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Abstract

We continue with the application of the linear model described in [2016WP22] to the Italian soccer championship. We consider the simulation taking the data from the final results of the 2016-2017 championship. We intend first to see if the model gives rise again to some discrepancies with the official final results and we want to study the reasons for that.

The problem of ranking a set of elements, namely giving a "rank" to the elements of the set, may arise in very different contexts and may be handled in some possible different ways, depending on the ways these elements are set in competition the ones against the others. In this working paper we deal with a so called even paired competition, where the pairings are evenly matched: in a national soccer championship actually each team is paired with every other team the same number of times.

A mathematically based ranking scheme can be easily defined in order to get the scores for all the teams. The underlined structure of the model depends on the existence and uniqueness of a particular eigenvalue of the preference matrix. At this point the Perron–Frobenius theorem is involved. In the previous working paper [2016WP22] we showed how in the ranking process some fundamental Linear Algebra concepts and results are important, mainly the dominant eigenvalue and a corresponding eigenvector. The linear ranking model was also applied to a first numerical simulation. This gave evidence of some discrepancies between the actual final placements of teams and the ones provided by the model. We want to go here into a more detailed study about this aspect.

Keywords. Ranking scheme, Linear transformation, Eigenvalues, Dominant eigenvalue

AMS Classification. 15A18

JEL Classification. C65, C69

1 Introduction

In [1] we first showed how a linear scheme in the ranking process directly takes to some fundamental Linear Algebra concepts and results, mainly the eigenvalues and eigenvectors of linear transformations and the Perron–Frobenius theorem. We applied also the linear ranking model to a numerical simulation, taking the data from the Italian soccer championship 2015-2016. Some interesting differences in the final ranking appeared, by comparing the actual placements of the teams at the end of the contest with the mathematical scores provided to teams by the theoretical model.

We want here to test if the same differences appear also in the soccer championship 2016-2017.

The model is taken from the interesting in-depth analysis on general ranking methods in [2], that was mainly concentrated on American football teams ranking. Within certain hypotheses the ranking problem may be easily formulated as a linear eigenvalue problem. In [3,4] the mathematical background for the linear approach was described, with reference both to the football teams case and the web page ranking, in the original definition of PageRank algorithm by Google. The linear mathematical approach involves some Linear Algebra notions and results. Among these, the concepts of eigenvalues and eigenvectors of a square matrix are fundamental, together with Perron–Frobenius theorem on the so called dominant eigenvalue.

We may generally describe a ranking problem as the need to assign a rank to the elements of a certain finite set. As a consequence of the ranking process we can get an order relation on the set, but the ranking scheme is a model for getting individual rankings to the elements, that is stronger than just having the order relation. Ranking schemes can be applied on a variety of situations, such as ranking soccer teams, or in general ranking teams or single individuals of a certain sport, ranking web pages, area of interest of search engines, ranking customers preferences, which marketing operators are interested in.

The aim of a mathematical ranking scheme may be to introduce some objectivity in the definition of the rank itself. It must be pointed out anyway that it is impossible to remove all the subjectivity, as in the process we have to give values to some parameters, and there is no a completely impartial way to do that.

The reasons why Perron–Frobenius theorem fits well in some ranking problems are quite straightforward and do apply more in general in each setting where a linear approach is taken to get the ranking method. In [3] and [4], as mentioned, some notes on web pages ranking were presented, by recalling that the original Google's Pagerank algorithm adopts a linear model to obtain such a ranking.

2 The linear ranking model

Let's briefly recall the model that can be used to define the ranking method.

Suppose we have a contest, a competition, with a number N of participants. We want to assign a score to the participants. We can do it in a sort of iterative steps. We may assume an initial rank for all the participants.[1]

Then we may think to update the ranks (the scores) by means of the information we can get from the results of the matches. This can be done each week taking the new last results or can be done at the end taking all the results of the year.

[1]The initial rank could be given, think of this year major soccer league for example, by the "strength" a team is supposed to have from its history. A team that won many previous championships is "strong", while a team that has passed to the major league from a secondary league is not.

The non objective part of the process is not actually on the initial ranks, that could be influenced by external and not updated information. The initial ranks will be modified anyway by the process itself taking the fresh information. It is instead in how we decide to use the results in the process. We could for example give the same "negative weight" to a team if it loses a match, but we could conversely give a "parametric weight" for a defeat: if a weak team loses against a strong team the weight could be small, while if a strong team loses against a weak team the weight should be big. We could also state that a strong rival gives me some (positive) score even if I lose not so badly the direct match with it. It is interesting to remark anyway that the model itself is automatically going to take care of these aspects in some way.

Anyway it is reasonable to assume that the score is based on the interactions with other participants and should depend on the outcome of the interaction and/or strength of its rivals. In the soccer or other sport contests the interaction is of course the match, where we have a final result saying who is the winner. In other contests, where there is no a real and direct interaction among participants, it may be tricky to define the weights of interactions.

Let's go into more mathematical details. We may define a vector of ranking values r, with positive components r_j, indicating the strength of the jth participant.[2]

The definition of the scores, that is the way in which we update the ranks after the results of all the matches, is a crucial step in the model as, how it happens in general, it may completely influence the model behaviour. By following [2], in the soccer case we may define a score for the ith participant as

$$s_i = \frac{1}{n_i} \sum_{j=1}^{N} a_{ij} r_j, \tag{1}$$

where N is the total number of participants in the contest, a_{ij} is a nonnegative number related to the outcome of the game between participant i and participant j, and n_i is the number of games played by participant i. This last parameter may be dropped in a contest like the Italian soccer championship. At the end all the teams must have played the same number of times. It may be different in other competitions.

Remark. This is a quite general definition, that may be applied in some different situations. We did not distinguish yet between even and uneven paired competitions. An even paired competition is the one in which each team plays against all the others. It is the case of the national Italian premier soccer league, where if there are N teams, each team plays $2(N-1)$ matches (home and away games) and there are $2(N-1) \cdot \frac{N}{2} = N(N-1)$ matches in whole. An uneven paired competition is the one in which each team does not play against all the others. It is the case for example of the European Champions League, where the pairings are not evenly matched. In this case the pairings are randomly chosen for the different phases of the event. The definition in (1) may be applied to both even and uneven pairings: in the even case the parameter n_i may be discarded, as the number of games is the same for all the teams, while the parameter may be important in the uneven case.

Remark. The linear structure of the scoring model is evident. The score of the ith team depends on all the ranks of the other teams by means of coefficients that characterizes the interaction between team i and other teams j. In the definition the division by n_i can be seen as a sort of normalization: it may be important in case some additional games are possible. But, apart

[2] Again, this can be an "initial guess", an initial assignment of ranks given to teams, based on the previous history. A natural question arises: how much important is this first assignment for the final result of the ranking process? We shall come back to this later on.

from this more technical aspect, the definition takes anyway to a classical linear model

$$s_i = \sum_{j=1}^{N} \frac{a_{ij}}{n_i} r_j = \sum_{j=1}^{N} b_{ij} r_j. \tag{2}$$

Some remarks on the coefficients a_{ij} (or b_{ij}): they are related to interactions between participants and they may take into account the different aspects of the specific field where we have the ranking problem. In the soccer case a simple and straightforward possibility is to set $a_{ij} = 1$ if team i won the game against team j, $a_{ij} = \frac{1}{2}$ if the game ended in a tie and $a_{ij} = 0$ if team i lost the game against team j.[3]

Equation (2) may get a matrix form: if s is the N-vector of scores and r is the N-vector of ranks, (2) is equivalent to

$$s = Ar \tag{3}$$

where A is the square matrix of the coefficients a_{ij} (let's take for simplicity $n_i = 1$ for each i). A is usually called *preference matrix* or *transition matrix*.

2.1 Eigenvalues of positive matrices

Let us recall some mathematical notions about eigenvalues and positive matrices.

Let A be an $n \times n$ matrix.[4]

Definition 1 *A scalar λ and a vector $x \neq 0$ satisfying $Ax = \lambda x$ are called, respectively, an* ***eigenvalue*** *and an* ***eigenvector*** *of A. The set $\sigma(A)$ of eigenvalues is called the* ***spectrum*** *of A.*

We have the following characterization for the eigenvalues of a matrix:

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad A - \lambda I \text{ is singular} \quad \Leftrightarrow \quad \det(A - \lambda I) = 0.$$

In the following we indicate with $R(A)$ the *range* (or the *image space*) of A, that is

$$R(A) = \{Ax \mid x \in \mathbb{R}^n\}.$$

We indicate also with $N(A)$ the *nullspace* (or the *kernel*) of A, that is

$$N(A) = \{x \mid Ax = 0\}.$$

Both $R(A)$ and $N(A)$ are subspaces of $\mathbb{R}^n$.

If λ is an eigenvalue of the matrix A, then

$$\{x \in \mathbb{R}^n \mid x \neq 0 \ \wedge \ x \in N(A - \lambda I)\}$$

is the set of all the eigenvectors associated with λ. $N(A-\lambda I)$ is called the **eigenspace** associated with λ.

[3] For mathematical reasons, that will be clear in the following, in order to apply the theory we have in mind, it is necessary to have positive or at least nonnegative coefficients. This is why a loss takes zero and not a negative number.

[4] In applications A is usually a real matrix, a matrix whose elements are real numbers. Nevertheless the definitions are usually given in the more general field of complex numbers $\mathbb{C}$. This is because the eigenvalues of a real matrix are not guaranteed to be real in general. They are guaranteed to exist in the complex filed.

Definition 2 *The number*

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the ***spectral radius*** *of* A.

It is a fundamental concept in Perron–Frobenius theory.

Remark. As λ might be complex, the absolute value has to be intended in the complex field but, in any case, $\rho(A)$ is real. The maximum exists because $\sigma(A)$ is a finite set.

This is a very important definition, also in applications, because the precise knowledge of all the eigenvalues is not required often, as many properties depend just on how large eigenvalues are, and then the knowledge of an upper bound may be enough.

Remark. The spectral radius is not necessarily an eigenvalue of A.[5]

Definition 3 *A vector* v *(a matrix* A*) with positive entries is said a positive vector (a positive matrix). We write* $v > 0$ *(*$A > 0$*) to say that* v *is a positive vector (*A *is a positive matrix).*

There are some important results on the spectral radius and the related eigenvalues of a positive matrix. Some of these results are almost immediate, some others are deeper and deeper. Perron's theorem is usually presented as a summary of all of these. To have a summary of the results that may be introductory to Perron' theorem see [1].

With reference to the second remark before, if the spectral radius is not necessarily an eigenvalue for a general matrix, it is indeed like that for a positive matrix.

Positive eigenpair If A is an $n \times n$ positive (real) matrix then

- $\rho(A) \in \sigma(A)$.
- If $Ax = \rho(A)x$ then $A|x| = \rho(A)|x|$ and $|x| > 0$.

Remarks. The first item says that $\rho(A)$ is an eigenvalue of A (it's a positive one by definition).

The second item says that if x is an eigenvector associated with the eigenvalue $\rho(A)$, then also $|x|$ is an eigenvector associated with the same eigenvalue[6] and moreover it is a positive eigenvector. It means, by the way, that all the eigenvectors associated with $\rho(A)$ do not have null components. These two results say then that a positive matrix has a positive (real) eigenvalue, that is the spectral radius, and we have some positive eigenvectors associated to this eigenvalue.

Two other important aspects in the statement and meaning of Perron's theorem are the multiplicities of an eigenvalue.

Definition 4 *If* $\lambda \in \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$

- *the* algebraic multiplicity *of* λ *is the number of times it is repeated as a root of the characteristic polynomial. In other words, alg mult*$(\lambda_i) = a_i$, $i = 1, \ldots, t$, *if and only if* $(x-\lambda_1)^{a_1} \cdots (x-\lambda_t)^{a_t} = 0$ *is the characteristic equation for the matrix* A. *If alg mult*$(\lambda) = 1$, λ *is called a* simple eigenvalue.

[5]The real matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$ and then the spectral radius is 1, that is not an eigenvalue. But, as an even simpler case, with real eigenvalues, we may consider the matrix $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, where the eigenvalues are 0 and -1, and again the spectral radius is 1, that is not an eigenvalue.

[6]It is opportune to notice that this is true in this context, talking about $\rho(A)$, while it is not true in general: in other words, if λ is any eigenvalue and x is an eigenvector associated to λ, it is not true in general that $|x|$ is also an eigenvector of λ.

- *The* geometric multiplicity *of* λ *is* $\dim N(A - \lambda I)$. *In other words, geo mult*(λ_i) *is the maximal number of linearly independent eigenvectors associated with* λ.

- *Eigenvalues* λ *such that alg mult*(λ) = *geo mult*(λ) *are called* semisimple eigenvalues *of the matrix* A.

Remarks. For every matrix A and for each $\lambda \in \sigma(A)$

$$geo\ mult(\lambda) \leq alg\ mult(\lambda).$$

It is well known that for the matrix to have fundamental properties it is important the two multiplicities are equal or not.

A strong result can be proved about the algebraic multiplicity of the spectrum of a positive matrix.

Multiplicities of $\rho(A)$ If A is an $n \times n$ positive matrix then *alg mult*$(\rho(A)) = 1$, that is the spectral radius of A is a simple eigenvalue of A.

Remark. The consequence is that

$$\dim N(A - \rho(A)I) = geo\ mult(\rho(A)) = alg\ mult(\rho(A)) = 1,$$

that is the eigenspace associated to the spectral radius is a one-dimensional subspace of $\mathbb{R}^n$.

Remark. By putting together the previous remark and the result in the Positive eigenpair step, we get that the eigenspace $N(A - \rho(A)I)$ can be spanned by some positive vector v. Then there is a unique eigenvector $p \in N(A - \rho(A)I)$ such that $p > 0$ and $\sum_i p_i = 1$ (of course $p = v/\|v\|_1$). This eigenvector p is called the *Perron vector* for the positive matrix A. The associated eigenvalue $\rho(A)$ is called the *Perron eigenvalue.*

The set of results we just summarised is the core of Perron's theorem. Further results can be proved in association with these and sometimes they are part of the statement of the theorem. We mention only one of these, a sort of uniqueness result: there are no nonnegative eigenvectors for a positive matrix A other than the Perron vector p and its positive multiples. We give anyway a condensed version of the main results in the following subsection.

2.2 Perron's theorem

Perron's theorem for positive matrices is usually presented as a collection of many results, all related to the eigenvalue with the largest absolute value (also called *leading eigenvalue*). The results were due to Oskar Perron (1907) and concerned just positive matrices. Later, as already mentioned, Georg Frobenius (1912) found the extension to certain classes of nonnegative matrices.

A statement of Perron's theorem in a short form can be the following.

Theorem 1 (Perron's theorem for a positive matrix) *Let* A *be an* $n \times n$ *positive matrix and let* $\rho(A)$ *be the spectral radius of* A. *Then the following properties hold.*

(i) $\rho(A) \in \sigma(A)$. *It is called the Perron eigenvalue.*

(ii) alg mult$(\rho(A)) = 1$.

(iii) There exists an eigenvector $x > 0$ *associated to* $\rho(A)$. *The Perron eigenvector* p *is the unique eigenvector such that* $p > 0$ *and* $\|p\|_1 = 1$. *The matrix does not have other positive eigenvalues, except for positive multiples of* p.

(iv) $\rho(A)$ is the only eigenvalue in the spectral circle of A.

Remark. An interesting property that relate the Perron eigenvector to the elements of the matrix is the following.

- The Perron eigenvalue $\rho(A)$ satisfies the inequalities

$$\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}.$$

From a theoretical point of view the Perron eigenvalue of a positive matrix is important for existence and uniqueness reasons. But it comes to be equally important in applications. Moreover, there is a powerful numerical method for the computation of the Perron eigenvalue based on the powers of the matrix A (the so called *Power method*).

In applications is usually non so easy to have matrices with all positive elements. Apart from cases in which negative elements are necessary, it is more likely to have matrices with nonnegative elements.

Definition 5 *A vector v (a matrix A) with nonnegative entries is said a nonnegative vector (a nonnegative matrix). We write $v \geq 0$ ($A \geq 0$) to say that v is a nonnegative vector (A is a nonnegative matrix).*

The obvious question is if Perron's theorem holds with a nonnegative matrix and the answer is no, not in all the various aspects of its thesis. Consider for example the real nonnegative matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, then the spectral radius is

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda| = \max(\lambda_1, \lambda_2) = 1.$$

It is still true $\rho(A)$ is an eigenvector of A, its algebraic multiplicity is 1, it has a positive eigenvector associated to it, but it is not the only eigenvalue in the spectral circle, as -1 is also.

Frobenius did some studies with the aim of extending Perron's theorem to nonnegative matrices and he obtained very interesting results. Here is a very short formulation of the main properties of a nonnegative matrix regarding positive eigenvalues and positive eigenvectors associated to them.

Theorem 2 (Perron-Frobenius for nonnegative matrices) *If the matrix A has nonnegative entries, then there exist a positive eigenvalue ρ and an associated eigenvector r with nonnegative entries. Furthermore, if the matrix A is irreducible, then there exist a unique positive eigenvalue ρ, simple and largest in absolute value. It has an eigenvector r with strictly positive entries.*

2.3 A simulation from a real problem

Here we want perform a realistic simulation, by taking the data from the Italian soccer 2016-17 championship. We write the transition matrix by setting a choice of "weights" in a classical 0/0.5/1 for the three possible results of the matches.

In the Italian soccer 2016-17 championship there are 20 teams. By setting the transition matrix A with

$$a_{ij} = \begin{cases} 1 & \text{if team } i \text{ defeats team } j \\ 0.5 & \text{if teams } i \text{ and } j \text{ tie the match} \\ 0 & \text{if team } i \text{ looses against team } j \end{cases}$$

we obtain the 20×20 matrix in appendix (section ...). By reading the elements by row we have the results in the home matches for the teams in the first column. By reading the elements by column we have the results in the away matches for the teams in the first row.

```
-->spec(A)
 ans  =

    10.276192
  - 0.4969956 + 1.6054656i
  - 0.4969956 - 1.6054656i
  - 2.0886799
  - 1.8317836 + 0.4743035i
  - 1.8317836 - 0.4743035i
  - 0.5974409 + 1.1216704i
  - 0.5974409 - 1.1216704i
    0.6489670 + 0.3703199i
    0.6489670 - 0.3703199i
    0.0370148 + 0.7519015i
    0.0370148 - 0.7519015i
    0.5334854
    0.3652981
  - 1.0657733 + 0.6807498i
  - 1.0657733 - 0.6807498i
  - 0.4404790 + 0.2368074i
  - 0.4404790 - 0.2368074i
  - 0.7966572 + 0.0649151i
  - 0.7966572 - 0.0649151i
```

If we ask Scilab to compute the eigenvalues of the matrix A, we get the solution here on the right. First of all it must be pointed out that A is not a positive matrix. As mentioned previously in the more general theory about nonnegative matrices and spectral radius, due to Frobenius, we know that in the case of an irreducible matrix the main results of Perron's theorem hold.

Our matrix A, that is nonnegative, is irreducible and it shows anyway an evident single real eigenvalue with maximum absolute value. We may also notice, together with many pairs of complex conjugate eigenvalues, there is a couple of small real positive eigenvalues.

```
r = R*eye(20,1)
 r  =

    0.2807986
    0.1420395
    0.2168076
    0.1435373
    0.1340696
    0.1141136
    0.2879021
    0.2144048
    0.2471512
    0.3631080
    0.2364787
    0.2513399
    0.2833684
    0.0947280
    0.0889082
    0.3153101
    0.2149577
    0.1543224
    0.2425359
    0.1898624
```

If we ask Scilab to compute the eigenspaces, associated to the largest eigenvalue $\rho(A) \approx 10.276$, we get the positive vector r here on the left as a normalized eigenvector. This vector has just positive components, hence it satisfies the properties of the Perron eigenvector. Worthwhile to say the Scilab output in this case is a normalized vector in the 2-norm ($\|r\|_2 = 1$).

By using the ∞-norm we get the alternative eigenvector here on the right, where we have divided the vector on the left by its largest component ($\|r\|_\infty = r_{10} = 0.3631080$).

```
r = r/0.3631080
 r  =

    0.7733197
    0.3911769
    0.5970885
    0.3953021
    0.3692280
    0.3142689
    0.7928827
    0.5904712
    0.6806547
    1.0000001
    0.6512627
    0.6921905
    0.7803971
    0.2608810
    0.2448533
    0.8683646
    0.5919937
    0.4250042
    0.6679443
    0.5228813
```

Now we may compare the placements given by the ranking vector together with the actual final placements we had at the end of the last championship. Remember that the official rules of the Italian championship gives 3 points to the winning team, 1 point to both teams for a tie and 0 points to the losing team.

	points	rank
JUV	91	1
ROM	87	0.868
NAP	86	0.780
ATA	72	0.773
LAZ	70	0.651
MIL	63	0.692
INT	62	0.681
FIO	60	0.793
TOR	53	0.668
SAM	48	0.592
CAG	47	0.597
SAS	46	0.425
UDI	45	0.523
CHI	43	0.395
BOL	41	0.391
GEN	36	0.590
CRO	34	0.369
EMP	32	0.314
PAL	26	0.261
PES	18	0.245

Table 1: Comparison with 3/1/0 point mechanism

Here is the comparison in Table 1.

A first and evident aspect is that the in the ranking model sometimes the relative deviation between the scores is not comparable with the the corresponding deviation of the final points. The first clear case is between Napoli (NAP) and Atalanta (ATA) teams. In the model the ordering is correct, but they are very near one to each other (1% of relative difference in the scores), while in the final classification they have more than 15% of relative difference.

The opposite behaviour is clear between Atalanta and Lazio (LAZ). This time the relative difference is higher in the model than in the final points.

The first misplacement in the ordering concerns the Lazio team. It has in the final classification a much higher placement than it should have looking at the model. In terms of placement it has four positions more that other teams, better placed in the model. In terms of final points it has al least 17 points more.

Conversely, going down in the classification, the Genoa (GEN) team is very badly misplaced with respect to the score it has in the model.

These are the two more interesting cases.

In [1] we tested if a different assignment of points could be important. The previous results are obtained with the actual assignment mechanism: 3 points for a win and just 1 point for a tie, while the matrix element setting is 1 for a win and the half for a tie. In this way it should be evident that teams with many ties and not so many wins are not favourite in the final real placement. In [1] we checked with a simulation of a real classification obtained by using the old fashioned assignment of points: 2 points for a win, 1 point for a tie and no points for a loss.

The result was that, apart from some slight modifications with respect to the other point assignment, the same misplacements we had before were still evident. The assignment rule of points did not seem to be so important.

It is not easy to tackle the problem from a general point of view. Given 1 point for the win and 0 points for the loss, if we set in the preference matrix α for the tie, there does not seem to be a sort of linearity in α on the solution, the eigenvector. In fact the characteristic polynomial that takes to the eigenvalues is not homogeneous in α.

We checked here with a simulation of a real classification obtained by using the old fashioned assignment of points: 2 points for a win, 1 point for a tie and no points for a loss.

In Table 2 there is the final classification we would have obtained together with the comparison with the model.

	points	rank
JUV	62	1
NAP	60	0.780
ROM	59	0.868
ATA	51	0.773
LAZ	49	0.651
MIL	45	0.692
FIO	44	0.793
INT	43	0.681
TOR	40	0.668
SAM	36	0.592
SAS	33	0.425
UDI	33	0.523
CAG	33	0.597
CHI	31	0.395
BOL	30	0.391
GEN	27	0.590
CRO	25	0.369
EMP	24	0.314
PAL	20	0.261
PES	15	0.245

Table 2: Comparison with 2/1/0 point mechanism

There are some modifications with respect to the previous case. These modifications are more important than the ones we had in [1]. It is interesting to observe that the placements of Roma and Napoli teams are interchanged. This shows that the point mechanism may be important.[7] It does not seem there are reasons for the existence of a points ratio coherent with the model, but this could be an interesting aspect to be studied. Anyway it is clear the model gives a final ranking that can be different from the real one, obtained by the official point mechanism. It is interesting to study if the linear model can be *better* that the official ranking method.

In [1] we investigated to identify some possible reasons for the inconsistencies between the model and the real life, by considering for example discrepancies among "home-points" and "away-points" the teams have collected in the matches. This is actually an aspect that supporters appreciate, as an away win is usually considered more valuable than a home win. In the same way a home win against a low rank team is perhaps as valuable as a home tie against a high rank team. But it is clear that this is just a subjective matter, as the point mechanism is not intended to take into account this aspect (maybe it should?). The obvious question is then: does the model does it? The answer is obviously no, it does not, as in the preference matrix the coefficients for a win is just 1, no important if it is an away or a home win.

Apart from the interchanged positions of Roma and Napoli, we still have the evident misplacement of the Lazio team, that has again a much higher placement than it should have looking at the model. The different points mechanism does not seem to be involved here.

Here the motivation could be related to the deeper aspects that make the difference in the model. To understand this it is worthwhile to remember a computational method that can be used to calculate eigenvalues and eigenvectors of a matrix, the so called *power method*.

[7] It is easy to realize that Roma team has a greater number of wins (28 against 26) but has also more lost matches and less tied matches. It is clear that the points ratio between wins and ties may be very significant, in some cases.

For the purpose of finding the leading eigenvalue of a matrix together with a corresponding eigenvector a great variety of methods have been designed. Let's consider a square $n \times n$ matrix A having n linearly independent eigenvectors associated to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and suppose

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

This means we are assuming that the dominant eigenvalue λ_1 has algebraic multiplicity one. Let's call $v^1, v^2, \ldots, v^n$ some corresponding eigenvectors. This means that

$$Av^i = \lambda_i v^i, \quad i = 1, 2, \ldots, n.$$

The power method, starting from any arbitrarily chosen vector $x^{(0)}$, builds up a sequence of vectors $\{x^{(k)}\}$ that converges to the eigenvector associated to the leading eigenvalue.

Suppose $x^{(0)}$ is an arbitrary vector in $\mathbb{R}^n$. We may write $x^{(0)}$ as a linear combination of $v^1, v^2, \ldots, v^n$ that, because the hypothesis of linear independence, is a basis of $\mathbb{R}^n$.

$$x^{(0)} = \sum_{i=1}^{n} \alpha_i v^i \quad \text{and suppose } \alpha_1 \neq 0.$$ [8]

Starting from $x^{(0)}$ we may build up the sequence

$$x^{(1)} = Ax^{(0)} \quad , \quad x^{(2)} = Ax^{(1)} \quad , \quad \ldots \quad , \quad x^{(k)} = Ax^{(k-1)} \quad , \quad \ldots$$

The following result holds:

Theorem 3 *For the sequence* $\{x^{(k)}\}_{k \in \mathbb{N}}$ *we have that*

$$\lim_{k \to \infty} \frac{x_j^{(k+1)}}{x_j^{(k)}} = \lambda_1 \quad \textit{and} \quad \lim_{k \to \infty} \frac{x^{(k)}}{x_j^{(k)}} = cv^1. \tag{4}$$

where j *is an index for which* $x_j^{(k)} \neq 0$*, for every value of* k*.*

In other words, apart from technicalities, if we are interested in finding the largest solution in module of the equation $Av = \lambda v$, together with a corresponding vector, we may use the sequence defined by $x^{(n)} = A^n x^{(0)}$, starting from an arbitrary vector $x^{(0)}$ in $\mathbb{R}^n$. If we see the equation $Av = \lambda v$ with the meaning it has inside the ranking model, the power method means that we start from *any* distribution of scores to the teams and we update the scores in a sequence of steps, each time considering more and more hidden relations inside the preference matrix. The method appears to be quite robust, because the solution is not affected by a possibly irrealistic initial distribution of scores. The data on which the iterative method operates are just the ones contained in the preference matrix, but they are not used just at a first level, as actually the official points mechanism does. They are fully taken into account. To be explicit: a team gets rank if it defeats a high rank team but also if it defeats a team that has defeated a high rank team, and so on.

The discrepancies between the theoretical model and the actual results may be explained with the remarks before. We may say that the linear model based on the leading Perron's eigenvalue could be a valuable one to understand the hidden aspects that the official points mechanism does not consider. Hence it can be adopted at the end of the contest to get a better insight.

[8] The condition means that we don't have to start from a point in the subspace spanned by the eigenvectors $v^2, v^3, \ldots, v^n$. We need $x^{(0)}$ to have a component in the subspace spanned by v^1.

Of course, during the championship, there is the need to produce, week by week, the official classification, by using a preference matrix that is quite sparse, at the beginning in particular. This can be a problem for the method, as a sparse matrix is not likely to be irreducible.

This can be a matter for a further study: the applicability of the method for a weekly definition of the ranking, in particular in the case where the transition matrix is not irreducible. The first trivial way to overcome the difficulty seems to be the definition of a preference matrix with positive elements (it means to give rank also in case of a defeat). Some simulations may be worthwhile to see if this takes to some inconsistency of the method.

Another interesting aspect to study could be a more general one: the interrelations between the preference matrix defined by the possible various rules and the properties of the matrix itself in terms of reducibility and/or lackness of a positive eigenvector.

Appendix

A All the results of the matches in the Italian premier league 2016-17

	ATA	BOL	CAG	CHI	CRO	EMP	FIO	GEN	INT	JUV	LAZ	MIL	NAP	PAL	PES	ROM	SAM	SAS	TOR	UDI
ATA		3–2	2–0	1–0	1–0	2–1	0–0	3–0	2–1	2–2	3–4	1–1	1–0	0–1	3–0	2–1	1–0	1–1	2–1	1–3
BOL	0–2		2–1	4–1	1–0	0–0	0–1	0–1	0–1	1–2	0–2	0–1	1–7	3–1	3–1	0–3	2–0	1–1	2–0	4–0
CAG	3–0	1–1		4–0	2–1	3–2	3–5	4–1	1–5	0–2	0–0	2–1	0–5	2–1	1–0	2–2	2–1	4–3	2–3	2–1
CHI	1–4	1–1	1–0		1–2	4–0	0–3	0–0	2–0	1–2	1–1	1–3	1–3	1–1	2–0	3–5	2–1	2–1	1–3	0–0
CRO	1–3	0–1	1–2	2–0		4–1	0–1	1–3	2–1	0–2	3–1	1–1	1–2	1–1	2–1	0–2	1–1	0–0	0–2	1–0
EMP	0–1	3–1	2–0	0–0	2–1		0–4	0–2	0–2	0–3	1–2	1–4	2–3	1–0	1–1	0–0	0–1	1–3	1–1	1–0
FIO	0–0	1–0	1–0	1–0	1–1	1–2		3–3	5–4	2–1	3–2	0–0	3–3	2–1	2–2	1–0	1–1	2–1	2–2	3–0
GEN	0–5	1–1	3–1	1–2	2–2	0–0	1–0		1–0	3–1	2–2	3–0	0–0	3–4	1–1	0–1	0–1	0–1	2–1	1–1
INT	7–1	1–1	1–2	3–1	3–0	2–0	4–2	2–0		2–1	3–0	2–2	0–1	1–1	3–0	1–3	1–2	1–2	2–1	5–2
JUV	3–1	3–0	4–0	2–0	3–0	2–0	2–1	4–0	1–0		2–0	2–1	2–1	4–1	3–0	1–0	4–1	3–1	1–1	2–1
LAZ	2–1	1–1	4–1	0–1	1–0	2–0	3–1	3–1	1–3	0–1		1–1	0–3	6–2	3–0	0–2	7–3	2–1	3–1	1–0
MIL	0–0	3–0	1–0	3–1	2–1	1–2	2–1	1–0	2–2	1–0	2–0		1–2	4–0	1–0	1–4	0–1	4–3	3–2	0–1
NAP	0–2	3–1	3–1	2–0	3–0	2–0	4–1	2–0	3–0	1–1	1–1	4–2		1–1	3–1	1–3	2–1	1–1	5–3	3–0
PAL	1–3	0–0	1–3	0–2	1–0	2–1	2–0	1–0	0–1	0–1	0–1	1–2	0–3		1–1	0–3	1–1	0–1	1–4	1–3
PES	0–1	0–3	1–1	0–2	0–1	0–4	1–2	5–0	1–2	0–2	2–6	1–1	2–2	2–0		1–4	1–1	1–3	0–0	1–3
ROM	1–1	3–0	1–0	3–1	4–0	2–0	4–0	3–2	2–1	3–1	1–3	1–0	1–2	4–1	3–2		3–2	3–1	4–1	4–0
SAM	2–1	3–1	1–1	1–1	1–2	0–0	2–2	2–1	1–0	0–1	1–2	0–1	2–4	1–1	3–1	3–2		3–2	2–0	0–0
SAS	0–3	0–1	6–2	1–3	2–1	3–0	2–2	2–0	0–1	0–2	1–2	0–1	2–2	4–1	0–3	1–3	2–1		0–0	1–0
TOR	1–1	5–1	5–1	2–1	1–1	0–0	2–1	1–0	2–2	1–3	2–2	2–2	0–5	3–1	5–3	3–1	1–1	5–3		2–2
UDI	1–1	1–0	2–1	1–2	2–0	2–0	2–2	3–0	1–2	1–1	0–3	2–1	1–2	4–1	3–1	0–1	1–1	1–2	2–2	

Reading key: By reading by row you have the home results of the team in the first column, while by reading by columns you have the away result of the team in the first row.

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