the context of categories of modules. They were extended to compactly generated triangulated categories T by various authors, in particular in [25] (see also [4] and Neeman's review [28] of [5] by Benson and Gnacadja). For an overview of purity in compactly generated triangulated categories we refer to [32, Ch. 17].

Consider the category of contravariant functors from T^{c} to the category Ab of abelian groups:

$$\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}} = ((\mathsf{T}^{\mathrm{c}})^{\mathrm{op}}, \mathsf{Ab}).$$

Definition 1.1. Let M, N be objects of T and $f\colon M \to N$ a morphism in T. The *restricted Yoneda functor* $Y\colon \mathsf{T} \to \mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is defined by

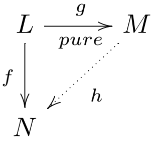
$$N \mapsto \mathrm{Hom}_{\mathsf{T}}(-, N)|_{\mathsf{T}^{\mathrm{c}}} \quad \text{and} \quad Y(f) = \mathrm{Hom}_{\mathsf{T}}(-, f)|_{\mathsf{T}^{\mathrm{c}}}.$$

Convention. Unless otherwise specified, the shorthand $(-, N)$ stands for the restricted *contravariant* functor $\mathrm{Hom}_{\mathsf{T}}(-, N)|_{\mathsf{T}^{\mathrm{c}}}$. However, the shorthand $(N, -)$ will be used for the *unrestricted covariant* functor $\mathrm{Hom}_{\mathsf{T}}(N, -)$, except briefly in Theorem 1.10 and Remark 1.11. Similarly for the action on morphisms.

A morphism $f\colon M \to N$ in T is a *pure-monomorphism* if $(-, f)$ is a monomorphism in $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$. We will take the following equivalent statements as the definition of a *pure-injective object* in T.

Proposition 1.2 *([25, §1.4]). Let T be a compactly generated triangulated category and let N be an object of T. Then the following statements are equivalent:*

(1) *The functor $(-, N)$ is an injective object of $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$.*
(2) *For every $M \in \mathsf{T}$ the induced morphism $(M, N) \to \big((-, M), (-, N)\big)$ is an isomorphism.*
(3) *The object N is injective over all pure-monomorphisms. In other words, for every pure-monomorphism $g\colon L \to M$ and every morphism $f\colon L \to N$, there exists a morphism $h\colon M \to N$ such that the following diagram commutes:*



It then follows (see [25, Cor. 1.9]) that the restricted Yoneda functor induces an equivalence of categories

$$\mathsf{Pinj}(\mathsf{T}) \simeq \mathsf{Inj}(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}).$$

where $\mathsf{Pinj}(\mathsf{T})$ is the full subcategory of pure-injective objects of T and $\mathsf{Inj}(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})$ is the full subcategory of injective objects in $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$.

1.2. The Ziegler spectrum of T

To define the compact open sets in the Ziegler spectrum, we require the notion of coherent functors from T to Ab. A functor $F\colon \mathsf{T} \to \mathsf{Ab}$ is *coherent* if there exists an exact sequence

$$(D, -) \to (C, -) \to F \to 0$$

for some $C, D \in \mathsf{T}^c$. The category of coherent functors from T to Ab will be denoted $\mathsf{Coh}(\mathsf{T})$.

Definition 1.3. The *Ziegler spectrum* $\mathrm{Zg}(\mathsf{T})$ of T is a topological space whose points are the isomorphism classes of indecomposable pure-injective objects in T. For each $F \in \mathsf{Coh}(\mathsf{T})$, we define the set

$$(F) := \{N \in \mathrm{Zg}(\mathsf{T}) \mid F(N) \neq 0\}.$$

These sets form a basis of (compact) open sets.

Definition 1.4. A full subcategory $\mathcal{D}$ of T is *definable* if it is of the form

$$\mathcal{D} = \{M \in \mathsf{T} \mid F_i(M) = 0\}$$

for some set $\{F_i \mid i \in I\}$ of coherent functors.

For a collection of objects $\mathsf{C} \subseteq \mathsf{T}$ the *definable subcategory generated by* C is

$$\langle \mathsf{C} \rangle := \{M \in \mathsf{T} \mid F \in \mathsf{Coh}(\mathsf{T}) \text{ and } F(C) = 0 \text{ for all } C \in \mathsf{C} \Longrightarrow F(M) = 0\}.$$

In particular, a subcategory $\mathcal{X}$ is definable if and only if $\langle \mathcal{X} \rangle = \mathcal{X}$.

In [26, §7] Krause shows that there is a natural bijection between the definable subcategories of T and the closed subsets of $\mathrm{Zg}(\mathsf{T})$. As a result, each closed subset of $\mathrm{Zg}(\mathsf{T})$ has the form $\mathcal{D} \cap \mathrm{Zg}(\mathsf{T})$ for some definable subcategory $\mathcal{D}$ of T.

1.3. The Ziegler spectrum of $\mathsf{Mod}\text{-}\mathsf{T}^c$

Since $\mathsf{Mod}\text{-}\mathsf{T}^c$ is a locally finitely presented abelian category we can make use of the corresponding definitions of purity, Ziegler spectrum and definable subcategory. Although T is not locally finitely presented, nor even necessarily finitely accessible, these definitions take exactly the same form for a compactly generated triangulated category as they do in those other cases; so the reader may obtain the corresponding definitions for $\mathsf{Mod}\text{-}\mathsf{T}^c$ by replacing T by $\mathsf{Mod}\text{-}\mathsf{T}^c$ and T^c by $\mathsf{mod}\text{-}\mathsf{T}^c$, where the last is the full subcategory of finitely presented functors.

Definition 1.5. A functor $F \in \mathsf{Mod}\text{-}\mathsf{T}^c$ is *absolutely pure* if every embedding $F \to G$ in $\mathsf{Mod}\text{-}\mathsf{T}^c$ is a pure-monomorphism. We say that F is *fp-injective* if it is injective over all embeddings $\tau\colon G \to H$ such that $\operatorname{coker} \tau$ is finitely presented.

Remark 1.6. Every functor in $\mathsf{Mod}\text{-}\mathsf{T}^c$ of the form $(-, M)$ for $M \in \mathsf{T}$ is absolutely pure; see [25, Lem. 1.6].

Proposition 1.7 *([32, Prop. 2.3.1]). For $F \in \mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$, the following are equivalent*

(1) *F is absolutely pure;*
(2) *F is fp-injective;*
(3) $\mathrm{Ext}^1(G, F) = 0$ *for each $G \in \mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$.*

Definition 1.5 and Proposition 1.7 work when T^{c} is replaced with any skeletally small preadditive category but since we are considering a triangulated category we have the following additional characterisation of absolutely pure objects, see [25, Lem. 2.7].

Proposition 1.8. *A functor $F \in \mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is absolutely pure if and only if it is flat. In particular, every absolutely pure functor F is a direct limit of representable functors.*

We will denote the full subcategory of $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ consisting of the absolutely pure objects by $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}$. Because $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is locally coherent, this is a definable subcategory (see [32, Thm. 3.4.24]), so we let $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ denote the closed subset $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}} \cap \mathrm{Zg}(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})$ with the topology induced from $\mathrm{Zg}(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})$.

1.4. The spaces $\mathrm{Zg}(\mathsf{T})$ and $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ are homeomorphic

In order to prove the main theorem of this section, we will need the following result.

Lemma 1.9 *([26, Lem. 7.2]). Let T be a compactly generated triangulated category. Then there is an equivalence of categories*

$$(\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Coh}(\mathsf{T}) \text{ given by } F \mapsto F^{\vee},$$

which is defined on objects $M \in \mathsf{T}$ by $F^{\vee}(M) := (F, (-, M))$.

Theorem 1.10. *The assignment $N \mapsto (-, N)$ defines a homeomorphism between topological spaces $\mathrm{Zg}(\mathsf{T})$ and $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$.*

Proof. The points of $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ are exactly the (isomorphism classes of) indecomposable injective objects in $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$. It therefore follows from the comment after Proposition 1.2 that the assignment $N \mapsto (-, N)$ is a bijection between the points of $\mathrm{Zg}(\mathsf{T})$ and $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$. It remains to show that this bijection is a homeomorphism.

We consider the topology on $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$. The compact open sets of $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ are, as in Definition 1.3, those of the form (G) where G is an object from a particular localisation of $(\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}, \mathsf{Ab})^{\mathrm{fp}}$ which is a category of functors on $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}$ and which we will denote by $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$; see [32, §12.3], especially [32, Prop. 12.3.21], for details. It is also the case that $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}$ is the category of exact functors on $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ and that $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$ is determined by $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}$ up to natural equivalence, see [32, §18.1.2]. So, by [33, Prop. 7.2] we have that $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}) \simeq (\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}})^{\mathrm{op}}$ and this equivalence allows us to describe the open sets of $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^c)$ directly in terms of isomorphism classes of objects from $\mathsf{mod}\text{-}\mathsf{T}^c$. One can also check that the equivalence $(\mathsf{mod}\text{-}\mathsf{T}^c)^{\mathrm{op}} \simeq \mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^c)$ takes an object $A \in (\mathsf{mod}\text{-}\mathsf{T}^c)^{\mathrm{op}}$ to the image of the representable functor $(A, -) \in (\mathsf{mod}\text{-}\mathsf{T}^c, \mathsf{Ab})^{\mathrm{fp}}$ in the localisation $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^c)$ (this follows from the comments before [33, Prop. 7.1] and the fact that $(A, -)$ is the dual of the functor $A \otimes_{\mathsf{T}^c} -$). It follows that the compact open sets are those of the form

$$(A) = \{(-, N) \in \mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^c) \mid (A, (-, N)) \neq 0\}$$

for $A \in \mathsf{mod}\text{-}\mathsf{T}^c$.

The equivalence in Lemma 1.9 gives us that, for each $F \in \mathsf{Coh}(\mathsf{T})$, there exists some $A \in \mathsf{mod}\text{-}\mathsf{T}^c$ such that $A^\vee = F$ and the corresponding compact open set in $\mathrm{Zg}(\mathsf{T})$ is

$$(F) := \{N \in \mathrm{Zg}(\mathsf{T}) \mid F(N) \neq 0\} = \{N \in \mathrm{Zg}(\mathsf{T}) \mid (A, (-, N)) \neq 0\}.$$

Thus $N \mapsto (-, N)$ defines a homeomorphism and we complete the proof. □

Remark 1.11. We recall how to define the unique extension, $\overrightarrow{F}$, of any functor $F \in (\mathsf{mod}\text{-}\mathsf{T}^c, \mathsf{Ab})^{\mathrm{fp}}$ to a functor from $\mathsf{Mod}\text{-}\mathsf{T}^c$ to Ab that commutes with direct limits. Let $M \in \mathsf{Mod}\text{-}\mathsf{T}^c$, then M can be written as a direct limit $M = \varinjlim M_\lambda$ with $M_\lambda \in \mathsf{mod}\text{-}\mathsf{T}^c$. Then

$$\overrightarrow{F}(M) := \varinjlim F(M_\lambda).$$

In order to define the sets (A) as above, one should note that the extension of $(A, -)$ from a functor on $\mathsf{mod}\text{-}\mathsf{T}^c$ to one on $\mathsf{Mod}\text{-}\mathsf{T}^c$ is still, since A is finitely presented, the functor $(A, -)$.

Recall that a typical object G in $\mathsf{Coh}(\mathsf{T})$ has a presentation $(D, -) \to (C, -) \to G \to 0$ where $C, D \in \mathsf{T}^c$. Since the Yoneda functor is full, there must be some $f : C \to D$ such that $G = \mathrm{coker}(f, -)$; then we will denote G by F_f. It will be useful to be able to explicitly describe the form of $F \in \mathsf{mod}\text{-}\mathsf{T}^c$ when $F^\vee = F_f$.

Corollary 1.12. *Consider the equivalence of Lemma 1.9 above. Suppose $F^\vee = F_f$ for some $f : C \to D$ in T^c. Then F is the kernel of the image of f under the restricted Yoneda functor, i.e. there is an exact sequence*

$$0 \longrightarrow F \longrightarrow (-, C) \xrightarrow{(-,f)} (-, D).$$

In particular, every finitely presented T^c-module is the kernel of a morphism between finitely generated projective T^c-modules.

Proof. Suppose, under the equivalence of Lemma 1.9, $F^\vee = F_f$ for some $f\colon C \to D$ in T^{c}. Note that $(-,D)^\vee = (D,-)$ whenever D is a compact object since $((-,D),(-,M)) \cong (D,M)$ for any M in T. We can rewrite the exact sequence

$$(D,-) \overset{(f,-)}{\longrightarrow} (C,-) \longrightarrow F_f \longrightarrow 0 \;\text{ as }\; (-,D)^\vee \overset{(-,f)^\vee}{\longrightarrow} (-,C)^\vee \longrightarrow F^\vee \longrightarrow 0$$

and so $0 \longrightarrow F \longrightarrow (-,C) \overset{(-,f)}{\longrightarrow} (-,D)$ is also exact. □

2. Cantor–Bendixson rank and Krull–Gabriel dimension

2.1. Cantor–Bendixson rank of a topological space

The following material is classical; see, for example, [32, §5.3.6]. Let T be a topological space, for instance $\mathrm{Zg}(\mathsf{T})$. A point $p \in T$ is *isolated* if $\{p\}$ is an open set and such points are assigned *Cantor–Bendixson rank* (or *CB-rank*) 0. The first *Cantor–Bendixson derivative* T' of T is the closed subset of T containing all the non-isolated points of T. With respect to the relative topology, T' itself may have isolated points, which are said to have CB rank 1. One continues this process inductively to obtain points of CB rank n for any $n \in \mathbb{N}$ and we write $T^{(n)} = (T^{(n-1)})'$, with $T = T^{(0)}$. If at some point $T^{(n+1)} = \emptyset$ but $T^{(n)} \neq \emptyset$ then we say that T has *CB rank n*.

Remark 2.1. The process above can be continued transfinitely. Since the topological spaces we consider turn out to have finite CB rank, we ignore this.

2.2. Localisation at Serre subcategories

A full subcategory S of an abelian category C is a *Serre subcategory* if, for every exact sequence $0 \to A \to B \to C \to 0$ in C, we have $A, C \in \mathsf{S}$ if and only if $B \in \mathsf{S}$.

For a Serre subcategory $\mathsf{S} \subseteq \mathsf{C}$, the quotient category C/S has the same objects as C and morphisms are given by

$$\mathrm{Hom}_{\mathsf{C}/\mathsf{S}}(A,B) := \varinjlim_{A/A' \in \mathsf{S}, B' \in \mathsf{S}} \mathrm{Hom}_{\mathsf{C}}(A', B/B').$$

2.3. Localisation with respect to hereditary torsion theories

For background on (hereditary) torsion theories and localisation with respect to torsion classes we refer the reader to [35, Ch. VI]; see also [32, Ch. 11]. We remind the reader of some key points that we will need here.

Let C be a Grothendieck abelian category. A *torsion class* A is a full subcategory C that is closed under quotients, extensions, direct sums and summands; objects of A are said to be *torsion*. A torsion class is *hereditary* if it is also closed under subobjects. It determines a *torsion pair* (A,B), where $\mathsf{B} = \mathsf{A}^{\perp} := \{B \in \mathsf{C} \mid \mathrm{Hom}(\mathsf{A}, B) = 0\}$. An object of B is said to be *torsionfree*. A torsion class comes equipped with a subfunctor t of the identity functor on C which sends an object C to $t(C)$, the largest torsion subobject of C. If the functor t commutes with direct limits, (A,B) is said to be of *finite type*.

A hereditary torsion theory determines, as in the subsection above, a *localisation functor*, $q\colon \mathsf{C} \to \mathsf{C}/\mathsf{A}$, which enjoys the following properties:

(1) q has a right adjoint $i\colon \mathsf{C}/\mathsf{A} \to \mathsf{C}$ which is a full embedding;
(2) if E is a torsionfree injective object then $E \cong i \circ q(E)$.

The composition $i \circ q$ is given (on objects) by

$$C \mapsto \pi^{-1}(t(E(C')/C')),$$

where $C' := C/tC$, $E(C')$ is the injective hull of C' and $\pi = \mathrm{coker}(C' \hookrightarrow E(C'))$.

Since $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is locally coherent, $\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is abelian and the $\varinjlim$-closure of each Serre subcategory of $\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$ is a hereditary finite-type torsion class.

Remark 2.2. We will also use the facts (see [19, Prop. 3.10], respectively [32, 11.1.31]) that if (A,B) is a torsion pair of finite type, if C is locally coherent and if C is a torsionfree absolutely pure object of C then $C \cong i \circ q(C)$ and $q(C)$ is an absolutely pure object of C/A.

2.4. Localisation with respect to definable subcategories

We refer to [32, §12.3, §12.4] for details of the following.

We saw in the proof of Theorem 1.10 that $\mathsf{fun}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}) \simeq (\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}})^{\mathrm{op}}$. It follows that each definable subcategory $\mathcal{D}$ of $\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}}$ is determined by the Serre subcategory

$$\mathsf{S}(\mathcal{D}) := \{F \in \mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}} \mid (F, D) = 0 \text{ for all } D \in \mathcal{D}\}$$

of $\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$ which then generates a hereditary, finite-type, torsion subclass in $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$

$$\mathsf{A}(\mathcal{D}) = \{A \in \mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}} \mid (A, D) = 0 \text{ for all } D \in \mathcal{D}\}.$$

We shall denote the corresponding localisation of $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$ by $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\mathcal{D}}$.

By [32, Prop. 11.1.29], the torsion theory $(\mathsf{A}(\mathcal{D}), \mathsf{B}(\mathcal{D}))$ is cogenerated by its indecomposable torsionfree injective objects, i.e. the set $\mathbf{X} := \mathcal{D} \cap \mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$, which is a closed subset of $\mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$. By abuse of notation we shall often write $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\mathbf{X}}$ for the localisation $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\mathcal{D}}$.

The category of finitely presented objects in $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\mathcal{D}}$ and the quotient of $\mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$ by $\mathsf{S}(\mathcal{D})$ coincide ([32, Cor. 11.1.34]). Again, by abuse of notation we denote this localisation by $(\mathsf{mod}\text{-}\mathsf{T}^c)_{\mathcal{D}}$ or $(\mathsf{mod}\text{-}\mathsf{T}^c)_{\mathbf{X}}$. Since $\mathsf{Coh}(\mathsf{T}) \simeq (\mathsf{mod}\text{-}\mathsf{T}^c)^{\mathrm{op}}$ by Lemma 1.9, we use similar notation for the corresponding localisations of $\mathsf{Coh}(\mathsf{T})$.

Given the homeomorphism $\mathrm{Zg}(\mathsf{T}) \cong \mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^c)$ of Theorem 1.10, there are parallel definitions and notations for localisations of $\mathsf{Coh}(\mathsf{T})$ at definable subcategories $\mathcal{D}$ of T and closed subsets $\mathbf{X} \subseteq \mathrm{Zg}(\mathsf{T})$.

2.5. Krull–Gabriel dimension

Recall from [14, Def. 2.1] that a *Krull–Gabriel filtration* of a (skeletally) small abelian category C consists of a properly increasing filtration by Serre subcategories

$$0 = \mathsf{C}_{-1} \subsetneq \mathsf{C}_0 \subsetneq \mathsf{C}_1 \subsetneq \cdots \subsetneq \mathsf{C}_n = \mathsf{C},$$

where, for each i, $\mathsf{C}_i/\mathsf{C}_{i-1}$ is the full subcategory of all objects of finite length in $\mathsf{C}/\mathsf{C}_{i-1}$. If there is such a filtration then the *Krull–Gabriel dimension* (or *KG-dimension*), $\mathrm{KGdim}(\mathsf{C})$, of C is n. For each i, we denote the localisation functor by $q_i \colon \mathsf{C} \to \mathsf{C}/\mathsf{C}_i$.

By abuse of notation, in the case of a compactly generated triangulated category T we shall write $\mathrm{KGdim}(\mathsf{T})$ for $\mathrm{KGdim}(\mathsf{Coh}(\mathsf{T}))$ and call it the *KG-dimension of* T.

Remark 2.3. More generally, ordinal-indexed Krull–Gabriel filtrations may be considered giving, as with CB-rank, an ordinal-valued KG-dimension. In the case that there is no such filtration, the KG-dimension is undefined.

Let L be a modular lattice and let L' be the quotient modular lattice obtained by collapsing all finite length intervals in L. Starting with $L_{-1} := L$ and defining $L_n = (L_{n-1})'$ for $n \in \mathbb{N}$, we define the *m-dimension* of L to be the least n such that $L_n = 0$. We recall the following lemma, see for example [23, Lem. 1.1], where $L_{\mathsf{C}}(X)$ denotes the lattice of subobjects of an object X of an abelian category C.

Lemma 2.4. *Let C be an abelian category and $X \in \mathsf{C}$ an object. Then q_i induces a natural isomorphism $L_{\mathsf{C}}(X)_i \cong L_{\mathsf{C}/\mathsf{C}_i}(q_i X)$ for all i.*

Remark 2.5. We highlight the following consequences of Lemma 2.4.

(1) For $X \neq 0$ the m-dimension of $L_{\mathsf{C}}(X)$ is n if and only if $q_{n-1}(X) \neq 0$ and has finite length in C/C_n.
(2) The Krull–Gabriel dimension of C is defined if and only if for each $X \in \mathsf{C}$ no subset of the lattice $L_{\mathsf{C}}(X)$ forms a densely ordered chain (see, for example, [32, Prop. 7.2.3, Prop. 13.2.1]).

Definition 2.6. The Ziegler spectrum of T satisfies the *isolation condition* if for every closed subset $\mathbf{X} \subseteq \mathrm{Zg}(\mathsf{T})$ and every point $M \in \mathbf{X}$ isolated in the relative topology on $\mathbf{X}$, there is a coherent functor F such that $(F) \cap \mathbf{X} = \{M\}$ and such that the image of F is simple in the localisation $\mathsf{Coh}(\mathsf{T})_{\mathbf{X}}$.

We have the following link between KG-dimension and the isolation condition, which we follow with the link between the CB-rank and KG-dimension.

Lemma 2.7 *([37, Lem. 8.11]). For a compactly generated triangulated category* T*, if* $\mathrm{KGdim}(\mathsf{T})$ *is defined, then* $\mathrm{Zg}(\mathsf{T})$ *satisfies the isolation condition.*

Proposition 2.8 *([37, Thm. 8.6], see also [31, Prop. 10.19]). Suppose* $\mathrm{Zg}(\mathsf{T})$ *satisfies the isolation condition. Then:*

(1) $\mathsf{Coh}(\mathsf{T})/\mathsf{Coh}(\mathsf{T})_i = \mathsf{Coh}(\mathsf{T})_{\mathbf{X}_i}$*, where* $\mathbf{X}_i$ *is the closed subset of* $\mathrm{Zg}(\mathsf{T})$ *containing the points of* $\mathrm{CB}\text{-rank} \geq i$*;*
(2) *there is a bijection between points* N *of CB-rank* i *and simple functors* F *in* $\mathsf{Coh}(\mathsf{T})_{\mathbf{X}_i}$ *given by* $(F') \cap \mathbf{X}_i = \{N\}$ *where* $F' \in \mathsf{Coh}(\mathsf{T})$ *is such that* $q_i F' = F$*.*

Here we are using the homeomorphism of Theorem 1.10 and the fact that the original result, proved for modules over rings, holds equally true for modules over small preadditive categories (the development of the theory in [32] accommodates this level of generality).

Corollary 2.9. *Suppose* T *is a compactly generated triangulated category. For an indecomposable pure-injective object* M *of* T *we have*

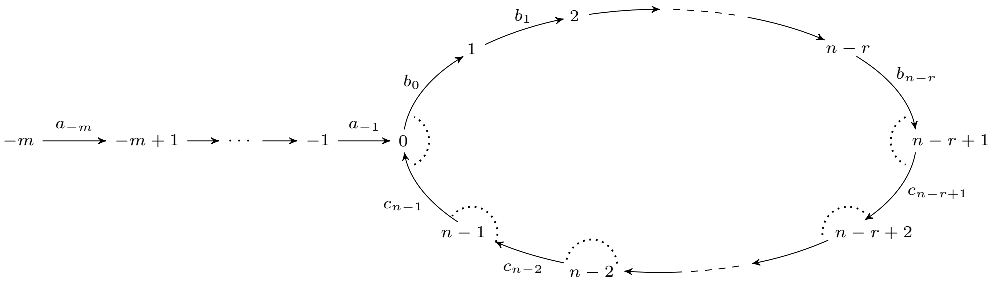
$$\mathrm{CB}\text{-rank} M \ \textit{in}\ \mathrm{Zg}(\mathsf{T}) = \mathrm{CB}\text{-rank}(-, M) \ \textit{in}\ \mathrm{Zg}(\mathsf{Abs}\text{-}\mathsf{T}^{\mathrm{c}})$$

If $\mathrm{Zg}(\mathsf{T})$ *satisfies the isolation condition then* $\mathrm{KGdim}(\mathsf{T}) = \mathrm{CB}\text{-rank}\mathrm{Zg}(\mathsf{T})$*, that is, if one is defined, then so is the other and their values are equal.*

Proof. This follows immediately from Theorem 1.10 and Proposition 2.8. □

3. The structure of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ for derived-discrete algebras

Let Λ be a derived-discrete algebra. In [7, Thm. A], Bobiński, Geiß and Skowroński obtained a canonical representative $\Lambda(r, n, m)$, with $1 \leq r \leq n$ and $m \geq 0$, given by the bound quiver $Q(r, n, m)$ below, for each derived-equivalence class of derived-discrete algebra.



In [7, Thm. B], the authors then used the triple (r, n, m) to describe the structure of the Auslander-Reiten (AR) quiver of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda(r, n, m))$. By [20], $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)^c \simeq \mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)$, and since $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$ is an algebraic triangulated category it is uniquely determined by its compact objects up to equivalence by results of, for example, Keller [21] and [22, Thm. 3.8]. Thus, from now on, we shall identify each derived-discrete algebra, Λ, with its representative $\Lambda(r, n, m)$ in the derived equivalence class and use this algebra to describe the structure of $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$.

3.1. String complexes

Since Λ is a gentle algebra the indecomposable complexes of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda) \simeq \mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda)$ can be described in terms of so-called *(homotopy) string complexes*; [3] (also [11]), and [6] for the terminology. In this section we describe a generalisation to string complexes with possibly unbounded (but still degreewise finite-dimensional) cohomology. We refer to [1, §2] for a summary of [3] using the terminology and notation of this article.

Remark 3.1. The main theorem of [3] asserts that the indecomposable complexes of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)$ are precisely the isoclasses of string and band complexes. However, the absence of band complexes is precisely the reason for discreteness, and we therefore make no further reference to band complexes.

We begin with a description of the string complexes in $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)$, which is an extension of [1, Lem. 7.1]; see also [16, Lem. 3]. The proof of the following lemma is a straightforward exercise in the combinatorics of the quiver of $\Lambda(r, n, m)$.

Lemma 3.2. *Each homotopy string of $\Lambda(r, n, m)$ is a (shifted) copy of a subword of w_l for $l \geq 0$ and w_∞, ${}_\infty w$ and ${}_\infty w_\infty$ or its inverse:*

$$w_l: \quad \bullet \xleftarrow{a_{-1}\cdots a_{-m}} \circ \xrightarrow{v_l} \circ \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_{n-r+1}} \bullet \xrightarrow[b_{n-r}\cdots b_0 a_{-1}\cdots a_{-m}]{} \bullet$$

$$w_\infty: \quad \bullet \xleftarrow{a_{-1}\cdots a_{-m}} \circ \xrightarrow{v} \circ \xrightarrow{v} \circ \xrightarrow{v} \circ \cdots\cdots>$$

$${}_\infty w: \quad \cdots\cdots> \circ \xrightarrow{v} \circ \xrightarrow{v} \circ \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_{n-r+1}} \bullet \xrightarrow[b_{n-r}\cdots b_0 a_{-1}\cdots a_{-m}]{} \bullet$$

$${}_\infty w_\infty: \quad \cdots\cdots> \circ \xrightarrow{v} \circ \xrightarrow{v} \circ \xrightarrow{v} \circ \cdots\cdots> ,$$

where v_l is the l-fold concatenation of $v\colon \bullet \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_{n-r+1}} \bullet \xrightarrow{b_{n-r}\cdots b_0} \bullet$ *. Note that for $\Lambda(n,n,m)$ we have $b_0 = c_0$ in our labelling convention. In the following we shall use the notation v_∞ and ${}_\infty v$ for the obvious one-sided infinite concatenations of v.*

The passage from a homotopy string w to the corresponding string complex P_w is first described in [3,6]. We follow the notation in [1, §2].

Definition 3.3. Let P_w be a string complex for $\Lambda = \Lambda(r,n,m)$. We say that P_w is

perfect if it is any shift of a string complex of a subword of w_l for $l \geq 0$;
left-infinite if it is any shift of a string complex of a left-infinite subword of ${}_\infty w$;
right-infinite if it is any shift of a string complex of a right-infinite subword of w_∞;
one-sided if it is left- or right-infinite;
two-sided if it is any shift of the string complex ${}_\infty w_\infty$.

We shall denote the set of all string complexes of Λ by $\mathsf{Str}\text{-}\Lambda$.

We have the following global statement regarding the dimensions of Hom-spaces between string complexes; cf. [10, Thm. 6.1] and [1, Thm. 7.4].

Proposition 3.4. *Let Λ be a derived-discrete algebra. Then, for $A, B \in \mathsf{Str}\text{-}\Lambda$, we have*

$$\dim \mathrm{Hom}_{\mathsf{K}}(A,B) \leq \begin{cases} 1 & \text{for } r > 1, \\ 2 & \text{for } r = 1. \end{cases}$$

Moreover, if one of $A, B \in \mathsf{Str}\text{-}\Lambda$ is nonperfect then $\dim \mathrm{Hom}_{\mathsf{K}}(A,B) \leq 1$.

Proof. For this we apply [1, Thm. 3.15] using arguments analogous to those in [1, §7]. □

To see which string complexes admit 2-dimensional Hom-spaces between them, we refer to [10, Prop. 6.2].

Corollary 3.5. *Let Λ be a derived-discrete algebra. Then the objects of* $\mathsf{Str}\text{-}\Lambda$ *are indecomposable.*

Proof. The perfect string complexes are known to be indecomposable by [3]. For any other string complex A we have $\mathrm{Hom}_{\mathsf{K}}(A, A) = \mathbf{k}$, whence A is indecomposable. □

3.2. Compact objects of K

To show that string complexes are pure-injective, we first need to identify the compact objects of K. This is a special case of a construction of Jørgensen [20]; see also [30] for a strengthening of this result.

Proposition 3.6. *Let $\Lambda = \Lambda(r, n, m)$ be a derived-discrete algebra.*

(1) *If $r \neq n$ then an indecomposable complex in* K *is compact if and only if it is a perfect string complex.*
(2) *If $r = n$ then an indecomposable complex in* K *is compact if and only if it is a perfect string complex or a right-infinite string complex.*

Proof. We follow Jørgensen's construction from [20]. Write $(-)^* = \mathrm{Hom}_\Lambda(-, \Lambda)$ for the standard duality with respect to Λ. If M is a finite-dimensional left Λ-module then M^* is a finite-dimensional right Λ-module. Let P_M be a projective resolution of M^*. Note that $P_M \in \mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda^{\mathrm{op}})$ and is quasi-isomorphic to M^* in $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda^{\mathrm{op}})$. We also consider the complex $P_M^* \in \mathsf{K}^{+,b}(\mathsf{proj}\text{-}\Lambda)$, and note that both P_M and P_M^* depend functorially on M in $\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda^{\mathrm{op}})$ and $\mathsf{K}^{+,b}(\mathsf{proj}\text{-}\Lambda)$, respectively; see [20, Construction 1.2].

Let $\mathsf{G} := \{\Sigma^i P_M^* \mid M \in \mathsf{mod}\text{-}\Lambda, i \in \mathbb{Z}\}$. [20, Thm. 2.4] asserts that G is a set of compact generators for K. Now consider the following two subcategories of K and $\mathsf{K}(\mathsf{Proj}\text{-}\Lambda^{\mathrm{op}})$, where the notation indicates the thick subcategory generated by a set of objects:

$$\mathsf{C} := \mathsf{thick}_{\mathsf{K}}(\mathsf{G}) \quad \text{and} \quad \mathsf{D} := \mathsf{thick}_{\mathsf{K}(\mathsf{Proj}\text{-}\Lambda^{\mathrm{op}})}(G^* \mid G \in \mathsf{G}).$$

Note that $\mathsf{C} = \mathsf{K}^c$, since G is a set of compact generators [27, Thm. 2.1.3]. Now, the proof of [20, Thm. 3.2] asserts that

$$\mathsf{C} \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathsf{D}$$

are quasi-inverse equivalences between triangulated categories. In the proof of [20, Thm. 3.2] it is shown that

$$\mathsf{D} = \mathsf{thick}_{\mathsf{K}(\mathsf{Proj}\text{-}\Lambda^{\mathrm{op}})}(P_N \mid P_N \text{ is a projective resolution of some } N \in \mathsf{mod}\text{-}\Lambda^{\mathrm{op}}).$$

In particular, this means that $\mathsf{D} = \mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda^{\mathrm{op}})$, the indecomposable complexes of which are precisely the perfect string complexes in $\mathsf{Str}\text{-}\Lambda^{\mathrm{op}}$ as well as any infinite string complexes arising from resolvable strings (in the terminology of [1]). For Λ derived-discrete, the image of the set of such string complexes in $\mathsf{Str}\text{-}\Lambda^{\mathrm{op}}$ under the equivalence $(-)^*$ is precisely the set of string complexes described in the proposition. □

Recall from [13,24] that an object N of a general compactly generated triangulated category is *endofinite* if for each $C \in \mathsf{T}^{\mathrm{c}}$ the $\mathrm{End}_{\mathsf{T}}(N)$-module $\mathrm{Hom}_{\mathsf{T}}(C, N)$ is of finite length.

Lemma 3.7 *([24, Thm. 1.2]). An endofinite object of a compactly generated triangulated category is pure-injective.*

Putting Propositions 3.4, 3.6 and Corollary 3.5 together with Lemma 3.7 gives:

Corollary 3.8. *Let Λ be a derived-discrete algebra. Then any string complex is endofinite and hence is an indecomposable pure-injective object of* K.

We now know that the objects of $\mathsf{Str}\text{-}\Lambda$ are indecomposable pure-injective objects of K. In Section 4 we shall see that $\mathsf{Str}\text{-}\Lambda$ is a complete list of indecomposable pure-injective objects of K. However, in order to do this we need to describe the 'AR quiver' of the additive category $\mathsf{add}(\mathsf{Str}\text{-}\Lambda)$ and the shapes of the Hom-hammocks, which we do in the next subsections.

3.3. The AR quiver of $\mathsf{Str}\text{-}\Lambda$ *for* Λ *of finite global dimension*

The AR quiver of $\mathsf{Str}\text{-}\Lambda$ is computed by a straightforward calculation using Bobiński's algorithm [6] (also, [1, §6] avoiding the Happel functor and including pictures). It has $8r$ components: $3r$ components sit in $\mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$, for which we keep the notation and co-ordinate system of [7]. In particular, the components are denoted $\mathcal{X}^0, \dots, \mathcal{X}^{r-1}$, $\mathcal{Y}^0, \dots, \mathcal{Y}^{r-1}$, and $\mathcal{Z}^0, \dots, \mathcal{Z}^{r-1}$. The components $\mathcal{X}^k$ and $\mathcal{Y}^k$ are of type $\mathbb{Z}A_\infty$. The components $\mathcal{Z}^k$ are of type $\mathbb{Z}A_\infty^\infty$. For each $k = 0, \dots, r-1$, we label the indecomposable objects of $\mathcal{X}^k$, $\mathcal{Y}^k$ and $\mathcal{Z}^k$ as follows:

$$X_{ij}^k \in \mathcal{X}^k \text{ with } i, j \in \mathbb{Z}, j \geq i; \quad Y_{ij}^k \in \mathcal{Y}^k \text{ with } i, j \in \mathbb{Z}, j \leq i; \quad Z_{ij}^k \in \mathcal{Z}^k \text{ with } i, j \in \mathbb{Z}.$$

The remaining $5r$ components consist of $4r$ components of type A_∞^∞ and r components of type A_1. The components of type A_∞^∞ will be denoted by $\mathcal{X}_\infty^k$, $\mathcal{X}_{-\infty}^k$, $\mathcal{Y}_\infty^k$ and $\mathcal{Y}_{-\infty}^k$, and the components of type A_1 by $\mathcal{Z}_\infty^k$. For each $k = 0, \dots, r-1$ we label the indecomposable objects of these components as follows:

$$X_{i,\infty}^k \in \mathcal{X}_\infty^k \text{ with } i \in \mathbb{Z}, \quad X_{-\infty,i}^k \in \mathcal{X}_{-\infty}^k \text{ with } i \in \mathbb{Z}, \quad Z_\infty^k \in \mathcal{Z}_\infty^k$$
$$Y_{\infty,i}^k \in \mathcal{Y}_\infty^k \text{ with } i \in \mathbb{Z}, \quad Y_{i,-\infty}^k \in \mathcal{Y}_{-\infty}^k \text{ with } i \in \mathbb{Z}.$$

We introduce the following notation to encode the action of the suspension on indecomposable objects.

Notation 3.9. For k as above and $a \in \mathbb{Z} \cup \{-\infty, \infty\}$ we set

$$a' = \begin{cases} a+r+m & \text{if } k = r-1 \text{ and } a \in \mathbb{Z}; \\ a & \text{otherwise}, \end{cases} \qquad a'' = \begin{cases} a+r-n & \text{if } k = r-1 \text{ and } a \in \mathbb{Z}; \\ a & \text{otherwise}, \end{cases}$$

For $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{\infty\}$ and $0 \le k < r$, suspension acts as follows:

$$\Sigma X^k_{a,b} = X^{k+1}_{a',b'}, \quad \Sigma Y^k_{a,b} = Y^{k+1}_{a'',b''}, \quad \Sigma Z^k_{a,b} = Z^{k+1}_{a',b''} \quad \text{and} \quad \Sigma Z^k_{\infty} = Z^{k+1}_{\infty}.$$

The category $\mathsf{K}^b(\text{proj-}\Lambda)$ admits a Serre functor $\mathbb{S}\colon \mathsf{K}^b(\text{proj-}\Lambda) \to \mathsf{K}^b(\text{proj-}\Lambda)$; see [15,18] for details. For $a, b \in \mathbb{Z}$ and $0 \le k < r$, the Serre functor acts as follows:

$$\mathbb{S}X^k_{a,b} = X^{k+1}_{a'-1,b'-1}, \quad \mathbb{S}Y^k_{a,b} = Y^{k+1}_{a''-1,b''-1}, \quad \text{and} \quad \mathbb{S}Z^k_{a,b} = Z^{k+1}_{a'-1,b''-1}.$$

Using the notation in Lemma 3.2, we fix the co-ordinate system as in [7] by identifying an object from each component. Since the components are determined up to suspension, it is sufficient, together with the action described above, to specify one each from $\mathcal{X}^0$, $\mathcal{X}^0_\infty$, $\Sigma\mathcal{X}^0_{-\infty}$, $\mathcal{Y}^0$, $\mathcal{Y}^0_\infty$, $\Sigma\mathcal{Y}^0_{-\infty}$ and $\mathcal{Z}^0$. We freely identify homotopy strings with the corresponding homotopy string complexes.

- Fix $Z^0_{0,0}$ to be the string complex arising as a projective resolution of simple module $S(0)$, i.e. corresponding to the homotopy string b_0 with $P(0)$ sitting in degree zero.
- For the $\mathcal{X}$ components, we have

$$X^0_{0,0} = \begin{cases} a_{-1} & \text{if } m > 0; \\ P(0) & \text{if } m = 0, \end{cases}$$

  where in each case $P(0)$ sits in degree 0. Note that in the first case the string complex is Σ^{-1} of a projective resolution of $S(-1)$.
  The object $X^0_{0,\infty}$ is the left-infinite string complex given by ${}_\infty v$, with $P(0)$ sitting in degree 0 and all higher degrees are zero.
  We have

$$\Sigma X^0_{-\infty,-1} = \begin{cases} c_{n-2} \cdots c_{n-r+1} b_{n-r} \cdots b_0 v_\infty & \text{if } r > 1; \\ v_\infty & \text{if } r = 1, \end{cases}$$

  where in the first case $P(n-1)$ sits in degree 0 and all lower degrees are zero, and in the second case $P(0)$ sits in degree 0 and all lower degrees are zero.

- For the $\mathcal{Y}$ components note that we must have $n > r$. We fix

$$Y^0_{0,0} = b_0 c_{n-1} \cdots c_{n-r+1} b_{n-r},$$

with $P(1)$ sitting in degree -1, i.e. $\Sigma^r Y^0_{0,0}$ is a projective resolution of $S(n-r)$. The object $Y^0_{\infty,0}$ is the right-infinite string complex given by $b_0 v_\infty$, with $P(1)$ sitting in degree -1 and all lower degrees zero. The object $\Sigma Y^0_{-1,-\infty}$ is the left-infinite string complex given by ${}_\infty v c_{n-1} \cdots c_{n-r+1} b_{n-r} \cdots b_1$ with $P(1)$ sitting in degree -1 and all higher degrees zero.

With this assignment, the positions of each string complex in the co-ordinate system are uniquely determined such that we have the following 'extended rays' and 'extended corays'; cf. [7, §3] and [10, Properties 2.2(5)].

Lemma 3.10. *There are sequences of objects and morphisms in* $\mathsf{Str}\text{-}\Lambda$*:*

$$X^k_{ii} \to X^k_{i,i+1} \to \cdots \to X^k_{i,\infty} \to \cdots \to Z^k_{i,i-1} \to Z^k_{i,i} \to Z^k_{i,i+1} \to \cdots \to \Sigma X^k_{-\infty,i-1} \to \cdots \to \Sigma X^k_{i-2,i-1} \to \Sigma X^k_{i-1,i-1},$$

$$Y^k_{ii} \to Y^k_{i+1,i} \to \cdots \to Y^k_{\infty,i} \to \cdots \to Z^k_{i-1,i} \to Z^k_{i,i} \to Z^k_{i+1,i} \to \cdots \to \Sigma Y^k_{i-1,-\infty} \to \cdots \to \Sigma Y^k_{i-1,i-2} \to \Sigma Y^k_{i-1,i-1},$$

$$\cdots \to X^k_{-\infty,i-1} \to X^k_{-\infty,i} \to X^k_{-\infty,i+1} \to \cdots \to Z^k_\infty \to \cdots \to Y^k_{\infty,i-1} \to Y^k_{\infty,i} \to Y^k_{\infty,i+1} \to \cdots,$$

$$\cdots \to Y^k_{i-1,-\infty} \to Y^k_{i,-\infty} \to Y^k_{i+1,-\infty} \to \cdots \to Z^k_\infty \to \cdots \to X^k_{i-1,\infty} \to X^k_{i,\infty} \to X^k_{i+1,\infty} \to \cdots,$$

such that any composition is nonzero. Moreover, $X^k_{i,\infty}$ *and* $\Sigma X^k_{-\infty,i-1} = X^{k+1}_{-\infty,i'-1}$ *are the only infinite string complexes admitting nontrivial morphisms from* X^k_{ii}*. Likewise,* $Y^k_{\infty,i}$ *and* $\Sigma Y^k_{i-1,-\infty} = Y^{k+1}_{i''-1,-\infty}$ *are the only infinite string complexes admitting nontrivial morphisms from* Y^k_{ii}*.*

The sequences in Lemma 3.10 whose leftmost parts consist of objects of $\mathcal{X} \cup \mathcal{X}_{-\infty}$ will be called *extended rays* in $\mathsf{Str}\text{-}\Lambda$ and those whose leftmost parts consist of objects of $\mathcal{Y} \cup \mathcal{Y}_{-\infty}$ will be called *extended corays.*

Proof. This is an exercise in homotopy string combinatorics using Bobiński's algorithm from [6] or [1, §6] and the description of morphisms between string complexes given in [1, Thm. 3.15], cf. [1, §7] for a more concrete description in this setting, and [10, Appendix B] using classical string combinatorics. In particular, for the vanishing statements, the finitely many objects X^k_{ii} and Y^k_{ii} up to shift are identified (as string modules) in [7]. Taking projective resolutions, one obtains string complexes. Then using the combinatorial description of the string complexes in Lemma 3.2 and [1], one can show that there are precisely two infinite string complexes admitting nontrivial morphisms from complexes on the mouths of the $\mathcal{X}$ and $\mathcal{Y}$ components. □

We now observe the positions of the different kinds of string complex.

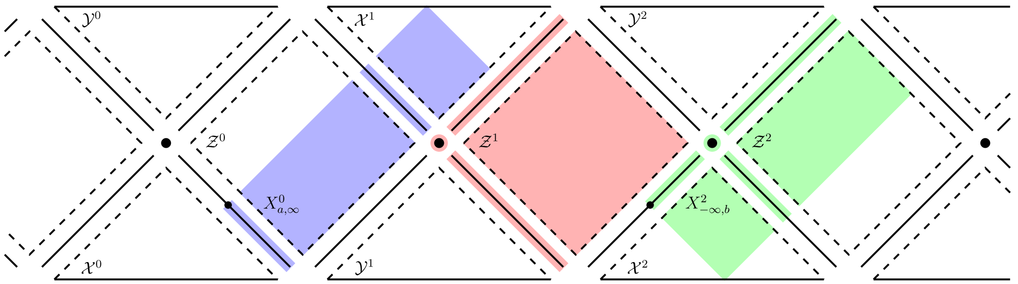


Fig. 1. The structure of $\mathsf{Str}\text{-}\Lambda$ when $\operatorname{gldim}\Lambda < \infty$. Triangular regions denote components of type $\mathbb{Z}A_\infty$ and diamond shaped regions those of type $\mathbb{Z}A_\infty^\infty$. Solid horizontal lines indicate boundaries or 'mouths' of the components. The solid line to the left of an $\mathcal{X}^k$ component is an $\mathcal{X}^k_{-\infty}$ component and to the right an $\mathcal{X}^k_\infty$ component; similarly for $\mathcal{Y}^k$, $\mathcal{Y}^k_{-\infty}$ and $\mathcal{Y}^k_\infty$. The $\mathcal{Z}^k_\infty$ components are indicated by the central dots to the left of the $\mathcal{Z}^k$ components. The shaded regions illustrate the Hom-hammocks for the various indecomposable objects indicated; cf. [10, §3].

Remark 3.11. Writing $\mathcal{X} := \mathsf{add}(\bigcup_{k=0}^{r-1} \mathcal{X}^k)$ etc., we have the following correspondences:

$$\mathsf{ind}(\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}) \xleftrightarrow{1-1} \{\text{perfect string complexes}\};$$
$$\mathsf{ind}(\mathcal{X}_\infty \cup \mathcal{Y}_{-\infty}) \xleftrightarrow{1-1} \{\text{left-infinite string complexes}\};$$
$$\mathsf{ind}(\mathcal{X}_{-\infty} \cup \mathcal{Y}_\infty) \xleftrightarrow{1-1} \{\text{right-infinite string complexes}\};$$
$$\mathsf{ind}(\mathcal{Z}_\infty) \xleftrightarrow{1-1} \{\text{two-sided string complexes}\}.$$

Moreover, in light of Proposition 3.6, when $\operatorname{gldim}\Lambda < \infty$ the compact indecomposable complexes in K are precisely those lying in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ components of $\mathsf{Str}\text{-}\Lambda$.

3.4. The AR quiver of $\mathsf{Str}\text{-}\Lambda$ *for* Λ *of infinite global dimension*

The AR quiver of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda) \simeq \mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda) \subset \mathsf{Str}\text{-}\Lambda$ has $2r$ components consisting of r components of type $\mathbb{Z}A_\infty$ denoted by $\mathcal{X}^0, \dots, \mathcal{X}^{r-1}$ and r components of type A_∞^∞ denoted by $\mathcal{Z}^0, \dots, \mathcal{Z}^{r-1}$. The full subcategory of perfect complexes $\mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$ is equal to $\mathsf{add}(\mathcal{X})$. We sketch the structure of $\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda)$ in Fig. 2; the co-ordinate system is that of [7]. For each $k = 0, \dots, r = n$, we label the indecomposable objects of $\mathcal{X}^k$ and $\mathcal{Z}^k$ as follows:

$$X^k_{ij} \in \mathcal{X}^k \text{ with } i, j \in \mathbb{Z}, j \geq i; \quad X^k_{i,\infty} \in \mathcal{Z}^k \text{ with } i \in \mathbb{Z}.$$

Again we fix the co-ordinate system: $X^0_{0,0}$ is given as in the finite global dimension case; see after Notation 3.9. The object $X^0_{0,\infty}$ is left-infinite string complex given by ${}_\infty v$, with $P(0)$ in degree 0 and all higher degrees zero. We have

$$\Sigma X^0_{-\infty,-1} = \begin{cases} c_{n-2} \cdots c_0 v_\infty & \text{if } r > 1; \\ v_\infty & \text{if } r = 1, \end{cases}$$

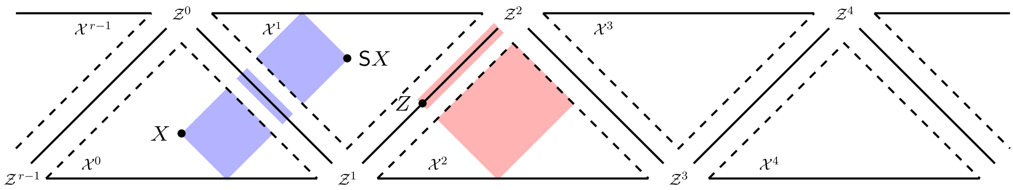
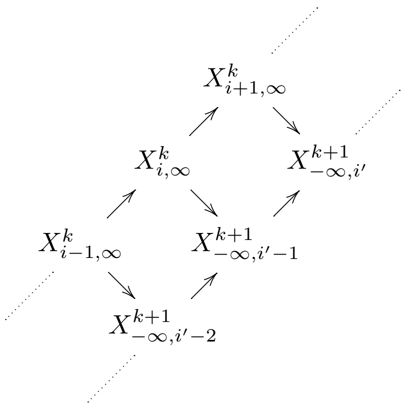


Fig. 2. The structure of $\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda)$ when $\operatorname{gldim}\Lambda = \infty$. Notation for components as in Fig. 1.

where in the first case $P(n-1)$ sits in degree 0 and all lower degrees are zero, and in the second case $P(0)$ sits in degree 0 and all lower degrees are zero.

The AR quiver of $\mathsf{Str}\text{-}\Lambda$ has $3r$ components. The r components $\mathcal{X}^k$ of $\mathsf{D}^b(\mathsf{mod}\text{-}\Lambda)$ of type $\mathbb{Z}A_\infty$ remain unchanged. The r components $\mathcal{Z}^k$ of type A_∞^∞ form the left-hand 'beams' of r 'ladder-type' components $\widetilde{\mathcal{Z}}^k$ where the objects $X^k_{i,\infty}$ lie on the left-hand 'beam' of the ladder. We follow Notation 3.9:



The remaining r components, denoted $\mathcal{Z}^k_\infty$, are of type A_1 as above, where the unique indecomposable complex in each such component is denoted Z^k_∞. The structure of $\mathsf{Str}\text{-}\Lambda$ for Λ of infinite global dimension is sketched in Fig. 3.

For $a \in \mathbb{Z} \cup \{-\infty\}$, $b \in \mathbb{Z} \cup \{\infty\}$ and $0 \le k < r$, we have

$$\Sigma X^k_{a,b} = X^{k+1}_{a',b'}\,,\ \Sigma Z^k_\infty = Z^{k+1}_\infty \text{ and } \mathbb{S}X^k_{a,b} = X^{k+1}_{a'-1,b'-1},$$

where, as before, $\mathbb{S}\colon \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda) \to \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$ is the Serre functor.

Remark 3.12. We have the following correspondences:

$$\mathsf{ind}(\mathcal{X}) \overset{1-1}{\longleftrightarrow} \{\text{perfect string complexes}\};$$

$$\{X^k_{i,\infty} \mid 0 \le k \le r, i \in \mathbb{Z}\} \overset{1-1}{\longleftrightarrow} \{\text{left-infinite string complexes}\};$$

$$\{X^k_{-\infty,i} \mid 0 \le k \le r, i \in \mathbb{Z}\} \overset{1-1}{\longleftrightarrow} \{\text{right-infinite string complexes}\};$$

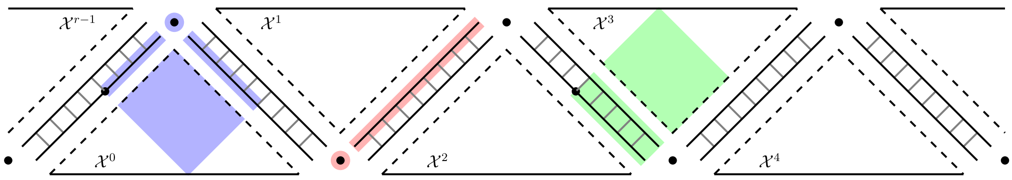


Fig. 3. The structure of $\mathsf{Str}\text{-}\Lambda$ when $\operatorname{gldim}\Lambda = \infty$. Notation for components is as in Fig. 1.

$$\mathsf{ind}(\mathcal{Z}_\infty) \overset{1-1}{\longleftrightarrow} \{\text{two-sided string complexes}\}.$$

Moreover, in light of Proposition 3.6, when $\operatorname{gldim}\Lambda = \infty$ the compact indecomposable complexes in K are those lying in $\mathcal{X}$ components and the 'righthand beam' of each of the $\widetilde{\mathcal{Z}}$ 'ladder' components, i.e. the objects $X^k_{-\infty,i}$ for $i \in \mathbb{Z}$ and $0 \le k < r$, in $\mathsf{Str}\text{-}\Lambda$.

3.5. Notation for Hom-hammocks

Before describing the structure of the Hom-hammocks in $\mathsf{Str}\text{-}\Lambda$ we first set up some notation. Let $A \in \mathsf{ind}(\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda))$, then we define the Hom-hammocks

$$\mathsf{F}^+(A) := \{X \in \mathsf{ind}(\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda)) \mid \operatorname{Hom}_{\mathsf{K}}(A, X) \neq 0\}$$
$$\mathsf{F}^-(A) := \{X \in \mathsf{ind}(\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda)) \mid \operatorname{Hom}_{\mathsf{K}}(X, A) \neq 0\}.$$

For $A \in \mathsf{Str}\text{-}\Lambda$, we define the Hom-hammocks

$$\mathsf{H}^+(A) := \{X \in \mathsf{Str}\text{-}\Lambda \mid \operatorname{Hom}_{\mathsf{K}}(A, X) \neq 0\} \quad \text{the } \textit{forward Hom-hammock} \text{ of } A;$$
$$\mathsf{H}^-(A) := \{X \in \mathsf{Str}\text{-}\Lambda \mid \operatorname{Hom}_{\mathsf{K}}(X, A) \neq 0\} \quad \text{the } \textit{backward Hom-hammock} \text{ of } A.$$

3.6. Hom-hammocks in $\mathsf{Str}\text{-}\Lambda$ for Λ of finite global dimension

The following proposition can be more easily understood by looking at Fig. 1 and natural extensions of the figures in [10, §3].

Proposition 3.13. *Suppose Λ is derived-discrete of finite global dimension. Let $a, b \in \mathbb{Z}$ and $0 \le k < r$. The forward Hom-hammocks of objects of $\mathsf{Str}\text{-}\Lambda$ are given by:*

$$\mathsf{H}^+(X^k_{a,b}) = \mathsf{F}^+(X^k_{a,b}) \cup \{X^k_{i,\infty} \mid a \le i \le b\} \cup \{X^{k+1}_{-\infty,j} \mid a' - 1 \le j \le b' - 1\};$$
$$\mathsf{H}^+(Y^k_{a,b}) = \mathsf{F}^+(Y^k_{a,b}) \cup \{Y^k_{\infty,j} \mid b \le j \le a\} \cup \{Y^{k+1}_{i,-\infty} \mid b'' - 1 \le i \le a'' - 1\};$$
$$\mathsf{H}^+(Z^k_{a,b}) = \mathsf{F}^+(Z^k_{a,b}) \cup \{X^{k+1}_{-\infty,j} \mid j \ge a' - 1\} \cup \{X^{k+1}_{i,\infty} \mid i \le a' - 1\}$$
$$\cup \{Z^{k+1}_{\infty}\} \cup \{Y^{k+1}_{i,-\infty} \mid i \ge b'' - 1\} \cup \{Y^{k+1}_{\infty,j} \mid j \le b'' - 1\};$$
$$\mathsf{H}^+(X^k_{a,\infty}) = \{X^k_{i,\infty} \mid i \ge a\} \cup \{Z^k_{i,j} \mid i \ge a \textit{ and } j \in \mathbb{Z}\} \cup \{X^{k+1}_{-\infty,j} \mid j \ge a' - 1\}$$

$$
\begin{aligned}
&\cup\{X^{k+1}_{i,j}\mid i\le a'-1 \textit{ and } j\ge a'-1\};\\
\mathsf{H}^+(X^k_{-\infty,b}) &= \{X^k_{-\infty,j}\mid j\ge b\}\cup\{X^k_{i,j}\mid i\le b \textit{ and } j\ge b\}\cup\{X^k_{i,\infty}\mid i\le b\}\\
&\cup\{Y^k_{\infty,j}\mid j\in\mathbb{Z}\}\cup\{Z^k_{i,j}\mid i\le b \textit{ and } j\in\mathbb{Z}\}\cup\{Z^k_\infty\};\\
\mathsf{H}^+(Y^k_{\infty,b}) &= \{Y^k_{\infty,j}\mid j\ge b\}\cup\{Z^k_{i,j}\mid i\in\mathbb{Z} \textit{ and } j\ge b\}\cup\{Y^{k+1}_{i,-\infty}\mid i\ge b''-1\}\\
&\cup\{Y^{k+1}_{i,j}\mid i\ge b''-1 \textit{ and } j\le b''-1\};\\
\mathsf{H}^+(Y^k_{a,-\infty}) &= \{Y^k_{i,-\infty}\mid i\ge a\}\cup\{Y^k_{i,j}\mid i\ge a \textit{ and } j\le a\}\cup\{Y^k_{\infty,j}\mid j\le a\}\\
&\cup\{X^k_{i,\infty}\mid i\in\mathbb{Z}\}\cup\{Z^k_{i,j}\mid i\in\mathbb{Z} \textit{ and } j\le a\}\cup\{Z^k_\infty\};\\
\mathsf{H}^+(Z^k_\infty) &= \{X^k_{i,\infty}\mid i\in\mathbb{Z}\}\cup\{Y^k_{\infty,j}\mid j\in\mathbb{Z}\}\cup\{Z^k_{i,j}\mid i,j\in\mathbb{Z}\}.
\end{aligned}
$$

The Hom-hammocks involving only perfect string complexes were established in [10, §3 & §6]; we extend these to left- and right-infinite and two-sided string complexes here. We start by identifying some distinguished triangles in K involving objects of $\mathsf{Str}\text{-}\Lambda$ analogous to those in [10, Properties 2.2(4)].

Lemma 3.14. *Let $a,b\in\mathbb{Z}$. The following are distinguished triangles in K:*

$$
\begin{aligned}
X^k_{a,a}\to X^k_{a,\infty}\to X^k_{a+1,\infty}\to\Sigma X^k_{a,a} \quad &\textit{and}\quad X^k_{a,a}\to X^{k+1}_{-\infty,a'-1}\to X^{k+1}_{-\infty,a'}\to\Sigma X^k_{a,a}\\
Y^k_{a,a}\to Y^k_{\infty,a}\to Y^k_{\infty,a+1}\to\Sigma Y^k_{a,a} \quad &\textit{and}\quad Y^k_{a,a}\to Y^{k+1}_{a''-1,-\infty}\to Y^{k+1}_{a'',-\infty}\to\Sigma Y^k_{a,a}\\
X^k_{-\infty,b}\to Z^k_\infty\to X^k_{b+1,\infty}\to\Sigma X^k_{-\infty,b} \quad &\textit{and}\quad Y^k_{a,-\infty}\to Z^k_\infty\to Y^k_{\infty,a+1}\to\Sigma Y^k_{a,-\infty}\\
X^k_{-\infty,b}\to Y^k_{\infty,j}\to Z^k_{b+1,j}\to\Sigma X^k_{-\infty,b} \quad &\textit{and}\quad Y^k_{a,-\infty}\to X^k_{i,\infty}\to Z^k_{i,a+1}\to\Sigma Y^k_{a,-\infty}
\end{aligned}
$$

Proof. From the form of the homotopy strings in Lemma 3.2, and therefore the straightforward form of the corresponding string complexes, the mapping cones of the morphisms in Lemma 3.10 are obtained by a standard computation; cf. [12]. □

We are now ready to prove Proposition 3.13. The arguments are in the spirit of [10, §3] so we just give a sketch.

Proof of Proposition 3.13. Let A and B be string complexes. We split the argument up into different cases.

Case $A\in\mathsf{ind}(\mathcal{X}\cup\mathcal{Y})$ and $B\in\mathsf{Str}\text{-}\Lambda$: In fact, we consider only the case $A\in\mathsf{ind}(\mathcal{X})$; the case $A\in\mathsf{ind}(\mathcal{Y})$ is analogous. Suppose $A=X^k_{a,b}$. We first deal with the non-vanishing statements. If B is a perfect string complex then this is [10, Prop. 3.4], so we assume that $B\in\mathsf{ind}(\mathcal{X}^k_\infty\cup\mathcal{X}^{k+1}_{-\infty})$. An induction on the height of $A=X^k_{a,b}$, $h(A)=b-a$, then gives the desired conclusion. Use Lemma 3.10 for the base step. For the inductive step, apply $\mathrm{Hom}_{\mathsf{K}}(-,X^k_{i,\infty})$ and $\mathrm{Hom}_{\mathsf{K}}(-,X^{k+1}_{-\infty,i})$ to the triangles from [10, Lem. 3.2]

$$
{}_0A\longrightarrow A\longrightarrow A''\xrightarrow{\ \varphi\ }\Sigma({}_0A)\quad\text{and}\quad A'\longrightarrow A\longrightarrow A_0\longrightarrow\Sigma A',
$$

where ${}_0A = X^k_{a,a}$, $A' = X^k_{a,b-1}$, $A_0 = X^k_{b,b}$ and $A'' = X^k_{a+1,b}$, and read off the Hom-spaces from the resulting long exact sequences.

The only subtleties occur when $r = 1$; we drop the superscripts in this case. We first note that the map $\varphi\colon A'' \to \Sigma({}_0A)$ occurring in the triangle above is the composition of the maps occurring in the extended ray in Lemma 3.10. When $b \neq a+1+m$ this is clear since $\dim \mathrm{Hom}_{\mathsf{K}}(X_{a+1,b}, X_{a+1+m,a+1+m}) = 1$ by [10, Prop. 6.2]. If $b = a+1+m$ then $\dim \mathrm{Hom}_{\mathsf{K}}(X_{a+1,b}, X_{a+1+m,a+1+m}) = 2$. A basis of $\mathrm{Hom}_{\mathsf{K}}(X_{a+1,a+1+m}, X_{a+1+m,a+1+m})$ is given by the finite composition of irreducible maps $f\colon X_{a+1,a+1+m} \to X_{a+1+m,a+1+m}$ and the infinite composition $g\colon X_{a+1,a+1+m} \to X_{a+1,\infty} \to X_{a+1+m,a+1+m}$ given in Lemma 3.10. Note that $g = fh$ for some $h\colon X_{a+1,a+1+m} \to X_{a+1,\infty} \to X_{a+1,a+1+m}$ by definition. Since $\dim \mathrm{Hom}_{\mathsf{K}}(X_{a+1,\infty}, X_{a+1,\infty}) = 1$, we have $h^2 = 0$. If φ is not a scalar multiple of g, then without loss of generality $\varphi = f + \lambda g = f + \lambda fh = f(1+\lambda h)$ for some scalar λ, where $1+\lambda h$ is an automorphism of $X_{a+1,a+1+m}$ with inverse $1-\lambda h$. In particular, the cone of φ is isomorphic to the cone of f, which is $\Sigma X_{a+1,a+m}$ (as it is one of the standard triangles of [10, Lem. 3.2]), which is a contraction. Therefore, up to scalars, $\varphi = g$ as claimed.

We must consider the case where $i = a$ or $i = a+1+m$. In the former case $\mathrm{Hom}_{\mathsf{K}}(X_{a,b}, X_{a,\infty}) \neq 0$ by Lemma 3.10. In the latter case, however, we cannot infer that $\mathrm{Hom}_{\mathsf{K}}(\Sigma({}_0A), B) = 0$. However, by the observation above that the map $\varphi\colon A'' = X_{a+1,b} \to \Sigma({}_0A) = X_{a+1+m,a+1+m} \to X_{a+1+m,\infty}$ factors as $X_{a+1,b} \to X_{a+1+m,\infty} \to X_{a+1+m,a+1+m}$ and the fact that the dimension $\dim \mathrm{Hom}_{\mathsf{K}}(X_{a+1+m,\infty}, X_{a+1+m,\infty}) = 1$, we conclude that the composition $X_{a+1,b} \to X_{a+1+m,a+1+m} \to X_{a+1+m,\infty}$ must be zero, for otherwise $X_{a+1+m,\infty}$ would be a direct summand of $X_{a+1+m,a+1+m}$. This now gives the required non-vanishing statements. The vanishing statements are obtained similarly by induction. Analogous arguments show the statements for $B = X^{k+1}_{i,-\infty}$.

Case $A \in \mathsf{ind}(\mathcal{Z})$ and $B \in \mathsf{Str}\text{-}\Lambda$: Let $A = Z^k_{a,b}$. We start with $B \in \mathsf{ind}(\mathcal{X}^{k+1}_{-\infty})$. Recalling Notation 3.9, Lemma 3.14 says that for each $t \in \mathbb{N}$ there is a distinguished triangle,

$$X^{k+1}_{-\infty,a'-1+t} \longrightarrow X^{k+1}_{-\infty,a'+t} \longrightarrow X^{k+1}_{a'+t,a'+t} \xrightarrow{-\Sigma f} \Sigma X^{k+1}_{-\infty,a'-1+t}. \tag{1}$$

By Lemma 3.10 $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, X^{k+1}_{-\infty,a'-1}) \neq 0$. Applying $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, -)$ to (1) allows one to read off $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, X^{k+1}_{-\infty,j}) \neq 0$ inductively for all $j \geq a'-1$ when $r > 1$.

For $r = 1$, we have the following long exact sequence, where we have dropped the superscript k because $r = 1$,

$$\begin{aligned}(Z_{a,b}, \Sigma^{-1}X_{a'+t,a'+t}) \xrightarrow{(Z_{a,b},f)} (Z_{a,b}, X_{-\infty,a'-1+t}) &\longrightarrow (Z_{a,b}, X_{-\infty,a'+t}) \\ &\longrightarrow (Z_{a,b}, X_{a'+t,a'+t}).\end{aligned}$$

For any $t \geq 0$, $\mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, X_{a'+t,a'+t}) = 0$. However,

$$\mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, \Sigma^{-1}X_{a'+t,a'+t}) = \mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, X_{a+t,a+t}) \neq 0 \text{ precisely when } t = m.$$

In this case, we cannot deduce the non-vanishing of $\mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, X_{-\infty,a'+m})$ immediately. However, using Lemma 3.10, observe that the map f factors as $X_{a+m,a+m} \to Z_{a+m,j} \to X_{-\infty,a+2m}$ for any $j \in \mathbb{Z}$. By [10, Prop. 3.6], $\mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, Z_{a+m,j}) = 0$ for $b - n < j < b$. Consider $g\colon Z_{a,b} \to X_{a+m,a+m}$. Since $n > r = 1$, we can take $j = b - 1$ and deduce that the composite $Z_{a,b} \xrightarrow{g} X_{a+m,a+m} \to Z_{a+m,j} \to X_{-\infty,a+2m}$ is zero, whence $\mathrm{Hom}_{\mathsf{K}}(Z_{a,b}, f)(g) = 0$. The non-vanishing statement now follows.

For the vanishing statements in the $\mathcal{X}^{k+1}_{-\infty}$ components, use (1) with $t \in \mathbb{Z}_{\le 0}$, using the one-dimensionality of the nonzero Hom-spaces to start the induction. The same argument applied to the $\mathcal{X}^{l}_{-\infty}$ components for $l \ne k+1 \bmod r$ shows $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, -) = 0$ on those components. Similar arguments can be used for the $\mathcal{Y}^{l}_{-\infty}$ components.

Next we consider $B \in \mathsf{ind}(\mathcal{X}^{k+1}_{\infty})$. By Lemma 3.14, for each $i \in \mathbb{Z}$ we have

$$\Sigma^{-1} Z^{k+1}_{i,b''} \xrightarrow{f} Y^{k+1}_{b''-1,-\infty} \longrightarrow X^{k+1}_{i,\infty} \longrightarrow Z^{k+1}_{i,b''}.$$

Applying the functor $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, -)$ to this triangle gives the long exact sequence,

$$(Z^k_{a,b}, \Sigma^{-1} Z^{k+1}_{i,b''}) \xrightarrow{(Z^k_{a,b}, f)} (Z^k_{a,b}, Y^{k+1}_{b''-1,-\infty}) \longrightarrow (Z^k_{a,b}, X^{k+1}_{i,\infty}) \longrightarrow (Z^k_{a,b}, Z^{k+1}_{i,b''}).$$

Now by [10, §3], $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, Z^{k+1}_{i,b''}) = 0$ for all $i \in \mathbb{Z}$. For $i < a'$, [10, §3] implies that $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, \Sigma^{-1} Z^{k+1}_{i,b''}) = 0$ giving the non-vanishing statement. For $i \ge a'$, one-dimensionality and Lemma 3.10 give that the map $(Z^k_{a,b}, f)$ is a surjection, whence we get the vanishing statements for the component $\mathcal{X}^{k+1}_{\infty}$. The vanishing statements for the $\mathcal{X}^{l}_{\infty}$ components for $l \ne k+1$ follow from the vanishing statements for the corresponding $\mathcal{Y}^{l}_{-\infty}$ components in an analogous manner. Similarly for the $\mathcal{Y}^{l}_{\infty}$ components.

Finally, the statements for Z^l_{∞} can be deduced by applying the functor $\mathrm{Hom}_{\mathsf{K}}(Z^k_{a,b}, -)$ to the triangles $X^l_{-\infty,a'-1} \to Z^l_{\infty} \to X^l_{a',\infty} \to \Sigma X^l_{-\infty,a'-1}$ from Lemma 3.14.

Case A is a nonperfect string complex and $B \in \mathsf{Str}\text{-}\Lambda$: If B is perfect, then this case is the dual of the two cases above, so we may assume that B is also nonperfect. Some of the non-vanishing statements are contained in Lemma 3.10. For the others and the vanishing statements, one needs to argue once from string combinatorics and [1, Thm. 3.15] for each $\mathcal{X}_{\pm\infty}$ and $\mathcal{Y}_{\pm\infty}$ component and then use the triangles from Lemma 3.14 for an induction. □

3.7. Hom-hammocks in Str-Λ *for* Λ *of infinite global dimension*

We now state the analogue of Proposition 3.13 for Λ of infinite global dimension. The proof is analogous to that for Proposition 3.13 and is therefore omitted. The statement is more easily understood with reference to a figure, the relevant illustrations are Figs. 2 and 3. Recall Notation 3.9.

Proposition 3.15. *Suppose Λ is derived-discrete of infinite global dimension. Let $a \le b \in \mathbb{Z}$ and $0 \le k < r$. The forward Hom-hammocks of objects of* Str-Λ *are given by:*

$$\mathsf{H}^+(X^k_{a,b}) = \mathsf{F}^+(X^k_{a,b}) \cup \{X^k_{-\infty,j} \mid a'-1 \le j \le b'-1\};$$
$$\mathsf{H}^+(X^k_{a,\infty}) = \mathsf{F}^+(X^k_{a,\infty}) \cup \{X^{k+1}_{-\infty,j} \mid j \ge a'-1\};$$
$$\mathsf{H}^+(X^k_{-\infty,b}) = \{X^k_{i,j} \mid i \le b,\, j \ge b\} \cup \{Z^k_\infty\} \cup \{X^k_{i,\infty} \mid i \le b\};$$
$$\mathsf{H}^+(Z^k_\infty) = \{Z^k_\infty\} \cup \{X^k_{i,\infty} \mid i \in \mathbb{Z}\}.$$

3.8. Factorisation properties

Finally, in order to determine how the simple functors isolate indecomposable pure-injective objects in the next section, we need to understand how morphisms factor in $\mathsf{Str}\text{-}\Lambda$. When $r > 1$, this is very straightforward.

Proposition 3.16. *Let $\Lambda = \Lambda(r, n, m)$ with $r > 1$. Suppose $A, C \in \mathsf{Str}\text{-}\Lambda$ with $C \in \mathsf{H}^+(A)$. Then any map $f\colon A \to C$ factors as $A \to B \to C$ for each $B \in \mathsf{H}^+(A) \cap \mathsf{H}^-(C)$.*

Proof. If $A, B, C \in \mathsf{ind}(\mathsf{K}^{-,b}(\mathsf{proj}\text{-}\Lambda))$ then this statement can be deduced from [10, §3] or [8, §4]. If A, B and C lie on the same extended ray or coray, then this is Lemma 3.10. The rest of the proof splits up into a case analysis building up from these rays and corays.

Case 1: $A \in \mathsf{ind}(\mathcal{X} \cup \mathcal{Y})$. We assume $A \in \mathsf{ind}(\mathcal{X}^k)$ for some $0 \le k < r$; the case $A \in \mathsf{ind}(\mathcal{Y}^k)$ is analogous. Suppose $C \in \mathsf{ind}(\mathcal{Z}^k \cup \mathcal{X}^{k+1})$. We verify the factorisation for $B \in \mathcal{X}^k_\infty$; the check for $B \in \mathcal{X}^{k+1}_{-\infty}$ is analogous. There exist $B' \in \mathsf{ind}(\mathcal{X}^k) \cap \mathsf{H}^+(A) \cap \mathsf{H}^-(C)$ and $C' \in \mathsf{ind}(\mathcal{Z}^k \cup \mathcal{X}^{k+1}) \cap \mathsf{H}^+(A) \cap \mathsf{H}^-(C)$ such that B', B and C' lie on the same extended ray or coray. The map $A \to C$ thus factors as $A \to B' \to C' \to C$ by the factorisation statements for maps between indecomposable perfect complexes. However, by Lemma 3.10, the map $B' \to C'$ factors as $B' \to B \to C'$ giving the desired statement. If $B, C \in \mathsf{ind}(\mathcal{X}^k_\infty)$ (or $B, C \in \mathsf{ind}(\mathcal{X}^{k+1}_{-\infty})$) then this is the dual of Case 3 below.

Case 2: $A \in \mathsf{ind}(\mathcal{Z})$. If $C \in \mathsf{ind}(\mathcal{X} \cup \mathcal{Y})$ then this is the dual to Case 1. Assume $A \in \mathsf{ind}(\mathcal{Z}^k)$. We only need to check the case that $C \in \mathsf{ind}(\mathcal{Z}^{k+1})$ (the other cases are covered by Cases 3 and 4 and their duals). Suppose $B \in \mathsf{ind}(\mathcal{X}^{k+1}_{-\infty})$. Then there exists $B' \in \mathsf{ind}(\mathcal{X}^{k+1}) \cap \mathsf{H}^+(A) \cap \mathsf{H}^+(B) \cap \mathsf{H}^-(C)$. In particular, the map $A \to C$ factorises as $A \to B' \to C$ and by above the map $A \to B'$ factorises as $A \to B \to B'$, as required.

Case 3: $A \in \mathsf{ind}(\mathcal{X}_{\pm\infty} \cup \mathcal{Y}_{\pm\infty})$. We treat the case $A \in \mathcal{X}^k_\infty$; the other cases are analogous. If C lies on the same extended ray as A, this is again Lemma 3.10, so assume that C lies on a different extended ray from A. We first cover the case that $B \in \mathsf{ind}(\mathcal{X}^k_\infty)$. Assume $A = X^k_{a,\infty}$ for some $a \in \mathbb{Z}$ and $B = X^k_{b,\infty}$ for some $b > a$. Inductively using triangles of the form

$$X^k_{i,i} \to X^k_{i,\infty} \to X^k_{i+1,\infty} \to \Sigma X^k_{i,i}$$

with $a \le i < b$ and using the fact that if $C \in \mathsf{H}^+(B)$ then by Proposition 3.13 and its dual, we have $\mathrm{Hom}_{\mathsf{K}}(X^k_{i,i}, C) = 0$, giving the desired factorisation. A similar argument holds if $B \in \mathsf{ind}(\mathcal{X}^{k+1}_{-\infty})$ using the analogous triangles. For B elsewhere, apply the same argument, but one should use the triangles from [10, Properties 2.2(4)] or [10, Lem. 3.2].

Case 4: $A = Z^k_\infty$. Apply the same argument as in Case 3 using the triangles $X^k_{-\infty,b} \to Z^k_\infty \to X^k_{b+1,\infty} \to \Sigma X^k_{-\infty,b}$ or $Y^k_{a,-\infty} \to Z^k_\infty \to X^k_{\infty,a+1} \to \Sigma Y^k_{a,-\infty}$ for appropriate choices of a and b to get factorisation through the objects of $\mathsf{ind}(\mathcal{X}^k_\infty \cup \mathcal{Y}^k_\infty)$. For $B \in \mathsf{ind}(\mathcal{Z}^k)$ one can get a factorisation $A \to B' \to C$ for an appropriate choice of $B' \in \mathsf{ind}(\mathcal{X}^k_\infty \cup \mathcal{Y}^k_\infty)$, which further factorises as $B' \to B \to C$, giving the desired factorisation. $\square$

When $r = 1$ the situation is a little more subtle. To make a clean statement we set up some notation. Let A be a string complex and $\mathcal{C}$ be a component of the AR quiver of $\mathsf{Str}\text{-}\Lambda$. If $\mathcal{C}$ is a component of the AR quiver of $\mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$ and $A \in \mathsf{ind}(\mathcal{C})$, we define,

$$\mathsf{H}^+_{\mathcal{C}}(A) = \{B \in \mathsf{H}^+(A) \cap \mathcal{C} \mid \text{there is a path in } \mathcal{C} \text{ from } A \text{ to } B\};$$

$$\mathsf{H}^+_{\mathbb{S}\mathcal{C}}(A) = \{B \in \mathsf{H}^+(A) \cap \mathcal{C} \mid \text{there is a path in } \mathcal{C} \text{ from } B \text{ to } \mathbb{S}A\},$$

where $\mathbb{S}\colon \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda) \to \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$ is the Serre functor (see Notation 3.9). The hammocks $\mathsf{H}^-_{\mathcal{C}}(A)$ and $\mathsf{H}^-_{\mathbb{S}\mathcal{C}}(A)$ are defined analogously. Note that $\mathsf{H}^+_{\mathbb{S}\mathcal{C}}(A) = \mathsf{H}^-_{\mathcal{C}}(\mathbb{S}A)$, explaining the notation. If $\mathcal{C}$ is a component of the AR quiver of $\mathsf{Str}\text{-}\Lambda \backslash \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$ or $A \notin \mathsf{ind}(\mathcal{C})$ we simply set $\mathsf{H}^+_{\mathcal{C}}(A) = \mathsf{H}^+(A) \cap \mathcal{C}$. For example,

$$\mathsf{H}^+_{\mathcal{X}}(X_{a,b}) = \{X_{i,j} \mid a \le i \le b \text{ and } j \ge b\};$$

$$\mathsf{H}^+_{\mathbb{S}\mathcal{X}}(X_{a,b}) = \{X_{i,j} \mid i \le a' - 1 \text{ and } a' - 1 \le j \le b' - 1\}.$$

We now state versions of Proposition 3.16 when $r = 1$. Drawing a picture with two copies of each component with the hammocks $\mathsf{H}^+_{\mathcal{C}}(A)$ and $\mathsf{H}^+_{\mathbb{S}\mathcal{C}}(A)$ drawn in the different copies of $\mathcal{C}$ should allow the reader to verify that the following statements are essentially the same as Proposition 3.16. While the statements are more complicated, the proofs proceed exactly as above and are therefore omitted. We make the statements for the finite global dimension case, the reader should have no difficulty obtaining the infinite global dimension statements from these.

Proposition 3.17. *Let $\Lambda = \Lambda(1, n, m)$ and $n > 1$. Suppose $A, C \in \mathsf{ind}(\mathcal{X})$ with $C \in \mathsf{H}^+_{\mathcal{X}}(A) \cap \mathsf{H}^+_{\mathbb{S}\mathcal{X}}(A)$. Then any nonisomorphism $f\colon A \to C$ is a linear combination of maps factoring as $f_1\colon A \to B_1 \to C$ and $f_2\colon A \to B_2 \to C$ for each $B_1 \in \mathsf{F}_1$ and $B_2 \in \mathsf{F}_2$, with f_1 a finite composition of irreducible maps and $f_2 \in \mathrm{rad}^\omega(A, C)$, where*

$$\mathsf{F}_1 = \mathsf{H}^+_{\mathcal{X}}(A) \cap \mathsf{H}^-_{\mathcal{X}}(C) \text{ and } \mathsf{F}_2 = \big(\mathsf{H}^+_{\mathcal{X}}(A) \cup \mathsf{H}^+_{\mathcal{X}_\infty}(A) \cup \mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A)\big) \cap \mathsf{H}^-(C).$$

Proposition 3.18. *Let $\Lambda = \Lambda(1, n, m)$ and $n > 1$. Suppose $A, C \in \mathsf{Str}\text{-}\Lambda$ with $C \in \mathsf{H}^+(A)$. Then any map $f\colon A \to C$ factors as $A \to B \to C$ for each $B \in \mathsf{F}$, where the factorisation region F is defined below.*

(1) *If* $A \in \mathcal{X}$ *and*
   (a) $C \in \mathsf{H}^+_{\mathcal{X}}(A) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{X}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+_{\mathcal{X}}(A) \cap \mathsf{H}^-_{\mathcal{X}}(C)$;
   (b) $C \in \mathsf{H}^+_{\mathbb{S}\mathcal{X}}(A) \backslash \mathsf{H}^+_{\mathcal{X}}(A)$ *then* $\mathsf{F} = \big(\mathsf{H}^+_{\mathcal{X}}(A) \cup \mathsf{H}^+_{\mathcal{X}_\infty}(A) \cup \mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A)\big) \cap \mathsf{H}^-_{\mathcal{X}}(C)$;
   (c) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+_{\mathcal{X}}(A) \cup \mathsf{H}^+_{\mathcal{X}_\infty}(A)) \cap \mathsf{H}^-(C)\big) \cup (\mathsf{H}^+_{\mathcal{Z}}(A) \cap \mathsf{H}^-_{\mathcal{Z}}(C))$;
   (d) $C \in \mathsf{H}^+_{\mathcal{X}_\infty}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+_{\mathcal{X}}(A) \cap \mathsf{H}^-_{\mathcal{X}}(C)) \cup (\mathsf{H}^+_{\mathcal{X}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{X}_\infty}(C))$;
   (e) $C \in \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{X}}(A)) \cap \mathsf{H}^-(C)$.
(2) *If* $A \in \mathcal{X}_\infty$ *and*
   (a) $C \in \mathsf{H}^+_{\mathcal{X}_\infty}(A)$ *then* $\mathsf{F} = \mathsf{H}^+_{\mathcal{X}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{X}_\infty}(C)$;
   (b) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+_{\mathcal{X}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{X}_\infty}(C)) \cup (\mathsf{H}^+_{\mathcal{Z}}(A) \cap \mathsf{H}^-_{\mathcal{Z}}(C))$;
   (c) $C \in \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathcal{X}}(A)) \cap \mathsf{H}^-(C)$;
   (d) $C \in \mathsf{H}^+_{\mathcal{X}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathcal{X}}(A)) \cap \mathsf{H}^-(C)\big) \cup (\mathsf{H}^+_{\mathcal{X}}(A) \cap \mathsf{H}^-_{\mathcal{X}}(C))$.
(3) *If* $A \in \mathcal{X}_{-\infty}$ *and*
   (a) $C \in \mathsf{H}^+(A) \cap (\mathcal{X}_{-\infty} \cup \mathcal{Z}_\infty \cup \mathcal{Y}_\infty)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big(\mathsf{H}^-_{\mathcal{X}_{-\infty}}(C) \cup \mathsf{H}^-_{\mathcal{Z}_\infty}(C) \cup \mathsf{H}^-_{\mathcal{X}_\infty}(C)\big)$;
   (b) $C \in \mathsf{H}^+_{\mathcal{X}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big(\mathsf{H}^-_{\mathcal{X}_{-\infty}}(C) \cup \mathsf{H}^-_{\mathcal{X}}(C)\big)$;
   (c) $C \in \mathsf{H}^+_{\mathcal{X}_\infty}(A)$ *then* $\mathsf{F} = \big(\mathsf{H}^+(A) \cap \mathsf{H}^-(C)\big) \backslash \mathcal{Z}$;
   (d) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big((\mathsf{H}^-(C) \backslash \mathcal{Z}) \cup \mathsf{H}^-_{\mathcal{Z}}(C)\big)$.
(4) *If* $A \in \mathcal{Y}$ *and*
   (a) $C \in \mathsf{H}^+_{\mathcal{Y}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+_{\mathcal{Y}}(A) \cap \mathsf{H}^-_{\mathcal{Y}}(C)$;
   (b) $C \in \mathsf{H}^+_{\mathbb{S}\mathcal{Y}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{Y}}(A)) \cap \mathsf{H}^-(C)\big) \cup (\mathsf{H}^+_{\mathbb{S}\mathcal{Y}}(A) \cap \mathsf{H}^-_{\mathcal{Y}}(C)$;
   (c) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+_{\mathcal{Y}}(A) \cup \mathsf{H}^+_{\mathcal{Y}_\infty}(A)) \cap \mathsf{H}^-(C)\big) \cup (\mathsf{H}^+_{\mathcal{Z}}(A) \cap \mathsf{H}^-_{\mathcal{Z}}(C))$;
   (d) $C \in \mathsf{H}^+_{\mathcal{Y}_\infty}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+_{\mathcal{Y}}(A) \cap \mathsf{H}^-_{\mathcal{Y}}(C)) \cup (\mathsf{H}^+_{\mathcal{Y}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{Y}_\infty}(C))$;
   (e) $C \in \mathsf{H}^+_{\mathcal{Y}_{-\infty}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{Y}}(A)) \cap \mathsf{H}^-(C)$.
(5) *If* $A \in \mathcal{Y}_\infty$ *and*
   (a) $C \in \mathsf{H}^+_{\mathcal{Y}_\infty}(A)$ *then* $\mathsf{F} = \mathsf{H}^+_{\mathcal{Y}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{Y}_\infty}(C)$;
   (b) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+_{\mathcal{Y}_\infty}(A) \cap \mathsf{H}^-_{\mathcal{Y}_\infty}(C)) \cup (\mathsf{H}^+_{\mathcal{Z}}(A) \cap \mathsf{H}^-_{\mathcal{Z}}(C))$;
   (c) $C \in \mathsf{H}^+_{\mathcal{Y}_{-\infty}}(A)$ *then* $\mathsf{F} = (\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathcal{Y}}(A)) \cap \mathsf{H}^-(C)$;
   (d) $C \in \mathsf{H}^+_{\mathcal{Y}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+(A) \backslash \mathsf{H}^+_{\mathcal{Y}}(A)) \cap \mathsf{H}^-(C)\big) \cup (\mathsf{H}^+_{\mathcal{Y}}(A) \cap \mathsf{H}^-_{\mathcal{Y}}(C))$.
(6) *If* $A \in \mathcal{Y}_{-\infty}$ *and*
   (a) $C \in \mathsf{H}^+(A) \cap (\mathcal{Y}_{-\infty} \cup \mathcal{Z}_\infty \cup \mathcal{X}_\infty)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big(\mathsf{H}^-_{\mathcal{Y}_{-\infty}}(C) \cup \mathsf{H}^-_{\mathcal{Z}_\infty}(C) \cup \mathsf{H}^-_{\mathcal{X}_\infty}(C)\big)$;
   (b) $C \in \mathsf{H}^+_{\mathcal{Y}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big(\mathsf{H}^-_{\mathcal{Y}_{-\infty}}(C) \cup \mathsf{H}^-_{\mathcal{Y}}(C)\big)$;
   (c) $C \in \mathsf{H}^+_{\mathcal{Y}_\infty}(A)$ *then* $\mathsf{F} = \big(\mathsf{H}^+(A) \cap \mathsf{H}^-(C)\big) \backslash \mathcal{Z}$;
   (d) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+(A) \cap \big((\mathsf{H}^-(C) \backslash \mathcal{Z}) \cup \mathsf{H}^-_{\mathcal{Z}}(C)\big)$.
(7) *If* $A \in \mathcal{Z}$ *and*
   (a) $C \in \mathsf{H}^+_{\mathcal{Z}}(A)$ *then* $\mathsf{F} = \mathsf{H}^+_{\mathcal{Z}}(A) \cap \mathsf{H}^-_{\mathcal{Z}}(C)$;
   (b) $C \in \mathsf{H}^+_{\mathcal{C}}(A)$, *where* $\mathcal{C} = \mathcal{X}_{-\infty}$ *or* $\mathcal{C} = \mathcal{Y}_{-\infty}$, *then* $\mathsf{F} = \big(\mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{C}}(A)\big) \cap \mathsf{H}^-(C)$;
   (c) $C = Z_\infty$ *then* $\mathsf{F} = \big(\mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A) \cup \mathsf{H}^+_{\mathcal{Y}_{-\infty}}(A)\big) \cap \mathsf{H}^-(C)$;
   (d) $C \in \mathsf{H}^+_{\mathcal{X}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{X}_{-\infty}}(A)) \cap \mathsf{H}^-(C)\big) \cup \big(\mathsf{H}^+(A) \cap \mathsf{H}^-_{\mathcal{X}}(C)\big)$;
   (e) $C \in \mathsf{H}^+_{\mathcal{Y}}(A)$ *then* $\mathsf{F} = \big((\mathsf{H}^+_{\mathcal{Z}}(A) \cup \mathsf{H}^+_{\mathcal{Y}_{-\infty}}(A)) \cap \mathsf{H}^-(C)\big) \cup \big(\mathsf{H}^+(A) \cap \mathsf{H}^-_{\mathcal{Y}}(C)\big)$;
   (f) $C \in \mathsf{H}^+_{\mathcal{C}}(A)$, *where* $\mathcal{C} = \mathcal{X}_\infty$ *or* $\mathcal{C} = \mathcal{Y}_\infty$, *then* $\mathsf{F} = \big(\mathsf{H}^+(A) \cap \mathsf{H}^-(C)\big) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{Z}}(A)$;
   (g) $C \in \mathsf{H}^+_{\mathbb{S}\mathcal{Z}}(A)$, *then* $\mathsf{F} = \big(\mathsf{H}^+(A) \cap \mathsf{H}^-(C)\big) \backslash \mathsf{H}^+_{\mathbb{S}\mathcal{Z}}(A)$.

3.9. Homotopy colimits

It is interesting to identify the indecomposable objects lying on the $\mathcal{X}^k_{\pm\infty}$ and $\mathcal{Y}^k_{\pm\infty}$ as homotopy colimits; see [29] for the definition.

Lemma 3.19. *Suppose Λ is derived-discrete of finite global dimension. Then:*

(1) *For each chain of objects and irreducible morphisms along a ray in an $\mathcal{X}$ component, $X^k_{a,a} \to X^k_{a,a+1} \to X^k_{a,a+2} \to \cdots$, we have $X^k_{a,\infty} \cong \underrightarrow{\mathsf{holim}}\, X^k_{a,j}$.*
(2) *For each chain of objects and irreducible morphisms along a coray in a $\mathcal{Y}$ component, $Y^k_{b,b} \to Y^k_{b+1,b} \to Y^k_{b+2,b} \to \cdots$, we have $Y^k_{\infty,b} \cong \underrightarrow{\mathsf{holim}}\, Y^k_{i,b}$.*
(3) *For each chain of objects and irreducible morphisms along a ray in a $\mathcal{Z}$ component, $Z^k_{a,j} \to Z^k_{a,j+1} \to Z^k_{a,j+2} \to \cdots$, we have $X^{k+1}_{-\infty,a'-1} \cong \underrightarrow{\mathsf{holim}}\, Z^k_{a,j}$ when $0 \le k < r-1$ and $X^0_{-\infty,a'-1} \cong \underrightarrow{\mathsf{holim}}\, Z^{r-1}_{a,j}$.*
(4) *For each chain of objects and irreducible morphisms along a coray in a $\mathcal{Z}$ component, $Z^k_{i,b} \to Z^k_{i+1,b} \to Z^k_{i+2,b} \to \cdots$, we have $Y^{k+1}_{b''-1,-\infty} \cong \underrightarrow{\mathsf{holim}}\, Z^k_{i,b}$ when $0 \le k < r-1$ and $Y^0_{b''-1,-\infty} \cong \underrightarrow{\mathsf{holim}}\, Z^{r-1}_{i,b}$.*
(5) *For each chain of objects and irreducible morphisms in an $\mathcal{X}_{-\infty}$ component, $X^k_{-\infty,a} \to X^k_{-\infty,a+1} \to X^k_{-\infty,a+2} \to \cdots$, we have $Z^k_\infty \cong \underrightarrow{\mathsf{holim}}\, X^k_{-\infty,j}$.*

Suppose Λ is derived-discrete of infinite global dimension. Then:

(1) *For each chain of objects and irreducible morphisms along a ray in an $\mathcal{X}$ component, $X^k_{a,a} \to X^k_{a,a+1} \to X^k_{a,a+2} \to \cdots$, we have $X^k_{a,\infty} \cong \underrightarrow{\mathsf{holim}}\, X^k_{a,j}$.*
(2) *For each chain of objects and irreducible morphisms in a $\mathcal{Z}$ component, $X^k_{-\infty,b} \to X^k_{-\infty,b+1} \to X^k_{-\infty,b+2} \to \cdots$, we have $Z^k_\infty \cong \underrightarrow{\mathsf{holim}}\, X^k_{-\infty,j}$.*

Proof. In each case, let P denote the object we wish to show is the homotopy colimit of the sequence with P^j denoting the module in degree j. It follows from [6] and [1, §6] that for each of the cases above, there is a subsequence of the form $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \cdots$ where, for each $i \ge 1$ and $j \in \mathbb{Z}$, the degree j parts of P_i and f_i are defined to be

$$P_i^j = \begin{cases} P^j & \text{for } -n_i \le j, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_i^j = \begin{cases} \mathrm{id}_{P^j} & \text{for } -n_i \le j, \\ 0 & \text{otherwise} \end{cases}$$

or

$$P_i^j = \begin{cases} P^j & \text{for } j < n_i, \\ Q_i & \text{for } j = n_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_i^j = \begin{cases} \mathrm{id}_{P^j} & \text{for } j < -n_i, \\ g_i & \text{for } j = n_i, \\ 0 & \text{otherwise} \end{cases}$$

where $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence in $\mathbb{N}$ and $g_i \colon Q_i \to P^{n_i}$ is a non-zero morphism. By [27, Lem. 2.8] (and Krause's observation after [26, Def. 2.2] that homology colimits are unique), it suffices to show that $\mathrm{Hom}_{\mathsf{K}}(C, P) \cong \varinjlim \mathrm{Hom}_{\mathsf{K}}(C, P_i)$ for each compact object C. For each $i \geq 1$ we can define a morphism $\mathrm{Hom}_{\mathsf{K}}(C, P_i) \to \mathrm{Hom}_{\mathsf{K}}(C, P)$ by $h \mapsto h \circ u_i$, where $u_i \colon P_i \to P$ is the obvious embedding. These morphisms induce an injective morphism $\varinjlim \mathrm{Hom}_{\mathsf{K}}(C, P_i) \to \mathrm{Hom}_{\mathsf{K}}(C, P)$. It remains to check that this morphism is surjective.

Let $h \colon C \to P$. The proof of Proposition 3.6 gives us that K^{c} is contained in $\mathsf{K}^{+,b}(\mathsf{proj}\text{-}\Lambda)$. Thus if Λ is of finite global dimension or P is not right-infinite nor two-sided then h is non-zero in only finitely many degrees. The remaining case is (5) and here the $X^{k}_{-\infty,i}$ are right-infinite. In each of these cases it is clear that there exists some $i \geq 1$ such that $h = h' u_i$ where $(h')^j = h^j$ for every degree $j \in \mathbb{Z}$. □

Corollary 3.20. *Let A be an algebra in the derived equivalence class of Λ. Then (by, for example, [21,22]) there exists a triangle equivalence $\Phi \colon \mathsf{K}(\mathsf{Proj}\text{-}\Lambda) \to \mathsf{K}(\mathsf{Proj}\text{-}A)$. The image $\Phi(P)$ of each homotopy colimit P described in Lemma 3.19 is an indecomposable pure-injective string complex.*

Proof. Note that, by [36], A is a gentle algebra. Moreover $\Phi(P)$ is indecomposable, pure-injective and is the homotopy colimit of a sequence of irreducible morphisms between indecomposable objects. Given the description of irreducible morphisms between string complexes over a gentle algebra in [1,6] and using arguments similar to those in the proof of Lemma 3.19, it is clear that this homotopy colimit will be a string complex. □

4. Krull–Gabriel dimension and simple functors

The Krull–Gabriel dimension of $\mathsf{D}(\mathsf{Mod}\text{-}\Lambda)$, that is, of $\mathsf{Coh}(\mathsf{D}(\mathsf{Mod}\text{-}\Lambda))$, was computed in [8]. In this section we follow the approach of [8] to compute the Krull–Gabriel dimension of $\mathsf{Coh}(\mathsf{K})$ and to identify the simple functors in $\mathsf{Coh}(\mathsf{K})/\mathsf{Coh}(\mathsf{K})_n$ for each $n \in \mathbb{N}$. The main result of this section is the following, where the first statement is Theorem A and the second statement is part of Theorem B. Note the contrast with $\mathrm{KGdim}(\mathsf{D}(\mathsf{Mod}\text{-}\Lambda)) = 1$ in the infinite global dimension case in [8].

Theorem 4.1. *Let Λ be a derived-discrete algebra. Then*

(1) $\mathrm{KGdim}(\mathsf{K}(\mathsf{Proj}\text{-}\Lambda)) = \mathrm{CB}\text{-}\mathrm{rank}(\mathrm{Zg}(\mathsf{K})) = 2$.
(2) *The objects of $\mathsf{Str}\text{-}\Lambda$ form a complete list of indecomposable, pure-injective complexes in K.*

Using the same argument as [32, Lem. 10.2.2], we have the following useful characterisation of coherent subfunctors of F_f.

Lemma 4.2. *Let* T *be a compactly generated triangulated category and let* F_f *be a coherent functor. Then, all coherent subfunctors of* F_f *have the form* $\operatorname{im}(h,-)/\operatorname{im}(f,-)$ *for some factorisation* $f = gh$ *of* f *in* T^{c}*.*

Lemma 4.3. *Let* Λ *be a derived-discrete algebra. Then* $\operatorname{KGdim}(\mathsf{K})$ *is defined.*

Proof. Let $F \in \mathsf{Coh}(\mathsf{K})$ and consider the lattice $L(F)$ of coherent subfunctors of F. By Lemma 4.2 and the descriptions of the Hom-hammocks in Propositions 3.13 and 3.15, it is clear that $L(F)$ has no densely ordered subset. By Remark 2.5, it follows that $\operatorname{KGdim}(\mathsf{K})$ is defined. $\square$

By Lemma 2.7 and Proposition 2.8, we have

$$\mathsf{Coh}(\mathsf{K})/\mathsf{Coh}(\mathsf{K})_n = \mathsf{Coh}(\mathsf{K})_{\mathbf{X}_n}$$

for all $n \geq 0$, where $\mathbf{X}_n$ is the closed subset of $\operatorname{Zg}(\mathsf{K})$ consisting of points with CB-rank greater than or equal to n. Let $q_n\colon \mathsf{Coh}(\mathsf{K}) \to \mathsf{Coh}(\mathsf{K})_{\mathbf{X}_n}$ denote the corresponding localisation functor. The following is immediate from Lemma 2.7 and Proposition 2.8.

Corollary 4.4. *The CB-rank of* $\operatorname{Zg}(\mathsf{K})$ *is defined and there is a natural bijection*

$$\{M \in \operatorname{Zg}(\mathsf{K}) \mid \text{CB -rank}(M) = n\} \overset{1-1}{\longleftrightarrow} \{F \in \mathsf{Coh}(\mathsf{K})_{\mathbf{X}_n} \mid F \text{ is simple}\}.$$

4.1. Cantor–Bendixson rank 0

It is well-known that the simple functors in $\mathsf{Coh}(\mathsf{K})$ correspond to the Auslander-Reiten triangles comprised of compact objects in K; see [2, §2].

Proposition 4.5. *The simple objects in* $\mathsf{Coh}(\mathsf{K})$ *are exactly those of the form* F_f *where* $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$ *is an Auslander-Reiten triangle. The point of* $\operatorname{Zg}(\mathsf{K})$ *isolated by* $F_f \cong (X,-)/\operatorname{im}(f,-)$ *is* X*.*

Corollary 4.6. *Let* Λ *be a derived-discrete algebra.*

(1) *If* $\operatorname{gldim}\Lambda = \infty$ *then* $\mathsf{ind}(\mathcal{X})$ *is the set of isolated points in* $\operatorname{Zg}(\mathsf{K})$*.*
(2) *If* $\operatorname{gldim}\Lambda < \infty$ *then* $\mathsf{ind}(\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})$ *is the set of isolated points in* $\operatorname{Zg}(\mathsf{K})$*.*

Proof. This now follows from Corollary 4.4 and the description of Auslander-Reiten triangles in [7] from which it follows that it is the indecomposable perfect complexes which begin Auslander-Reiten sequences. $\square$

4.2. Cantor–Bendixson rank 1

We begin by identifying certain morphisms that give rise to simple functors in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$.

Definition 4.7. Let Λ be a derived-discrete algebra and recall Notation 3.9.

(1) Suppose $\operatorname{gldim}\Lambda < \infty$. For $0 \le k < r$ consider the morphisms

$$h\colon X^k_{i,j} \to X^k_{i+1,j} \oplus Z^k_{i,t} \qquad h\colon Y^k_{i,j} \to Y^k_{i,j+1} \oplus Z^k_{t,j}$$
$$h\colon Z^k_{i,j} \to Z^k_{i+1,j} \oplus X^{k+1}_{t,i'-1} \qquad h\colon Z^k_{i,j} \to Z^k_{i,j+1} \oplus Y^{k+1}_{j''-1,t}$$

where $X^k_{i+1,i}$ and $Y^k_{i,i+1}$ are defined to be the zero module. Morphisms of this form with $h = \binom{h_1}{h_2}$, where h_1 is irreducible whenever its target is nonzero and $h_2 \neq 0$, will be called 1-*simple morphisms*.

(2) Suppose $\operatorname{gldim}\Lambda = \infty$. For $0 \le k < r$ consider the morphisms

$$h\colon X^k_{i,j} \to X^k_{i+1,j} \oplus X^{k+1}_{-\infty,i'-1} \qquad h\colon X^k_{-\infty,j} \to X^k_{-\infty,j+1} \oplus X^k_{i,j}$$

where $X^k_{i+1,i}$ is defined to be the zero module. Morphisms of this form with $h = \binom{h_1}{h_2}$, where h_1 is irreducible whenever its target is nonzero and $h_2 \neq 0$, will be called 1-*simple morphisms*.

From Lemma 4.2, which correlates subfunctors of F_h with factorisations of h, and our description of morphisms, it is clear that $q_0(F_h)$ is a simple functor if h is a 1-simple morphism. Since these functors are simple, it follows from Corollary 4.4 that there are corresponding indecomposable pure-injective objects with CB-rank equal to 1.

Proposition 4.8. *Let Λ be a derived-discrete algebra.*

(1) *If* $\operatorname{gldim}\Lambda = \infty$ *then the set* $\mathsf{ind}(\widetilde{\mathcal{Z}})$ *is a complete list of indecomposable, pure-injective complexes of CB rank* 1*.*
(2) *If* $\operatorname{gldim}\Lambda < \infty$ *then the set* $\mathsf{ind}(\mathcal{X}_\infty \cup \mathcal{X}_{-\infty} \cup \mathcal{Y}_\infty \cup \mathcal{Y}_{-\infty})$ *is a complete list of indecomposable, pure-injective complexes of CB-rank* 1*.*

Lemma 4.9. *In the set-up of Proposition 4.8, the complexes listed in each case are indecomposable, pure-injective and of CB-rank* 1*.*

Proof. For each of these complexes there is a 1-simple morphism f such that the open set corresponding to the functor $(X,-)/\operatorname{im}(f,-)$, where X is the domain of f, contains just that complex and complexes of CB-rank 0. So this is immediate from Proposition 2.8 together with the description of the Hom-hammocks in Propositions 3.13 and 3.15. □

To prove that the list is complete, we first need a definition and some preliminary results.

Definition 4.10. We define an equivalence relation $\sim$ on the set of 1-simple morphisms as follows:

(1) Suppose $\operatorname{gldim}\Lambda < \infty$. For $0 \leq k, l < r$ we set

$$[h\colon X^k_{i,j} \to X^k_{i+1,j} \oplus Z^k_{i,t}] \sim [g\colon X^l_{a,b} \to X^l_{a+1,b} \oplus Z^l_{a,c}] \iff k = l \text{ and } i = a;$$
$$[h\colon Y^k_{i,j} \to Y^k_{i,j+1} \oplus Z^k_{t,j}] \sim [g\colon Y^l_{a,b} \to Y^l_{a,b+1} \oplus Z^l_{c,b}] \iff k = l \text{ and } j = b;$$
$$[h\colon Z^k_{i,j} \to Z^k_{i+1,j} \oplus X^{k+1}_{t,i'-1}] \sim [g\colon Z^l_{a,b} \to Z^l_{a+1,b} \oplus X^{l+1}_{c,a'-1}] \iff k = l \text{ and } i = a;$$
$$[h\colon Z^k_{i,j} \to Z^k_{i,j+1} \oplus Y^{k+1}_{j''-1,t}] \sim [g\colon Z^l_{a,b} \to Z^l_{a,b+1} \oplus Y^{l+1}_{b''-1,c}] \iff k = l \text{ and } j = b.$$

(2) Suppose $\operatorname{gldim}\Lambda = \infty$. For $0 \leq k, l < r$ we set

$$[h\colon X^k_{i,j} \to X^k_{i+1,j} \oplus X^{k+1}_{-\infty,i'-1}] \sim [g\colon X^l_{a,b} \to X^l_{a+1,b} \oplus X^{l+1}_{-\infty,a'-1}]$$
$$\iff k = l \text{ and } i = a;$$
$$[h\colon X^k_{-\infty,j} \to X^k_{-\infty,j+1} \oplus X^k_{i,j}] \sim [g\colon X^l_{-\infty,b} \to X^l_{-\infty,b+1} \oplus X^l_{a,b}]$$
$$\iff k = l \text{ and } j = b.$$

From the description of the Hom-hammocks in Propositions 3.13 and 3.15, we see that (F_h) and (F_g) contain the same pure-injective with CB-rank 1 exactly when $h \sim g$. So the following corollary is immediate from Corollary 4.4 and inspection of the Hom-hammocks.

Corollary 4.11. *Let h and g be 1-simple morphisms. Then, $q_0(F_h) = q_0(F_g)$ if and only if $h \sim g$.*

Next we will prove that all simple functors in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$ arise from 1-simple morphisms.

Lemma 4.12. *Let T be a compactly generated triangulated category and let $f\colon A \to B$ be a morphism in T^{c}. If $q\colon \mathsf{Coh}(\mathsf{T}) \to \mathsf{Coh}(\mathsf{T})_{\mathbf{X}}$ is a localisation functor and $q(F_f)$ is simple, then there exists some $g\colon C \to D$ in T^{c} such that $q(F_f) = q(F_g)$ and C is indecomposable.*

Proof. Suppose $A = A' \oplus A''$ where $A', A'' \in \mathsf{T}^{\mathrm{c}}$ are nonzero. Since $(A, -) \cong (A', -) \oplus (A'', -)$, we have $F_f \cong \big((A', -) \oplus (A'', -)\big)/\operatorname{im}(f, -)$ where we identify $\operatorname{im}(f, -)$ with its image in $(A', -) \oplus (A'', -)$. Consider the subfunctors of F_f

$$^{(\operatorname{im}(f,-)+A)}\big/_{\operatorname{im}(f,-)} \cong {}^{(A',-)}\big/_{\operatorname{im}(f,-)\cap(A',-)} = H'$$

and

$$^{(\mathrm{im}(f,-)+A)}/_{\mathrm{im}(f,-)} \cong {}^{(A'',-)}/_{\mathrm{im}(f,-)\cap(A'',-)} = H''.$$

The sum of these subfunctors is F_f so, since $q(F_f)$ is simple, the image of at least one of them under q equals $q(F_f)$. Hence either $q(F_f) \cong q(H')$ or $q(F_f) \cong q(H'')$.

Now, $\mathrm{im}(f,-)\cap(A',-) \subseteq (A',-)$ is a finitely generated subfunctor so, by Lemma 4.2, there exists some $g\colon A' \to B'$ in T^{c} such that $\mathrm{im}(g,-) \cong \mathrm{im}(f,-)\cap(A',-)$ and similarly with A'' in place of A. As T^{c} is Krull-Schmidt, the result follows. □

Proposition 4.13. *Suppose $q_0(F_f)$ is simple in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$, then there exists a 1-simple morphism h such that $q_0(F_f) = q_0(F_h)$.*

The proof for the finite global dimension case can be found in [8]; our proof applies to both infinite and finite global dimension.

Proof. If $f = gh$, then F_h is a factor of F_f. Thus, for any such h, we have $q_0(F_h) \cong q_0(F_f)$ if and only if $q_0(F_h) \neq 0$. We have already observed that if h is a 1-simple morphism, then $q_0(F_h) \neq 0$ so it remains to show that we always have a factorisation $f = gh$ where h is 1-simple. By Lemma 4.12 we may assume that $f\colon A \to \bigoplus_{i=1}^{n} B_i$ where $A, B_1, \ldots, B_n$ are indecomposable objects in T^{c}.

We observe that the Hom-hammock structure, combined with Propositions 3.16, 3.17 and 3.18, implies that there is a 1-simple morphism through which f factors, except in the following cases:

(1) $A = X^k_{i,j}$ for some $0 \le k < r$ and $i \le j$ and $B_l = X^k_{i,j+t}$ for some $t \ge 0$ and $1 \le l \le n$.
(2) $A = Y^k_{i,j}$ for some $0 \le k < r$ and $j \le i$ and $B_l = Y^k_{i+t,j}$ for some $t \ge 0$ and $1 \le l \le n$.
(3) $A = Z^k_{i,j}$ for some $0 \le k < r$ and $i, j \in \mathbb{Z}$ and $B_p = Z^k_{i+t,j}$, $B_q = Z^k_{i,j+s}$ for some $t, s \ge 0$ and $1 \le p, q \le n$.

We argue, by contradiction, that none of these cases arise. In each of these cases, by inspection of the Hom-hammocks, the set of indecomposable compact objects C for which $F_f(C) \neq 0$ is finite. By Corollary 4.6, the open set (F_f) contains only finitely many isolated points of $\mathrm{Zg}(\mathsf{K})$. It follows from Lemma 4.3, that the CB-rank of $\mathrm{Zg}(\mathsf{K})$ is defined and so the isolated points are dense (see, for example, [32, Lem. 5.3.36]). But each isolated point, being of finite endolength, is closed (see [32, Thm. 5.1.12]) so there are no other points in (F_f). Since these are the only points on which F_f is nonzero and since their direct sum is of finite endolength, it follows that F_f is finite length and $q_0(F_f) = 0$ which is a contradiction. □

The completeness statement in Proposition 4.8 now follows from the corollary below.

Corollary 4.14. *The simple objects in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$ are in natural one-to-one correspondence with the $\sim$-equivalence classes of 1-simple morphisms.*

4.3. Cantor–Bendixson rank 2

We obtain $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_1}$ by localising $\mathsf{Coh}(\mathsf{K})$ at the Serre subcategory consisting of the functors F such that $q_0(F)$ has finite length.

Remark 4.15. Since the isolation condition holds and using Corollary 3.8, we may apply a similar argument to the one contained in the proof of Proposition 4.13 to obtain that an object $q_0(F)$ in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$ is finite length if and only if (F) contains finitely many points of CB-rank 1.

Proposition 4.16. *Let Λ be a derived-discrete algebra. An object $q_1(F)$ in $\mathsf{Coh}(\mathsf{K})_{\mathbf{X}_1}$ is simple if and only if*

(1) $q_1(F) = q_1((X^k_{-\infty,j}, -))$ *for some j, k, when Λ has infinite global dimension;*
(2) $q_1(F) = q_1((Z^k_{i,j}, -))$ *for some i, j, k, when Λ has finite global dimension.*

Proof. We give an argument for the case where Λ has infinite global dimension; the case where Λ has finite global dimension can be found in [8].

Note that, by Remark 4.15 and the description of Hom-hammocks in Proposition 3.15, $q_1((X^k_{i,j}, -)) = 0$ and $q_1((X^k_{-\infty,j}, -)) \neq 0$ for all $i, j \in \mathbb{Z}$ and $0 \leq k < r$. It follows from this and Lemma 4.12 that any simple object will be of the form $q_1(F_f)$ where $f: X^k_{-\infty,j} \to B$ for some compact object B, $j \in \mathbb{Z}$ and $0 \leq k < r$. It remains to show that $q_1((X^k_{-\infty,j}, -))$ is simple. Any coherent subobject of $q_1((X^k_{-\infty,j}, -))$ will (by Lemma 4.2) be the image under q_1 of some $\mathrm{im}(f, -) \subseteq (X^k_{-\infty,j}, -)$ where $f: X^k_{-\infty,j} \to \bigoplus_{i=1}^n B_i$ with $B_1, \ldots, B_n$ indecomposable. Note that $\mathrm{im}(f, -)$ is the sum of the $\mathrm{im}(\pi_i f, -)$ where π_i is the projection to B_i. If $B_i = X^k_{-\infty,t}$ for some $1 \leq i \leq n$ and $t \geq j$, then Remark 4.15 and Proposition 3.15 give us that $q_1(F_f) = 0$ and so $q_1((X^k_{-\infty,j}, -)) = q_1(\mathrm{im}(f, -))$. So consider the case where $B_i \in \mathcal{X}^{k+1}$ for each $1 \leq i \leq n$. Then there is an epimorphism $(\bigoplus_{i=1}^n B_i, -) \to \mathrm{im}(f, -)$ and so $q_1(\mathrm{im}(f, -)) = 0$. □

Corollary 4.17. *Let Λ be a derived-discrete algebra of either finite or infinite global dimension. The set $\mathcal{Z}_\infty$ is a complete list of all indecomposable, pure-injective complexes of CB-rank* 2.

Proof. We have already seen that the objects Z^k_∞ are indecomposable and pure-injective and it is clear from the Hom-hammocks that the Hom-functors $(X^k_{-\infty,j}, -)$ (in the infinite global dimension case) and $(Z^k_{i,j}, -)$ (in the finite global dimension case) isolate these points in $\mathbf{X}_2$. □

This now completes the proof of Theorem 4.1 for the derived-discrete algebras $\Lambda(r, n, m)$. We now extend the proof to an arbitrary derived-discrete algebra A in the derived equivalence class of $\Lambda(r, n, m)$.

Proof of Theorem 4.1. Let A be a derived-discrete algebra in the derived equivalence class of $\Lambda(r,n,m)$. The Cantor–Bendixson analysis carried out in sections 4.1, 4.2 and 4.3 is purely categorical and therefore shows the first statement for arbitrary A.

By arguing as in Proposition 3.4, Corollary 3.5 and Corollary 3.8, any string complex in $\mathsf{K}(\mathsf{Proj}\text{-}A)$ is indecomposable and pure-injective.

We now need to see there are no further indecomposable pure-injective objects that are not string complexes. At the categorical level, a complete list of indecomposable pure-injective objects is given in Corollary 4.6, Proposition 4.8 and Corollary 4.17. By Proposition 3.6, which holds for a general derived-discrete algebra up to suitable choice of orientation of the strings, the indecomposable compact objects of $\mathsf{K}(\mathsf{Proj}\text{-}A)$ are string complexes. So we need only consider indecomposable non-compact pure-injective objects. But each such object lies in an $\mathcal{X}_{\pm\infty}$, $\mathcal{Y}_{\pm\infty}$, $\mathcal{Z}_{\infty}$ component in the case A has finite global dimension, or lies in a $\mathcal{Z}_{\infty}$ component or has the form $X^k_{i,\infty}$ in the case A is of infinite global dimension. By Lemma 3.19 each of these is realised as a homotopy colimit of compact objects, whence by Corollary 3.20, they are string complexes. □

Although we do not do it here explicitly, using these methods and results, it is possible to obtain a complete description of the topology of the Ziegler spectrum of K.

5. Indecomposable complexes with compact support

Before showing that all indecomposable objects of K are pure-injective, we establish some preliminary results which hold for Λ of both finite and infinite global dimension. For an object M of a compactly generated triangulated category T, its *support* is the Ziegler-closed subset $\operatorname{supp}(M) := \langle M\rangle \cap \operatorname{Zg}(\mathsf{T})$, where $\langle M\rangle$ is the definable subcategory generated by M; see Definition 1.4. Recall the definition of localisation at a definable subcategory from Section 2.4.

Lemma 5.1. *Let M be an object in a compactly generated triangulated category T. If $\operatorname{supp}(M)$ contains a compact object C such that $\{C\} = (F) \cap \operatorname{supp}(M)$ for some functor $F \in \mathsf{Coh}(\mathsf{T})$ whose image is simple in $\mathsf{Coh}(\mathsf{T})_{\operatorname{supp}(M)}$, then C is a direct summand of M.*

Proof. Let $q\colon \mathsf{Coh}(\mathsf{T}) \to \mathsf{Coh}(\mathsf{T})_{\operatorname{supp}(M)}$ be the localisation functor and $i\colon \mathsf{Coh}(\mathsf{T})_{\operatorname{supp}(M)} \hookrightarrow \mathsf{Coh}(\mathsf{T})$ be its right adjoint. Recall from Section 2.4, that there is a parallel and compatible localisation $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\operatorname{supp}(M)}$, whose localisation functor and canonical embedding we again denote by q and i.

Let F be as in the statement of the lemma. Since F is nonzero in $\mathsf{Coh}(\mathsf{T})_{\operatorname{supp}(M)}$, $F(M) \neq 0$ and, since $C \in (F)$, $F(C) \neq 0$. By Lemma 1.9, $F = G^{\vee}$ for some functor $G \in \mathsf{mod}\text{-}\mathsf{T}^{\mathrm{c}}$ and $q(G)$ must also be simple in $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\operatorname{supp}(M)}$. Also by that result, $(G,(-,M)) \neq 0$ and $(G,(-,C)) \neq 0$.

Since $(-,M)$ and $(-,C)$ are torsionfree it follows that $(q(G), q(-,M)) \neq 0$ and $(q(G),(-,C)) \neq 0$, so there are embeddings $k\colon q(G) \hookrightarrow q(-,M)$ and $j\colon q(G) \hookrightarrow q(-,C)$.

Since C is an indecomposable pure-injective, by [32, Prop 11.1.31] $q(-,C)$ is an indecomposable injective object of $(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})_{\mathrm{supp}(M)}$, thus $q(-,C)$ is the injective hull of $q(G)$ and so j must embed $q(G)$ as the simple socle of $q(-,C)$. The cokernel of j is finitely presented so, since $q(-,M)$ is fp-injective by Remarks 1.6 and 2.2, there exists some morphism $h\colon q(-,C) \to q(-,M)$ such that $k = hj$. As $q(-,C)$ has simple essential socle, h must be a monomorphism.

Identifying h with its image under i, we can regard h as a monomorphism from $(-,C)(\cong iq(-,C))$ to $(-,M)$ in $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$. Since C is compact, Yoneda's lemma says that h must be induced by some $h'\colon C \to M$ in T and by definition this must be a pure monomorphism. But C is pure-injective so we must conclude that h' splits and C is a direct summand of M. □

Corollary 5.2. *Let M be an indecomposable object of K and suppose $C \in \mathrm{supp}(M)$ is a compact object which is isolated in $\mathrm{supp}(M)$. Then $C = M$.*

Proof. By Lemma 4.3, $\mathrm{KGdim}(\mathsf{K}) < \infty$ whence the isolation condition holds for $\mathrm{Zg}(\mathsf{K})$ by Lemma 2.7. Therefore there exists a functor $F \in \mathsf{Coh}(\mathsf{K})$ such that $\{C\} = (F) \cap \mathrm{supp}(M)$ and $q(F) \in (\mathsf{Coh}(\mathsf{K}))_{\mathrm{supp}(M)}$ is simple. Hence, by Lemma 5.1, C is a direct summand of M and hence $C = M$. □

Corollary 5.3. *Let M be an indecomposable object of K. If $C \in \mathrm{supp}(M)$ is compact then M is compact.*

Proof. If $C \in \mathsf{K}^b(\mathsf{proj}\text{-}\Lambda)$, then C is isolated in $\mathrm{Zg}(\mathsf{K})$ and so is also isolated in $\mathrm{supp}(M)$, so Lemma 2.7 applies. Otherwise, without loss of generality, assume that $\mathrm{supp}(M) \subseteq \mathrm{Zg}(\mathsf{K})'$, i.e. C is a compact but nonperfect object, in particular, Λ must have infinite global dimension. By Proposition 4.8 C has CB-rank 1, hence is isolated in $\mathrm{Zg}(\mathsf{K})'$ and hence in $\mathrm{supp}(M)$; now apply Corollary 5.2. □

6. Indecomposable complexes for infinite global dimension derived-discrete algebras

Recall notation and the structure of $\mathsf{add}(\mathsf{Str}\text{-}\Lambda)$ from Section 3.4 in the case $\mathrm{gldim}\,\Lambda = \infty$.

Setup 6.1. Let Λ be a derived-discrete algebra with $\mathrm{gldim}\,\Lambda = \infty$. Consider the noncompact, left-infinite string complexes $X^k_{i,\infty}$ forming the left-hand beams of the ladder type AR components $\tilde{\mathcal{Z}}^k$. For each $0 \le k < r$, there is a sequence of irreducible morphisms

$$\cdots \longrightarrow X^k_{i-1,\infty} \xrightarrow{t^k_{i-1}} X^k_{i,\infty} \xrightarrow{t^k_i} X^k_{i+1,\infty} \longrightarrow \cdots$$

We define $X^k_\infty := \bigoplus_{i\in\mathbb{Z}} X^k_{i,\infty}$ and write $\widetilde{X} := \bigoplus_{k=0}^{r-1} X^k_\infty$.

In Theorem 6.8 we obtain the pure-injectivity of each indecomposable complex of K as a corollary of the fact that $\widetilde{X}$ is Σ-pure-injective – a strengthening of the definition of pure-injectivity.

Definition 6.2. An object of T is Σ-*pure-injective* if the coproduct $N^{(I)}$ is pure-injective for any (possibly infinite) set I. Equivalently, N is Σ-pure-injective if and only if for each $C \in \mathsf{T}^{\mathrm{c}}$, $\mathrm{Hom}_{\mathsf{T}}(C, N)$ satisfies the descending chain condition on $\mathrm{End}_{\mathsf{T}}(N)$-submodules.

Remark 6.3. There are convenient references for the equivalence above and related results in the context of module and functor categories, hence which apply directly to $\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}}$. In view of the equivalence, $\mathsf{Pinj}(\mathsf{T}) \simeq \mathsf{Inj}(\mathsf{Mod}\text{-}\mathsf{T}^{\mathrm{c}})$ after Proposition 1.2, between objects and, Lemma 1.9, between concepts involving definability, these references thus apply equally to compactly generated triangulated categories. We will use the facts that any object of finite endolength is Σ-pure-injective (see [32, Cor. 4.4.24]), that a direct sum of finitely many Σ-pure-injective objects is Σ-pure-injective (see [32, Lem. 4.4.26]), and that if M is Σ-pure-injective then every object in the definable subcategory generated by M is Σ-pure-injective (see [32, Prop. 4.4.27]).

Lemma 6.4. *Suppose we are in the situation of Setup 6.1. If C is a perfect string complex, then $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty)$ satisfies the descending chain condition on $\mathrm{End}_{\mathsf{K}}(X^k_\infty)$-submodules.*

Proof. Since C is compact, $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty})$. By Proposition 3.15, only finitely many of the $X^k_{i,\infty}$ admit nontrivial morphisms from C, making the right-hand side of the isomorphism above a finite direct sum. Moreover, by Proposition 3.4, each $\mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty})$ is a finite-dimensional $\mathbf{k}$-vector space, whence $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty)$ is finite-dimensional. It follows that $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty)$ satisfies the descending chain condition on $\mathrm{End}_{\mathsf{K}}(X^k_\infty)$-submodules. □

We now turn to the case that C is a nonperfect compact string complex, i.e. by Remark 3.12, C is a right-infinite string complex.

Lemma 6.5. *Suppose we are in the situation of Setup 6.1. Suppose $C = X^l_{-\infty,b}$ for some $0 \le l < r$ and $b \in \mathbb{Z}$. Then $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty)$ satisfies the descending chain condition on $\mathrm{End}_{\mathsf{K}}(X^k_\infty)$-submodules.*

Proof. By Proposition 3.4, we have $\dim \mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty}) \le 1$. By Proposition 3.15 there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty}) = 0$ for all $i > N$; assuming $\dim \mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty}) = 1$ for some i, we may choose N minimal. Starting with $i = N$, we can use Propositions 3.16 and 3.18 to define a family of morphisms $c_i \colon C \to X^k_{i,\infty}$ such that for each $j \ge 1$ we have $c_{i+j} = t^k_{i+j-1} \cdots t^k_i c_i$. By compactness of C and the one-dimensionality of $\mathrm{Hom}_{\mathsf{K}}(C, X^k_{i,\infty})$, we have $\{c_i \mid c_i \neq 0,\, i \in \mathbb{Z}\}$ is a basis for the Hom-space $\mathrm{Hom}_{\mathsf{K}}(C, X^k_\infty)$.

Suppose $M \subseteq \mathrm{Hom}_{\mathsf{K}}(C, X_\infty^k)$ is an $\mathrm{End}_{\mathsf{K}}(X_\infty^k)$-submodule. If $c_i \in M$ then $c_{i+1} = t_i^k c_i \in M$ for each $i \in \mathbb{Z}$. Therefore, if M is a proper submodule of $\mathrm{Hom}_{\mathsf{K}}(C, X_\infty^k)$ the set $\{i \mid c_i \in M\}$ has a minimal element i_0, so the dimension of M is finite (namely $N - i_0 + 1$) and the result follows. □

Proposition 6.6. *Suppose we are in the situation of Setup 6.1. Then the direct sum of all left-infinite string complexes, $\widetilde{X} := \bigoplus_{k=0}^{r-1} X_\infty^k$, is Σ-pure-injective.*

Proof. By Lemmas 6.4 and 6.5, $\mathrm{Hom}_{\mathsf{K}}(C, X_\infty^k)$ satisfies the descending chain condition for each $0 \le k < r$ and each indecomposable compact object C of K. Since the functor $\mathrm{Hom}_{\mathsf{K}}(-, X_\infty^k)$ commutes with finite direct sums, it follows that $\mathrm{Hom}_{\mathsf{K}}(C, X_\infty^k)$ satisfies the descending chain condition for each compact object C of K. By the equivalent formulation of Σ-pure-injectivity in Definition 6.2, it follows that X_∞^k is Σ-pure-injective. By Remark 6.3, $\widetilde{X}$ is Σ-pure-injective. □

Corollary 6.7. *Let Λ be a derived-discrete algebra with* $\operatorname{gldim} \Lambda = \infty$. *If M is an indecomposable object of* K *such that* $\operatorname{supp}(M)$ *contains only noncompact objects, then M is Σ-pure-injective.*

Proof. Let $Z := \widetilde{X} \oplus \big(\bigoplus_{k=0}^{r-1} Z_\infty^k\big)$ be the direct sum of all noncompact objects in the Ziegler spectrum $\mathrm{Zg}(\mathsf{K})$, see Remark 3.12. Then Z is Σ-pure-injective since it is a finite direct sum of Σ-pure-injective objects (by Proposition 6.6 and 3.8). Then the definable subcategory $\langle Z \rangle$ contains only Σ-pure-injective objects. But by assumption $\langle M \rangle \subseteq \langle Z \rangle$ and therefore M is Σ-pure-injective. □

Putting this together gives Theorem B in the infinite global dimension case.

Theorem 6.8. *Suppose Λ is a derived-discrete algebra with* $\operatorname{gldim} \Lambda = \infty$. *The objects of* Str-Λ *form a complete list of indecomposable objects of* K. *In particular, each indecomposable object of* K *is pure-injective.*

Proof. Suppose M is an indecomposable object of K. If $\operatorname{supp}(M)$ contains a compact pure-injective then M itself is compact by Corollary 5.3 and therefore pure-injective; see Proposition 3.6 and Remark 3.12. So we may assume that $\operatorname{supp}(M)$ contains no compact pure-injective objects, whence Corollary 6.7 tells us that M is in fact Σ-pure-injective. Now Theorem 4.1 tells us that Str-Λ is a complete list of indecomposable, pure-injective complexes in K. □

7. Indecomposable complexes for finite global dimension derived-discrete algebras

In this section, we complete the proof that all indecomposable complexes in K for a derived-discrete algebra Λ are pure-injective by treating the finite global dimension case. We refer to Section 3.3 for the structure of the AR quiver whose vertices are the indecomposable pure-injective complexes. We start by showing that any indecomposable complex whose support contains only pure-injective complexes of CB rank 2 is already on our list.

Lemma 7.1. *Let M be an indecomposable object in K. If every indecomposable pure-injective in* $\operatorname{supp}(M)$ *has Cantor–Bendixson rank* 2*, then M is Σ-pure-injective.*

Proof. By Corollary 4.17, there are only finitely many pure-injective objects with Cantor–Bendixson rank 2. By Corollary 3.8 each has finite endolength, hence is Σ-pure-injective. Thus, $\operatorname{supp}(M)$ is a finite set in which each object is Σ-pure-injective.

Let N be the direct sum of one copy of each object of $\operatorname{supp}(M)$. Then N is Σ-pure-injective, hence every object in the definable subcategory that N generates, in particular M, is Σ-pure-injective by Remark 6.3. □

We shall invoke the following setup.

Setup 7.2. Let $\mathbf{X}_0 \subset \mathrm{Zg}(\mathsf{K})$ be the closed set of non-isolated points and consider the localisation adjoint pair from Section 2.4,

$$\mathsf{Mod}\text{-}\mathsf{K}^{\mathrm{c}} \underset{i}{\overset{q}{\rightleftarrows}} (\mathsf{Mod}\text{-}\mathsf{K}^{\mathrm{c}})_{\mathbf{X}_0},$$

where the canonical inclusion i is right adjoint to the localisation functor q.

Let M be an indecomposable complex in K. By Corollary 5.3 and Lemma 7.1, we may assume that $(-, M)$ is torsionfree and $\operatorname{supp}(M)$ contains no compact objects and at least one object of CB rank 1. Let $N \in \operatorname{supp}(M)$ be such a complex of CB rank 1. By Section 3.3 and Proposition 4.8, N lies in one of the A_∞^∞ components of $\mathsf{add}(\mathsf{Str}\text{-}\Lambda)$.

We must consider four cases: N is either $X^k_{a,\infty}, Y^k_{\infty,a}, X^k_{-\infty,a}$ or $Y^k_{a,-\infty}$ for some $a \in \mathbb{Z}$ and $0 \leq k \leq r-1$. Recall from Lemma 3.10 that we have the following sequences of morphisms such that every composition is non-zero:

$$X^k_{a,a} \to X^k_{a,a+1} \to \cdots X^k_{a,\infty} \cdots \to Z^k_{a,a-1} \to Z^k_{a,a} \to Z^k_{a,a+1} \to \cdots$$

$$Y^k_{a,a} \to Y^k_{a+1,a} \to \cdots Y^k_{\infty,a} \cdots \to Z^k_{a-1,a} \to Z^k_{a,a} \to Z^k_{a+1,a} \to \cdots$$

$$\cdots \to \Sigma^{-1} Z^k_{a+1,a} \to \Sigma^{-1} Z^k_{a+1,a+1} \to \Sigma^{-1} Z^k_{a+1,a+2} \to \cdots X^k_{-\infty,a} \cdots \to X^k_{a-1,a} \to X^k_{a,a}$$

$$\cdots \to \Sigma^{-1} Z^k_{a,a+1} \to \Sigma^{-1} Z^k_{a+1,a+1} \to \Sigma^{-1} Z^k_{a+2,a+1} \to \cdots Y^k_{a,-\infty} \cdots \to Y^k_{a,a-1} \to Y^k_{a,a}.$$

Let

$$\mathcal{L} := \begin{cases} \mathbb{Z}_{\geq a} \times \mathbb{Z} & \text{if } N = X^k_{a,\infty} \text{ or } Y^k_{\infty,a} \\ \mathbb{Z} \times \mathbb{Z}_{\geq a} & \text{if } N = X^k_{-\infty,a} \text{ or } Y^k_{a,-\infty} \end{cases}$$

with the ordering $(i,j) \leq (s,t)$ whenever $i \leq s$ and $j \geq t$ with equality exactly when $i = s$ and $j = t$. Then, for each $(i,j) \in \mathcal{L}$, fix a morphism $g_{ij}\colon C_i \to D_j$ and let $G_{ij} := (-, C_i)/\ker(-, g_{ij})$. In each case C_i and D_j are defined as follows:

(1) If $N = X^k_{a,\infty}$, then let $C_i := X^k_{a,i}$ and $D_j := Z^k_{a,j}$.
(2) If $N = Y^k_{\infty,a}$, then let $C_i := Y^k_{i,a}$ and $D_j := Z^k_{j,a}$.
(3) If $N = X^k_{-\infty,a}$, then let $C_i := \Sigma^{-1} Z^k_{a+1,i}$ and $D_j := X^k_{j,a}$.
(4) If $N = Y^k_{a,-\infty}$, then let $C_i := \Sigma^{-1} Z^k_{i,a+1}$ and $D_j := Y^k_{a,j}$.

Lemma 7.3. *Suppose we are in the situation of Setup 7.2 and let $(i,j) \in \mathcal{L}$. Then we have the following embeddings*

$$G_{ij} \hookrightarrow G_{st} \hookrightarrow G_{s't'} \hookrightarrow (-, N)$$

for all $(i,j) \leq (s,t) \leq (s',t')$. Moreover, we have $\bigcup_{(i,j)\leq(s,t)} G_{st} \cong (-, N)$.

Proof. For $r > 1$, using that Hom-spaces are either one-dimensional or zero and the factorisation properties described in Section 3.8, one can check that $G_{ij}(C) \hookrightarrow G_{st}(C) \hookrightarrow G_{s't'}(C) \hookrightarrow (C, N)$ for all $C \in \mathsf{K}^{\mathrm{c}}$. When $r = 1$, note that the composition of two maps each factoring through an object in an $\mathcal{X}_\infty$ component is zero (see, for example, the argument in the $r = 1$ case of the proof of Proposition 3.13). In particular, this means that $\dim G_{ij}(C) \leq 1$, and one can use the same argument as in the $r > 1$ case.

For the second claim note that, for every $C \in \mathsf{K}^{\mathrm{c}}$, there exists some $(s,t) \geq (i,j)$ such that $G_{st}(C) \cong (C, N)$. To see this one should consider the diagrams in Fig. 4. □

Lemma 7.4. *In the situation of Setup 7.2, the functor $q(-, N)$ is injective with simple socle.*

Proof. Since N has CB-rank 1, by Proposition 2.8, there is a functor $S' \in \mathsf{Coh}(\mathsf{K})$ whose image is simple in $\mathsf{Coh}(\mathsf{K})/\mathsf{Coh}(\mathsf{K})_0 = \mathsf{Coh}(\mathsf{K})_{\mathbf{X}_0}$ and such that N is the unique point of CB-rank 1 in the open set (S'). Then we continue as in the proof of Lemma 5.1. □

Lemma 7.5. *Suppose we are Setup 7.2.*

(1) *If $N \in \mathcal{X}_\infty \cup \mathcal{Y}_\infty$, then qG_{aj} is simple for all $j \in \mathbb{Z}$. That is qG_{aj} is isomorphic to the simple socle of $q(-, N)$.*
(2) *If $N \in \mathcal{X}_{-\infty} \cup \mathcal{Y}_{-\infty}$, then qG_{ia} is simple for all $i \in \mathbb{Z}$. That is qG_{ia} is isomorphic to the simple socle of $q(-, N)$.*

Proof. By Corollary 1.12, it suffices to exhibit 1-simple morphisms h_∞ and $h_{-\infty}$ such that $qG_{aj} \cong q\ker(-, h_\infty)$ and $qG_{ia} \cong q\ker(-, h_{-\infty})$.

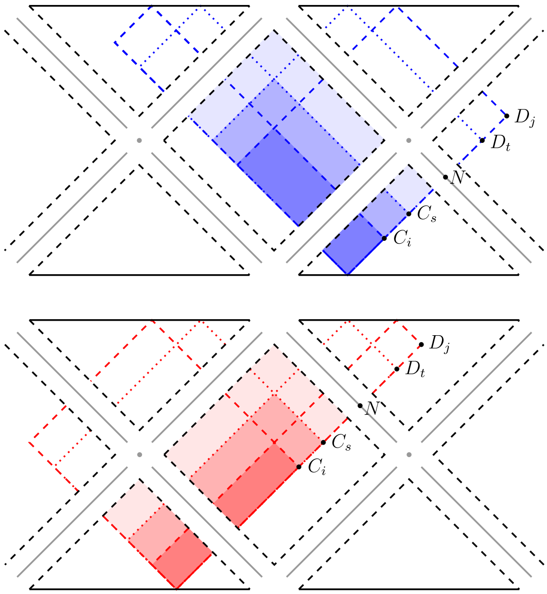


Fig. 4. In both pictures the darkest shaded region shows the compact objects C such that $G_{ij}(C) \neq 0$. The intermediate shaded region together with the darkest shaded region shows the compact objects C such that $G_{st}(C) \neq 0$. The three shaded regions together show the compact objects C such that $(C, N) \neq 0$.

(1) If $N = X^k_{a,\infty}$, then let $h_\infty \colon Z^k_{a,j} \to Z^k_{a+1,j} \oplus X^{k+1}_{a'-1,a'-1}$. If $N = Y^k_{\infty,a}$, then let $h_\infty \colon Z^k_{j,a} \to Z^k_{j,a+1} \oplus Y^{k+1}_{a''-1,a''-1}$. Then $\ker(-, h_\infty) \subseteq G_{aj}$ and $G_{aj}/\ker(-, h_\infty)$ is only non-zero on $X^k_{a,a}$ (respectively on $Y^k_{a,a}$). We may therefore conclude that $qG_{aj} \cong q\ker(-, h_\infty)$ cf. Remark 4.15.

(2) If $N = X^k_{-\infty,a}$, then let $h_{-\infty} \colon X^k_{a,a} \to Z^k_{a,i}$. If $N = Y^k_{a,-\infty}$ then let $h_{-\infty} \colon Y^k_{a,a} \to Z^k_{i,a}$. Comparing Hom-hammocks, we have that $G_{ia} \cong \ker(-, h_{-\infty})$ and hence $qG_{ia} \cong q\ker(-, h_{-\infty})$. □

We are now ready to prove the main theorem of this section, which is Theorem B in the finite global dimension case.

Theorem 7.6. *Suppose Λ is a derived-discrete algebra such that* $\operatorname{gldim} \Lambda < \infty$. *The objects of* Str-Λ *form a complete list of indecomposable objects of* K. *In particular, each indecomposable object of* K *is pure-injective.*

Proof. Theorem 4.1 tells us that Str-Λ is a complete list of indecomposable, pure-injective objects of K. We need to check that there are no further indecomposable objects.

Suppose, for a contradiction, $M \in \mathsf{ind}(\mathsf{K})$ is not a string complex. By Corollary 5.3 and Lemma 7.1, we may assume that M satisfies the hypotheses of Setup 7.2.

By Lemma 7.4, there is an embedding $qS \hookrightarrow q(-, N)$ where qS is simple and, arguing as in the proof of Lemma 5.1, there is also an embedding $qS \hookrightarrow q(-, M)$. Next, we will argue that this extends to a split monomorphism $(-, N) \hookrightarrow (-, M)$. We argue explicitly for the case where $N \in \mathcal{X}_\infty \cup \mathcal{Y}_\infty$; the argument for $N \in \mathcal{X}_{-\infty} \cup \mathcal{Y}_{-\infty}$ can be obtained by replacing every instance of G_{aj} with G_{ia}.

Indeed, by Lemma 7.5 we have $qS = qG_{aj}$ and so there is an embedding $qG_{aj} \hookrightarrow q(-, M)$ for each $j \in \mathbb{Z}$. Moreover, for every $(i, j) \in \mathcal{L}$, we have an embedding $qG_{aj} \to qG_{ij}$ with a finitely presented cokernel (as both qG_{aj} and qG_{ij} are finitely presented in $(\mathsf{Mod}\text{-}\mathsf{K}^{\mathrm{c}})_{\mathbf{X}_0}$). By Remarks 1.6 and 2.2, we have that $q(-, M)$ is fp-injective and so there exists a factorisation

$$\begin{array}{ccc} qG_{aj} & \hookrightarrow & qG_{ij} \\ \downarrow & \swarrow{\scriptstyle \exists} & \\ q(-, M). & & \end{array}$$

By Lemma 7.3, we obtain an embedding $q(-, N) \hookrightarrow q(-, M)$. Now using Remark 2.2, we can deduce that there is a nontrivial monomorphism $(-, N) \hookrightarrow (-, M)$. Pure-injectivity of N now means that $(-, N) \hookrightarrow (-, M)$ is a split monomorphism.

Thus, we have a pure epimorphism $f\colon M \to N$ such that the induced $(-, f)\colon (-, M) \to (-, N)$ is split by a morphism $\tau\colon (-, N) \to (-, M)$, that is, $(-, f)\tau = 1_{(-,N)}$. We want to find some $h : N \to M$ with $fh = 1_N$.

The next part of the argument requires some homotopy string combinatorics; see Section 3. Since N lies in an $\mathcal{X}^k_{\pm\infty}$ or $\mathcal{Y}^k_{\pm\infty}$ component, N is a one-sided string complex (Definition 3.3). For convenience of exposition, we assume that $N = P_w$ is a right-infinite string complex, where w is a right-infinite subword of w_∞. Moreover, we may assume without loss of generality that the projective Λ-module sitting in cohomological degree n, N^n, is zero for all $n < 0$ and $N^0 \neq 0$. An analogous argument holds in the case that N is a left-infinite string complex.

There is a sequence of finite subwords of w corresponding to a sequence of perfect complexes $C_i = P_{v_i}$ in which $C_i^n = 0$ for each $n < 0$ and $C_i^0 \neq 0$,

$$v_0 \subset v_1 \subset v_2 \subset \cdots \subset w \quad \longleftrightarrow \quad C_0 \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} \cdots \to N, \tag{2}$$

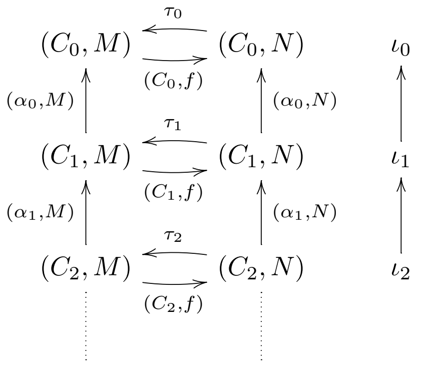
together with a graph map $\iota_i\colon C_i \to N$, such that each α_i is a graph map and each morphism $\iota_i\colon C_i \to N$ factors as

$$\begin{array}{ccc} C_i & \xrightarrow{\alpha_i} & C_{i+1}. \\ & {\scriptstyle \iota_i}\searrow & \downarrow{\scriptstyle \iota_{i+1}} \\ & & N \end{array}$$

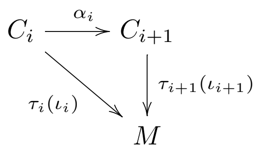
We remark that the sequence (2) is a subsequence of indecomposable complexes lying on an appriopriately chosen ray or coray in an $\mathcal{X}^k$, $\mathcal{Y}^k$ or $\mathcal{Z}^k$ component.

Define length $\iota_i = \max\{n \mid (\iota_i)^n \neq 0\}$. In particular, this means that $(\iota_i)^n = \mathrm{id}_{C^n} = \mathrm{id}_{N^n}$ for all $n \leq \operatorname{length} \iota_i - 1$ by [1, §3].

Applying $(-,f)\colon (-,M) \to (-,N)$ to each $\alpha_i\colon C_i \to C_{i+1}$ gives a commutative tower in which $\tau_i\colon (C_i,N) \to (C_i,M)$ is the splitting map:



From the commutativity of the squares in the tower above, we get



We define a map $h\colon N \to M$ componentwise $h^n\colon N^n \to M^n$ by

$$h^n := h_i^n := (\tau_i(\iota_i))^n \circ ((\iota_i)^n)^{-1} \text{ for all } n < \operatorname{length}\iota_i,$$

noting that since ι_i is a graph map that $(\iota_i)^n$ is an isomorphism for each $n < \operatorname{length}\iota_i$. We first need to check that this is well defined, i.e. if $j \geq i$ then $h_i^n = h_j^n$ for $n < \operatorname{length}\iota_i$. Writing $\alpha_{ij} := \alpha_{j-1}\cdots\alpha_i$, observe that $\operatorname{length}\alpha_{ij} = \operatorname{length}\alpha_i = \operatorname{length}\iota_i$ and that for $n < \operatorname{length}\alpha_i$, the n^{th}-component $(\alpha_{ij})^n$ is an isomorphism. Hence,

$$\begin{aligned} h_j^n &= (\tau_j(\iota_j))^n \circ ((\iota_j)^n)^{-1} \\ &= \big((\tau_j(\iota_j))^n \circ (\alpha_{ij})^n\big) \circ \big(((\alpha_{ij})^n)^{-1} \circ ((\iota_j)^n)^{-1}\big) \\ &= (\tau_i(\iota_i))^n \circ ((\iota_i)^n)^{-1} = h_i^n. \end{aligned}$$

Next, we have to show that h is a cochain map. By choosing i sufficiently large, it is clear that $h^{n+1}d_N^n = d_M^n h^n$ for all $n < \operatorname{length}\iota_i - 1$, in particular, h is a cochain map.

Finally, we show that $fh = \mathrm{id}_N$, for which it is enough to show that, for each n, we have $f^n \circ h^n = \mathrm{id}_{N^n}$. By the splitting we have $(C_i,f)\big(\tau_i(\iota_i)\big) = \iota_i$, that is, $f \circ \tau_i(\iota_i) = \iota_i$ which, in the n^{th} component, becomes $f^n \circ (\tau_i(\iota_i))^n = (\iota_i)^n$, whence $f^n \circ (\tau_i(\iota_i))^n \circ ((\iota_i)^n)^{-1} = \mathrm{id}_{N^n}$, that is $f^n \circ h^n = \mathrm{id}_{N^n}$, as required. We have therefore found a split monomorphism $h\colon N \hookrightarrow M$, hence $M = N$. $\square$

Remark 7.7. We finish by remarking that, although every indecomposable complex over a derived-discrete algebra is pure-injective, there are complexes which are neither pure-injective nor a direct sum of indecomposable objects. One may see this directly or use [4, Thm. 9.3] or [25, Thm. 2.10].

Acknowledgments

The first author kindly acknowledges The University of Manchester for their hospitality on a research visit there where this work was begun. The second author was supported by a School of Mathematics, University of Manchester, Scholarship. The last two named authors acknowledge the financial support of the EPSRC through grant EP/K022490/1. We would also like to thank the referee for his/her detailed reading, corrections and extensive suggestions.

References

[1] K.K. Arnesen, R. Laking, D. Pauksztello, Morphisms between indecomposable complexes in the bounded derived category of a gentle algebra, J. Algebra 467 (2016) 1–46.
[2] M. Auslander, Representation theory of Artin algebras II, Comm. Algebra 1 (1974) 269–310.
[3] V. Bekkert, H.A. Merklen, Indecomposables in derived categories of gentle algebras, Algebr. Represent. Theory 6 (3) (2003) 285–302.
[4] A. Beligiannis, Relative homological algebra and purity in triangulated categories, J. Algebra 227 (1) (2000) 268–361.
[5] D.J. Benson, G.Ph. Gnacadja, Phantom maps and purity in modular representation theory I, Fund. Math. 161 (1–2) (1999) 37–91.
[6] G. Bobiński, The almost split triangles for perfect complexes over gentle algebras, J. Pure Appl. Algebra 215 (4) (2011) 642–654.
[7] G. Bobiński, C. Geiß, A. Skowroński, Classification of discrete derived categories, Cent. Eur. J. Math. 2 (1) (2004) 19–49 (electronic).
[8] G. Bobiński, H. Krause, The Krull–Gabriel dimension of discrete derived categories, Bull. Sci. Math. 139 (3) (2015) 269–282.
[9] N. Broomhead, D. Pauksztello, D. Ploog, Discrete derived categories II: the silting pairs CW complex and the stability manifold, J. Lond. Math. Soc. 93 (2) (2016) 273–300.
[10] N. Broomhead, D. Pauksztello, D. Ploog, Discrete derived categories I: homomorphisms, autoequivalences and t-structures, Math. Z. 285 (1–2) (2017) 39–89.
[11] I. Burban, Y. Drozd, Derived categories of nodal algebras, J. Algebra 272 (1) (2004) 46–94.
[12] I. Çanakçi, D. Pauksztello, S. Schroll, Mapping cones in the bounded derived category of a gentle algebra, arXiv:1609.09688, 2016.
[13] W.W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (5) (1994) 1641–1674.
[14] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary Artin algebra and applications to the structure of exact sequences, Manuscripta Math. 54 (1–2) (1985) 83–106.
[15] C. Geiß, I. Reiten, Gentle algebras are Gorenstein, in: Representations of Algebras and Related Topics, in: Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133.
[16] Y. Han, C. Zhang, Brauer–Thrall type theorems for derived module categories, Algebr. Represent. Theory 19 (6) (2016) 1369–1386.
[17] Z. Han, The Homotopy Categories of Injective Modules of Derived Discrete Algebras, PhD thesis, Bielefeld University, 2013.
[18] D. Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[19] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. Lond. Math. Soc. (3) 74 (3) (1997) 503–558.

[20] P. Jørgensen, The homotopy category of complexes of projective modules, Adv. Math. 193 (1) (2005) 223–232.

[21] B. Keller, Deriving DG categories, Ann. Sci. Éc. Norm. Supér. (4) 27 (1) (1994) 63–102.

[22] B. Keller, On differential graded categories, in: International Congress of Mathematicians, vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[23] H. Krause, Generic modules over Artin algebras, Proc. Lond. Math. Soc. (3) 76 (2) (1998) 276–306.

[24] H. Krause, Decomposing thick subcategories of the stable module category, Math. Ann. 313 (1) (1999) 95–108.

[25] H. Krause, Smashing subcategories and the telescope conjecture—an algebraic approach, Invent. Math. 139 (1) (2000) 99–133.

[26] H. Krause, Coherent functors in stable homotopy theory, Fund. Math. 173 (1) (2002) 33–56.

[27] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. 9 (1) (1996) 205–236.

[28] A. Neeman. Review of, D. Bensen, G.Ph. Gnacadja, Phantom maps and purity in modular representation theory I, in: Mathematical Reviews, 1999.

[29] A. Neeman, Triangulated Categories, Ann. of Math. Stud., vol. 148, Princeton University Press, Princeton, NJ, 2001.

[30] A. Neeman, The homotopy category of flat modules, and Grothendieck duality, Invent. Math. 174 (2) (2008) 255–308.

[31] M. Prest, Model Theory and Modules, London Mathematical Society Lecture Note Series, vol. 130, Cambridge University Press, Cambridge, 1988.

[32] M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009.

[33] M. Prest, Categories of imaginaries for definable additive categories, arXiv:1202.0427, 2012.

[34] Y. Qin, Jordan–Hölder theorems for derived categories of derived discrete algebras, J. Algebra 461 (2016) 295–313.

[35] B. Stenström, Rings of Quotients, Die Grundlehren der Mathematischen Wissenschaften, vol. 217, Springer-Verlag, New York–Heidelberg, 1975.

[36] D. Vossieck, The algebras with discrete derived category, J. Algebra 243 (1) (2001) 168–176.

[37] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic 26 (2) (1984) 149–213.