UNIVERSITÀ di VERONA | Department of ECONOMICS

Working Paper Series
Department of Economics
University of Verona

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WP Number: 21 November 2019

ISSN: 2036-2919 (paper), 2036-4679 (online)

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November, 2019

Abstract

A mathematical program with complementarity constraints (MPCC) is an optimization problem with equality/inequality constraints in which a complementarity type constraint is considered in addition. This complementarity condition modifies the feasible region so as to remove many of those properties that are usually important to obtain the standard optimality conditions, e.g., convexity and constraint qualifications. In the literature, these problems have been tackled in many different ways: methods that introduce a parameter in order to relax the complementarity constraint, modified simplex methods that use an appropriate rule for choosing the non basic variable in order to preserve complementarity. We introduce a decomposition method of the given problem in a sequence of parameterized problems, that aim to force complementarity. Once we obtain a feasible solution, by means of duality results, we are able to eliminate a set of parameterized problems which are not worthwhile to be considered. Furthermore, we provide some bounds for the optimal value of the objective function and we present an application of the proposed technique in a non trivial example.

Keywords Mathematical programs with complementarity constraints, duality, decomposition methods

AMS Classification 49M27, 65K05, 90C30, 90C33, 90C46

JEL Classification C61

1 Introduction

In the field of equilibrium models, mathematical programs with complementarity constraints (MPCC) form a class of important, but extremely difficult, problems. MPCC's constitute a subclass of the well-known mathematical programs with equilibrium constraints (MPEC), widely studied in recent years. Their relevance comes from many applications which arise, for example, in economics and structural engineering [3, 6]. Their difficulty is due to the presence of the complementarity constraints, because the feasible region may not enjoy some fundamental properties: it may be not convex, even not connected and such that many of the standard constraint qualifications are violated at any feasible point. This last lack implies that the usual KKT conditions may not be fulfilled at an optimal solution, even in the linear case [7], that is the one considered in the present work.

In this paper we consider a reformulation of MPCC by means of a family of parameterized linear problems whose minimization leads to an optimal solution of MPCC. Exploiting the classic tools of the duality theory, we propose an iterative method which explores the set of parameters, excluding at each step a subset of them, by means of a suitable cut; indeed, the optimal values of the linear problems associated with such a subset are proved to be greater than or equal to the optimal value related to the current parameter. A similar approach can be found in [5, 9] where different kinds of cuts are considered. The method that we propose is different from the classic relaxation and penalty methods for MPEC (see e.g. [4, 8]), and allows us to define an algorithm which can be implemented in an interactive way taking advantage of some devices that speed up the solving procedure, owing to the decomposition of the given problem in a sequence of parameterized problems.

The paper is organized as follows. In Section 2, we introduce the problem and describe the decomposition method of MPCC in a family of parameterized problems. In Section 3, we exploit the classic tools of duality theory to obtain a necessary condition for improving the current solution and, moreover, we establish a sufficient optimality condition. In Section 4, still using duality, we obtain upper and lower bounds of the optimum value of the problem; in such a way, we define a sequence of nested intervals, which allows us to stop the iterative procedure, when their width is below a fixed tolerance. Finally, in Section 5, the iterative method is described and illustrated by means of an example. In Section 6, we conclude with some final remarks and the description of further developments.

2 A decomposition approach

We consider the following constrained minimization problem, whose objective function is linear and having a linear complementarity constraint besides affine ones:

$$(P) \qquad \begin{array}{l} f^0 := \min(\langle c, x\rangle + \langle d, y\rangle) \\ \text{s.t.} \quad (x, y) \in K := \{(x, y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0, \langle x, y\rangle = 0\}, \end{array}$$

where $A, B \in \mathbb{R}^{m\times n}$, $c, d \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$.

We will assume that the feasible set K nonempty and a global minimum point of P exists; call it (x^0, y^0). Let us introduce the following penalized form for the gradient of the objective function of P:

$$\begin{aligned} c(\alpha) &= (c_j(\alpha_j) := c_j + \rho_j\alpha_j, j = 1, ..., n), \\ d(\alpha) &= (d_j(\alpha_j) := d_j + \sigma_j(1 - \alpha_j), j = 1, ..., n), \end{aligned}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \Delta := \{0,1\}^n$. For our purposes, we will assume that ρ_j and σ_j, $j = 1, ..., n$, are large enough positive constants; the meaning of this assumption will be clear inside the proof of Theorem 1.

The given problem P can be associated with a family $\{P(\alpha)\}_{\alpha\in\Delta}$ of subproblems

$$P(\alpha) \qquad \begin{array}{l} f^{\downarrow}(\alpha) := \min\ [f(x,y;\alpha) := \langle c(\alpha), x\rangle + \langle d(\alpha), y\rangle] \\ \text{s.t. } (x,y) \in R := \{(x,y) \in \mathbb{R}^{2n} : Ax + By \geq b, x \geq 0, y \geq 0\}. \end{array}$$

Let us suppose that $\forall\alpha \in \Delta$ the objective function of $P(\alpha)$ is bounded from below on R; hence, $\forall\alpha \in \Delta$ there exists a minimum point, say $(x(\alpha), y(\alpha))$, of $P(\alpha)$. A sufficient condition for the boundedness of $f(\cdot,\cdot;\alpha)$ on R, $\forall\alpha \in \Delta$, is that $\langle c, x\rangle + \langle d, y\rangle$ is bounded from below on R, which obviously implies that the objective function of P is bounded from below on K, which in turn yields that a global minimum point of P exists.

We have the following result.

Theorem 1. *It holds*

$$f^0 = \min_{\alpha\in\Delta} f^{\downarrow}(\alpha) = \min_{\alpha\in\Delta} \min_{(x,y)\in R} [f(x,y;\alpha) = \langle c(\alpha), x\rangle + \langle d(\alpha), y\rangle]. \tag{1}$$

Proof. Suppose that (x^0, y^0) is a minimum point of P, so that $f^0 = \langle c, x^0\rangle + \langle d, y^0\rangle$. Recalling that Δ is a finite set, let $i = 1$ and $\alpha^i \in \Delta := \{\alpha^1, \alpha^2, \dots, \alpha^{2^n}\}$. From the definition of $c(\alpha)$ and $d(\alpha)$, we have

$$f(x,y;\alpha^i) = \langle c, x\rangle + \langle d, y\rangle + \sum_{j=1}^{n} \rho_j \alpha_j^i x_j + \sum_{j=1}^{n} \sigma_j (1-\alpha_j^i) y_j, \ \forall (x,y) \in R.$$

Let $vertR$ be the set of vertices of R. Consider $(\bar{x}, \bar{y}) \in vertR$; if $(\bar{x}, \bar{y})$ is such that

$$\sum_{j=1}^{n} \rho_j \alpha_j^i \bar{x}_j + \sum_{j=1}^{n} \sigma_j (1-\alpha_j^i) \bar{y}_j = 0 \tag{2}$$

(which happens independently on the positive constants ρ and σ), then $\langle \bar{x}, \bar{y}\rangle = 0$ and $(\bar{x}, \bar{y})$ is a feasible solution of P. Therefore, $f^0 \leq f(\bar{x}, \bar{y}; \alpha^i)$; choose $\rho^i, \sigma^i \in \mathbb{R}^n$ arbitrarily, for example $\rho^i = \sigma^i = (1, \dots, 1) \in \mathbb{R}^n$. Otherwise, if $(\bar{x}, \bar{y})$ is a vertex of R such that (2) is not fulfilled, we can choose $\rho^i = (\rho_1^i, \dots, \rho_n^i)$ and $\sigma^i = (\sigma_1^i, \dots, \sigma_n^i)$ such that $f^0 \leq f(\bar{x}, \bar{y}; \alpha^i)$. Noticing that $vertR$ is a finite set, it is possible to find a couple (ρ^i, σ^i) such that

$$f^0 \leq f(x,y;\alpha^i), \ \forall (x,y) \in vertR. \tag{3}$$

Since a minimum point of $P(\alpha)$ is attained in the set $vertR$, from (3), it follows $f^0 \leq f^{\downarrow}(\alpha^i)$. Now, consider α^{i+1}; if $f^0 \leq f(x,y;\alpha^{i+1})$, $\forall (x,y) \in vertR$, with $\rho = \rho^i$ and $\sigma = \sigma^i$, then set $\rho^{i+1} = \rho^i$ and $\sigma^{i+1} = \sigma^i$; in such a way, we obtain $f^0 \leq f^{\downarrow}(\alpha^{i+1})$. Otherwise, choose ρ^{i+1} and σ^{i+1} by increasing the previous vectors $\rho = \rho^i$ and $\sigma = \sigma^i$, in order to get $f^0 \leq f^{\downarrow}(\alpha^{i+1})$. Set $i = i + 1$ and repeat this procedure for all $i = 2, .., 2^n - 1$. It turns out that

$$f^0 \leq \min_{\alpha\in\Delta} f^{\downarrow}(\alpha). \tag{4}$$

Moreover, starting again from (x^0, y^0) minimum point of P, let us define the following vector $\alpha^0 = \alpha(x^0, y^0)$:

$$\alpha^0 := \begin{cases} \alpha_j^0 = 0, & \text{if } x_j^0 > 0 \\ \alpha_j^0 = 1, & \text{if } x_j^0 = 0 \end{cases}$$

The above vector α^0 belongs to Δ and $f^0 = f(x^0, y^0; \alpha^0)$. Since $(x^0, y^0) \in R$ is a feasible solution of the problem $P(\alpha^0)$, then

$$f^0 = f(x^0, y^0; \alpha^0) \geq f^{\downarrow}(\alpha^0) \geq \min_{\alpha \in \Delta} f^{\downarrow}(\alpha). \quad (5)$$

Inequalities (4) and (5) imply (1) and this concludes the proof. □

Based on Theorem 1, we can propose a decomposition approach for solving problem P. In fact, Theorem 1 establishes that the minimum f^0 of problem P can be achieved by first determining $f^{\downarrow}(\alpha)$ $\forall \alpha \in \Delta$, and secondly by minimizing $f^{\downarrow}(\alpha)$ with respect to $\alpha \in \Delta$; in other words, equation (1) decomposes the problem P in a sequence of subproblems $P(\alpha)$.

In view of the definition of α, ρ_j and σ_j, $j = 1, ..., n$, and of problem $P(\alpha)$, by setting $\alpha_j = 1$ or $\alpha_j = 0$ one would expect $x_j = 0$ or $y_j = 0$, respectively, in an optimal solution of $P(\alpha)$; however, in general, an optimal solution of $P(\alpha)$ will not necessarily comply with such an expectation for a given $\alpha \in \Delta$. Anyway, if such an expectation does not occur for a given $\alpha \in \Delta$, then $f^{\downarrow}(\alpha) > f^0$, provided we choose ρ and σ large enough. Consequently, we should work with the subset $\bar{\Delta}$ of Δ whose elements fulfill the following definition.

Definition 1. Let $\bar{\Delta}$ be the set of all $\alpha \in \Delta$ such that the system

$$Ax + By \geq b, \; x \geq 0, \; y \geq 0$$

is possible, when one sets $x_j = 0$ for $\alpha_j = 1$ and $y_j = 0$ for $\alpha_j = 0, j = 1, ..., n$.

Clearly, if a minimum point of P exists, then the set $\bar{\Delta}$ is nonempty.

Unfortunately, the cardinality of Δ, and also of $\bar{\Delta}$, is in general so large that the above outlined decomposition is not, by itself, of use. We aim at solving P through the penalized problems $P(\alpha)$'s, by running as less as possible on $\alpha \in \Delta$: at step k, having solved $P(\alpha^k)$, we try to determine α^{k+1}, such that

$$f(x(\alpha^k), y(\alpha^k), \alpha^k) > f(x(\alpha^{k+1}), y(\alpha^{k+1}), \alpha^{k+1}). \quad (6)$$

An initial effort in this direction is described in the first part of next section.

3 Optimality conditions

Suppose that we have solved, at step k, the problem $P(\alpha^k)$; we try to determine α^{k+1}, such that (6) holds. To this aim, we introduce the dual problem of $P(\alpha)$ given by:

$$P^*(\alpha) \qquad \begin{array}{ll} \max & < \lambda, b > \\ \text{s.t.} & \lambda A \leq c(\alpha), \quad \lambda B \leq d(\alpha), \quad \lambda \geq 0. \end{array}$$

Let $R^*(\alpha)$ denote the feasible region of $P^*(\alpha)$.

Theorem 2. *If $\alpha^{k+1} \in \Delta^k := \{\alpha \in \Delta : \bar{\lambda} A \leq c(\alpha), \bar{\lambda} B \leq d(\alpha)\}$, where $\bar{\lambda}$ is a maximum point of $P^*(\alpha^k)$, then*

$$f(x(\alpha^k), y(\alpha^k), \alpha^k) \leq f(x(\alpha^{k+1}), y(\alpha^{k+1}), \alpha^{k+1}).$$

Proof. Observe that $\bar{\lambda}$ is a feasible solution of $P^*(\alpha^{k+1})$ so that the maximum of such a problem is greater than or equal to that of $P^*(\alpha^k)$:

$$\max_{\lambda \in R^*(\alpha^{k+1})} \langle \lambda, b \rangle \geq \langle \bar{\lambda}, b \rangle := \max_{\lambda \in R^*(\alpha^k)} \langle \lambda, b \rangle.$$

By the strong duality theorem, the minimum of $P(\alpha^{k+1})$ is greater than or equal to that of $P(\alpha^k)$. □

By the previous theorem, we infer that a necessary condition for (6) to hold is that

$$\alpha^{k+1} \notin \Delta^k = \{\alpha \in \Delta : \bar{\lambda} A \leq c(\alpha), \bar{\lambda} B \leq d(\alpha)\}. \tag{7}$$

In other words, having an optimal solution of $P(\alpha^k)$ for some $\alpha^k \in \Delta$, and, thus, a feasible solution to the given problem P, a necessary condition for obtaining a problem $P(\alpha^{k+1})$ with a minimum $f^{\downarrow}(\alpha^{k+1}) < f^{\downarrow}(\alpha^k)$ (namely, a feasible solution of P "better" than the current one), is that α^{k+1} belongs to the set $\Delta \setminus \Delta^k$.

Theorem 2 provides a criterion for eliminating from considerations subsequent to the k-th one, the subset Δ^k; observe that Δ^k cannot be empty as it contains at least α^k.

We need to deepen the analysis of (7); to this aim, let us reconsider the inequalities in the definition of Δ^k:

$$\bar{\lambda} A \leq c(\alpha) \quad \text{and} \quad \bar{\lambda} B \leq d(\alpha).$$

Taking into account the definition of $c(\alpha)$ and $d(\alpha)$, the two inequalities can be rewritten in the form:

$$\frac{\langle \bar{\lambda}, A^j \rangle - c_j}{\rho_j} \leq \alpha_j, \quad j = 1, \ldots, n; \tag{8a}$$

$$\frac{\langle \bar{\lambda}, B^j \rangle - d_j}{\sigma_j} \leq 1 - \alpha_j, \quad j = 1, \ldots, n, \tag{8b}$$

where A^j and B^j are the j-th column of A and B, respectively.

Remark 1. If the left-hand side of the j-th inequality of (8a) is non positive, then the inequality is fulfilled for any choice of α_j, i.e. both for $\alpha_j = 0$ and $\alpha_j = 1$. Otherwise, let us suppose that it is positive; in such a case, since $\rho_j > 0$ is large enough, we can assume - without any loss of generality - that $\frac{\langle \bar{\lambda}, A^j \rangle - c_j}{\rho_j} < 1$. Hence, the j-th inequality of (8a) is fulfilled if $\alpha_j = 1$, but not if $\alpha_j = 0$. Obviously, we can make analogous remarks on the j-th inequality of (8b) and we obtain that if $\frac{\langle \bar{\lambda}, B^j \rangle - d_j}{\sigma_j}$ is positive, then the j-th inequality of (8b) is satisfied if $\alpha_j = 0$, but not if $\alpha_j = 1$.

Let us introduce the following sets of indexes

$$I_x(\bar{\lambda}) := \left\{ j = 1, \ldots, n : \frac{\langle \bar{\lambda}, A^j \rangle - c_j}{\rho_j} > 0 \right\} = \left\{ j = 1, \ldots, n : \langle \bar{\lambda}, A^j \rangle - c_j > 0 \right\} \tag{9}$$

$$I_y(\bar{\lambda}) := \left\{ j = 1, \ldots, n : \frac{\langle \bar{\lambda}, B^j \rangle - d_j}{\sigma_j} > 0 \right\} = \left\{ j = 1, \ldots, n : \langle \bar{\lambda}, B^j \rangle - d_j > 0 \right\} \tag{10}$$

where, both in (9) and in (10), the second equality is true because $\rho_j > 0$ and $\sigma_j > 0$, $j = 1, \ldots, n$.

Now we establish some results expressed in terms of the sets $I_x(\bar{\lambda})$ and $I_y(\bar{\lambda})$. In the next results we assume that $\alpha^k \in \Delta$ and $\bar{\lambda}$ is a maximum point of $P^*(\alpha^k)$.

Proposition 1. *$I_x(\bar{\lambda}) \cap I_y(\bar{\lambda}) = \varnothing$.*

Proof. Inequalities (8) are fulfilled by α^k because $\bar{\lambda}$ is an optimal solution of $P^*(\alpha^k)$. They can be equivalently written as:

$$\frac{\langle \bar{\lambda}, A^j \rangle - c_j}{\rho_j} \leq \alpha_j \leq 1 - \frac{\langle \bar{\lambda}, B^j \rangle - d_j}{\sigma_j}, \quad j = 1, \ldots, n. \tag{11}$$

If, *ab absurdo*, $I_x(\bar{\lambda}) \cap I_y(\bar{\lambda}) \neq \varnothing$, then there exists at least one index $j \in I_x(\bar{\lambda}) \cap I_y(\bar{\lambda})$ such that the j-th inequalities of (11) are simultaneously satisfied neither by $\alpha_j^k = 0$ nor by $\alpha_j^k = 1$ and this is a contradiction. □

Theorem 3. (**sufficient optimality condition**) If

$$I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing, \tag{12}$$

then an optimal solution $(\bar{x}^k, \bar{y}^k)$ of the problem $P(\alpha^k)$ is an optimal solution of problem P.

Proof. $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$ is equivalent to

$$\frac{\langle \bar{\lambda}, A^j \rangle - c_j}{\rho_j} \leq 0 \quad \text{and} \quad \frac{\langle \bar{\lambda}, B^j \rangle - d_j}{\sigma_j} \leq 0, \quad \forall\, j = 1, \ldots, n.$$

Then all $\alpha \in \Delta$ satisfy the inequalities (11); i.e., $\Delta = \Delta^k$ which implies $\Delta \setminus \Delta^k = \varnothing$. Hence, there is no way to improve the current solution $(\bar{x}^k, \bar{y}^k)$, so that it is an optimal solution of P. □

The following result establishes a condition equivalent to the necessary condition (7). Notice that we are interested only in the case where $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. Indeed, if $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$, the optimality of the current solution is proved by Theorem 3.

Theorem 4. *Suppose that $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. Then*

$$\alpha \notin \{\alpha \in \Delta : \bar{\lambda} A \leq c(\alpha), \bar{\lambda} B \leq d(\alpha)\} \tag{13}$$

iff

$$\sum_{i \in I_x(\bar{\lambda})} \alpha_i - \sum_{j \in I_y(\bar{\lambda})} \alpha_j \leq |I_x(\bar{\lambda})| - 1. \tag{14}$$

Proof. Let us consider the inequalities in (13) in the equivalent form (8). Therefore (13) holds if at least one of the inequalities of (8) is not fulfilled. Observe that, if $i \notin I_x(\bar{\lambda})$ or $j \notin I_y(\bar{\lambda})$, then the i-th inequality in (8a) or the j-th in (8b) is fulfilled whatever α_i or α_j may be, respectively. Then the system (8) is equivalent to the following

$$\begin{cases} \alpha_i > 0, & i \in I_x(\bar{\lambda}) \\ 1 - \alpha_j > 0, & j \in I_y(\bar{\lambda}). \end{cases}$$

The above system is possible iff

$$\sum_{i \in I_x(\bar{\lambda})} \alpha_i + \sum_{j \in I_y(\bar{\lambda})} (1 - \alpha_j) = |I_x(\bar{\lambda})| + |I_y(\bar{\lambda})|,$$

so that it is impossible iff

$$\sum_{i \in I_x(\bar{\lambda})} \alpha_i + \sum_{j \in I_y(\bar{\lambda})} (1 - \alpha_j) \leq |I_x(\bar{\lambda})| + |I_y(\bar{\lambda})| - 1,$$

which coincides with (14). □

Define the relaxed problem of P obtained by dropping the complementarity constraints, i.e.

$$(RP) \qquad \begin{cases} \min(\langle c, x \rangle + \langle d, y \rangle) \quad \text{s.t.} \\ Ax + By \geq b, \quad x \geq 0, \quad y \geq 0 \end{cases}$$

and denote by $\bar{f}$ the optimal value of RP (possibly $-\infty$).

The next result deepens the meaning of the sufficient optimality condition given by Theorem 3.

Proposition 2. *Let $\alpha \in \Delta$ and $\bar{\lambda}$ be an optimal solution of $P^*(\alpha)$. If $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$, then $f^0 = \bar{f}$.*

Proof. The assumption $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$ is equivalent to say that $\bar{\lambda}A \leq c$, $\bar{\lambda}B \leq d$. Therefore, since $\bar{\lambda} \geq 0$, we have that $\bar{\lambda}$ is a feasible solution for the dual of RP: if it were $\bar{f} = -\infty$, we achieve a contradiction. Suppose that $\bar{f} > -\infty$; then, by strong duality $f^{\downarrow}(\alpha) = \langle \bar{\lambda}, b \rangle$ and, by weak duality,

$$f^{\downarrow}(\alpha) = \langle \bar{\lambda}, b \rangle \leq \bar{f},$$

which implies $f^0 := \min_{\alpha \in \Delta} f^{\downarrow}(\alpha) \leq \bar{f}$. On the other hand, $f^0 \geq \bar{f}$ and this completes the proof. □

4 Lower and upper bounds for the minimum

The sufficient optimality condition (12) given in Theorem 3 is a very restrictive condition. Indeed, it directly implies that the minimum value of the relaxed problem RP is equal to the one of P, as proved by Proposition 2.

Therefore, in this section, we propose an alternative iterative approach that leads, not only to a different sufficient optimality condition, but mainly to the possibility to evaluate the difference between the current value of the objective function of P and its minimum value, i.e., $f^{\downarrow}(\alpha^k) - f^0$. We will achieve this purpose by defining a finite sequence of lower and upper bounds of the minimum of P.

We restrict our attention to the case where it is possible to find vectors of upper bounds, say $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$, for x and y respectively, in such a way that the optimal value of problem P does not change. Therefore, alternatively to $\{P(\alpha)\}_{\alpha \in \Delta}$, we can associate with the given problem P the following family $\{Q(\alpha)\}_{\alpha \in \Delta}$ of subproblems:

$$Q(\alpha) \qquad \begin{array}{ll} \min(\langle c, x \rangle + \langle d, y \rangle) \\ \text{s.t.} \;\; Ax + By \geq b, \\ \qquad 0 \leq x_j \leq X_j(1 - \alpha_j), \quad j = 1, \ldots, n \\ \qquad 0 \leq y_j \leq Y_j \alpha_j, \quad j = 1, \ldots, n. \end{array}$$

Denote by $S(\alpha)$ the feasible set of $Q(\alpha)$. The dual of $Q(\alpha)$, say $Q^*(\alpha)$, is given by:

$$Q^*(\alpha) \qquad \begin{array}{ll} \max \left(\langle \lambda, b \rangle - \sum_{j=1}^n \mu_j X_j (1 - \alpha_j) - \sum_{j=1}^n \nu_j Y_j \alpha_j \right) \\ \text{s.t.} \;\; \lambda A - \mu \leq c \\ \qquad \lambda B - \nu \leq d \\ \qquad \lambda \geq 0, \mu \geq 0, \nu \geq 0. \end{array}$$

An optimal basic solution of $Q^*(\alpha)$ can be immediately derived from an optimal basic solution of $P^*(\alpha)$, as it proved by the following proposition.

Proposition 3. *If $\bar{\lambda}$ is an optimal basic solution of $P^*(\alpha)$, then the vector $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$, where*

$$\bar{\mu}_j = max\{0, \bar{\lambda}A_j - c_j\}, \quad j = 1, \ldots, n, \tag{15a}$$

$$\bar{\nu}_j = max\{0, \bar{\lambda}B_j - d_j\}, \quad j = 1, \ldots, n \tag{15b}$$

is an optimal basic solution of $Q^*(\alpha)$.

Proof. Let us compare problems $P^*(\alpha)$ and $Q^*(\alpha)$. By construction, $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a feasible basic solution of $Q^*(\alpha)$. Moreover, $\forall \alpha \in \Delta$ we have that

$$\langle \bar{\lambda}, b \rangle - \sum_{j=1}^{n} \bar{\mu}_j X_j (1 - \alpha_j) - \sum_{j=1}^{n} \bar{\nu}_j Y_j \alpha_j \leq \langle \bar{\lambda}, b \rangle;$$

in other words, $\langle \bar{\lambda}, b \rangle$ is an upper bound of the feasible values of the objective function of $Q^*(\alpha)$. Now, we will prove that the objective function of $Q^*(\alpha)$ assumes at $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ exactly the value $\langle \bar{\lambda}, b \rangle$ and hence $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ is an optimal solution of $Q^*(\alpha)$. To this aim, we rewrite (15) as follows:

$$\bar{\lambda} A_j - c_j > 0 \Rightarrow \bar{\mu}_j = \bar{\lambda} A_j - c_j; \quad \bar{\lambda} A_j - c_j \leq 0 \Rightarrow \bar{\mu}_j = 0, \; j = 1, \ldots, n, \tag{16a}$$

$$\bar{\lambda} B_j - d_j > 0 \Rightarrow \bar{\nu}_j = \bar{\lambda} B_j - d_j; \quad \bar{\lambda} B_j - d_j \leq 0 \Rightarrow \bar{\nu}_j = 0, \; j = 1, \ldots, n. \tag{16b}$$

When $\alpha_j = 0$, ν_j does not affect the objective function of $Q^*(\alpha)$; on the other hand, if $\alpha_j = 0$, the j-th inequality of $\bar{\lambda} A \leq c(\alpha)$ in $P^*(\alpha)$ becomes $\bar{\lambda} A_j \leq c_j$ and by the second implication of (16a) we obtain $\bar{\mu}_j = 0$. Similar results hold for $\alpha_j = 1$: μ_j does not affect the objective function of $Q^*(\alpha)$ and $\bar{\nu}_j = 0$. In conclusion, in the objective function of $Q^*(\alpha)$, we have $\sum_{j=1}^{n} \bar{\mu}_j X_j (1 - \alpha_j) + \sum_{j=1}^{n} \bar{\nu}_j Y_j \alpha_j = 0$ and this completes the proof. □

Let us observe that the feasible set of $Q^*(\alpha)$ does not depend on α; call this set $S^*, \forall \alpha \in \Delta$, and notice that $S^* \neq \varnothing$ by Proposition 3. Denote by $V := vertS^*$, the set of all vertices of S^*, or, equivalently, of all basic solutions of $Q^*(\alpha)$. Then, we have the following result.

Theorem 5. *The minimum f^0 of problem P equals the minimum of the problem:*

$$\begin{array}{ll} \min\limits_{\alpha, f} & f \\ \text{s.t.} & \alpha \in \Delta \\ & f \geq \left(\langle \lambda^h, b \rangle - \sum_{j=1}^{n} \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^{n} \nu_j^h Y_j \alpha_j \right), \; (\lambda^h, \mu^h, \nu^h) \in V. \end{array} \tag{17}$$

Proof. The following relations are readily seen to hold:

$$\begin{aligned} f^0 &= \min_{\alpha \in \Delta} \min_{(x,y) \in S(\alpha)} (\langle c, x \rangle + \langle d, y \rangle) \\ &= \min_{\alpha \in \Delta} \max_{(\lambda, \mu, \nu) \in S^*} \left(\langle \lambda, b \rangle - \sum_{j=1}^{n} \mu_j X_j (1 - \alpha_j) - \sum_{j=1}^{n} \nu_j Y_j \alpha_j \right) \\ &= \min_{\alpha \in \Delta} \max_{(\lambda^h, \mu^h, \nu^h) \in V} \left(\langle \lambda^h, b \rangle - \sum_{j=1}^{n} \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^{n} \nu_j^h Y_j \alpha_j \right). \end{aligned} \tag{18}$$

The last equality and the introduction of the scalar variable f prove the thesis of the theorem. □

Remark 2. Observe that if $\bar{V}$ is any subset of V, then by (18) we have:

$$f^0 \geq \min_{\alpha \in \Delta} \max_{(\lambda^h, \mu^h, \nu^h) \in \bar{V}} \left(\langle \lambda^h, b \rangle - \sum_{j=1}^{n} \mu_j^h X_j (1 - \alpha_j) - \sum_{j=1}^{n} \nu_j^h Y_j \alpha_j \right)]. \tag{19}$$

Therefore, the minimum in (19) is a lower bound t of f^0. At every α met in the solution process, an optimal basic solution of $Q(\alpha)$ and, hence, an optimal basic solution of $Q^*(\alpha)$ is available. Let V_k be the set of the basic solutions of $Q^*(\alpha^k)$ considered until step k. Thus, V_k which is initially empty, gains a new element. Every time this happens, problem (19) may be solved to find a new lower bound on f^0, say it t_k. Moreover, the minimum $f^{\downarrow}(\alpha^k)$ of problem $P(\alpha^k)$ for every $\alpha^k \in \Delta$, is obviously an upper bound on the optimal value f^0; let T_k be the minimum of all the previously found upper bounds. Obviously, at k-th step, the equality $T_k = t_k$ is a sufficient condition for optimality; moreover, $|T_k - t_k|$ is un upper bound of the difference between the current value of the objective function of P and its minimum. We can decide to stop the iterative procedure if such a difference is small enough, in a sense that can be specified case by case according to the meaning of the given complementarity problem.

5 The iterative method

The analysis developed in the previous sections allows us to define an iterative method for the minimization of the problem P.

General Algorithm

Step 0) (initialization).
Consider RP, the relaxed problem of P, and let $\bar{f}$ be the optimal value of RP (possibly $-\infty$). If $\bar{f} > -\infty$, let $(\bar{x}, \bar{y})$ be an optimal solution of RP. If $\langle \bar{x}, \bar{y} \rangle = 0$, then $(\bar{x}, \bar{y})$ is an optimal solution of P too; hence $\rightarrow$ **STOP**. Otherwise (i.e., if $\bar{f} = -\infty$ or $\langle \bar{x}, \bar{y} \rangle > 0$), set $k = 0$, $\Delta_0 = \Delta$, $T_{-1} = +\infty$, $t_{-1} = \bar{f}$. Go to Step 1.

Step 1) (solution of problem $P(\alpha^k)$).
Choose $\alpha^k \in \Delta_k$ and solve $P(\alpha^k)$. Let $(x(\alpha^k), y(\alpha^k))$ be an optimal solution of $P(\alpha^k)$ with optimal value $f^{\downarrow}(\alpha^k)$. If $f^{\downarrow}(\alpha^k) < T_{k-1}$ then $T_k = f^{\downarrow}(\alpha^k)$; otherwise $T_k = T_{k-1}$. Go to Step 2.

Step 2) (computation of a cut on the set Δ).
Let $\bar{\lambda}$ be an optimal solution of $P^*(\alpha^k)$. If $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$, then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. Otherwise, by means of inequality (13), determine the set $\{\alpha \in \Delta : \bar{\lambda} A \leq c(\alpha), \bar{\lambda} B \leq d(\alpha)\}$, that gives the vectors α to be rejected in the sequel. Go to Step 3.

Step 3) (computation of a lower bound).
Determine an optimal solution $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ of the problem $Q^*(\alpha^k)$ and let t the minimum in (18) obtained by adding the (basic) solution $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ to the set $\bar{V}$. If $t > t_{k-1}$, then $t_k = t$; otherwise $t_k = t_{k-1}$. If $T_k = t_k$ then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. Otherwise, let $\Delta_{k+1} = \Delta_k \setminus \{\alpha \in \Delta : \bar{\lambda} A \leq c(\alpha), \bar{\lambda} B \leq d(\alpha)\}$. If $\Delta_{k+1} = \varnothing$, then $(x(\alpha^k), y(\alpha^k))$ is an optimal solution of P; hence $\rightarrow$ **STOP**. (Remark that $\Delta_{k+1} = \varnothing$ is implied by $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) = \varnothing$). Otherwise, set $k = k + 1$ and go to Step 1.

Remark 3. The following observations are worth noting:

(a) The algorithm ends in a finite numbers of steps because there are at most 2^n problems $P(\alpha)$ to be solved.

(b) The decomposition method requires to process the set of binary vectors $\alpha \in \Delta$. As a binary vector is equivalent to the binary representation of an integer number, the enumeration of all the α's can be obtained starting from $\alpha^0 = (0, \dots, 0)$ and by adding each time the binary unit to α^i in order to obtain α^{i+1}.

(c) In the formulation of problem $Q(\alpha)$ and hence of its dual $Q^*(\alpha)$, we need to choose the values of the upper bounds $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ of which we have assumed the existence. The selection of such vectors is a crucial aspect of the method. Even if it is valid only for the particular case, a suggestion for this choice is given in the subsequent Example 5.1 at Step 3 of Iteration 1.

(d) If, at Step 3, we can prove that $f^{\downarrow}(\alpha) \geq f^{\downarrow}(\alpha^k)\ \forall \alpha \in \Delta_{k+1}$, then $f^{\downarrow}(\alpha^k)$ is the optimum value of P and the current solution $(x(\alpha^k), y(\alpha^k))$ is an optimal solution. Suppose for example that $|I_x(\bar{\lambda})| + |I_y(\bar{\lambda})| = 1$; from the inequality (14) it follows that there is a unique index j such that any α to be considered in the sequel has $\alpha_j = 0$ if $|I_x(\bar{\lambda})| = 1$, and $\alpha_j = 1$ if $|I_y(\bar{\lambda})| = 1$. If, by solving RP with the additional condition $x_j = 0$ when $\alpha_j = 1$ or $y_j = 0$ when $\alpha_j = 0$, we obtain a minimum greater than or equal to $f^{\downarrow}(\alpha^k)$, the current solution is an optimal one. Otherwise, such a minimum is a lower bound of f^0 and it will replace the current lower bound if it is better.

(e) Recall that in the proposed decomposition method we should work with the subset $\bar{\Delta}$ of Δ, introduced in Definition 1. If a vector $\alpha \notin \bar{\Delta}$, the optimal value of the corresponding $P(\alpha)$ is of the same magnitude of ρ_j's and σ_j's. In this case, we skip Step 1 and we generate a new α. We refer to this case as a *null step*.

Example 1. Let us apply the iterative method to the following problem P:

$$\min(2x_1 + 2x_2 + x_3 + 2x_4 + 2y_1 + 2y_2 + 2y_3 + 2y_4)$$

$$\begin{cases} x_1 + x_4 + y_1 + y_2 + y_3 \geq 20 \\ x_1 + x_2 + y_1 + y_3 \geq 14 \\ x_2 + x_3 + y_1 \geq 10 \\ x_2 + y_3 + y_4 \geq 10 \\ x_1 + x_3 + y_4 \geq 5 \\ \langle x, y \rangle = 0 \\ x \geq 0, y \geq 0 \end{cases} \tag{20}$$

For the solution of some of the steps, the numerical software MATLAB has been used.

Step 0) The solution of the relaxed problem RP is $\bar{x} = \left(\frac{5}{2}, 0, \frac{5}{2}, 0\right)$, $\bar{y} = \left(\frac{15}{2}, 0, 10, 0\right)$, with optimal value $\bar{f} = \frac{85}{2}$. As $\langle \bar{x}, \bar{y} \rangle > 0$ and hence the complementarity condition is not satisfied, set $k = 0$, $\Delta_0 = \Delta = \{0,1\}^4$, $T_{-1} = +\infty$, $t_{-1} = \bar{f} = \frac{85}{2}$. Goto Step 1.

Iteration 1 with $\mathbf{k = 0}$

Step 1) Let's choose in the binary ordering a first $\alpha^0 = (0,0,0,0)$. Let's solve $P(\alpha^0)$. We get

$$x(\alpha^0) = (5, 10, 0, 15)\ ,\ y(\alpha^0) = (0,0,0,0), \quad \text{with } f^{\downarrow}(\alpha^0) = 60.$$

As $f^{\downarrow}(\alpha^0) < T_{k-1}$ then $T_0 = 60$. Go to Step 2.

Step 2) The optimal solution of $P^{\star}(\alpha^0)$ is $\bar{\lambda} = (2,0,1,1,0)$. The sets of indexes defined in (9) and (10) are $I_x(\bar{\lambda}) = \varnothing$ and $I_y(\bar{\lambda}) = \{1,3\}$. Since $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$, according to Theorem 4 the subsequent α's to be considered must satisfy the inequality $\alpha_1 + \alpha_3 \geq 1$ (see inequality (14)). Therefore, we can disregard the following α's

$$\alpha^0 = (0,0,0,0)\ ;\ (0,0,0,1)\ ;\ (0,1,0,0)\ ;\ (0,1,0,1).$$

Go to Step 3.

Step 3) Let us consider $\{Q(\alpha)\}_{\alpha \in \Delta}$ where the upper bounds are

$$X = \left(20, 10, 20, \tfrac{31}{2}\right) \qquad \text{and} \qquad Y = \left(20, \tfrac{31}{2}, 25, 10\right).$$

The values of the upper bounds can be determined by combining the inequality

$$2x_1 + 2x_2 + x_3 + 2x_4 + 2y_1 + 2y_2 + 2y_3 + 2y_4 \leq 60,$$

given by the objective function less than or equal to its current value, with the inequalities coming from the constraints. For example, by considering the first constraint together with the above inequality, we get

$$40 + 2x_2 + x_3 + 2y_4 \leq 2(x_1 + x_4 + y_1 + y_2 + y_3) + 2x_2 + x_3 + 2y_4 \leq 60,$$

that gives the bounds $x_2 \leq 10$, $x_3 \leq 20$ and $y_4 \leq 10$. Similar bounds for the other variables may be obtained by means either of other constraints, taken singularly, or linear combinations of them. The optimal solution of $Q^*(\alpha^0)$, obtained from (15), is

$$(\bar{\lambda}; \bar{\mu}; \bar{\nu}) = (2, 0, 1, 1, 0; 0, 0, 0, 0; 1, 0, 1, 0).$$

Therefore, problem (17) is

$$t = \min_{\alpha, f} \ f \quad \text{s.t.} \quad \alpha \in \Delta \ , \ f \geq 60 - Y_1\alpha_1 - Y_3\alpha_3$$

with solution $t = 15$. Since $t = 15 < t_{-1}$, set $t_0 = t_{-1} = \frac{85}{2}$. As $T_0 \neq t_0$, let us continue in Step 3. $\Delta_1 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta \ : \ \alpha_1 + \alpha_3 \geq 1\}$. As $\Delta_1 \neq \varnothing$, set $k = 1$ and go to Step 1.

Iteration 2 with k = 1
Step 1) We have to choose $\alpha^1 \in \Delta_1$. As $\alpha^0 + 1 = (0, 0, 0, 1) \notin \Delta_1$, following the binary ordering on vectors α, we have $\alpha^1 = (0, 0, 1, 0) \in \Delta_1$. The optimal solution of $P(\alpha^1)$ is

$$x(\alpha^1) = (5, 10, 0, 15) \ , \ y(\alpha^1) = (0, 0, 0, 0), \quad \text{with } f^{\downarrow}(\alpha^1) = 60.$$

As $f^{\downarrow}(\alpha^1) = T_0$ then $T_1 = T_0 = 60$. Goto Step 2.
Step 2) By solving $P^{\star}(\alpha^1)$ we get $\bar{\lambda} = (2, 0, 2, 0, 0)$. The sets of indexes defined in (9) and (10) are $I_x(\bar{\lambda}) = \{3\}$ and $I_y(\bar{\lambda}) = \{1\}$; $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. The subsequent vectors $\alpha \in \Delta_1$ to be considered must satisfy the additional inequality $\alpha_3 - \alpha_1 \leq 0$. Therefore, in the following analysis we can disregard the following α's

$$\alpha^1 = (0, 0, 1, 0) \ ; \ (0, 0, 1, 1) \ ; \ (0, 1, 1, 0) \ ; \ (0, 1, 1, 1).$$

Go to Step 3.
Step 3) The optimal solution of $Q^*(\alpha^1)$ is $(\bar{\lambda}; \bar{\mu}; \bar{\nu}) = (2, 0, 2, 0, 0; 0, 0, 1, 0; 2, 0, 0, 0)$. Therefore, problem (17) becomes

$$t = \min_{\alpha, f} \ f \quad \text{s.t.} \quad \alpha \in \Delta \ , \ f \geq 60 - Y_1\alpha_1 - Y_3\alpha_3 \ ; \ f \geq 60 - X_3(1 - \alpha_3) - 2Y_1\alpha_1$$

with solution $t = 20$; observe that this value of t ($t = 20$) improves the previous one ($t = 15$). Since $t = 20 < t_0 = \frac{85}{2}$, set $t_1 = t_0$. As $T_1 \neq t_1$, let us continue in Step 3. $\Delta_2 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta \ : \ \alpha_1 + \alpha_3 \geq 1 \ \wedge \ \alpha_3 - \alpha_1 \leq 0\}$. As $\Delta_2 \neq \varnothing$, set $k = 2$ and go to Step 1.

Iteration 3 with k = 2
Step 1) We have to choose $\alpha^2 \in \Delta_2$. Following the binary ordering on α's the first element of Δ not already discarded is $\alpha^2 = (1, 0, 0, 0) \in \Delta_2$. The optimal solution of $P(\alpha^2)$ is

$$x(\alpha^2) = (0, 10, 5, 16) \ , \ y(\alpha^2) = (4, 0, 0, 0), \quad \text{with } f^{\downarrow}(\alpha^2) = 65.$$

As $f^{\downarrow}(\alpha^2) > T_1$ then $T_2 = T_1 = 60$. Go to Step 2.
Step 2) By solving $P^{\star}(\alpha^2)$ we get $\bar{\lambda} = (2, 0, 0, 2, 1)$. The sets of indexes defined in (9) and (10) are $I_x(\bar{\lambda}) = \{1\}$ and $I_y(\bar{\lambda}) = \{3, 4\}$; $I_x(\bar{\lambda}) \cup I_y(\bar{\lambda}) \neq \varnothing$. The subsequent vectors $\alpha \in \Delta_2$ to be considered must satisfy the additional inequality $\alpha_1 - \alpha_3 - \alpha_4 \leq 0$. Therefore, we have to disregard the following α's

$$\alpha^2 = (1, 0, 0, 0) \ ; \ (1, 1, 0, 0).$$

Go to Step 3.
Step 3) The optimal solution of $Q^*(\alpha^2)$ takes to the problem (17)

$$t = \min_{\alpha, f} \ f \qquad \text{s.t.} \quad \begin{cases} \alpha \in \Delta \\ f \geq 60 - Y_1\alpha_1 - Y_3\alpha_3 \\ f \geq 60 - X_3(1 - \alpha_3) - 2Y_1\alpha_1 \\ f \geq 65 - X_1(1 - \alpha_1) - 2Y_3\alpha_3 - Y_4\alpha_4 \end{cases}$$

with solution $t = 20$. Since $t = 20 < t_1$, set $t_2 = t_1 = \frac{85}{2}$. As $T_2 \neq t_2$, let us continue in Step 3. $\Delta_3 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta \ : \ \alpha_1 + \alpha_3 \geq 1 \ \wedge \ \alpha_3 - \alpha_1 \leq 0 \ \wedge \ \alpha_1 \leq \alpha_3 + \alpha_4\}$. As $\Delta_3 \neq \varnothing$, set $k = 3$ and go to Step 1.

Iteration 4 with k = 3
Step 1) We have to choose $\alpha^3 \in \Delta_3$. Following the binary ordering on α's the first element of Δ not already discarded is $\alpha^3 = (1, 0, 0, 1) \in \Delta_3$. The optimal solution of $P(\alpha^3)$ is

$$x(\alpha^3) = (0, 0, 0, 0) \ , \ y(\alpha^3) = (20, 0, 0, 10), \quad \text{with } f^{\downarrow}(\alpha^3) = 60.$$

As $f^{\downarrow}(\alpha^3) = T_2$ then $T_3 = T_2 = 60$. Go to Step 2.
Step 2) Let us solve $P^{\star}(\alpha^3)$. We obtain the sets of indexes $I_x(\bar{\lambda}) = \varnothing$ and $I_y(\bar{\lambda}) = \{3\}$, that take to the inequality $\alpha_3 \geq 1$.
We are now under the assumption of item d) of Remark 3. We solve the relaxed problem RP with the additional condition $x_3 = 0$, getting a minimum value equal to 45. This is a lower bound of the optimal value f^0, better than the current one $t_2 = \frac{85}{2}$. Hence we set $t_3 = 45$. Go to Step 3.
Step 3) From now on, for the sake of simplicity, in this example we drop in Step 3 the part related to the computation of the lower bound. As $T_3 \neq t_3$, let us continue in Step 3. $\Delta_4 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta : \alpha_1 + \alpha_3 \geq 1 \ \wedge \ \alpha_3 - \alpha_1 \leq 0 \ \wedge \ \alpha_1 \leq \alpha_3 + \alpha_4 \ \wedge \ \alpha_3 \geq 1\}$.
As $\Delta_4 \neq \varnothing$, set $k = 4$ and go to Step 1.

Iteration 5 with k = 4
Step 1) We have to choose $\alpha^4 \in \Delta_4$. Following the binary ordering on α's the first element of Δ not already discarded is $\alpha^4 = (1, 0, 1, 0) \in \Delta_4$. The solution of $P(\alpha^4)$ produces a null step (see item e) of Remark 3). The next vector, denoted again by α^4, is $(1, 0, 1, 1)$. By solving $P(\alpha^4)$, we get

$$x(\alpha^4) = (0, 0, 0, 0) \ , \ y(\alpha^4) = (10, 0, 10, 5), \quad \text{with } f^{\downarrow}(\alpha^4) = 50.$$

As $f^{\downarrow}(\alpha^4) < T_3$ then $T_4 = f^{\downarrow}(\alpha^4) = 50$. Go to Step 2.
Step 2) Let us solve $P^{\star}(\alpha^4)$. We obtain the sets of indexes $I_x(\bar{\lambda}) = \{1, 3\}$ and $I_y(\bar{\lambda}) = \varnothing$, that take to the inequality $\alpha_1 + \alpha_3 \leq 1$. This inequality leads to discard the last two vectors in Δ_4, namely

$$(1, 1, 1, 0) \ ; \ (1, 1, 1, 1).$$

Go to Step 3.
Step 3) (Recall that we are dropping the first part of this step.) Set $t_4 = t_3 = 45$. As $T_4 \neq t_4$, let us continue in Step 3. The set Δ_5 is empty; hence $\rightarrow$ **STOP**. The current solution $(x(\alpha^4), y(\alpha^4))$ is an optimal solution of P.

6 Conclusions

We have introduced a decomposition method for a linear problem with complementarity constraints in a sequence of parameterized problems. By means of suitable cuts we have proposed an iterative method that leads to an optimal solution of the given problem or to an approximation of it providing an estimate of the error. In particular the considered method avoids the use of constraint qualifications which are hard to be determined in such kind of problems. The iterative method has been described by means of a non trivial example; for the solution of the example, at some steps, the numerical software MATLAB has been used. A completely unified MATLAB code fully implementing the whole algorithm is still in progress. This is a possible further development, together with some numerical testing. A full implementation will allow us to try out the method presented in the paper on standard test complementarity problems and to compare it with other existing methods.

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