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Abstract

The problem of ranking a set of elements, namely giving a "rank" to the elements of the set, may arise in very different contexts and may be handled in some possible different ways, depending on the ways these elements are set in competition the ones against the others. For example there are contexts in which we deal with an even paired competition, in the sense the pairings are evenly matched: if we think for example of a national soccer championship, each team is paired with every other team the same number of times. Sometimes we may deal with an uneven paired competition: think for example of the UEFA Champions League, in which the pairings are not fully covered, but just some pairings are set, by means of a random selection process for example. Mathematically based ranking schemes can be used and may show interesting connections between the ranking problems and classical theoretical results.

In this working paper we first show how a linear scheme in the ranking process directly takes to some fundamental Linear Algebra concepts and results, mainly the eigenvalues and eigenvectors of linear transformations and Perron–Frobenius theorem. We apply also the linear ranking model to a numerical simulation taking the data from the Italian soccer championship 2015-2016. We finally point out some interesting differences in the final ranking by comparing the actual placements of the teams at the end of the contest with the mathematical scores provided to teams by the theoretical model.

Keywords. Ranking scheme, Linear transformation, Eigenvalues, Dominant eigenvalue

AMS Classification. 15A18

JEL Classification. C65, C69

1 Introduction

In [1] an interesting in-depth analysis on possible ranking methods, mainly concentrated on football teams ranking, may be found. Within certain hypotheses the ranking problem may be easily formulated as a linear eigenvalue problem. In [2,3] the mathematical background for the linear approach was described, with reference both to the football teams case and the web page ranking, in the original definition of PageRank algorithm by Google. The linear mathematical approach involves some Linear Algebra notions and results. Among these, the concepts of eigenvalues and eigenvectors of a square matrix are fundamental, together with Perron–Frobenius theorem on the so called dominant eigenvalue.

We may generally describe a ranking problem as the need to assign a rank to the elements of a certain finite set. As a consequence of the ranking process we can get an order relation on the set, but the ranking scheme is a model for getting individual rankings to the elements, that is stronger than just having the order relation. Ranking schemes can be applied on a variety of situations, such as ranking soccer teams, or in general ranking teams or single individuals of a certain sport, ranking web pages, area of interest of search engines, ranking customers preferences, which marketing operators are interested in.

The aim of a mathematical ranking scheme is to introduce some objectivity in the definition of the rank itself. It must be pointed out anyway that it is impossible to remove all the subjectivity, as in the process we have to give values to some parameters, and there is no a completely impartial way to do that.

The reasons why Perron–Frobenius theorem fits well in some ranking problems are quite straightforward and do apply more in general in each setting where a linear approach is taken to get the ranking method. In [2] and [3], as mentioned, some notes on web pages ranking were presented, by recalling that the original Google's Pagerank algorithm adopts a linear model to obtain such a ranking.

2 A linear ranking model

Let's have a look at a linear way to define a ranking method and the step in which the theorem becomes important.

Suppose we have a contest, a competition, with a number N of participants to which we want to assign a score. We may assume an initial rank for all the participants.[1]

Then we may think to regularly update the ranks (the scores) each week taking into consideration the results of the weekly matches. Here it comes the non objective part of the process. It is worthwhile to stress that the choices we make here are going to condition the final results of the ranking scheme. For example we could give the same "negative weight" to a team if it loses a match, but we could conversely give a "parametric weight" for a defeat: if a weak team loses against a strong team the weight could be small, while if a strong team loses against a weak team the weight should be big. We could also state that a strong rival gives me some (positive) score even if I lose not so badly the direct match with it.

Anyway it is reasonable to assume that the score is based on the interactions with other participants and should depend on the outcome of the interaction and/or strength of its rivals. In the soccer or other sports context the interaction is of course the match, where we have a final result saying who is the winner. In other contexts, where there is no a real and direct interaction among participants, it may be tricky to define the weights of interactions.

Let's go into more mathematical details. We may define a vector of ranking values r, with positive components r_j, indicating the strength of the jth participant.[2]

The definition of the scores, that is the way in which we update the ranks after the results of all the matches, is a crucial step in the model as, how it happens in general, it completely influences the model behaviour. By following [1], in the soccer case we may define a score for

[1] The initial rank could be given, think of this year major soccer league for example, by the "strength" a team is supposed to have from its history. A team that won many previous championships is "strong", while a team that has passed to the major league from a secondary league is not.

[2] Again, this can be an "initial guess", an initial assignment of ranks given to teams, based on the previous history. A natural question arises: how much important is this first assignment for the final result of the ranking process? We shall come back to this later on.

the ith participant as

$$s_i = \frac{1}{n_i} \sum_{j=1}^{N} a_{ij} r_j, \tag{1}$$

where N is the total number of participants in the contest, a_{ij} is a nonnegative number related to the outcome of the game between participant i and participant j, and n_i is the number of games played by participant i.

Remark. This is a quite general definition, that may be applied in some different situations. We did not distinguish yet between even and uneven paired competitions. An even paired competition is the one in which each team plays against all the others. It is the case of the national Italian premier soccer league, where if there are N teams, each team plays $2(N-1)$ matches (home and away games) and there are $2(N-1) \cdot \frac{N}{2} = N(N-1)$ matches in whole. An uneven paired competition is the one in which each team does not play against all the others. It is the case for example of the European Champions League, where the pairings are not evenly matched. In this case the pairings are randomly chosen for the different phases of the event. The definition in (1) may be applied to both even and uneven pairings: in the even case the parameter n_i may be discarded, as the number of games is the same for all the teams, while the parameter may be important in the uneven case.

Remark. The linear structure of the scoring model is evident. The score of the ith team depends on all the ranks of the other teams by means of coefficients that characterizes the interaction between team i and other teams j. In the definition the division by n_i can be seen as a sort of normalization: it may be important in case some additional games are possible. But, apart from this more technical aspect, the definition takes anyway to a classical linear model

$$s_i = \sum_{j=1}^{N} \frac{a_{ij}}{n_i} r_j = \sum_{j=1}^{N} b_{ij} r_j. \tag{2}$$

Some remarks on the coefficients a_{ij} (or b_{ij}): they are related to interactions between participants and they may take into account the different aspects of the specific field where we have the ranking problem. In the soccer case a simple and straightforward possibility is to set $a_{ij} = 1$ if team i won the game against team j, $a_{ij} = \frac{1}{2}$ if the game ended in a tie and $a_{ij} = 0$ if team i lost the game against team j. For mathematical reasons, that will be clear in the following, in order to apply the theory we have in mind, it is necessary to have positive or at least nonnegative coefficients. This is why a loss takes zero and not a negative number.

Equation (2) may be presented in matrix form: if s is the N-vector of scores and r is the N-vector of ranks, (2) is equivalent to

$$s = Ar \tag{3}$$

where A is the square matrix of the coefficients a_{ij} (let's take for simplicity $n_i = 1$ for each i). A is usually called *preference matrix* or *transition matrix*.

2.1 A numerical example

Let's make a numerical example. Suppose we have a competition with $N = 4$ teams (A-B-C-D) and suppose we give them the initial rank represented by the vector

$$r^{(0)} = \begin{pmatrix} 10 \\ 8 \\ 5 \\ 3 \end{pmatrix}.$$

Suppose we have the following results after the first two matches:[3]

1st match	outcome	2nd match	outcome
A – C	1	A – D	1
B – D	1	B – C	X

With the choice 1/0.5/0 for the coefficients, the matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0.5 & 1 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

After the first two matches the scoring is given by the vector

$$s^{(2)} = Ar^{(0)} = \begin{pmatrix} 8 \\ 5.5 \\ 4 \\ 0 \end{pmatrix}.$$

The meaning is obviously that the A team takes the full rank of C and D teams, having defeated them, while the B team takes the full rank of D team, having defeated it, but just half of the rank of C team, as a consequence of the tie. Analogously for the other two teams. To see how the scoring has changed we may compare the two normalized vectors $r^{(0)}$ and $s^{(2)}$. By using the ∞-norm[4] we have

$$\frac{r^{(0)}}{\|r^{(0)}\|_\infty} = \begin{pmatrix} 1 \\ 0.8 \\ 0.5 \\ 0.3 \end{pmatrix} \quad \text{and} \quad \frac{s^{(2)}}{\|s^{(2)}\|_\infty} = \begin{pmatrix} 1 \\ 0.69 \\ 0.5 \\ 0 \end{pmatrix}.$$

We could have used the 1-norm instead.[5]

$$\frac{r^{(0)}}{\|r^{(0)}\|_1} = \begin{pmatrix} 0.38 \\ 0.32 \\ 0.19 \\ 0.12 \end{pmatrix} \quad \text{and} \quad \frac{s^{(2)}}{\|s^{(2)}\|_1} = \begin{pmatrix} 0.46 \\ 0.31 \\ 0.23 \\ 0 \end{pmatrix}.$$

The relative scores are significant, and they are the same with the two norms. Team B made its rank worse because of the loss against a weaker team and it may be noticed that the relative rank may change also in comparison with teams not yet played against.

Now suppose that after the end of the competition (3 matches) we have the outcomes

1st match	outcome	2nd match	outcome	3rd match	outcome
A – C	1	A – D	1	A – B	X
B – D	1	B – C	X	C – D	2

[3] We are assuming the competition is similar to a usual Italian soccer championship: each week there are $\frac{N}{2}$ matches and all the teams play a match against another. We are also using the standard notation for soccer matches: 1 means the home team wins, 2 for the the win of the guest team, and X for the tie.

[4] We recall that if $x \in \mathbb{R}^n$ the ∞-norm of x is defined by $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$.

[5] We recall that if $x \in \mathbb{R}^n$ the 1-norm of x is defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$.

The matrix A is then

$$A = \begin{pmatrix} 0 & 0.5 & 1 & 1 \\ 0.5 & 0 & 0.5 & 1 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

After all the matches of the contest have been played, the scoring is given by the vector

$$s^{(3)} = Ar^{(0)} = \begin{pmatrix} 12 \\ 10.5 \\ 4 \\ 5 \end{pmatrix} \quad \text{with} \quad \frac{s^{(3)}}{\|s^{(3)}\|_\infty} = \begin{pmatrix} 1 \\ 0.88 \\ 0.33 \\ 0.42 \end{pmatrix}.$$

We see that the relative ranks have changed accordingly to the results of the last matches.

It may be interesting to see the effects of a light modification in the a_{ij} coefficients. We could assume that not all the wins are of the same "value": in other terms a win against a weaker team is not as valuable as a win against an equal or higher level team. In the same way a tie against a weaker team is less valuable than a tie against equal or higher level teams. And again with losses: we could set that a loss against equal or lower level teams is not valuable at all, but a loss against a higher level team is almost inevitable, and hence worthy of a (very small) value.[6] For example we could implement the previous point of view by assuming the following matrix

$$B = \begin{pmatrix} 0 & 0.4 & 0.8 & 0.8 \\ 0.5 & 0 & 0.4 & 0.8 \\ 0.1 & 0.5 & 0 & 0 \\ 0.1 & 0.1 & 1 & 0 \end{pmatrix}. \tag{5}$$

After all the matches of the contest, with this new scoring assumptions and with the same initial rank values $r^{(0)}$ the scoring would be given by the vector

$$\sigma^{(3)} = Br^{(0)} = \begin{pmatrix} 9.6 \\ 9.4 \\ 5 \\ 6.8 \end{pmatrix} \quad \text{with} \quad \frac{\sigma^{(3)}}{\|\sigma^{(3)}\|_\infty} = \begin{pmatrix} 1 \\ 0.98 \\ 0.52 \\ 0.71 \end{pmatrix}.$$

Remarks. It is clear that the new a_{ij} coefficients tend to penalize the Team A, that is the strongest team in the competition, while tend to enforce the scores of the weak teams C and D.

Now let's go back to the general aspects: the reason why eigenvalues are involved. We have seen that the scoring model can be applied for obtaining an "update" of an initial ranking by means of the results of the matches, stored in the elements of a matrix. If we look for a "solution" of the model, a sort of the "right" ranking vector, we may think that if we apply the scoring process to this ranking vector, we must obtain some scores that are proportional to the ranks.[7]

Following [1], it is very reasonable to assume that the score of a team is proportional to its strength (its rank), as defined by (2). Let's call λ the proportionality constant. By taking the matrix equation (3) and $s = \lambda r$, we get

$$Ar = \lambda r \tag{6}$$

[6] It could be more reasonable to use negative values, thinking that a loss against a weaker team deserves a negative value, but our initial assumptions take just nonnegative elements for the matrix A. This is important from a mathematical point of view.

[7] In fact asking that the scoring is equal to the ranking is a sort of stretch that does not consider one important aspect, already evident in the numerical example: the significant thing is not the actual values of the scores, but the relative scores. Hence it is reasonable, almost necessary, to assume that the actual solution is determined up to a constant.

```
-->A=[0,0.5,1,1; 0.5,0,0.5,1;0,0.5,0,0;0,0,1,0]
 A  =
    0.     0.5    1.     1.
    0.5    0.     0.5    1.
    0.     0.5    0.     0.
    0.     0.     1.     0.

-->spec(A)
 ans  =

    1.1558535
  - 0.3743167 + 0.6253205i
  - 0.3743167 - 0.6253205i
  - 0.4072202

-->[R,diagevals] = spec(A)
 diagevals  =
    1.1558535    0                          0                          0
    0          - 0.3743167 + 0.6253205i     0                          0
    0            0                        - 0.3743167 - 0.6253205i     0
    0            0                          0                        - 0.4072202
 R  =

  - 0.7004418    0.3039539 + 0.1108692i     0.3039539 - 0.1108692i     0.8550967
  - 0.6195192    0.6163412                  0.6163412                - 0.1522354
  - 0.2679921  - 0.2171813 - 0.3628156i   - 0.2171813 + 0.3628156i     0.1869202
  - 0.2318565  - 0.2740931 + 0.5113840i   - 0.2740931 - 0.5113840i   - 0.4590151
```

Table 1: Scilab output for matrix A

that is

$$\sum_{j=1}^{N} a_{ij} r_j = \lambda r_i.$$

This is the usual equation that takes to the definition of the eigenvalues λ of the matrix A and the corresponding eigenvectors r. The meaning of this is: if we assume the linear relation among scores and the proportionality between ranks and scores, then a possible ranking vector must be an eigenvector of the coefficient matrix A, the preference matrix, and the corresponding eigenvalue is the scalar for the proportion.

But of course, at the moment, we cannot be sure a meaningful solution exists: the matrix A might have just negative eigenvalues ore we might have for example all the eigenvectors with positive and negative components at the same time. Here is the importance of having a matrix with nonnegative entries together with Perron–Frobenius theorem.

Before tackling some general mathematical aspects regarding positive matrices and eigenvalues, we can solve numerically the example with four teams. With the help of a scientific software[8] we get Table 1.

At the first prompt the command `A=[...]` just defines the matrix A. At the second prompt the `spec(A)` command gives the eigenvalues of A, namely

$$\lambda_1 = 1.1558535 \quad , \quad \lambda_{2,3} = -0.3743167 \pm 0.6253205i \quad , \quad \lambda_4 = -0.4072202.$$

The first eigenvalue is real and positive, then we have a pair of complex conjugate eigenvalues and finally a real and negative one.

[8] We have been using Scilab for the numerical computations.

At the third prompt again the `spec(A)` command, in a more complete form, gives the diagonal matrix similar to A (the eigenvalues are in the main diagonal) and a second matrix where the columns are normalized eigenvectors, respectively associated to the corresponding eigenvalues of the matrix. Scilab gives one of the possible generators of the corresponding eigenspace.

We are interested in the first column v^1 of this matrix. It is a negative eigenvector associated to the positive real eigenvalue.[9] Of course $-v^1$ is a positive solution for our problem:

$$r = \begin{pmatrix} 0.7004418 \\ 0.6195192 \\ 0.2679921 \\ 0.2318565 \end{pmatrix}.$$

It is a normalized vector (with the euclidean 2-norm) and it can be taken as the ranking vector for the problem. Apart from the specific single values, the order of the ranks and the relative ratios are the same as before. For comparison, the normalized vectors by means of the ∞-norm and the 1-norm are

$$\frac{r}{\|r\|_\infty} = \begin{pmatrix} 1 \\ 0.88 \\ 0.38 \\ 0.33 \end{pmatrix} \quad \text{and} \quad \frac{r}{\|r\|_1} = \begin{pmatrix} 0.38 \\ 0.34 \\ 0.15 \\ 0.13 \end{pmatrix}.$$

With the matrix B defined in (5) we get the output in Table 2.

Remark. With the matrix B we have again a more well-balanced contest. This is quite reasonable, if we compare the elements in the transition matrices A and B.

The positive eigenvector we are interested in is

$$r = \begin{pmatrix} 0.6419594 \\ 0.6058645 \\ 0.3051498 \\ 0.3573512 \end{pmatrix} \quad \text{with} \quad \frac{r}{\|r\|_\infty} = \begin{pmatrix} 1 \\ 0.94 \\ 0.48 \\ 0.56 \end{pmatrix} \quad \text{and} \quad \frac{r}{\|r\|_1} = \begin{pmatrix} 0.34 \\ 0.32 \\ 0.16 \\ 0.19 \end{pmatrix}.$$

2.2 Positive matrices vs. eigenvalues

Let us go back to the technical aspects, related to the sign of a matrix and the fundamental results regarding positive matrices and their eigenvalues/eigenvectors. Here is a detailed summary of the relevant concepts and results.

Definition 1 *A vector v (matrix A) with positive entries is said a positive vector (positive matrix). We write $v > 0$ ($A > 0$) to say that v is a positive vector (A is a positive matrix).*

Now let us recall the classical definitions of eigenvalues/eigenvectors and some other related definitions.

Let A be an $n \times n$ matrix.[10] Scalars λ and vectors $x \neq 0$ satisfying $Ax = \lambda x$ are called, respectively, **eigenvalues** and **eigenvectors** of A. The set $\sigma(A)$ of eigenvalues is called the **spectrum** of A.

[9] The fact it is negative is not significant: it is just one possible eigenvector.

[10] In many applications A is a real matrix, a matrix whose elements are real numbers. Nevertheless the definitions are usually given in the more general field of complex numbers $\mathbb{C}$. This is because the eigenvalues of a real matrix are not guaranteed to be real in general. They are guaranteed to exist in the complex filed.

```
-->B=[0,0.4,0.8,0.8; 0.5,0,0.4,0.8;0.1,0.5,0,0;0.1,0.1,1,0]
 B  =

    0.     0.4    0.8    0.8
    0.5    0.     0.4    0.8
    0.1    0.5    0.     0.
    0.1    0.1    1.     0.

-->spec(B)
 ans  =

    1.2031082
  - 0.4275828
  - 0.3877627 + 0.5705332i
  - 0.3877627 - 0.5705332i

-->[R,diagevals] = spec(B)
 diagevals  =

    1.2031082    0            0                          0
    0          - 0.4275828    0                          0
    0            0          - 0.3877627 + 0.5705332i     0
    0            0            0                        - 0.3877627 - 0.5705332i
 R  =

    0.6419594    0.8413420    0.0205178 - 0.3311567i     0.0205178 + 0.3311567i
    0.6058645  - 0.2819028  - 0.3253763 - 0.4799555i   - 0.3253763 + 0.4799555i
    0.3051498    0.1328800  - 0.1965252 + 0.4151231i   - 0.1965252 - 0.4151231i
    0.3573512  - 0.4416078    0.5854382                  0.5854382
```

Table 2: Scilab output for matrix B

We have the following characterization for the eigenvalues of a matrix:

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad A - \lambda I \text{ is singular} \quad \Leftrightarrow \quad \det(A - \lambda I) = 0.$$

In the following we indicate with $R(A)$ the *range* (or the *image space*) of A, that is

$$R(A) = \big\{ Ax \;\big|\; x \in \mathbb{R}^n \big\}.$$

We indicate also with $N(A)$ the *nullspace* (or the *kernel*) of A, that is

$$N(A) = \big\{ x \;\big|\; Ax = 0 \big\}.$$

Both $R(A)$ and $N(A)$ are subspaces of $\mathbb{R}^n$.
If λ is an eigenvalue of the matrix A, then

$$\big\{ x \in \mathbb{R}^n \;\big|\; x \neq 0 \ \wedge \ x \in N(A - \lambda I) \big\}$$

is the set of all the eigenvectors associated with λ. $N(A-\lambda I)$ is called the **eigenspace** associated with λ. A fundamental concept in Perron–Frobenius theory is the following: the number

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the **spectral radius** of A.

Remark. As λ might be complex, the absolute value has to be intended in the complex field but, in any case, $\rho(A)$ is real. The maximum exists because $\sigma(A)$ is a finite set.

This is a very important definition, also in applications, because the precise knowledge of all the eigenvalues is not required often, as many properties depend just on how large eigenvalues are, and then the knowledge of an upper bound may be enough.

The definition of the spectral radius clearly does not say $\rho(A)$ is an eigenvalue of A and this is the crucial point.[11] There are some important results on the spectral radius and the related eigenvalues of a positive matrix. Some of these results are almost immediate, some others are deeper and deeper. Perron's theorem is usually presented as a summary of all of these. Let us review some of these results to fully describe the theoretical properties we have in Perron's theorem. If v is a vector (M is a matrix) we indicate with $|v|$ ($|M|$) the vector (matrix) of the absolute values.

Positive eigenpair If A is an $n \times n$ positive (real) matrix then

- $\rho(A) \in \sigma(A)$.
- If $Ax = \rho(A)x$ then $A|x| = \rho(A)|x|$ and $|x| > 0$.

Remarks. The first item says that $\rho(A)$ is an eigenvalue of A (of course it's a positive one). We have already noticed that there are counterexamples with matrices that are non positive.

The second item says that if x is an eigenvector associated with the eigenvalue $\rho(A)$, then also $|x|$ is an eigenvector associated with the same eigenvalue[12] and moreover it is a positive eigenvector. It means, by the way, that all the eigenvectors associated with $\rho(A)$ do not have null components. These two results say then that a positive matrix has a positive (real) eigenvalue, that is the spectral radius, and we have some positive eigenvectors associated to this eigenvalue.

Important properties hold with regard to the multiplicity of this eigenvalue.

Index of $\rho(A)$ If A is an $n \times n$ positive matrix then

- $\rho(A)$ is the only eigenvalue of A on the spectral circle.
- $index(\rho(A)) = 1$.

Remarks. The spectral circle of the matrix A is the set

$$\big\{\alpha \in \mathbb{C} \;\big|\; |\alpha| = \rho(A)\big\}.$$

The first item says that, apart from $\rho(A)$, we do not have eigenvalues with absolute value $\rho(A)$. Thinking of real eigenvalues, it means that $-\rho(A)$ is not an eigenvalue of A. Remark that the item do not say anything about the multiplicity of the eigenvalue $\rho(A)$.

The second item takes to recall some more definitions and concepts. First of all the index of a square matrix. The concept is related to the so called *range-nullspace decomposition*: for every singular square matrix there exists a positive integer k such that the rank and the nullspace of A^k, $R(A^k)$ and $N(A^k)$, are complementary subspaces in $\mathbb{R}^n$, in the sense

$$\mathbb{R}^n = R(A^k) \oplus N(A^k).$$

[11] The real matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$ and then the spectral radius is 1, that is not an eigenvalue. But, as an even simpler case, with real eigenvalues, we may consider the matrix $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, where the eigenvalues are 0 and -1, and again the spectral radius is 1, that is not an eigenvalue.

[12] It is opportune to notice that this is true in this context, talking about $\rho(A)$, while it is not true in general: in other words, if λ is any eigenvalue and x is an eigenvector associated to λ, it is not true in general that $|x|$ is also an eigenvector of λ.

The smallest positive integer k for which $\mathbb{R}^n$ is the direct sum of the range and the nullspace of A^k is called the **index** of A. For nonsingular matrices we set $index(A) = 0$.

The index of A can be proven to be the smallest nonnegative integer k such that any one of the three following statements is true:

- $rank(A^k) = rank(A^{k+1})$.
- $R(A^k) = R(A^{k+1})$.
- $N(A^k) = N(A^{k+1})$.

For singular matrices $index(A)$ is the smallest positive integer k such that either of the following two statements is true:

- $R(A^k) \cap N(A^k) = 0$.
- $\mathbb{R}^n = R(A^k) \oplus N(A^k)$.

Now, the index of an eigenvalue λ is by definition the index of the matrix $A - \lambda I$. As said before, it can be proved that $index(\rho(A)) = 1$. In general, if an eigenvalue has index 1 we cannot say it has multiplicity one.[13] Let us recall the fundamental definitions regarding the multiplicity of eigenvalues.

If $\lambda \in \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$

- the *algebraic multiplicity* of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $alg\ mult(\lambda_i) = a_i$, $i = 1, \ldots, t$, if and only if $(x-\lambda_1)^{a_1} \cdots (x-\lambda_t)^{a_t} = 0$ is the characteristic equation for the matrix A. If $alg\ mult(\lambda) = 1$, λ is called a *simple eigenvalue*.
- The *geometric multiplicity* of λ is $\dim N(A - \lambda I)$. In other words, $geo\ mult(\lambda_i)$ is the maximal number of linearly independent eigenvectors associated with λ.
- Eigenvalues λ such that $alg\ mult(\lambda) = geo\ mult(\lambda)$ are called *semisimple eigenvalues* of the matrix A.

Remarks. For every matrix A and for each $\lambda \in \sigma(A)$

$$geo\ mult(\lambda) \leq alg\ mult(\lambda).$$

Then every simple eigenvalue is also semisimple, but the converse is not true. Semisimple eigenvalues are important in order the matrix is diagonalizable.

There is an interesting relation among the index and the multiplicity of eigenvalues. As a consequence of the Jordan form we have that the index of an eigenvalue λ is 1 if and only if λ is a semisimple eigenvalue. This allows us to say that the spectral radius is a semisimple eigenvalue.

But a stronger result can be proved.

[13] By trivially taking the identity matrix $I_{n\times n}$ in $\mathbb{R}^n$, the only eigenvalue $\lambda = 1$ has multiplicity n and $index(1) = index(I_{n\times n} - I_{n\times n}) = index(0_{n\times n}) = 1$, as $R(0_{n\times n}) = \{0\}$ and $N(0_{n\times n}) = \mathbb{R}^n$.

Multiplicities of $\rho(A)$ If A is an $n \times n$ positive matrix then $alg\ mult(\rho(A)) = 1$, that is the spectral radius of A is a simple eigenvalue of A.

Remark. The consequence is that

$$\dim N(A - \rho(A)I) = geo\ mult(\rho(A)) = alg\ mult(\rho(A)) = 1,$$

that is the eigenspace associated to the spectral radius is a one-dimensional subspace of $\mathbb{R}^n$.

Remark. By putting together the previous remark and the result in the Positive eigenpair step, we get that the eigenspace $N(A - \rho(A)I)$ can be spanned by some positive vector v. Then there is a unique eigenvector $p \in N(A - \rho(A)I)$ such that $p > 0$ and $\sum_i p_i = 1$ (of course $p = v/\|v\|_1$). This eigenvector p is called the *Perron vector* for the positive matrix A. The associated eigenvalue $\rho(A)$ is called the *Perron eigenvalue*.

A couple of further results can be proved, that are usually part of Perron's theorem. The first is that there are no nonnegative eigenvectors for a positive matrix A other than the Perron vector p and its positive multiples.

The second is an interesting formula that gives the spectral radius an analytical property.

Collatz–Wielandt formula If A is an $n \times n$ positive matrix then for the Perron eigenvalue $\rho(A)$ we have that

$$\rho(A) = \max_{x \in Q} f(x)$$

where

$$f(x) = \min_{x_i \neq 0} \frac{[Ax]_i}{x_i} \quad \text{and} \quad Q = \big\{x \mid x \geq 0 \ \wedge \ x \neq 0\big\}.$$

Remark. The function $f(x)$ is defined in the whole $\mathbb{R}^n$ apart from the origin. It is homogeneous with degree 0 ($f(\alpha x) = f(x)$ for every $x \in \mathbb{R}^n \setminus \{0\}$).

Consider for example the matrix

$$\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}.$$

The function f is defined in $Q = \big\{(x_1, x_2) \mid x_1, x_2 \geq 0\big\} \setminus \big\{0, 0\big\}$. The homogeneity can be useful to simplify the computations. In $\mathbb{R}^2$ we may reduce to only one variable because we may consider the function just for example on the segment $\big\{(t, 1-t) \mid 0 \leq t \leq 1\big\}$.

For $t = 0$ we have

$$f(0,1) = \frac{\left[\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_2}{\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_2} = \frac{5}{1} = 5.$$

For $t = 1$ we have

$$f(1,0) = \frac{\left[\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_1}{\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_1} = \frac{4}{1} = 4.$$

For $0 < t < 1$ we have

$$f(t, 1-t) = \min\left(\frac{\left[\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} t \\ 1-t \end{pmatrix}\right]_1}{\left[\begin{pmatrix} t \\ 1-t \end{pmatrix}\right]_1}, \frac{\left[\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} t \\ 1-t \end{pmatrix}\right]_2}{\left[\begin{pmatrix} t \\ 1-t \end{pmatrix}\right]_2}\right) = \min\left(2 + \frac{2}{t}, 4 + \frac{1}{1-t}\right).$$

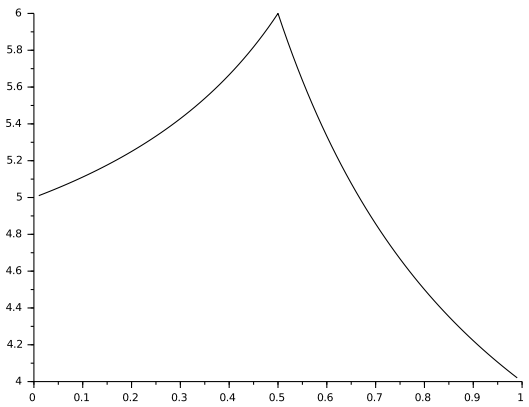

Figure 1: The Collatz–Wielandt function

By comparing the two rational functions in the interval $(0, 1)$ we find that

$$f(t, 1-t) = \begin{cases} 4 + \frac{1}{1-t} & 0 < t < \frac{1}{2} \\ 2 + \frac{2}{t} & \frac{1}{2} < t < 1. \end{cases}$$

In Figure 1 there is the graphic of the Collatz–Wielandt function for the example. We see that the maximum value is 6, attained at $t = \frac{1}{2}$. A direct computation of the eigenvalues shows that the matrix has real eigenvalues 3 and 6.

2.3 Perron's theorem

Perron's theorem for positive matrices is usually presented as a set of many properties, all related to the matrix eigenvalue with the largest absolute value (also called leading eigenvalue). The results were due to Oskar Perron (1907) and concerned just positive matrices. Later, as already mentioned, Georg Frobenius (1912) found the extension to certain classes of nonnegative matrices.

We summarize then some possible statements of Perron's theorem. Some of them were presented in the previous section.

Theorem 1 (Perron's theorem for a positive matrix) *Let A be an $n \times n$ positive matrix and let $\rho(A)$ be the spectral radius of A. Then the following properties hold.*

(i) $\rho(A) > 0$.

(ii) $\rho(A) \in \sigma(A)$. *It is called the Perron eigenvalue.*

(iii) $alg\ mult(\rho(A)) = 1$.

(iv) There exists an eigenvector $x > 0$ associated to $\rho(A)$. The Perron eigenvector p is the unique eigenvector such that $p > 0$ and $\|p\|_1 = 1$. The matrix does not have other positive eigenvalues, except for positive multiples of p.

(v) $\rho(A)$ is the only eigenvalue in the spectral circle of A.

There are many other important properties associated to the Perron eigenvalue. One of them is the already mentioned Collatz–Wielandt formula. Some other are the following.

```
-->Bpos=[0.01,0.4,0.8,0.8; 0.5,0.01,0.4,0.8;0.1,0.5,0.01,0.01;0.1,0.1,1,0.01]
 Bpos  =

    0.01    0.4     0.8     0.8
    0.5     0.01    0.4     0.8
    0.1     0.5     0.01    0.01
    0.1     0.1     1.      0.01

-->[R,diagevals] = spec(Bpos)
 diagevals  =

    1.2162167    0            0                          0
    0          - 0.4173454    0                          0
    0            0          - 0.3794356 + 0.5660844i     0
    0            0            0                        - 0.3794356 - 0.5660844i
 R  =

    0.6415709  - 0.8412212    0.0180994 - 0.3298451i     0.0180994 + 0.3298451i
    0.6050689    0.2739041  - 0.3290958 - 0.4789124i   - 0.3290958 + 0.4789124i
    0.3069680  - 0.1340713  - 0.1972672 + 0.4128303i   - 0.1972672 - 0.4128303i
    0.3578395    0.4464843    0.5864046                  0.5864046
```

Table 3: Scilab output for matrix B_{pos}

- The Perron eigenvalue $\rho(A)$ satisfies the inequalities

$$\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}.$$

The Perron eigenvalue is also important in the growing of powers of matrix A. Moreover there is a powerful numerical method for the computation of the Perron eigenvalue based on the powers of matrix A (the so called Power method). We intend to consider in a further study the Power method inside the ranking model, to see if it gives some insight to the model itself.

In order to apply the strong results of Perron's theorem to our practical problem of ranking, we must have a positive transition matrix. This maybe appears to be a too strong condition, as the diagonal elements of the matrix cannot be positive. In fact they should be related to the result of the match of a team against itself. It is not reasonable to have $a_{ii} \neq 0$. It can be interesting though to make a simulation on the effect of forcing those elements to be nonzero. Let us take the example seen before with the transition matrix B defined in (5) and let us give all the null elements some small positive value (matrix B_{pos}). Scilab output is in Table 3.

The positive eigenvector is

$$r_{\text{pos}} = \begin{pmatrix} 0.6415709 \\ 0.6050689 \\ 0.3069680 \\ 0.3578395 \end{pmatrix} \quad \text{with} \quad \frac{r_{\text{pos}}}{\|r_{\text{pos}}\|_\infty} = \begin{pmatrix} 1 \\ 0.94 \\ 0.48 \\ 0.56 \end{pmatrix} \quad \text{and} \quad \frac{r_{\text{pos}}}{\|r_{\text{pos}}\|_1} = \begin{pmatrix} 0.34 \\ 0.32 \\ 0.16 \\ 0.19 \end{pmatrix}.$$

Up to the second decimal digit the new solution is equal to the previous one and $\|r - r_{\text{pos}}\|_2$ is in the order of 10^{-3} (this is also the relative variation as the 2-norm of r is one).

A general study on the variation of eigenvalues/eigenvectors following a small variation in the null elements of the transition matrix should be necessary, but we do not tackle this aspect in this working paper. Some results on the continuity of eigenvalues are easy to be stated, but they do not mean continuity of eigenvectors.

2.4 A simulation from a real problem

Here we want to try a more realistic simulation, by taking the data from the Italian soccer 2015-16 championship. We intend to write the transition matrix, setting a choice of "weights" in two different ways, similar to the ones we used in our previous example with four teams.

In the Italian soccer 2015-16 championship there are 20 teams. By setting the transition matrix A with

$$a_{ij} = \begin{cases} 1 & \text{if team } i \text{ defeats team } j \\ 0.5 & \text{if teams } i \text{ and } j \text{ tie the match} \\ 0 & \text{if team } i \text{ looses against team } j \end{cases}$$

we obtain the 20×20 matrix in appendix (section B). By reading the elements by row we have the results in the home matches for the teams in the first column. By reading the elements by column we have the results in the away matches for the teams in the first row.

```
-->spec(A)
 ans  =

   10.441499
   1.0014714 + 0.3595954i
   1.0014714 - 0.3595954i
   0.0067272 + 1.431486i
   0.0067272 - 1.431486i
   0.1818987 + 1.144175i
   0.1818987 - 1.144175i
 - 2.0648274 + 0.2894272i
 - 2.0648274 - 0.2894272i
 - 1.2565733 + 0.9997415i
 - 1.2565733 - 0.9997415i
 - 1.4969516 + 0.8407335i
 - 1.4969516 - 0.8407335i
 - 1.735093
 - 1.259439
   0.1899274 + 0.6031604i
   0.1899274 - 0.6031604i
   0.1445207
 - 0.3574159 + 0.4615436i
 - 0.3574159 - 0.4615436i
```

If we ask Scilab to compute the eigenvalues of the matrix A, we get the solution here on the right. First of all it must be pointed out that A is not a positive matrix. As mentioned previously there is a more general theory about nonnegative matrices and spectral radius, due to Frobenius. We are not going into details on this subject, but we just say that the thesis of Perron's theorem does not hold for every nonnegative matrix. Only a particular class of this kind of matrices satisfies part of those properties, not all of them.

Our matrix A, that is nonnegative, shows anyway an evident single eigenvalue with maximum absolute value. We may also notice many pairs of complex conjugate eigenvalues, a couple of real negative eigenvalues and finally another small real positive eigenvalue.

```
r = R*eye(20,1)
 r  =

   0.1995248
   0.1643252
   0.1528436
   0.2002577
   0.1932821
   0.2557750
   0.1390425
   0.2221020
   0.2769960
   0.3460334
   0.2029033
   0.2289038
   0.3438883
   0.1397745
   0.3105056
   0.1772138
   0.2513688
   0.1641741
   0.1641179
   0.1598767
```

If we ask Scilab to compute the eigenspaces we get the positive vector r here on the left as a normalized eigenvector associated to the largest eigenvalue $\rho(A) \approx 10.44$. This vector has just positive components, hence it satisfies the properties of the Perron eigenvector (apart from the normalization step, because Scilab adopts the 2-norm).

By using the ∞-norm, as before, this takes to the alternative eigenvector

$$\frac{r}{\|r\|_\infty} = \big(0.58, 0.47, 0.41, 0.58, 0.56, 0.74, 0.40, 0.64, 0.80, 1, 0.59, 0.66, 0.99,$$

$$0.40, 0.90, 0.51, 0.73, 0.47, 0.47, 0.46\big)^T$$

Now the interesting thing is a comparison between the placements given by the ranking vector and the actual final placements we had at the end of the last championship. Remember that the official rules of the Italian championship gives 3 points to the winning team, 1 point to both teams for a tie and 0 points to the losing team.

Here is the comparison.

	points	rank
JUV	91	1
NAP	82	0.99
ROM	80	0.90
INT	67	0.80
FIO	64	0.74
SAS	61	0.73
MIL	57	0.66
LAZ	54	0.59
CHI	50	0.58
EMP	46	0.56
GEN	46	0.64
ATA	45	0.58
TOR	45	0.47
BOL	42	0.47
SAM	40	0.51
PAL	39	0.40
UDI	39	0.47
CAR	38	0.41
FRO	31	0.40
VER	28	0.46

As we can see there are some "misplacements". For the first 7 teams the final official placement is coherent with the mathematical ranking. Then we have the first and apparently the most important inconsistency, the one concerning the team of Genoa (GEN), that has an official placement certainly worse than it should have by considering it theoretical ranking: it has at least 8 points less that it should. A slight downward misplacement can be detected for the Atalanta (ATA) and Sampdoria (SAM) teams, and a quite relevant downward misplacement also for the Verona (VER) and Udinese (UDI) teams.

The first possible interpretation for these inconsistencies could be lying in the point assignment mechanism: 3 points for a win and just 1 point for a tie. Remember that the matrix element setting is 1 for a win and the half for a tie. In this way it should be apparent that teams with many ties and not so many wins are not favourite in the final real placement.

This can be checked with a simulation of a real classification obtained by using the old fashioned assignment of points: 2 points for a win, 1 point for a tie and no points for a loss.
This is the final classification we would have obtained.

	points	rank
JUV	62	1
NAP	57	0.99
ROM	57	0.90
INT	47	0.80
FIO	46	0.74
SAS	45	0.73
MIL	42	0.66
LAZ	39	0.59
CHI	37	0.58
ATA	34	0.58
EMP	34	0.56
GEN	33	0.64
TOR	33	0.47
BOL	31	0.47
SAM	30	0.51
CAR	29	0.41
PAL	29	0.40
UDI	29	0.47
FRO	23	0.40
VER	23	0.46

There are some slight modifications with respect to the previous case, but the same misplacements we had before are still evident: Genoa is still overestimated by the mathematical model, together with Sampdoria, Udinese and Verona. The assignment rule of points does not seem to be so important.

While trying to identify some other possible reasons for the inconsistencies, restricting ourselves to the more accessible and general data, it is interesting to remark the discrepancies among "home-points" and "away-points" the teams have collected in the matches.

By taking back the real placements with the current assignment rule (3-1-0), in Table 4 the home and away points are given together with the ratio between the two.

Let us focus on the Genoa and Milan teams, that are likely the most evident and relevant case of inconsistency. They have almost the same mathematical ranking and 11 points of difference in the real placement. Another evident case is the Verona team, that has a ranking comparable with that of teams with 10 points more.

	ATA	BOL	CAR	CHI	EMP	FIO	FRO	GEN	INT	JUV
home	30	20	23	29	27	38	22	33	41	50
away	15	22	15	21	19	26	9	13	26	41
ratio	2.00	0.91	1.53	1.38	1.42	1.46	2.44	2.54	1.58	1.22

	LAZ	MIL	NAP	PAL	ROM	SAM	SAS	TOR	UDI	VER
home	33	33	51	22	44	28	32	24	22	18
away	21	24	31	17	36	12	29	21	17	10
ratio	1.57	1.38	1.65	1.29	1.22	2.33	1.10	1.14	1.29	1.80

Table 4: Home and away points

Both Genoa and Verona teams have large ratios, but again this is not the point, as our settings for the matrix elements do not take into consideration this aspect, namely the difference between

home and away matches. We are going to leave this study to a further work.

We finish this working paper by testing in this large 20×20 example the same simple matrix modification we have used previously, intended to obtain a positive transition matrix. We just replace all the null value in the matrix A with a small positive value (matrix A_{pos}). The Scilab output for the usual requests (eigenvalues and Perron eigenvector) is the following.

```
-->spec(Apos)
 ans  =

    10.525752
    0.9922719 + 0.3572352i
    0.9922719 - 0.3572352i
    0.0088839 + 1.4231788i
    0.0088839 - 1.4231788i
    0.1771070 + 1.1303748i
    0.1771070 - 1.1303748i
  - 2.0496968 + 0.2876677i
  - 2.0496968 - 0.2876677i
  - 1.2409994 + 0.9871947i
  - 1.2409994 - 0.9871947i
  - 1.4827154 + 0.8394363i
  - 1.4827154 - 0.8394363i
  - 1.7125023
  - 1.2403283
    0.1840837 + 0.5972932i
    0.1840837 - 0.5972932i
    0.1455438
  - 0.3481676 + 0.4541784i
  - 0.3481676 - 0.4541784i
```

```
rpos = R*eye(20,1)
 rpos  =

    0.2001446
    0.1650125
    0.1542842
    0.2007000
    0.1937678
    0.2554628
    0.1407845
    0.2223212
    0.2763984
    0.3442746
    0.2039178
    0.2290315
    0.3421790
    0.1415532
    0.3091348
    0.1784367
    0.2505696
    0.1653376
    0.1650773
    0.1603617
```

Table 5: Scilab output for matrix A_{pos}

where on the left we have the eigenvalues (the first is the largest one) and on the right the normalized associated eigenvector. If we compare the two results, we have a relative deviation of 0.8 percent on the eigenvalue and a relative deviation ($\|r - r_{\text{pos}}\|_2$) in the order of 10^{-3} on the eigenvector.

The discrepancies between the theoretical model and the actual results seem worthwhile to be investigated. Another interesting point, that we intend to deal with, is the case of the ranking where the transition matrix has not a positive eigenvector. Actually the reason why in our example we have a positive solution, even if the matrix is not positive, is because it is irreducible. As we have mentioned, there is a more general theory for nonnegative matrices that keeps some of the properties of the Perron eigenvector for a special subclass of matrices. In a ranking problem of an even paired competition it seems fairly reasonable to end up with an irreducible matrix, as the matrix seems to be rather "dense", even with a very simple rule for the transitions. It could be interesting though to study some particular rules in the assignment of the matrix elements that take either to a reducible matrix or to a lack of a positive eigenvector. The case of an uneven paired competition seems also likely to take to a reducible matrix, as the density should be rather low.

Appendix

A All the results of the matches in the Italian premier league 2015-16

	ATA	BOL	CAR	CHI	EMP	FIO	FRO	GEN	INT	JUV	LAZ	MIL	NAP	PAL	ROM	SAM	SAS	TOR	UDI	VER
ATA		2–0	3–0	1–0	0–0	2–3	2–0	0–2	1–1	0–2	2–1	2–1	1–3	3–0	3–3	2–1	1–1	0–1	1–1	1–1
BOL	3–0		0–0	0–1	2–3	1–1	1–0	2–0	0–1	0–0	2–2	0–1	3–2	0–1	2–2	3–2	0–1	0–1	1–2	0–1
CAR	1–1	1–2		1–2	1–0	0–1	2–1	4–1	1–2	2–3	1–3	0–0	0–0	1–1	1–3	2–1	1–3	2–1	2–1	0–0
CHI	1–0	0–0	1–0		1–1	0–0	5–1	1–0	0–1	0–4	4–0	0–0	0–1	3–1	3–3	1–1	1–1	1–0	2–3	1–1
EMP	0–1	0–0	3–0	1–3		2–0	1–2	2–0	0–1	1–3	1–0	2–2	2–2	0–0	1–3	1–1	1–0	2–1	1–1	1–0
FIO	3–0	2–0	2–1	2–0	2–2		4–1	1–0	2–1	1–2	1–3	2–0	1–1	0–0	1–2	1–1	3–1	2–0	3–0	1–1
FRO	0–0	1–0	2–1	0–2	2–0	0–0		2–2	0–1	0–2	0–0	2–4	1–5	0–2	0–2	2–0	0–1	1–2	2–0	3–2
GEN	1–2	0–1	1–2	3–2	1–0	0–0	4–0		1–0	0–2	0–0	1–0	0–0	4–0	2–3	2–3	2–1	3–2	2–1	2–0
INT	1–0	2–1	1–1	1–0	2–1	1–4	4–0	1–0		0–0	1–2	1–0	2–0	3–1	1–0	3–1	0–1	1–2	3–1	1–0
JUV	2–0	3–1	2–0	1–1	1–0	3–1	1–1	1–0	2–0		3–0	1–0	1–0	4–0	1–0	5–0	1–0	2–1	0–1	3–0
LAZ	2–0	2–1	0–0	4–1	2–0	2–4	2–0	2–0	2–0	0–2		1–3	0–2	1–1	1–4	1–1	0–2	3–0	2–0	5–2
MIL	0–0	0–1	0–0	1–0	2–1	2–0	3–3	2–1	3–0	1–2	1–1		0–4	3–2	1–3	4–1	2–1	1–0	1–1	1–1
NAP	2–1	6–0	1–0	3–1	5–1	2–1	4–0	3–1	2–1	2–1	5–0	1–1		2–0	0–0	2–2	3–1	2–1	1–0	3–0
PAL	2–2	0–0	2–2	1–0	0–1	1–3	4–1	1–0	1–1	0–3	0–3	0–2	0–1		2–4	2–0	0–1	1–3	4–1	3–2
ROM	0–2	1–1	5–1	3–0	3–1	4–1	3–1	2–0	1–1	2–1	2–0	1–1	1–0	5–0		2–1	2–2	3–2	3–1	1–1
SAM	0–0	2–0	5–2	0–1	1–1	0–2	2–0	0–3	1–1	1–2	2–1	0–1	2–4	2–0	2–1		1–3	2–2	2–0	4–1
SAS	2–2	0–2	1–0	1–1	3–2	1–1	2–2	0–1	3–1	1–0	2–1	2–0	2–1	2–2	0–2	0–0		1–1	1–1	1–0
TOR	2–1	2–0	0–0	1–2	0–1	3–1	4–2	3–3	0–1	1–4	1–1	1–1	1–2	2–1	1–1	2–0	1–3		0–1	0–0
UDI	2–1	0–1	1–2	0–0	1–2	2–1	1–0	1–1	0–4	0–4	0–0	2–3	3–1	0–1	1–2	1–0	0–0	1–5		2–0
VER	2–1	0–2	1–2	3–1	0–1	0–2	1–2	1–1	3–3	2–1	1–2	2–1	0–2	0–1	1–1	0–3	1–1	2–2	1–1	

Reading key: By reading by row you have the home results of the team in the first column, while by reading by columns you have the away result of the team in the first row.

B The corresponding transition matrix

	ATA	BOL	CAR	CHI	EMP	FIO	FRO	GEN	INT	JUV	LAZ	MIL	NAP	PAL	ROM	SAM	SAS	TOR	UDI	VER
ATA	0	1	1	1	0.5	0	1	0	0.5	0	1	1	0	1	0.5	1	0.5	0	0.5	0.5
BOL	1	0	0.5	0	0	0.5	1	1	0	0.5	0.5	0	1	0	0.5	1	0	0	0	0
CAR	0.5	0	0	0	1	0	1	1	0	0	0	0.5	0.5	0.5	0	1	0	1	1	0.5
CHI	1	0.5	1	0	0.5	0.5	1	1	0	0	1	0.5	0	1	0.5	0.5	0.5	1	0	0.5
EMP	0	0.5	1	0	0	1	0	1	0	0	1	0.5	0.5	0.5	0	0.5	1	1	0.5	1
FIO	1	1	1	1	0.5	0	1	1	1	0	0	1	0.5	0.5	0	0.5	1	1	1	0.5
FRO	0.5	1	1	0	1	0.5	0	0.5	0	0	0.5	0	0	0	0	1	0	0	1	1
GEN	0	0	0	1	1	0.5	1	0	1	0	0.5	1	0.5	1	0	0	1	1	1	1
INT	1	1	0.5	1	1	0	1	1	0	0.5	0	1	1	1	1	1	0	0	1	1
JUV	1	1	1	0.5	1	1	0.5	1	1	0	1	1	1	1	1	1	1	1	0	1
LAZ	1	1	0.5	1	1	0	1	1	1	0	0	0	0	0.5	0	0.5	0	1	1	1
MIL	0.5	0	0.5	1	1	1	0.5	1	1	0	0.5	0	0	1	0	1	1	1	0.5	0.5
NAP	1	1	1	1	1	1	1	1	1	1	1	0.5	0	1	0.5	0.5	1	1	1	1
PAL	0.5	0.5	0.5	1	0	0	1	1	0.5	0	0	0	0	0	0	1	0	0	1	1
ROM	0	0.5	1	1	1	1	1	1	0.5	1	1	0.5	1	1	0	1	0.5	1	1	0.5
SAM	0.5	1	1	0	0.5	0	1	0	0.5	0	1	0	0	1	1	0	0	0.5	1	1
SAS	0.5	0	1	0.5	1	0.5	0.5	0	1	1	1	1	1	0.5	0	0.5	0	0.5	0.5	1
TOR	1	1	0.5	0	0	1	1	0.5	0	0	0.5	0.5	0	1	0.5	1	0	0	0	0.5
UDI	1	0	0	0.5	0	1	1	0.5	0	0	0.5	0	1	0	0	1	0.5	0	0	1
VER	1	0	0	1	0	0	0	0.5	0.5	1	0	1	0	0	0.5	0	0.5	0.5	0.5	0

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