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Abstract. Vector Optimization Problems have been intensively investigated by carrying out the analysis in the image space (namely, the space where the objective functions run) and several interesting results have been achieved. Here, exploiting some of these results and taking into account previous results, we present a scalarization method for Vector Optimization Problems. All the vector optima are found starting from the scalar problem and varying a parameter in the image space. In the special case of two objective functions, the method requires only one parameter and some calculus advantage are obtained. If the objective functions are convex, the method shrinks to a known one.

Keywords: Vector Optimization, Scalarization, Image Space, Bicriteria Optimization.

AMS Classification: 90C, 49K.

1 Introduction

Several theoretical aspects of Vector Optimization - like optimality conditions, duality, penalization - have been developed by exploiting a general scheme based on separation or alternative theorems and by carrying out the analysis in the image space [5, 7]. Moreover, as collateral result of this general approach, in [5, 7] it has been defined a scalarization method for Vector Optimization Problems (for short, VOPs). Afterwards, the authors of [5, 7] got acquainted with the existence of a previous analogous result [3, 4]. This latter result has been obtained by a way which is straighter and simpler than the former; the proof of the former is more complicated because it is a by-product of a general scheme which aims to embrace several existing developments and to stimulate new ones in the theory of VOPs and Vector Variational Inequalities.

In this paper, we start with the result of [5, 7], taking into account [3, 4], and we present a scalarization approach for a VOP with two objective functions. In the convex case, this approach recovers a known one, due to Benson [1, 2].

Since the main scope of the present paper is not the existence of extrema, in what follows the assumptions on their existence will be understood.

Let the positive integers ℓ, m, n, k and the cone $C \subseteq \mathbb{R}^\ell$ be given. In the sequel, it will be assumed that C is convex, closed and pointed – so that it expresses a partial order – with apex at the origin and with $\text{int}C \neq \varnothing$, namely with nonempty interior.

Consider the vector–valued functions $f : \mathbb{R}^n \to \mathbb{R}^\ell$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^k$ and the subset $X \subseteq \mathbb{R}^n$. Let $C_0 := C \setminus \{O\}$; we will consider the following vector minimization problem, which is called *generalized Pareto problem*:

$$\min_{C_0} f(x), \quad x \in K := \{x \in X : g(x) \geq 0, h(x) = 0\}, \tag{1.1}$$

where $\min_{C_0}$ denotes vector minimum with respect to the cone C_0: $x^0 \in K$ is a (global) vector minimum point (for short, v.m.p.) of (1.1), iff

$$f(x^0) \not\geq_{C_0} f(x) , \quad \forall x \in K, \tag{1.2}$$

where the inequality means $f(x^0) - f(x) \notin C_0$. At $C = \mathbb{R}^\ell_+$, (1.1) becomes the classic *Pareto VOP*.

Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^\ell$ and $C^* := \{c^* \in \mathbb{R}^\ell : \langle c, c^* \rangle \geq 0, \ \forall c \in C\}$ be the (positive) polar cone of C.

2 Scalarization

Fix $p \in \text{int}C^*$ and introduce the following scalar problem:

$$\min\langle p, f(x)\rangle, \quad x \in K \cap R(\xi), \tag{2.1}$$

where $\xi = (\xi_1, \ldots, \xi_\ell) \in f(X) \subseteq \mathbb{R}^\ell$ and $R(\xi) := \{x \in X : f(x) \in \xi - C\}$. For every $\xi \in f(X)$, let $x^0(\xi)$ be a minimum point (for short, m.p.) of (2.1).

Theorem 2.1. [3, 5, 7]. If $x^0(\xi)$ is a minimum point of (2.1) for any $\xi \in f(X)$, then $x^0(\xi)$ is a vector minimum point of (1.1). If x^0 is a vector minimum point of (1.1), then x^0 is a minimum point of (2.1) for $\xi = f(x^0)$.

Consider the following "fixed point" condition:

$$f(x^0(\xi)) = \xi, \quad \forall \xi \in f(X). \tag{2.2}$$

As observed by Corley in [3], for any solution ξ to (2.2), $x^0(\xi)$ is a v.m.p. of (1.1) by the first part of Theorem 2.1. The second part guarantees that all v.m.p.s of (1.1) are obtained, i.e. no v.m.p. of (1.1) is excluded by condition (2.2). Moreover, condition (2.2) eliminates the redundancy (i.e., the possibility that different values of ξ imply the same solution of (1.1)). The exclusion of the redundancy by means of (2.2) is proved by the following reasoning: if, *ab absurdo*, $\xi^1 \neq \xi^2$ implies $x^0(\xi^1) = x^0(\xi^2)$, then we have $f(x^0(\xi^1)) = f(x^0(\xi^2))$ and hence, by (2.2), we obtain $\xi^1 = \xi^2$.

Based on the previous results, it is possible to define a method in order to find all the v.m.p.s of (1.1). Fix any $p \in \text{int}C^*$; p will remain fixed in the sequel. Then choose any $\xi \in f(X)$ and solve the (scalar) problem (2.1). We find a solution x^0 (if any); x^0 is a solution of (1.1) too. If we want to give the possibility of running implicitly through the set of v.m.p.s – this happens, for instance, when a given function must be optimized over the set of v.m.p.s of (1.1) – we have to move $\xi \in f(X)$ in (2.1) starting with $\xi = \xi^0 = f(x^0)$ and maintaining a solution, say $x^0(\xi)$, to (2.1). The above method is different from that of Corley, who in [3] proposed first to find the solutions to (2.1) for all the values of parameter ξ, namely to find the function or multifunction $x^0(\xi)$, and next to equate $f(x^0(\xi)) = \xi$. In our proposal, (2.2) could be added to the parametric solution of (2.1), if we want to eliminate the redundancy.

3 The case of two objective functions

In this section we analyse the particular case $\ell = 2$. The method described in the previous section becomes simple since it is possible to handle only one parameter. In the convex case, it will be compared with the method of Benson [1].

Consider problem (1.1) with $\ell = 2$ and $C = \mathbb{R}^2_+$ and define the problem

$$\min f_1(x), \quad x \in K \cap \{x \in X : f_2(x) \le \xi_2\} \tag{3.1}$$

for each $\xi_2 \in \Xi$, where Ξ is the set of all $\xi_2 \in f_2(X)$ such that $K \cap \{x \in X : f_2(x) \le \xi_2\} \ne \varnothing$. Let $\xi_1(\xi_2) := f_1(x^0(\xi_2))$, where $x^0(\xi_2)$ is a m.p. of (3.1). Fix any $p \in \text{int}C^*$ and for every $\xi_2 \in \Xi$ consider the following problem

$$\min[p_1 f_1(x)) + p_2 f_2(x)], \quad x \in K \cap \{x \in X : f_1(x) \le \xi_1(\xi_2)), f_2(x) \le \xi_2\}. \tag{3.2}$$

Since $C = \mathbb{R}^2_+$, then $p = (p_1, p_2) \in \text{int}C^*$ is equivalent to $p_1 > 0$, $p_2 > 0$; in the sequel this condition will be understood.

The following proposition shows the equivalence between (1.1) and problem (3.2) which depends on the scalar parameter $\xi_2 \in \Xi$.

Proposition 3.1. If $x^0(\xi_2)$ is a minimum point of (3.2) for any $\xi_2 \in \Xi$, then $x^0(\xi_2)$ is a vector minimum point of (1.1). If x^0 is a vector minimum point of (1.1), then x^0 is a minimum point of (3.2) for $\xi_2^0 = f_2(x^0)$.

Proof. Suppose that $x^0(\xi_2)$ (for short, x^0) be a m.p. of (3.2) for any ξ_2 and set $\xi_1 := \xi_1(\xi_2)$. Then x^0 is a m.p. of (2.1) for (ξ_1, ξ_2) and hence, by Theorem 2.1, x^0 is a vector minimum point of (1.1).
To prove the second statement, let x^0 be a v.m.p. of (1.1) and consider (3.2) for $\xi_2^0 = f_2(x^0)$ and $\xi_1^0 := \xi_1(\xi_2^0)$. A feasible point x of (3.2) is such that $f_1(x) \le \xi_1(\xi_2^0) = \min_{f_2(x) \le \xi_2^0} f_1(x)$ and $f_2(x) \le \xi_2^0 = f_2(x^0)$. Now, observe that, if x^0 is a v.m.p. of (1.1), then x^0 is a minimum point of (3.1) with $\xi_2 = f_2(x^0)$. In fact, if *ab absurdo* x^0 is not a minimum point of (3.1), then there exists $\hat{x}$ such that $f_1(\hat{x}) < f_1(x^0)$ and $f_2(\hat{x}) \le f_2(x^0)$ and this contradicts the optimality of x^0 for (1.1). Therefore we have that $\xi_1(\xi_2^0) = f_1(x^0)$; hence a feasible point of (3.2) is such that $f_1(x) \le f_1(x^0)$ and $f_2(x) \le f_2(x^0)$. Since x^0 is a v.m.p. of (1.1), then x is a feasible point of (3.2) iff $f_1(x) = f_1(x^0)$ and $f_2(x) = f_2(x^0)$ and this implies that the objective function of (3.2) is constant for all feasible x. Since x^0 is feasible, the proof is complete. □

Let $m_i := \min_{x \in K} f_i(x),\ i = 1, 2$, and $M_2 := \min_{\substack{x \in K \\ f_1(x) = m_1}} f_2(x)$; obviously, it is $m_2 \le M_2$ and hence the interval $I := [m_2, M_2] \subset \Xi$ is well defined.

Proposition 3.2. If x^0 is a vector minimum point of (1.1), then $\xi_2^0 = f_2(x^0) \le M_2$.

Proof. Let x^0 be a v. m. p. of (1.1) and suppose, *ab absurdo*, that $\xi_2^0 = f_2(x^0) > M_2 := \min_{\substack{x \in K \\ f_1(x) = m_1}} f_2(x) = f_2(\tilde{x})$. Hence we have $f_1(\tilde{x}) = m_1 \le f_1(x^0)$ and $f_2(\tilde{x}) < f_2(x^0)$, and this contradicts the optimality of x^0 to (1.1). □

Proposition 3.1 shows that problem (1.1) can be solved by means of the parametric problem (3.2) and Proposition 3.2 shows that it is enough that the parameter ξ_2 runs in the interval $I := [m_2, M_2]$.

Moreover, let us observe that in (3.2) the inequality $f_1(x) \le \xi_1(\xi_2)$ can be equivalently substituted with $f_1(x) = \xi_1(\xi_2)$, since $\xi_1(\xi_2)$ is the minimum value of f_1 on the set of points $x \in K$ such that $f_2(x) \le \xi_2$; hence problem (3.2) is equivalent to:

$$p_1\xi_1(\xi_2) + \min(p_2 f_2(x)), \quad x \in K \cap \{x \in X : f_1(x) = \xi_1(\xi_2), f_2(x) \le \xi_2\}. \tag{3.3}$$

The above approach holds without any assumption on (1.1). Nevertheless, in some cases, we can have some advantages in solving problem (3.3), like under the hypotheses of the following proposition.

Proposition 3.3. Suppose that f_1 and f_2 be strictly concave functions on the convex set K. The solution set of (3.1) equals the solution set of

$$\min f_1(x), \quad x \in conv(K \cap \{x \in X : f_2(x) \le \xi_2\}), \tag{3.4}$$

where *conv* denotes the convex hull.

Proof. Since the feasible region $S(\xi_2) := K \cap \{x \in X : f_2(x) \le \xi_2\}$ of (3.1) is strictly contained in that of (3.4), it is enough to prove that, if x_0 is a m.p. of (3.1), then $\forall x \in convS(\xi_2) \setminus S(\xi_2)$, it turns out $f_1(x) > f_1(x^0)$. *Ab absurdo*, suppose that $\exists \tilde{x} \in convS(\xi_2) \setminus S(\xi_2)$ such that $f_1(x^0) \ge f_1(\tilde{x})$. Since $\tilde{x} \in convS(\xi_2)$, then there exist $x^1, \dots, x^r \in S(\xi_2)$ and $\alpha_1, \dots, \alpha_r > 0$, $\sum_{i=1}^r \alpha_i = 1$, with $r \le n+1$ such that $\tilde{x} = \sum_{i=1}^r \alpha_i x^i$. From the strict concavity of f_1 and the optimality of x^0, we have

$$f_1(x^0) \ge f_1(\tilde{x}) = f_1\left(\sum_{i=1}^r \alpha_i x^i\right) > \sum_{i=1}^r \alpha_i f_1(x^i) \ge \sum_{i=1}^r \alpha_i f_1(x^0) = f_1(x^0),$$

and this complete the proof. □

As a consequence of the previous result, if f_1 and f_2 are strictly concave and K is a polytope, then the set of m.p. of (3.1) is finite and this could simplify the solution of (3.3).

At last, observe that, in the present case, the fixed point condition (2.2) reduces to:

$$f_2(x^0(\xi_1(\xi_2), \xi_2)) = \xi_2. \tag{3.5}$$

4 The convex case

In this section, it will proved that, if a convexity assumption is made for problem (1.1), then it is enough to solve problem (3.1) to obtain all the v.m.p. of (1.1).

First of all, observe that, without any hypothesis, a m.p. of (3.2) is always a m.p. of (3.1). This result is contained in the proof of Proposition 3.1, where it is shown that a v.m.p. of (1.1) (that is equivalent to (3.2)) is a m.p. of (1.1); nevertheless, it can proved directly:

Proposition 4.1. If $x^0(\xi_2)$ is a minimum point of (3.2) then $x^0(\xi_2)$ is a minimum point of (3.1).

Proof. Let $x^0(\xi_2)$ (for short, x^0) be a m.p. of (3.2); then x^0 is such that $f_1(x^0) \leq \xi_1(\xi_2) = \min_{f_2(x)\leq\xi_2} f_1(x)$ and $f_2(x^0) \leq \xi_2$. Hence x^0 is a feasible point of (1.1); moreover, since $f_1(x^0) \leq \min_{f_2(x)\leq\xi_2} f_1(x)$, it turns out $f_1(x^0) = \min_{f_2(x)\leq\xi_2} f_1(x)$. □

Proposition 4.2. Suppose that f_1 and f_2 are convex functions on the convex set K. In problem (3.1) let $\xi_2 \in [m_2, M_2]$; if $x^0(\xi_2)$ is a minimum point of (3.1), then $x^0(\xi_2)$ is a minimum point of (3.2).

Proof. Fix $\xi_2 \in [m_2, M_2]$ and let $x^0(\xi_2)$ (for short, x^0) be a m.p. of (3.1). *Ab absurdo*, suppose that x^0 is not a m.p. of (3.2). Then there exists $\tilde{x} \in K$ such that

$$f_1(\tilde{x}) \leq \xi_1(\xi_2) = \min_{\substack{x\in K\\ f_2(x)\leq\xi_2}} f_1(x) = f_1(x^0), \tag{4.1}$$

$$f_2(\tilde{x}) \leq \xi_2, \tag{4.2}$$

$$f_1(\tilde{x}) + f_2(\tilde{x}) < f_1(x^0) + f_2(x^0). \tag{4.3}$$

Since from (4.1) we have $f_1(\tilde{x}) = f_1(x^0)$, (4.3) implies

$$f_2(\tilde{x}) < f_2(x^0). \tag{4.4}$$

The optimality of x^0 to (3.1) and the convexity of f_1, f_2 and K imply [1] [6] the existence of $\theta^0, \lambda^0 \in \mathbb{R}$, $\theta^0, \lambda^0 \geq 0$ and $(\theta^0, \lambda^0) \neq (0,0)$ such that

$$\theta^0(f_1(x^0) - f_1(x)) + \lambda^0(\xi_2 - f_2(x)) \leq 0 \quad \forall x \in K;$$

or, equivalently, since $\lambda^0(\xi_2 - f_2(x^0)) = 0$,

$$\theta^0 f_1(x^0) - \lambda^0(\xi_2 - f_2(x^0)) \leq \theta^0 f_1(x) - \lambda^0(\xi_2 - f_2(x)) \quad \forall x \in K. \tag{4.5}$$

Firstly, suppose that $\lambda^0 > 0$. Since $f_1(\tilde{x}) = f_1(x^0)$ and $f_2(\tilde{x}) < f_2(x^0)$, we have

$$\theta^0 f_1(\tilde{x}) - \lambda^0(\xi_2 - f_2(\tilde{x})) < \theta^0 f_1(x^0) - \lambda^0(\xi_2 - f_2(x^0))$$

which contradicts (4.5). Suppose, now, that $\lambda^0 = 0$ and hence $\theta^0 > 0$. From (4.5), we have $f_1(x^0) \leq f_1(x)$ $\forall x \in K$, i.e., x^0 minimizes f_1 on the whole set K. The same is for $\tilde{x}$, because of $f_1(\tilde{x}) = f_1(x^0)$; hence $f_1(\tilde{x}) = f_1(x^0) = m_1$. Moreover, the feasibility of x^0 for (3.1) implies

$$f_2(x^0) \leq \xi_2 \leq M_2 := \min_{\substack{x \in K \\ f_1(x) = m_1}} f_2(x);$$

therefore it is $f_2(x^0) \leq f_2(x)$ $\forall x \in K$ such that $f_1(x) = m_1$. In particular, it turns out that $f_2(x^0) \leq f_2(\tilde{x})$ and this contradicts (4.4). □

From Proposition 4.1 and Proposition 4.2 we have the following theorem that is exactly the result of Benson:

Theorem 4.1. (Theorem 2.1 of [1]). Assume that f_1 and f_2 are convex functions on the convex set K. Then x^0 is a vector minimum point of (1.1) iff x^0 is a minimum point of (3.1) for some $\xi_2 \in [m_2, M_2]$.

Remark. Note that if we assume the strict convexity of f_1 then we have the uniqueness of the minimum point of (3.1) for every $\xi_2 \in I$. This fact shortens the proof of Proposition 4.2 because the uniqueness is contradicted by (4.1) and (4.2). Finally. observe that whatever condition on (1.1) guaranteeing the linear separation of (3.1) in the image space permits to solve problem (1.1) by means of (3.1) and this generalizes the result of Benson.

The following nonconvex example shows that may exist points which are solutions to (3.1) for some $\xi_2 \in I$, but not v.m.p.s of (1.1), so that, without convexity assumption, the sufficiency of Theorem 4.1 (i.e., Proposition 4.2) does not hold.

[1] This property is called "image linear separation".

Example 4.1. Let us consider in (1.1) the following positions: $f_1(x) = -x^2 + 2x + 3$, $f_2(x) = -\frac{1}{3}x^2 + 2x + 1$, $K = \{x \in \mathbb{R} : 0 \leq x \leq 3\}$; f_1 and f_2 are strictly concave functions. The solution set is $\{0\} \cup (2, 3]$. It turns out $m_1 = 0$, $m_2 = 1$, $M_2 = 4$ and hence $I = [1, 4]$. For each $\xi_2 \in [1, 4]$, problem (3.1) is $\min_{\substack{x\in[0,3] \\ f_2(x)\leq\xi_2}} f_1(x)$. If we choose $\xi_2 = \frac{11}{3}$, the feasible region of (3.1) is $K = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$ and hence (3.1) assumes its minimum value at $x = 0$ and $x = 2$; the latter is not a vector minimum point of (1.1). If we exchange the role of f_1 and f_2, we obtain $M_1 = \min_{\substack{x\in[0,3] \\ f_2(x)=m_2}} f_1(x) = 3$ and hence $I = [0, 3]$. For every $\xi_1 \in [0, 3]$, we have to solve the problem $\min_{\substack{x\in[0,3] \\ f_1(x)\leq\xi_1}} f_2(x)$. When ξ_1 runs in $[0, 3)$, the interval $(2, 3]$ of v.m.p. is obtained; if $\xi_1 = 3$, the problem becomes $\min_{x\in\{0\}\cup[2,3]} f_2(x)$ and it assumes its minimum value at $x = 0$. Hence the procedure finds exactly the solution set of the given problem; this proves that the arrangement of f_1 and f_2 is not indifferent.

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