

RECIPROCAL CLASS OF JUMP PROCESSES

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ABSTRACT. Processes having the same bridges as a given reference Markov process constitute its *reciprocal class*. In this paper we study the reciprocal class of compound Poisson processes whose jumps belong to a finite set $\mathcal{A} \subset \mathbb{R}^d$. We propose a characterization of the reciprocal class as the unique set of probability measures on which a family of time and space transformations induces the same density, expressed in terms of the *reciprocal invariants*. The geometry of \mathcal{A} plays a crucial role in the design of the transformations, and we use tools from discrete geometry to obtain an optimal characterization. We deduce explicit conditions for two Markov jump processes to belong to the same class. Finally, we provide a natural interpretation of the invariants as short-time asymptotics for the probability that the reference process makes a cycle around its current state.

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INTRODUCTION

For a given \mathbb{R}^d -valued stochastic process $X = (X_t)_{t \in [0,1]}$ and $I \subseteq [0,1]$ we let \mathcal{F}_I be the σ -field generated by the random variables $(X_s : s \in I)$. We say that X is a *reciprocal process* if for every $0 \leq s < t \leq 1$ the σ -fields $\mathcal{F}_{[0,s] \cup [t,1]}$ and $\mathcal{F}_{[s,t]}$ are independent *conditionally* to $\mathcal{F}_{\{s,t\}}$. Comparing this notion to that of Markov process ($\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{(t,1]}$ independent conditionally to $\mathcal{F}_{\{t\}}$) it is not hard to show that every Markov process is reciprocal, but non-Markov reciprocal processes exist (see e.g. [25]).

The notion of reciprocal process is very natural in many respects. On one hand it emerges when one solves dynamic problem such as stochastic control problems or stochastic differential equations with boundary constraints, i.e. constraints on the joint distribution at the boundary times $t = 0$ and $t = 1$; this point of view has actually inspired the whole theory of reciprocal

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processes, that originated from ideas in [28] and [1] and led to several developments (see e.g. [34, 33, 10]). On the other hand it is a special case of the more general notion of *Markov random field* ([8]), which provides a Markov property for families of random variables (X_r) indexed by $r \in \mathbb{R}^d$.

The systematic study of reciprocal processes has initiated with the Gaussian case: covariance functions giving rise to reciprocal Gaussian processes have been thoroughly studied and characterized ([13, 14, 15, 4, 3, 19]). A more ambitious aim has been that of describing reciprocal processes in terms of *infinitesimal characteristics*, playing the role that infinitesimal generators play for Markov processes; this has led to the introduction of a second order stochastic calculus ([16, 31, 17]).

In this paper we consider a related problem. Suppose we are given a *reference* Markov process, whose law on its path space will be denoted by \mathbb{P} . For simplicity, we assume X to be the canonical process on its path space. A probability \mathbb{Q} is said to belong to the *reciprocal class* of \mathbb{P} if for every $0 \leq s < t \leq 1$ and $A \in \mathcal{F}_{[s,t]}$ we have

$$\mathbb{Q}(A|\mathcal{F}_{[0,s] \cup [t,1]}) = \mathbb{P}(A|\mathcal{F}_{[0,s] \cup [t,1]}) = \mathbb{P}(A|X_s, X_t), \quad (0.1)$$

where the last equality is an immediate consequence of the fact that X , being Markov, is a reciprocal process under \mathbb{P} . In particular, X is a reciprocal process also under \mathbb{Q} . The elements of the reciprocal class of \mathbb{P} are easy to characterize from a measure-theoretic point of view. Denote by \mathbb{P}^{xy} the *bridge* of \mathbb{P} from x to y , i.e. the probability obtained by conditioning \mathbb{P} on $\{(X_0, X_1) = (x, y)\}$; a probability \mathbb{Q} is in the reciprocal class of \mathbb{P} if and only if it is a mixture of the bridges of \mathbb{P} , i.e.

$$\mathbb{Q} = \int \mathbb{P}^{xy} \mu(dx, dy)$$

for some probability μ on $\mathbb{R}^d \times \mathbb{R}^d$.

Along the lines of what we have discussed above, it is desirable to characterize the probability measures belonging to the reciprocal class of \mathbb{P} by infinitesimal characteristics. One first question in this direction is the following. Assume \mathbb{P} is the law of a Markov process with infinitesimal generator L . Given another Markov generator L' , under what conditions the laws of the Markov processes generated by L' belong to the reciprocal class of \mathbb{P} ? This question is well understood for \mathbb{R}^d -valued diffusion processes with smooth coefficients, and it has motivated the search for the so-called *reciprocal invariants*: the collection of reciprocal invariants forms a functional $F(L)$ of the coefficients of the generator, such that the following statement holds: the laws of the processes generated by L and L' belong to the same reciprocal class if and only if $F(L) = F(L')$. Explicit expressions for the reciprocal invariants of diffusion processes can be found in [16, 6, 20]. For pure jump processes with values in a discrete state space, the understanding of reciprocal invariants is very limited, except for some special cases like counting processes ([7, 25]) or for pure jump processes with independent increments under very restrictive assumption on the geometry of jumps (called incommensurability), see Chapter 8 in [22].

In this paper we consider possibly time-inhomogeneous compound Poisson processes with jumps in a finite set \mathcal{A} , considerably weakening the assumptions in [22]. Our analysis reveals how reciprocal invariants are related to the geometric and graph-theoretic structure induced by the jumps. Close ideas apply to other context where a similar structure emerges, in particular to random walks of graphs, which will be treated in a forthcoming paper. To make clearer the improvement with respect to [22], we note that the assumption there imply that the corresponding graph structure in *acyclic*; In our framework, we will see that *cycles* are exactly the main parameter in the collection of reciprocal invariants, see Proposition 3.4.

The basic tool for identifying the reciprocal invariants is provided by functional equations, called duality formulae or integration by parts formulae, which represent a subject of independent interest. They have provided in particular useful characterizations of Poisson processes ([29] [21]). The idea of restricting the class of test functions in the duality formula in order to characterize the whole reciprocal class has appeared for the first time in [26, 27] in the framework of diffusions. In this paper we make explicit a functional equation containing a difference operator, which only depend on reciprocal invariants and characterize the reciprocal class of compound Poisson processes.

The paper is organized as follows. In Section 1 we set up the necessary notations and provide the relevant definitions. In Section 2 we define suitable transformations on the path space, and compute the density of the image of the law of the process under these transformations. This will allow in Section 3 to derive the duality formulae and to identify the reciprocal invariant. At the end of Section 3 we also give an interpretation of the reciprocal invariants in the time-homogeneous case, in terms on the asymptotic probability the process follows a given cycle. This could be extended to other contexts, e.g. to Markov diffusions, providing an alternative to the physical interpretation given in [6]. These extensions will be the subject of a forthcoming work.

1. FRAMEWORK. SOME DEFINITIONS AND NOTATIONS

We consider \mathbb{R}^d -valued random processes with finitely many types of jumps, chosen in a given set

$$\mathcal{A} = \{a^1, \dots, a^A\} \subseteq \mathbb{R}^d \quad (1.1)$$

of cardinality A . We associate to \mathcal{A} the matrix $\mathbf{A} = (a_i^j)_{1 \leq i \leq d, 1 \leq j \leq A} \in \mathbb{R}^{d \times A}$. Their paths are elements of $\mathbb{D}([0, 1], \mathbb{R}^d)$, the space of right continuous with left limits functions on $[0, 1]$ (usually called in french *càdlàg* paths), equipped with its canonical filtration $(\mathcal{F}_t)_{t \in [0, 1]}$: $\mathcal{F}_{[s, t]}$ is defined by $\sigma((X_r)_{r \in [s, t]})$

In this setting, paths can be described by the counting processes corresponding to each type of jumps. It is therefore natural to introduce the following random variables:

Definition 1.1. *Let define $\mathbf{N} = (\mathbf{N}_t)_{0 \leq t \leq 1}$, where $\mathbf{N}_t := (N_t^1, \dots, N_t^A)$ and, for any $j \in \{1, \dots, A\}$, N_t^j counts how many times the jump a^j has occurred*

up to time t :

$$N_t^j(\omega) = \sum_{s \leq t} \mathbf{1}_{\{\omega_s - \omega_{s-} = a^j\}}.$$

The total amount of jumps up to time t , $|N|_t$, is given by the sum of the coordinates of \mathbf{N}_t , that is $|N|_t := \sum_{j=1}^A N_t^j$.

The i -th jump time of type a^j is:

$$T_i^j := \inf \left\{ t \in [0, 1] : N_t^j = i \right\} \wedge 1.$$

Finally, the i^{th} jump time of the process is:

$$T_i := \inf \{ t \in [0, 1] : |N|_t = i \} \wedge 1.$$

Then, we can express the canonical process as $X_t = X_0 + \sum_j a^j N_t^j$, which leads to introduce the following set Ω of paths indeed carried by the processes we consider here:

$$\begin{aligned} \Omega &= \left\{ \omega : |N|_1(\omega) < +\infty \text{ and } X_t(\omega) = X_0(\omega) + \mathbf{A}\mathbf{N}_t(\omega), 0 \leq t \leq 1 \right\} \\ &\subseteq \mathbb{D}([0, 1], \mathbb{R}^d). \end{aligned}$$

We also define the set \mathcal{S} of possible initial and final values (X_0, X_1) for paths in Ω :

$$\mathcal{S} := \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \exists \mathbf{n} \in \mathbb{N}^A \text{ such that } y = x + \mathbf{A}\mathbf{n} \right\}.$$

For any measurable space \mathcal{X} , we will denote by $\mathcal{M}(\mathcal{X})$ the set of all non negative measures on \mathcal{X} and by $\mathcal{P}(\mathcal{X})$ the subset of probability measures on \mathcal{X} . $\mathcal{B}(\mathcal{X})$ is the set of bounded measurable functions on \mathcal{X} .

A general element of $\mathcal{P}(\Omega)$ will be denoted by \mathbb{Q} . Concerning its time projections, we use the notations

$$\mathbb{Q}_t := \mathbb{Q} \circ X_t^{-1} \text{ and } \mathbb{Q}_{01} := \mathbb{Q} \circ (X_0, X_1)^{-1}$$

for the law at time t , resp. the joint law at time 0 and 1.

As reference process, we will consider in this paper a time-inhomogeneous compound Poisson process denoted by \mathbb{P}_ν^x , where $x \in \mathbb{R}^d$ is a fixed initial position and ν is a regular jump measure belonging to the set

$$\begin{aligned} \mathcal{J} &:= \left\{ \nu \in \mathcal{M}(\mathcal{A} \times [0, 1]), \nu(dx, dt) = \sum_{j=1}^A \delta_{a^j}(dx) \otimes \nu^j(t) dt, \right. \\ &\quad \left. \nu^j(\cdot) \in C([0, 1], \mathbb{R}_+), 1 \leq j \leq A \right\}. \end{aligned} \quad (1.2)$$

Heuristically, the process with law \mathbb{P}_ν^x leaves its current state at rate $\sum_{j=1}^A \nu^j$ and when it jumps, it chooses the jump a^j with probability $\nu^j / \sum_{j=1}^A \nu^j$. More precisely, under \mathbb{P}_ν^x the canonical process X satisfies $X_0 = x$ a.s. and it has independent increments whose distribution is determined by its characteristic function

$$\mathbb{P}_\nu^x \left(\exp(i\lambda \cdot (X_t - X_s)) \right) = \exp \left(\sum_{j=1}^A (e^{i\lambda \cdot a^j} - 1) \int_s^t \nu^j(r) dr \right), \lambda \in \mathbb{R}^d, \quad (1.3)$$

where $\lambda \cdot x$ is the scalar product in \mathbb{R}^d . Note that here, as well as in the rest of the paper, for $\mathbb{Q} \in \mathcal{P}(\Omega)$ and $F : \Omega \rightarrow \mathbb{R}$, we write $\mathbb{Q}(F)$ for $\int F d\mathbb{Q}$. The properties of \mathbb{P}_ν^x are well known, in particular its semimartingale characteristics $(0, \nu, 0)$, see e.g. Chapters II and III of [12].

1.1. Reciprocal classes. We first define a bridge of the compound Poisson process \mathbb{P}_ν^x .

Definition 1.2. For $(x, y) \in \mathcal{S}$ and $\nu \in \mathcal{J}$, \mathbb{P}_ν^{xy} , the bridge of the compound Poisson process from x to y , is given by the probability measure on Ω :

$$\mathbb{P}_\nu^{xy} := \mathbb{P}_\nu^x(\cdot | X_1 = y).$$

Remark 1.3. Note that \mathbb{P}_ν^{xy} is well defined as soon as $(x, y) \in \mathcal{S}$, since in that case $\mathbb{P}_\nu^x(X_1 = y) > 0$.

The reciprocal class associated to the jump measure $\nu \in \mathcal{J}$ is now defined as the set of all possible mixtures of bridges in the family $(\mathbb{P}_\nu^{xy})_{(x,y) \in \mathcal{S}}$.

Definition 1.4. Let $\nu \in \mathcal{J}$. Its associated reciprocal class is the following set of probability measures on Ω :

$$\mathfrak{R}(\nu) := \left\{ \mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{Q}(\cdot) = \int_{\mathcal{S}} \mathbb{P}_\nu^{xy}(\cdot) \mathbb{Q}_{01}(dxdy) \right\}.$$

Let us describe the specific structure of any probability measure in the reciprocal class $\mathfrak{R}(\nu)$.

Proposition 1.5. Let $\mathbb{Q} \in \mathcal{P}(\Omega)$. Define then the compound Poisson process $\mathbb{P}_\nu^{\mathbb{Q}}$ with the same dynamics as \mathbb{P}_ν^x but the same initial distribution as \mathbb{Q} by $\mathbb{P}_\nu^{\mathbb{Q}} = \int_{\mathbb{R}^d} \mathbb{P}_\nu^x(\cdot) \mathbb{Q}_0(dx)$. Then the following assertions are equivalent:

- i) $\mathbb{Q} \in \mathfrak{R}(\nu)$
- ii) \mathbb{Q} is absolutely continuous with respect to $\mathbb{P}_\nu^{\mathbb{Q}}$ and the density $d\mathbb{Q}/d\mathbb{P}_\nu^{\mathbb{Q}}$ is $\sigma(X_0, X_1)$ -measurable.

Proof. i) \Rightarrow ii)

We first prove that \mathbb{Q}_{01} is absolutely continuous with respect to $(\mathbb{P}_\nu^{\mathbb{Q}})_{01}$.

Let us consider a Borel set $O \subseteq \mathbb{R}^d \times \mathbb{R}^d$ such that $\mathbb{Q}_{01}(O) > 0$. There exists $\mathbf{n} \in \mathbb{N}^A$ such that $\mathbb{Q}_{01}(O \cap \{\mathbf{N}_1 = \mathbf{n}\}) > 0$. We can rewrite this event in a convenient way:

$$\{(X_0, X_1) \in O\} \cap \{\mathbf{N}_1 = \mathbf{n}\} = \{X_0 \in \tau_{\mathbf{n}}^{-1}(O)\} \cap \{\mathbf{N}_1 = \mathbf{n}\}$$

where $\tau_{\mathbf{n}} : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is the map $x \mapsto (x, x + \mathbf{A}\mathbf{n})$. Since $\mathbb{Q}_{01}(O \cap \{\mathbf{N}_1 = \mathbf{n}\}) > 0$, $\mathbb{Q}_0(\tau_{\mathbf{n}}^{-1}(O)) = (\mathbb{P}_\nu)_0(\tau_{\mathbf{n}}^{-1}(O)) > 0$.

A simple application of the Markov property of $\mathbb{P}_\nu^{\mathbb{Q}}$ implies that

$$\begin{aligned} (\mathbb{P}_\nu^{\mathbb{Q}})_{01}(O) &\geq \mathbb{P}_\nu^{\mathbb{Q}}((X_0, X_1) \in O \cap \{\mathbf{N}_1 = \mathbf{n}\}) \\ &= (\mathbb{P}_\nu^{\mathbb{Q}})_0(\tau_{\mathbf{n}}^{-1}(O)) \mathbb{P}_\nu^{\mathbb{Q}}(\mathbf{N}_1 = \mathbf{n}) > 0. \end{aligned}$$

Therefore we can conclude that $\mathbb{Q}_{01} \ll (\mathbb{P}_\nu^{\mathbb{Q}})_{01}$ and we denote by h the density function $d\mathbb{Q}_{01}/d(\mathbb{P}_\nu^{\mathbb{Q}})_{01}$.

Finally, let us choose any $F \in \mathcal{B}(\Omega)$. By hypothesis:

$$\begin{aligned} \mathbb{Q}(F) &= \mathbb{Q}(\mathbb{Q}[F | X_0, X_1]) = \mathbb{Q}(\mathbb{P}_\nu^{X_0, X_1}(F)) \\ &= \mathbb{P}_\nu^{\mathbb{Q}}(\mathbb{P}_\nu^{X_0, X_1}(F) h(X_0, X_1)) = \mathbb{P}_\nu^{\mathbb{Q}}(\mathbb{P}_\nu^{X_0, X_1}(F h(X_0, X_1))) \\ &= \mathbb{P}_\nu^{\mathbb{Q}}(F h(X_0, X_1)), \end{aligned}$$

which leads to the conclusion that

$$\frac{d\mathbb{Q}}{d\mathbb{P}_\nu^\mathbb{Q}} = \frac{d\mathbb{Q}_{01}}{d(\mathbb{P}_\nu^\mathbb{Q})_{01}} = h(X_0, X_1).$$

This proves *ii*).

ii) \Rightarrow *i*) Suppose that there exists a non negative measurable function h such that $d\mathbb{Q}/d\mathbb{P}_\nu^\mathbb{Q} = h(X_0, X_1)$. It is a general result in the framework of reciprocal processes that, in that case, $\mathbb{Q} \in \mathfrak{R}(\nu)$, see e.g. Theorem 2.15 in [18]. For sake of completeness, we recall shortly the arguments. Let us take three measurable test functions ϕ, ψ, F .

$$\begin{aligned} \mathbb{Q}(\phi(X_0)\psi(X_1)F) &= \mathbb{P}_\nu^\mathbb{Q}(\phi(X_0)\psi(X_1)h(X_0, X_1)F) \\ &= \mathbb{P}_\nu^\mathbb{Q}\left(\phi(X_0)\psi(X_1)h(X_0, X_1)\mathbb{P}_\nu^\mathbb{Q}(F|X_0, X_1)\right) \\ &= \mathbb{Q}\left(\phi(X_0)\psi(X_1)\mathbb{P}_\nu^\mathbb{Q}(F|X_0, X_1)\right). \end{aligned}$$

Thus $\mathbb{Q}(F|X_0, X_1) = \mathbb{P}_\nu^\mathbb{Q}(F|X_0, X_1)$ \mathbb{Q} -a.s. for arbitrary functions F which implies that

$$\mathbb{Q}(\cdot|X_0 = x, X_1 = y) = \mathbb{P}_\nu^{xy} \quad \mathbb{Q}_{01}\text{-a.s.}$$

and the decomposition written in Definition 1.4 follows. \square

2. THE TIME AND SPACE TRANSFORMATIONS

In this section we define two families of transformations on the path space Ω , and we analyze their action on the probability measures of the reciprocal class $\mathfrak{R}(\nu)$. This will later provide (in Theorem 3.3) a characterization of $\mathfrak{R}(\nu)$ as the set of probability measures under which a functional equation holds true. As a consequence, we will see that $\mathfrak{R}(\nu)$ depends on ν only through a family of specific functionals of ν , that we call *reciprocal invariants*.

2.1. Time Changes. We consider the set \mathcal{U} of all regular diffeomorphisms of the time interval $[0, 1]$, parametrized by the set \mathcal{A} :

$$\mathcal{U} = \left\{ u \in C^1(\{1, \dots, A\} \times [0, 1]; [0, 1]), u(\cdot, 0) \equiv 0, u(\cdot, 1) \equiv 1, \min_{j \in A, t \in [0, 1]} \dot{u}(j, t) > 0 \right\}.$$

With the help of each $u \in \mathcal{U}$ we construct a transformation of the reference compound Poisson process by time changes acting separately on each component process $N^j, j = 1, \dots, A$.

Definition 2.1. Let $u \in \mathcal{U}$. We define the time-change transformation π_u by:

$$\begin{aligned} \pi_u : \Omega &\longrightarrow \mathbb{D}([0, 1], \mathbb{R}^d) \\ \pi_u(\omega)(t) &:= \omega(0) + \sum_{j=1}^A a^j N_{u(j,t)}^j(\omega), \quad 0 \leq t \leq 1. \end{aligned}$$

Remark 2.2. We cannot a priori be sure that π_u takes values in Ω since it may happen that jumps sincronize, i.e. $u^{-1}(j, T_i^j) = u^{-1}(j', T_{i'}^{j'})$ for some j, j' . However it is easy to see that this happens with zero probability under \mathbb{P}_ν^x .

We now define a family of maps called *reciprocal time-invariants*.

Definition 2.3. *The reciprocal time-invariant associated to ν is the function:*

$$\Xi^\nu : \{1, \dots, A\} \times [0, 1]^2 \rightarrow \mathbb{R}_+, \quad \Xi^\nu(j, s, t) := \frac{\nu^j(t)}{\nu^j(s)}.$$

Remark 2.4. Note that in the time-homogeneous case $\Xi^\nu \equiv 1$.

In the next proposition we shall prove that the image of \mathbb{P}_ν^x under the above time change π_u is absolutely continuous with respect to \mathbb{P}_ν^x , and that its density is indeed a function of the reciprocal time-invariant Ξ^ν .

Proposition 2.5. *The following functional equation holds under \mathbb{P}_ν^x : For all $u \in \mathcal{U}$,*

$$\mathbb{P}_\nu^x(F \circ \pi_u) = \mathbb{P}_\nu^x\left(F \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right), \quad \forall F \in \mathcal{B}(\Omega). \quad (2.1)$$

Proof. We first observe that, for every fixed $j \in \{1, \dots, A\}$ the process

$$N_t^j \circ \pi_u - \int_0^t \nu^j(u(j, s)) \dot{u}(j, s) ds \quad (2.2)$$

is a \mathbb{P}_ν^x -martingale w.r.t. to its natural filtration $\tilde{\mathcal{F}}$. Indeed, for any $s \leq t$ and any $F \tilde{\mathcal{F}}_s$ -measurable, by applying the basic properties of processes with independent increments, we obtain:

$$\begin{aligned} \mathbb{P}_\nu^x(F (N_t^j - N_s^j) \circ \pi_u) &= \mathbb{P}_\nu^x(F) \int_{u(j, s)}^{u(j, t)} \nu^j(r) dr \\ &= \mathbb{P}_\nu^x(F) \int_s^t \nu^j(u(j, r)) \dot{u}(j, r) dr. \end{aligned}$$

Therefore $N_t^j \circ \pi_u$ is an inhomogeneous Poisson process with intensity

$$\nu^j(u(j, \cdot)) \dot{u}(j, \cdot).$$

Moreover, if $j \neq j'$, $N^j \circ \pi_u$ and $N^{j'} \circ \pi_u$ are independent processes under \mathbb{P}_ν^x , because the processes N^j and $N^{j'}$ are independent and π_u acts separately on each component. This implies that the image of \mathbb{P}_ν^x under π_u , $\mathbb{P}_\nu^x \circ \pi_u^{-1}$, is a compound Poisson process whose jump measure is given by $\sum_{j=1}^A \delta_{a^j}(dx) \otimes \nu^j(u(j, t)) \dot{u}(j, t) dt$.

We can now apply the Girsanov theorem (see e.g. Theorem 5.35, Chapter III of [12]) to get the density of the push-forward measure $\mathbb{P}_\nu^x \circ \pi_u^{-1}$ w.r.t.

\mathbb{P}_ν^x :

$$\frac{d\mathbb{P}_\nu^x \circ \pi_u^{-1}}{d\mathbb{P}_\nu^x} = \exp \left[\sum_{j=1}^A \left(\int_0^1 (\nu^j(u(j,t))\dot{u}(j,t) - \nu^j(t)) dt + \int_0^1 \log(\Xi^\nu(j,t,u(j,t))\dot{u}(j,t)dN_t^j) \right) \right].$$

With the change of variable $t = u^{-1}(t')$ we have for any j

$$\int_0^1 \nu^j(u(j,t))\dot{u}(j,t) dt = \int_0^1 \nu^j(t') dt'.$$

Therefore the first integral disappears and the conclusion follows. \square

Remark 2.6. In the recent work [7], the authors establish a differential version of the equation (2.1), in the context of counting processes (i.e. $\mathcal{A} = \{1\}$) with a possibly space-dependent intensity. Such a formula is inspired by the Malliavin calculus for jump processes developed in [2] putting in duality a differential operator and the stochastic integral. We can, without getting into the details, relate the functional equation (2.1) with the formula proved there as follows: First consider a smooth function v satisfying the loop condition $\int_0^1 v_t dt = 0$ and define the function $u_\varepsilon^t := t + \varepsilon \int_0^t v_s ds$, $\varepsilon > 0$. Note that, for ε sufficiently small, $u^\varepsilon \in \mathcal{U}$. If we then apply (2.1) to a smooth functional of the jump times, we obtain after some elementary operations:

$$\frac{1}{\varepsilon} \mathbb{P}_\nu^x(F \circ \pi_{u^\varepsilon} - F) = \frac{1}{\varepsilon} \mathbb{P}_\nu^x \left(F \left(\exp \left(\int_{[0,1]} \log \Xi^\nu(1,t,u^\varepsilon(t)) \dot{u}_t^\varepsilon dN_t \right) - 1 \right) \right).$$

If we now let ε tend to 0, we obtain the duality formula

$$\mathbb{P}_\nu^x(D_v F) = \mathbb{P}_\nu^x \left(F \int_0^1 v_t dN_t \right) + \mathbb{P}_\nu^x \left(F \int_0^1 v_t \int_t^1 \frac{\dot{v}(s)}{\nu(s)} dN_s dt \right)$$

where $D_v F = \lim_{\varepsilon \rightarrow 0} (F \circ \pi_{u^\varepsilon} - F) / \varepsilon$.

This formula can then be extended to space-dependent intensities.

2.2. Space transformations. The transformations π_u introduced in the previous section, when acting on a given path, change the jump times leaving unchanged the total number of jumps of each type. We now introduce transformations that modify the total number of jumps; these transformations act on the counting variable \mathbf{N}_1 taking its values in \mathbb{N}^A , which we embed into \mathbb{Z}^A to take advantage of the lattice structure.

2.2.1. Shifting a Poisson random vector. We first consider a multivariate Poisson distribution $\mathbf{p}_\lambda \in \mathcal{P}(\mathbb{N}^A)$ where $\lambda = (\lambda^1, \dots, \lambda^A) \in \mathbb{R}_+^A$:

$$\forall \mathbf{n} \in \mathbb{N}^A, \quad \mathbf{p}_\lambda(\mathbf{n}) = \exp \left(- \sum_{j=1}^A \lambda^j \right) \prod_{j=1}^A \frac{(\lambda^j)^{n^j}}{n^j!}. \quad (2.3)$$

We first give a straightforward multidimensional version of Chen's characterization of a Poisson random variable. He introduced it to estimate the rate of convergence of sum of dependent trials to the Poisson distribution (see the original paper [5] and Chapter 9 in [30] for a complete account of Chen's method).

Proposition 2.7. *Let $\lambda \in (\mathbb{R}_+)^A$. Then $\rho \in \mathcal{P}(\mathbb{N}^A)$ is the multivariate Poisson distribution \mathfrak{p}_λ if and only if*

$$\forall \mathbf{e}^j, j = 1, \dots, A, \quad \rho(f(\mathbf{n} + \mathbf{e}^j)) = \lambda^j \rho(f(\mathbf{n})n^j), \quad \forall f \in \mathcal{B}(\mathbb{N}^A),$$

where \mathbf{e}^j denote the j -th vector of the canonical basis of \mathbb{Z}^A .

One can interpret this characterization as the computation of the density of the image measure by any shift along the canonical basis of \mathbb{N}^A .

Now we consider as more general transformations multiple left- and right-shifts, acting simultaneously on each coordinate, that is, we shift by vectors $\mathbf{v} \in \mathbb{Z}^A$.

Definition 2.8. *Let $\mathbf{v} \in \mathbb{Z}^A$. We define the \mathbf{v} -shift by*

$$\begin{aligned} \theta_{\mathbf{v}} : \mathbb{Z}^A &\longrightarrow \mathbb{Z}^A \\ \mathbf{z} &\longmapsto \theta_{\mathbf{v}}(\mathbf{z}) = \mathbf{z} + \mathbf{v}. \end{aligned}$$

Consider the image of \mathfrak{p}_λ under $\theta_{\mathbf{v}}$. It is a probability measure whose support is no more included in \mathbb{N}^A since there may be $\mathbf{z} \in \mathbb{N}^A$ such that $\theta_{\mathbf{v}}(\mathbf{z}) \notin \mathbb{N}^A$. Therefore we only compute the density of its absolutely continuous component, appearing in the Radon-Nykodim decomposition:

$$\mathfrak{p}_\lambda \circ \theta_{\mathbf{v}}^{-1} = \mathfrak{p}_\lambda^{\mathbf{v},ac} + \mathfrak{p}_\lambda^{\mathbf{v},sing}. \quad (2.4)$$

A version of the density of the absolutely continuous component is given by

$$\frac{d\mathfrak{p}_\lambda^{\mathbf{v},ac}}{d\mathfrak{p}_\lambda}(\mathbf{n}) = \lambda^{-\mathbf{v}} \prod_{j=1}^A \frac{n^j!}{(n^j - v^j)!} \mathbb{1}_{\{n^j \geq v^j\}}, \quad \text{where } \lambda^{\mathbf{v}} := \prod_{j=1}^A (\lambda^j)^{v^j}.$$

In view of obtaining a change of measure formula as in Proposition [2.7](#) we define

$$G_{\mathbf{v}}(\mathbf{n}) := \prod_{j=1}^A \frac{n^j!}{(n^j - v^j)!} \mathbb{1}_{\{n^j \geq v^j\}}. \quad (2.5)$$

Let us now consider the space $\mathcal{B}^\sharp(\mathbb{Z}^A) \subseteq \mathcal{B}(\mathbb{Z}^A)$ consisting of test functions with support in \mathbb{N}^A :

$$\mathcal{B}^\sharp(\mathbb{Z}^A) := \{f \in \mathcal{B}(\mathbb{Z}^A) : f(\mathbf{z}) = 0 \ \forall \mathbf{z} \notin \mathbb{N}^A\}.$$

Then, the considerations above can be summarized in the following formula:

$$\mathfrak{p}_\lambda(f \circ \theta_{\mathbf{v}}) = \lambda^{-\mathbf{v}} \mathfrak{p}_\lambda(f G_{\mathbf{v}}), \quad \forall f \in \mathcal{B}^\sharp(\mathbb{Z}^A). \quad (2.6)$$

Example 2.9. Let $\mathcal{A} = \{-1, 1\}$. We call n^- (rather than n^1) and n^+ (rather than n^2) the counting variables for the jumps -1 and 1 respectively. The same convention is adopted for the intensity vector $\lambda = (\lambda^-, \lambda^+)$. Then, for $\mathbf{v} = (1, 1)$ (resp. $\mathbf{v} = (1, -1)$) and for any $f \in \mathcal{B}^\sharp(\mathbb{Z}^2)$,

$$\begin{aligned} \mathfrak{p}_\lambda\left(f(n^- + 1, n^+ + 1)\right) &= \frac{1}{\lambda^- \lambda^+} \mathfrak{p}_\lambda\left(f(n^-, n^+) n^- n^+\right), \\ \mathfrak{p}_\lambda\left(f(n^- + 1, n^+ - 1)\right) &= \frac{\lambda^+}{\lambda^-} \mathfrak{p}_\lambda\left(f(n^-, n^+) \frac{n^-}{n^+ + 1}\right). \end{aligned}$$

2.2.2. *Lattices and conditional distributions.* We now consider, associated to a measure $\mu \in \mathcal{P}(\mathbb{N}^A)$, the following set of probability measures on \mathbb{N}^A :

$$\mathcal{R}_{\mathbf{A}}(\mu) := \left\{ \rho \in \mathcal{P}(\mathbb{N}^A) : \rho(\cdot) = \int \mu(\cdot | \sigma(\mathbf{A})) d\rho_{\sigma(\mathbf{A})} \right\},$$

where the σ -algebra $\sigma(\mathbf{A})$ is generated by the application $\mathbf{z} \mapsto \mathbf{A}\mathbf{z}$ defined on \mathbb{Z}^A , and the measure $\rho_{\sigma(\mathbf{A})}$ is the projection of ρ on $\sigma(\mathbf{A})$.

The set $\mathcal{R}_{\mathbf{A}}(\mu)$ presents strong analogies with a reciprocal class as introduced in Definition 1.4. Indeed, one can prove an analogous to Proposition 1.5 that is

$$\rho \in \mathcal{R}_{\mathbf{A}}(\mu) \text{ if and only if } \rho \ll \mu \text{ and } \frac{d\rho}{d\mu} \text{ is } \sigma(\mathbf{A})\text{-measurable.}$$

Our first goal is to characterize $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$ using the formula (2.6) computed for a suitably chosen set of shift vectors \mathbf{v} . The right set will be the following sublattice of \mathbb{Z}^A :

$$\ker_{\mathbb{Z}}(\mathbf{A}) := \ker(\mathbf{A}) \cap \mathbb{Z}^A. \quad (2.7)$$

Let us observe that if two paths $\omega, \tilde{\omega} \in \Omega$ have the same initial and final values, $(X_0, X_1)(\omega) = (X_0, X_1)(\tilde{\omega})$, then $\mathbf{N}_1(\omega) - \mathbf{N}_1(\tilde{\omega}) \in \ker_{\mathbb{Z}}(\mathbf{A})$. The next statement clarifies the role of $\ker_{\mathbb{Z}}(\mathbf{A})$.

Proposition 2.10. *Let $\rho \in \mathcal{P}(\mathbb{N}^A)$. Then $\rho \in \mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$ if and only if*

$$\forall \mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A}), \quad \rho(f \circ \theta_{\mathbf{c}}) = \frac{1}{\lambda^{\mathbf{c}}} \rho(f G_{\mathbf{c}}) \quad \forall f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A), \quad (2.8)$$

where $G_{\mathbf{c}}$ is defined in (2.5).

Proof. (\Rightarrow) Let $f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A)$ and $\mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A})$. By definition of $\ker_{\mathbb{Z}}(\mathbf{A})$ and $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$ we can choose a version of the density $h = \frac{d\rho}{d\mathfrak{p}_\lambda}$ such that $h \circ \theta_{\mathbf{c}} = h$. Applying the formula (2.6) we obtain:

$$\begin{aligned} \rho(f \circ \theta_{\mathbf{c}}) &= \mathfrak{p}_\lambda((f \circ \theta_{\mathbf{c}})h) = \rho((fh) \circ \theta_{\mathbf{c}}) \\ &= \lambda^{-\mathbf{c}} \mathfrak{p}_\lambda(fh) = \lambda^{-\mathbf{c}} \rho(fh) \end{aligned}$$

(\Leftarrow) Let $\mathbf{n}, \mathbf{m} \in \mathbb{N}^A$ be such that $\mathbf{A}\mathbf{n} = \mathbf{A}\mathbf{m}$. Set $f := \mathbf{1}_{\mathbf{n}}$, $\mathbf{c} := \mathbf{n} - \mathbf{m}$. Then

$$\rho(\mathbf{m}) = \rho(f \circ \theta_{\mathbf{c}}) = \lambda^{-\mathbf{c}} G_{\mathbf{c}}(\mathbf{n}) \rho(\mathbf{n}).$$

Since, by (2.6), the same relation holds under \mathfrak{p}_λ , we have

$$\frac{d\rho}{d\mathfrak{p}_\lambda}(\mathbf{m}) = \frac{d\rho}{d\mathfrak{p}_\lambda}(\mathbf{n}),$$

which completes the proof. \square

Example 2.11. Resuming Example 2.9, we verify that, in this case, $\ker_{\mathbb{Z}}(\mathbf{A}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbb{Z}$. Proposition 2.10 tells us that a probability distribution ρ on \mathbb{N}^2 satisfies

$$\rho(\cdot | n^+ - n^- = x) = \mathfrak{p}_\lambda(\cdot | n^+ - n^- = x) \quad \forall x \in \mathbb{Z}$$

if and only if, for all k in \mathbb{N}^* and for all $f \in \mathcal{B}^{\sharp}(\mathbb{Z}^2)$,

$$\rho(f(n^- + k, n^+ + k)) = \frac{1}{(\lambda^+ \lambda^-)^k} \rho(f(n^-, n^+) \prod_{i=0}^{k-1} (n^- - i)(n^+ - i))$$

and

$$\rho\left(f(n^- - k, n^+ - k)\right) = (\lambda^+ \lambda^-)^k \rho\left(f(n^-, n^+)\right) \prod_{i=1}^k \frac{1}{(n^- + i)(n^+ + i)}.$$

Taking Proposition 2.10 as a characterization of $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$ is rather unsatisfactory, since we do not exploit the lattice structure of $\ker_{\mathbb{Z}}(\mathbf{A})$. It is then natural to look for improvements, by imposing (2.8) to be satisfied only for shifts in a smaller subset of $\ker_{\mathbb{Z}}(\mathbf{A})$, but still characterizing $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$. In particular, since $\ker_{\mathbb{Z}}(\mathbf{A})$ is a sublattice of \mathbb{Z}^A , one wants to understand if restricting to a basis suffices. We answer affirmatively if $\ker_{\mathbb{Z}}(\mathbf{A})$ satisfy a simple geometrical condition, see Proposition 2.15. However, in general this is false, and we construct the Counterexample 2.12. Before doing so, let us recall that $\mathcal{L} \subseteq \mathbb{R}^A$ is a *lattice* if there is a set of linearly independent vectors $\{b_1, \dots, b_l\}$ such that:

$$\mathcal{L} = \left\{ \sum_{k=1}^l z_k b_k, z_k \in \mathbb{Z}, k = 1, \dots, l \right\}. \quad (2.9)$$

Any set $\{z_1, \dots, z_l\}$ satisfying (2.9) is a *basis* for \mathcal{L} . Since any discrete subgroup of \mathbb{R}^A is a lattice (see e.g. Proposition 4.2 in [23]) then $\ker_{\mathbb{Z}}(\mathbf{A})$ is a lattice.

The equations (2.8) essentially tell us that, if $\mathbf{n} \in \mathbb{N}^A$ is such that $\mathbf{m} := \theta_{-\mathbf{c}} \mathbf{n}$ is also an element of \mathbb{N}^A , then $\rho(\mathbf{m}) = \rho(\mathbf{n}) \mathfrak{p}_\lambda(\mathbf{m}) / \mathfrak{p}_\lambda(\mathbf{n})$. If we let \mathbf{c} vary in a lattice basis it may happen that the "network" of relations constructed in this way is not enough to capture the structure of conditional probabilities. In the next paragraph, we will indeed reformulate this problem as a connectivity problem for a certain family of graphs, and propose a solution in this framework using *generating sets* of lattices.

Counterexample 2.12. Let $\mathcal{A} = \{3, 4, 5\}$. Then

$$\ker_{\mathbb{Z}}(\mathbf{A}) = \{ \mathbf{c} \in \mathbb{Z}^3 : 3c_1 + 4c_2 + 5c_3 = 0 \}.$$

We define three vectors

$$f = (-3, 1, 1), \quad g = (1, -2, 1), \quad h = (2, 1, -2).$$

Note that $\{f, g, h\} \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$. We also define

$$\mathbf{n}_f := (3, 0, 0), \quad \mathbf{n}_g := (0, 2, 0), \quad \mathbf{n}_h := (0, 0, 2).$$

Moreover, we observe that

$$\text{if, for some } \mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A}), \theta_{\mathbf{c}} \mathbf{n}_f \in \mathbb{N}^3 \text{ then } \mathbf{c} = f. \quad (2.10)$$

This can be checked with a direct computation. The analogous statement also holds for g and h , i.e.

$$\theta_{\mathbf{c}} \mathbf{n}_g \in \mathbb{N}^3 \Rightarrow \mathbf{c} = g, \quad \theta_{\mathbf{c}} \mathbf{n}_h \in \mathbb{N}^3 \Rightarrow \mathbf{c} = h.$$

Let us now consider any basis $\ker_{\mathbb{Z}}^*(\mathbf{A})$ of $\ker_{\mathbb{Z}}(\mathbf{A})$. Since $\ker_{\mathbb{Z}}(\mathbf{A})$ is two dimensional, at least one vector, f or g or h , does not belong to $\ker_{\mathbb{Z}}^*(\mathbf{A})$. We assume w.l.o.g that $f \notin \ker_{\mathbb{Z}}^*(\mathbf{A})$. For any $0 < \varepsilon < 1$, $\lambda \in \mathbb{R}_+^3$, we define

the probability measure $\rho \in \mathcal{P}(\mathbb{N}^3)$ as a mixture between the degenerate measure $\delta_{\mathbf{n}_f}$ and \mathfrak{p}_λ as follows:

$$\rho = \varepsilon \delta_{\mathbf{n}_f} + (1 - \varepsilon) \mathfrak{p}_\lambda. \quad (2.11)$$

Note that $\rho \notin \mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$. Indeed any version of the density must be such that:

$$\frac{d\rho}{d\mathfrak{p}_\lambda}(\mathbf{n}_f) = \frac{\varepsilon}{\mathfrak{p}_\lambda(\mathbf{n}_f)} + (1 - \varepsilon), \quad \frac{d\rho}{d\mathfrak{p}_\lambda}(\theta_{\mathbf{c}_f} \mathbf{n}_f) = 1 - \varepsilon.$$

But, on the other hand, identity (2.8) is satisfied for any $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$. Let us pick any test function $f = \mathbb{1}_{\{\mathbf{z}=\bar{\mathbf{n}}\}}$, where $\bar{\mathbf{n}} \in \mathbb{N}^3$ and $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$. There are two possibilities:

- Either $\theta_{-\mathbf{c}}\bar{\mathbf{n}} \in \mathbb{Z}^3 \setminus \mathbb{N}^3$. In this case (2.8) is satisfied by ρ because both sides of the equality are zero, the left side because $\theta_{-\mathbf{c}}\bar{\mathbf{n}} \notin \mathbb{N}^A$, $\rho(\mathbb{N}^A) = 1$ and the right side because $G_{\mathbf{c}}(\bar{\mathbf{n}}) = 0$.
- Or $\theta_{-\mathbf{c}}\bar{\mathbf{n}} \in \mathbb{N}^A$. In this case, thanks to (2.10) and $f \notin \ker_{\mathbb{Z}}^*(\mathbf{A})$ we have $\bar{\mathbf{n}} \neq \mathbf{n}_f$ and $\theta_{-\mathbf{c}}\bar{\mathbf{n}} \neq \mathbf{n}_f$. Therefore, by (2.11),

$$\begin{aligned} \rho(\mathbb{1}_{\{\theta_{\mathbf{c}}\mathbf{z}=\bar{\mathbf{n}}\}}) &= \frac{\rho(\theta_{-\mathbf{c}}\bar{\mathbf{n}})}{\rho(\bar{\mathbf{n}})} \rho(\mathbb{1}_{\{\mathbf{z}=\bar{\mathbf{n}}\}}) = \frac{\mathfrak{p}_\lambda(\theta_{-\mathbf{c}}\bar{\mathbf{n}})}{\mathfrak{p}_\lambda(\bar{\mathbf{n}})} \rho(\mathbb{1}_{\{\mathbf{z}=\bar{\mathbf{n}}\}}) \\ &= \lambda^{-\mathbf{c}} \rho(\mathbb{1}_{\{\mathbf{z}=\bar{\mathbf{n}}\}} G_{\mathbf{c}}(\mathbf{z})) \end{aligned}$$

which is equivalent to (2.8).

We thus obtain an example of a set \mathcal{A} such that, for any $\lambda \in \mathbb{R}_+^3$ and any basis $\ker_{\mathbb{Z}}^*(\mathbf{A})$ of $\ker_{\mathbb{Z}}(\mathbf{A})$ we can construct a probability measure ρ which satisfies (2.8) for $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$ and $f \in \mathcal{B}^\sharp(\mathbb{Z}^A)$ but does not belong to $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_\lambda)$.

2.2.3. Generating sets of lattices and conditional distributions. We first define the *foliation* that the lattice $\ker_{\mathbb{Z}}(\mathbf{A})$ induces on \mathbb{N}^A : given $\mathbf{n} \in \mathbb{N}^A$, let define the leaf containing \mathbf{n} by

$$\mathfrak{F}_{\mathbf{A},\mathbf{n}} := \{\mathbf{n} + \ker_{\mathbb{Z}}(\mathbf{A})\} \cap \mathbb{N}^A. \quad (2.12)$$

Fix now $\Gamma \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$. Γ induces a graph structure on each leaf (see e.g. [23]):

Definition 2.13. For $\Gamma \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$ and $\mathbf{n} \in \mathbb{N}^A$ we define $\mathcal{G}(\mathfrak{F}_{\mathbf{A},\mathbf{n}}, \Gamma)$ as the graph whose vertex set is $\mathfrak{F}_{\mathbf{A},\mathbf{n}}$ and whose edge set is given by

$$\{(\mathbf{m}, \mathbf{m}') \in \mathfrak{F}_{\mathbf{A},\mathbf{n}} \times \mathfrak{F}_{\mathbf{A},\mathbf{n}} : \exists \mathbf{c} \in \Gamma \text{ with } \mathbf{m} = \theta_{\mathbf{c}}(\mathbf{m}')\}.$$

We are now ready to introduce the notion of generating set for $\ker_{\mathbb{Z}}(\mathbf{A})$.

Definition 2.14. The set Γ is a generating set for $\ker_{\mathbb{Z}}(\mathbf{A})$ if, for all $\mathbf{n} \in \mathbb{N}^A$, $\mathcal{G}(\mathfrak{F}_{\mathbf{A},\mathbf{n}}, \Gamma)$ is a connected graph for the graph structure we have just defined.

We now recall, following Chapters 10 and 11 of the recent book [11] - in which an extensive study of generating sets for lattices is done - that there exists a finite generating set for any given lattice. Finding a generating set for a lattice $\ker_{\mathbb{Z}}(\mathbf{A})$ is indeed closely related to the algebraic problem of finding a set of generators for the *lattice ideal* associated to $\ker_{\mathbb{Z}}(\mathbf{A}) \subseteq \mathbb{Z}^A$, see Lemma 11.3.3 in [11].

Any generating set contains a lattice basis, but in general it might be much larger. Figure 1 illustrates a case where $\{(2, -4, 2), (0, -5, 4)\}$, basis of $\ker_{\mathbb{Z}}(\mathbf{A})$, is not a generating set since the graph $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \Gamma)$ is not connected for $\mathbf{n} = (6, 1, 2)$.

Computing explicitly generating sets is a hard task. We give here two simple conditions on $\ker_{\mathbb{Z}}(\mathbf{A})$ under which a lattice basis is also a generating set. The proof is given in the Appendix.

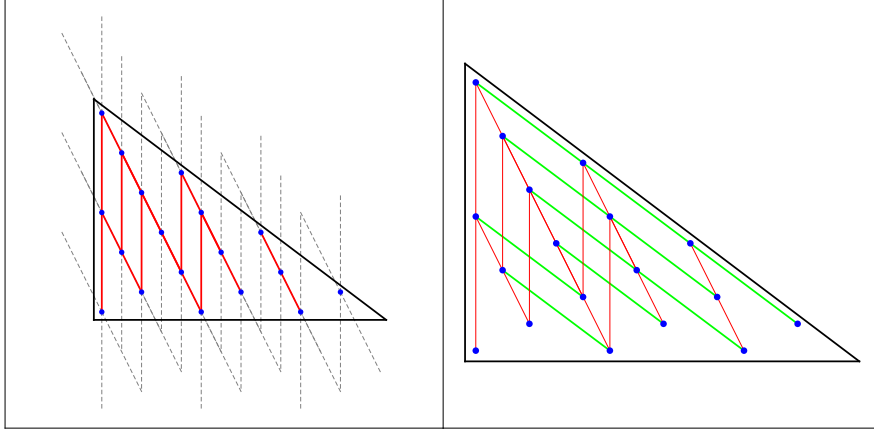


FIGURE 1. $\mathcal{A} = \{3, 4, 5\}$ and $\ker_{\mathbb{Z}}^*(\mathbf{A}) = \{(2, -4, 2), (0, -5, 4)\}$. Left: Projection on the x_1x_2 plane of $G := \mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \ker_{\mathbb{Z}}^*(\mathbf{A}))$ for $\mathbf{n} = (6, 1, 2)$. The red lines are the edges of G , while the dashed lines represent edges that are not in G because one endpoints does not belong to \mathbb{N}^3 . The graph $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \ker_{\mathbb{Z}}^*(\mathbf{A}))$ has three connected components. Right: Adding the vector $(4, -3, 0)$ to $\ker_{\mathbb{Z}}^*(\mathbf{A})$ turns G into a connected graph.

Proposition 2.15. *Let $\ker_{\mathbb{Z}}^*(\mathbf{A})$ be a basis of $\ker_{\mathbb{Z}}(\mathbf{A})$. Suppose that one of the following conditions holds:*

i) The basis $\ker_{\mathbb{Z}}^(\mathbf{A})$ contains an element $\bar{\mathbf{c}}$ such that each coordinate $\bar{c}^j, j = 1, \dots, A$ is positive.*

ii) Each vector of the basis $\ker_{\mathbb{Z}}^(\mathbf{A})$ is an element of \mathbb{N}^A .*

Then, the basis $\ker_{\mathbb{Z}}^(\mathbf{A})$ is a generating set.*

In the next theorem we show how one can use generating sets to characterize the set of probability measures $\mathcal{R}_{\mathbf{A}}(\mu)$. Even though we are interested here in the case $\mu = \mathfrak{p}_{\lambda}$ the statement is proven in a slightly more general context. In the same spirit as in (2.4), we consider the Radon-Nykodim decomposition of the image of μ by $\theta_{\mathbf{c}}, \mu \circ \theta_{\mathbf{c}}^{-1} = \mu_{\mathbf{c}}^{ac} + \mu_{\mathbf{c}}^{sing}$, and the density of $\mu_{\mathbf{c}}^{ac}$ with respect to μ :

$$G_{\mathbf{c}}^{\mu} := \frac{d\mu_{\mathbf{c}}^{ac}}{d\mu}. \quad (2.13)$$

Theorem 2.16. *Let $\mathbf{A} \in \mathbb{R}^{d \times A}$ be any matrix and the lattice $\ker_{\mathbb{Z}}(\mathbf{A})$ be defined as before by $\ker_{\mathbb{Z}}(\mathbf{A}) := \ker(\mathbf{A}) \cap \mathbb{Z}^A$. Assume that Γ is a generating set of $\ker_{\mathbb{Z}}(\mathbf{A})$ and let μ, ρ be two probability measures on \mathbb{N}^A . Suppose moreover that $\mu(\mathbf{n}) > 0$ for all $\mathbf{n} \in \mathbb{N}^A$. Then $\rho \in \mathcal{R}_{\mathbf{A}}(\mu)$ if and only if*

$$\forall \mathbf{c} \in \Gamma, \quad \rho(f \circ \theta_{\mathbf{c}}) = \rho(f G_{\mathbf{c}}^{\mu}) \quad \forall f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A), \quad (2.14)$$

where $G_{\mathbf{c}}^{\mu}$ is defined by (2.13).

Proof. (\Rightarrow) goes along the same lines of Proposition 2.10 since $\Gamma \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$.

(\Leftarrow) Let $\mathbf{n}, \mathbf{m} \in \mathbb{N}^A$ be such that $\mathbf{A}\mathbf{n} = \mathbf{A}\mathbf{m}$ and assume that $\rho(\mathbf{n}) > 0$. Then $\mathbf{m} \in \mathfrak{F}_{\mathbf{A}, \mathbf{n}}$ (see (2.12)). Since Γ is a generating set for $\ker_{\mathbb{Z}}(\mathbf{A})$ there exists a path from \mathbf{m} to \mathbf{n} included in $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \Gamma)$ i.e. there exists $\mathbf{c}_1, \dots, \mathbf{c}_K \in \Gamma$ such that, if we define recursively:

$$w_0 = \mathbf{m}, \quad w_k = \theta_{\mathbf{c}_k} w_{k-1},$$

then $w_k \in \mathbb{N}^A$ for all k and $w_K = \mathbf{n}$. We can choose $f^k = \mathbb{1}_{\{\mathbf{z}=w_k\}}$ and apply (2.14) for $\mathbf{c} = \mathbf{c}_k$:

$$\rho(w_{k-1}) = \frac{\mu(w_{k-1})}{\mu(w_k)} \rho(w_k)$$

which, since μ is a positive probability on \mathbb{N}^A , offers an inductive proof that $\rho(w_k) > 0$. Therefore one obtains

$$\frac{\rho(\mathbf{m})}{\rho(\mathbf{n})} = \prod_{k=1}^K \frac{\rho(w_{k-1})}{\rho(w_k)} = \prod_{k=1}^K \frac{\mu(w_{k-1})}{\mu(w_k)} = \frac{\mu(\mathbf{m})}{\mu(\mathbf{n})}$$

which is equivalent to $d\rho/d\mu(\mathbf{n}) = d\rho/d\mu(\mathbf{m})$, which completes the proof. \square

As consequence of Theorem 2.16 we obtain the following statement, which improves Proposition 2.10

Corollary 2.17. *Let $\rho \in \mathcal{P}(\mathbb{N}^A)$ and Γ be a generating set of $\ker_{\mathbb{Z}}(\mathbf{A})$ defined by (2.7). Then $\rho \in \mathcal{R}_{\mathbf{A}}(\mathfrak{p}_{\lambda})$ if and only if*

$$\forall \mathbf{c} \in \Gamma, \quad \rho(f \circ \theta_{\mathbf{c}}) = \frac{1}{\lambda_{\mathbf{c}}} \rho(f G_{\mathbf{c}}), \quad \forall f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A), \quad (2.15)$$

where $G_{\mathbf{c}}$ is defined in (2.5).

Example 2.18. We continue Examples (2.9) and (2.11), illustrating Corollary 2.17. For given λ^-, λ^+ the probability measure ρ belongs to $\mathcal{R}_{\mathbf{A}}(\mathfrak{p}_{\lambda})$ if and only if

$$\rho(f(n^- + 1, n^+ + 1)) = \frac{1}{\lambda^- \lambda^+} \rho(f(n^-, n^+) n^+ n^-) \quad \forall f \in \mathcal{B}^{\sharp}(\mathbb{Z}^2).$$

This improves Example 2.11, where we obtained a redundant characterization.

3. CHARACTERIZATION OF THE RECIPROCAL CLASS

3.1. Main result. We present here our main result: the reciprocal class $\mathfrak{R}(\nu)$ associated to a compound Poisson process with jump measure ν is characterized as the set of all probabilities for which a family of transformations induces the same density, expressed in terms of the reciprocal invariants. We have already introduced in the previous section the family of reciprocal time-invariants. Let us now introduce the family of reciprocal space-invariants.

Definition 3.1. Let ν be a jump measure in \mathcal{J} as defined in (1.2). For any $\mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A})$ we call **reciprocal space-invariant** $\Phi_{\nu}^{\mathbf{c}}$ the positive number

$$\Phi_{\nu}^{\mathbf{c}} := \prod_{j=1}^A \left(\int_0^1 \nu^j(t) dt \right)^{-c^j}.$$

Remark 3.2. In the time homogeneous case, $\nu_t^j \equiv \nu^j$, $\Phi_{\nu}^{\mathbf{c}} = 1 / \prod_{j=1}^A (\nu^j)^{c^j}$.

We can now use these invariants to characterize the reciprocal class.

Theorem 3.3. Let $\nu \in \mathcal{J}$ and $\mathbb{Q} \in \mathcal{P}(\Omega)$. Then \mathbb{Q} belongs to the reciprocal class $\mathfrak{R}(\nu)$ if and only if

i) For all $u \in \mathcal{U}$ and all $F \in \mathcal{B}(\Omega)$,

$$\mathbb{Q}(F \circ \pi_u) = \mathbb{Q}\left(F \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^{\nu}(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right). \quad (3.1)$$

ii) There exists a generating set $\Gamma \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$ such that for every $\mathbf{c} \in \Gamma$ and every $f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A)$, the following identity holds:

$$\rho(f \circ \theta_{\mathbf{c}}) = \Phi_{\nu}^{\mathbf{c}} \rho(f \circ G_{\mathbf{c}}), \quad (3.2)$$

where $\rho := \mathbb{Q} \circ \mathbf{N}_1^{-1} \in \mathcal{P}(\mathbb{N}^A)$ is the law of \mathbf{N}_1 under \mathbb{Q} .

Remark 3.4. Note that identities similar to (3.2) hold for any $t \in]0, 1]$, i.e. any $\mathbb{Q} \in \mathfrak{R}(\nu)$ satisfies (we assume a time homogeneous ν , for simplicity):

$$\mathbb{Q}(f \circ \theta_{\mathbf{c}}(\mathbf{N}_t)) = \Phi_{\nu}^{\mathbf{c}} (1-t)^{-|\mathbf{c}|} \mathbb{Q}((f \circ G_{\mathbf{c}})(\mathbf{N}_t)), \quad \forall f \in \mathcal{B}^{\sharp}(\mathbb{Z}^A), 0 < t \leq 1, \quad (3.3)$$

where $|\mathbf{c}| := \sum_{j=1}^A c^j$. However, the identities (3.3) do not contain enough information to characterize the reciprocal class as the time-invariants do not appear.

Proof. (\Rightarrow) Let $\mathbb{Q} \in \mathfrak{R}(\nu)$ and $\mathbb{P}_{\nu}^{\mathbb{Q}}$ be constructed as in Proposition 1.5. Since there is no ambiguity, we write \mathbb{P}_{ν} rather than $\mathbb{P}_{\nu}^{\mathbb{Q}}$. An application of the same proposition gives that $\mathbb{Q} \ll \mathbb{P}_{\nu}$, and $h := \frac{d\mathbb{Q}}{d\mathbb{P}_{\nu}}$ is $\sigma(X_0, X_1)$ -measurable. Consider now $u \in \mathcal{U}$. By definition of u , for any j , $N_1^j \circ \pi_u = N_1^j$, so that $(X_0, X_1) \circ \pi_u = (X_0, X_1)$, \mathbb{P}_{ν} -a.s..

We then consider $F \in \mathcal{B}(\Omega)$ and apply Proposition [2.5](#) under the measure \mathbb{P}_ν , which leads to

$$\begin{aligned} \mathbb{Q}(F \circ \pi_u) &= \mathbb{P}_\nu\left((F \circ \pi_u)h(X_0, X_1)\right) = \mathbb{P}_\nu\left((Fh(X_0, X_1)) \circ \pi_u\right) \\ &= \mathbb{P}_\nu\left(Fh(X_0, X_1) \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right) \\ &= \mathbb{Q}\left(F \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right). \end{aligned}$$

In a similar way, if $\mathbf{c} \in \Gamma$, since $\Gamma \subseteq \ker_{\mathbb{Z}}(\mathbf{A})$ we have that $\mathbf{A}(\theta_{\mathbf{c}}\mathbf{N}_1) = \mathbf{A}\mathbf{N}_1$. We observe that $\mathbb{P}_\nu(\mathbf{N}_1 \in \cdot | X_0 = x) = \mathfrak{p}_\lambda$, where

$$\lambda^j := \int_0^1 \nu^j(t) dt. \quad (3.4)$$

For $f \in \mathcal{B}^\sharp(\mathbb{Z}^A)$ and $\mathbf{c} \in \Gamma$ we use Proposition [2.10](#), observing that \mathbf{N}_1 has law \mathfrak{p}_λ and is independent of X_0 , to obtain

$$\begin{aligned} \rho(f \circ \theta_{\mathbf{c}}) &= \mathbb{Q}(f \circ \theta_{\mathbf{c}}(\mathbf{N}_1)) \\ &= \mathbb{P}_\nu(h(X_0, X_1) f \circ \theta_{\mathbf{c}} \circ \mathbf{N}_1) \\ &= \mathbb{P}_\nu(h(X_0, X_0 + \mathbf{A}(\theta_{\mathbf{c}}\mathbf{N}_1)) f \circ \theta_{\mathbf{c}} \circ \mathbf{N}_1) \\ &= \mathbb{P}_\nu\left(\mathbb{P}_\nu^{X_0}(h(X_0, X_0 + \mathbf{A}(\theta_{\mathbf{c}}\mathbf{N}_1)) f \circ \theta_{\mathbf{c}} \circ \mathbf{N}_1)\right) \\ &= \Phi_\nu^{\mathbf{c}} \mathbb{P}_\nu(h(X_0, X_1)(fG_{\mathbf{c}}) \circ \mathbf{N}_1) = \Phi_\nu^{\mathbf{c}} \rho(fG_{\mathbf{c}}) \end{aligned}$$

and *ii*) is now proven.

(\Leftarrow) We will show that \mathbb{Q} satisfies *ii*) of Proposition [1.5](#), which is equivalent to $\mathbb{Q} \in \mathfrak{R}(\nu)$. We divide the proof in three steps. In a first step, we refer to the Appendix for the proof of the absolute continuity of \mathbb{Q} w.r.t. to $\mathbb{P}_\nu^{\mathbb{Q}}$, since it is quite technical. In a second step we prove that the density is $\sigma(X_0, \mathbf{N}_1)$ -measurable and in a third one we prove that this density is indeed $\sigma(X_0, X_1)$ -measurable. For sake of clarity, since there is no ambiguity, we denote by \mathbb{P}_ν the probability $\mathbb{P}_\nu^{\mathbb{Q}}$.

Step 1: Absolute continuity.

See the Appendix.

Step 2: The density $H := \frac{d\mathbb{Q}}{d\mathbb{P}_\nu}$ is invariant under time change.

We show that, for any $u \in \mathcal{U}$, H is π_u -invariant, i.e. $H \circ \pi_u = H$ \mathbb{P}_ν -a.s.. Since $\mathbb{P}_\nu(\Omega) = 1$ we have that π_u is \mathbb{P}_ν -a.s. invertible. Applying the identity

(2.1) under \mathbb{P}_ν we obtain, for any $F \in \mathcal{B}(\Omega)$:

$$\begin{aligned} \mathbb{P}_\nu(F H \circ \pi_u) &= \mathbb{P}_\nu((F \circ \pi_u^{-1} H) \circ \pi_u) \\ &= \mathbb{P}_\nu\left(F \circ \pi_u^{-1} H \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right) \\ &= \mathbb{Q}\left(F \circ \pi_u^{-1} \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right)\right) \\ &= \mathbb{Q}(F) = \mathbb{P}_\nu(F H) \end{aligned}$$

which gives us the desired invariance, since F is arbitrary.

We claim that this implies that H is $\sigma(X_0, \mathbf{N}_1)$ -measurable, i.e. that there exists a function $h : \mathbb{R} \times \mathbb{N}^A \rightarrow \mathbb{R}^+$ such that

$$H = \frac{d\mathbb{Q}}{d\mathbb{P}_\nu} = \frac{d\mathbb{Q} \circ (X_0, \mathbf{N}_1)^{-1}}{d\mathbb{P}_\nu \circ (X_0, \mathbf{N}_1)^{-1}} = h(X_0, \mathbf{N}_1) \quad \mathbb{P}_\nu\text{-a.s.}$$

This is true since, given any two $\omega, \omega' \in \Omega$ with the same initial state and the same number of jumps of each type, one can construct $u \in \mathcal{U}$ such that $\omega' = \pi_u(\omega)$.

Step 3: The density H is invariant under shifts in Γ .

Let us recall that $\mathbb{P}_\nu(\mathbf{N}_1 \in \cdot | X_0 = x) = \mathfrak{p}_\lambda$, where λ is given by (3.4). Under our assumption we might apply Corollary 2.17 to $\mathfrak{p}_\lambda = \mathbb{P}_\nu(\mathbf{N}_1 \in \cdot | X_0 = x)$ and $\rho^x = \mathbb{Q}(\mathbf{N}_1 \in \cdot | X_0 = x)$. We obtain that the conditional density $\frac{d\rho^x}{d\mathfrak{p}_\lambda}$ is \mathbf{AN}_1 -measurable \mathbb{Q}_0 -a.s. and, by mixing over the initial condition, that $\frac{d\mathbb{Q} \circ (X_0, \mathbf{N}_1)^{-1}}{d\mathbb{P}_\nu \circ (X_0, \mathbf{N}_1)^{-1}} = \frac{d\mathbb{Q}}{d\mathbb{P}_\nu}$ is $\sigma(X_0, \mathbf{AN}_1) = \sigma(X_0, X_1)$ -measurable. \square

3.2. Comparing reciprocal classes through invariants. In what follows and in the next subsections, we consider jump measures $\nu \in \mathcal{J}$ which are time-homogeneous. In that case we identify ν with the vector $(\nu^1(0), \dots, \nu^A(0)) \in \mathbb{R}_+^A$.

We present in Proposition 3.5 a set of explicit necessary and sufficient conditions for two compound Poisson processes \mathbb{P}_ν^x and $\mathbb{P}_{\tilde{\nu}}^x$ to have the same bridges, or equivalently, to belong to the same reciprocal class. A more implicit condition has been presented by R. Murr in his PhD thesis, see [22], Proposition 7.50. Our result is in the spirit of [7], where two counting processes were shown to have the same bridges if and only if their reciprocal (time-)invariants coincide.

In the next proposition we denote by $\ker_{\mathbb{Z}}(\mathbf{A})^\perp$ the orthogonal complement of the affine hull of $\ker_{\mathbb{Z}}(\mathbf{A})$, and the logarithm of the vector $\nu \in \mathbb{R}_+^A$, denoted by $\log(\nu)$, has to be understood componentwise.

Proposition 3.5. *Let $x \in \mathbb{R}^d$, $\nu, \tilde{\nu} \in \mathbb{R}_+^A$ and $\ker_{\mathbb{Z}}^*(\mathbf{A})$ be a lattice basis of $\ker_{\mathbb{Z}}(\mathbf{A})$. The following assertions are equivalent:*

- i) $\mathbb{P}_{\tilde{\nu}}^x \in \mathfrak{R}(\nu)$.
- ii) For every $\mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A})^*$ the equality $\Phi_\nu^{\mathbf{c}} = \Phi_{\tilde{\nu}}^{\mathbf{c}}$ holds.
- iii) There exists $v \in \ker_{\mathbb{Z}}(\mathbf{A})^\perp$ such that $\log(\tilde{\nu}) = \log(\nu) + v$.

Proof. $i) \Rightarrow ii)$ By applying (3.2) and the trivial fact that $\mathbb{P}_\nu^x \in \mathfrak{R}(\tilde{\nu})$, we have

$$\Phi_\nu^c \mathbb{P}_\nu^x(f) = \mathbb{P}_\nu^x(f \circ \theta_c \circ \mathbf{N}_1) = \Phi_\nu^c \mathbb{P}_\nu^x(f), \quad \forall f \in \mathcal{B}^\sharp(\mathbb{Z}^A), \quad (3.5)$$

and $ii)$ follows.

$ii) \Rightarrow i)$ Observe that since $\ker_{\mathbb{Z}}(\mathbf{A})^*$ is a lattice basis, any $\mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A})$ can be written as an integer combination of the elements of $\ker_{\mathbb{Z}}^*(\mathbf{A})$, i.e. $\mathbf{c} = \sum_{\mathbf{c}^* \in \ker_{\mathbb{Z}}^*(\mathbf{A})} z_{\mathbf{c}^*} \mathbf{c}^*$, $z_{\mathbf{c}^*} \in \mathbb{Z}$. Therefore all the reciprocal space-invariants coincide since

$$\Phi_\nu^c = \prod_{\mathbf{c}^* \in \ker_{\mathbb{Z}}^*(\mathbf{A})} (\Phi_\nu^{\mathbf{c}^*})^{z_{\mathbf{c}^*}} = \prod_{\mathbf{c}^* \in \ker_{\mathbb{Z}}^*(\mathbf{A})} (\Phi_{\tilde{\nu}}^{\mathbf{c}^*})^{z_{\mathbf{c}^*}} = \Phi_{\tilde{\nu}}^c, \quad \forall \mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A}). \quad (3.6)$$

With a similar argument as above one proves that the identity (3.2) is satisfied under $\mathbb{P}_{\tilde{\nu}}^x$. The functional equation (3.1) is trivially satisfied by $\mathbb{P}_{\tilde{\nu}}^x$ because $\Xi^\nu \equiv \Xi^{\tilde{\nu}} = 1$. The conclusion follows by applying Theorem 3.3

$ii) \Leftrightarrow iii)$ We just observe that the equality $\Phi_\nu^c = \Phi_{\tilde{\nu}}^c$ is equivalent to

$$\sum_{j=1}^A \log(\nu^j) c^j = \sum_{j=1}^A \log(\tilde{\nu}^j) c^j.$$

Since a lattice basis $\ker_{\mathbb{Z}}^*(\mathbf{A})$ of $\ker_{\mathbb{Z}}(\mathbf{A})$ is a linear basis of the affine hull of $\ker_{\mathbb{Z}}(\mathbf{A})$ $ii)$ is equivalent to the fact that $\log(\nu)$ and $\log(\tilde{\nu})$ have the same projection onto $\ker_{\mathbb{Z}}(\mathbf{A})$, which is equivalent to $iii)$. \square

Example 3.6. Continuing on Example 2.18, two time-homogeneous compound Poisson processes with jumps in $\mathcal{A} = \{-1, 1\}$ and rate $\nu = (\nu^-, \nu^+)$ resp. $\tilde{\nu} = (\tilde{\nu}^-, \tilde{\nu}^+)$ have the same bridges if and only if

$$\nu^- \nu^+ = \tilde{\nu}^- \tilde{\nu}^+.$$

Example 3.7. Let $\mathcal{A} = \{-1, 3\}$ and define two time-homogeneous compound Poisson processes with jumps in \mathcal{A} and rate $\nu = (\nu^-, \nu^+)$ resp. $\tilde{\nu} = (\tilde{\nu}^-, \tilde{\nu}^+)$. They have the same bridges if and only if

$$(\nu^-)^3 \nu^+ = (\tilde{\nu}^-)^3 \tilde{\nu}^+.$$

Example 3.8. Let $\mathcal{A} = \{a^1, \dots, a^6\}$ be the vertices of an hexagon, see the Figure 2:

$$a^i = \left(\cos\left(\frac{2\pi}{6}(i-1)\right), \sin\left(\frac{2\pi}{6}(i-1)\right) \right) \in \mathbb{R}^2, \quad i = 1, \dots, 6. \quad (3.7)$$

Then a basis of $\ker_{\mathbb{Z}}(\mathbf{A})$ is:

$$\ker_{\mathbb{Z}}^*(\mathbf{A}) = \{\mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_5, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_6\}. \quad (3.8)$$

By Proposition 3.5, \mathbb{P}_ν^x with jump rates $\nu = (\nu^1, \dots, \nu^6)$ belongs to $\mathfrak{R}(\tilde{\nu})$ if and only if

$$\begin{cases} \nu^1 \nu^4 = \tilde{\nu}^1 \tilde{\nu}^4, \\ \nu^2 \nu^5 = \tilde{\nu}^2 \tilde{\nu}^5, \\ \nu^1 \nu^3 \nu^5 = \tilde{\nu}^1 \tilde{\nu}^3 \tilde{\nu}^5, \\ \nu^2 \nu^4 \nu^6 = \tilde{\nu}^2 \tilde{\nu}^4 \tilde{\nu}^6 \end{cases}$$

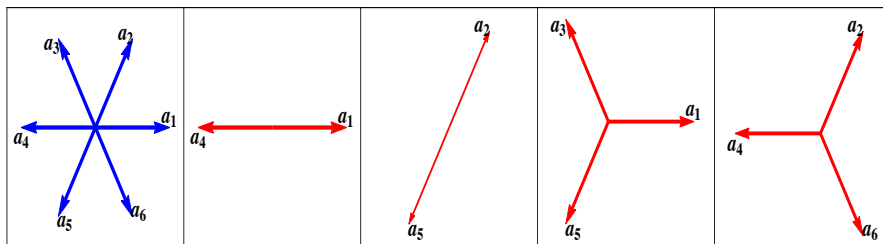


FIGURE 2. A representation of the vectors of \mathcal{A} and of the incidence vectors of $\ker_{\mathbb{Z}}^*(\mathbf{A})$

3.3. An interpretation of the reciprocal space-invariants. We aim at an interpretation of the space-invariants for a time-homogeneous jump measure $\nu \in \mathcal{J}$ under the geometrical assumption *ii*) of Proposition 2.15.

$\ker_{\mathbb{Z}}(\mathbf{A})$ admits a lattice basis $\ker_{\mathbb{Z}}^*(\mathbf{A})$ included in \mathbb{N}^A . (3.9)

A lattice basis satisfying (3.9) is a generating set for $\ker_{\mathbb{Z}}(\mathbf{A})$. Therefore it is sufficient to interpret the invariants $\Phi_{\nu}^{\mathbf{c}}$ for $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$.

Assumption (3.9) is not only natural in view of the interpretation we will give in Proposition 3.12 but it is satisfied in many interesting situations. One can prove that this is the case when $\mathcal{A} \subseteq \mathbb{Z}$ and \mathcal{A} contains at least one negative and one positive jump. Assumption (3.9) also holds in several situations when $d > 1$, e.g. in the setting of Example 3.8

In the context of diffusions, various physical interpretation of the reciprocal invariants have been given, mainly based on analogies with Stochastic Mechanics, see [9], [20], [31] and [32]. Regarding jump processes, the only interpretation known to the authors was given by R. Murr [22]. Inspired by [24] he related the reciprocal time-invariant associated to a counting process (the space-invariants trivialize) with a stochastic control problem, whose cost function is expressed in terms of the invariant.

We propose here a different interpretation of the invariants as infinitesimal characteristics, based on the short-time expansions for the probability that the process makes a cycle around its current state. We believe this interpretation to be quite general, and we are currently working on various extensions.

To be precise, let us define the concept of cycle we use here. In the rest of this section, a basis $\ker_{\mathbb{Z}}^*(\mathbf{A})$ satisfying (3.9) is fixed.

Definition 3.9. A cycle is a finite sequence $\gamma := (x_k)_{0 \leq k \leq l}$ such that

- i) $x_k - x_{k-1} \in \mathcal{A}$, $1 \leq k \leq l$,
- ii) $x_l = x_0 = 0$.

To each cycle γ we can associate an element $\mathbf{N}(\gamma) \in \ker_{\mathbb{Z}}(\mathbf{A}) \cap \mathbb{N}^A$ by counting how many times each jump occurred in the cycle, thus neglecting the order at which they occurs:

$$\mathbf{N}(\gamma)^j := \#\{k : x_k - x_{k-1} = a^j\}, \quad 1 \leq j \leq A,$$

where $\#E$ denotes the number of elements of a finite set E . Note that, for a given $\mathbf{c} \in \ker_{\mathbb{Z}}(\mathbf{A})$, we can construct a cycle γ such that $\mathbf{N}(\gamma) = \mathbf{c}$ if and

only if $\mathbf{c} \in \mathbb{N}^A$. Therefore, under assumption (3.9), $\mathbf{N}^{-1}(\mathbf{c})$ is non empty for any $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$.

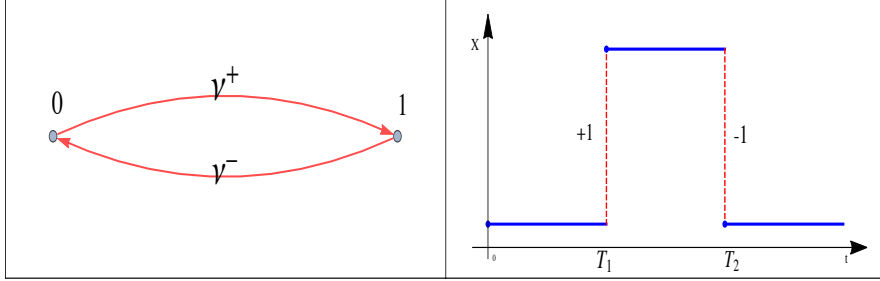


FIGURE 3. Here $\mathcal{A} = \{-1, 1\}$ and $\ker_{\mathbb{Z}}(\mathbf{A}) = (1, 1)\mathbb{Z}$. Left: A representation of the cycle $\gamma = \{0, 1, 0\}$ satisfying $\mathbf{N}(\gamma) = (1, 1)$. Right: A typical path in L_{ε}^{γ} . The probability of L_{ε}^{γ} is equivalent to $(\nu^+ \nu^-) \varepsilon^2$ over the whole reciprocal class, as $\varepsilon \rightarrow 0$.

Definition 3.10. We define the trace $\Upsilon_{\varepsilon}(\omega)$ of a path $\omega \in \Omega$ as the ordered sequence formed by the displacements from the initial position up to time ε :

$$\Upsilon_{\varepsilon}(\omega) = (0, X_{T_1} - X_0, \dots, X_{T_{|\mathbf{N}(\gamma)|\varepsilon}} - X_0).$$

The subset of paths whose trace coincides with a given cycle γ over a small time interval $[0, \varepsilon]$ is denoted by

$$L_{\varepsilon}^{\gamma} := \{\omega : \Upsilon_{\varepsilon}(\omega) = \gamma\}.$$

Finally, we introduce the usual time-shift operator on the canonical space:

$$\tau_t : \mathbb{D}([0, 1], \mathbb{R}^d) \longrightarrow \mathbb{D}([0, 1-t], \mathbb{R}^d), \quad \tau_t(\omega)_s = \omega_{t+s}, \quad \forall 0 \leq s \leq 1-t.$$

The following short-time expansion holds under the compound Poisson process.

Proposition 3.11. Let $\nu \in \mathcal{J}$ be a time-homogeneous jump measure, $x, x_0 \in \mathbb{R}^d$. Then for any time $t \geq 0$, $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$ and any cycle γ with $\mathbf{N}(\gamma) = \mathbf{c}$, we have:

$$\mathbb{P}_{\nu}^{x_0}(\tau_t(X) \in L_{\varepsilon}^{\gamma} | X_t = x) = \frac{1}{\Phi_{\nu}^{\mathbf{c}} |\mathbf{c}|!} \varepsilon^{|\mathbf{c}|} + o(\varepsilon^{|\mathbf{c}|}) \quad \text{as } \varepsilon \rightarrow 0$$

where $|\mathbf{c}| = \sum_{j=1}^A c^j$.

Proof. First observe that w.l.o.g. $t = 0$, the general result following from the Markov property of $\mathbb{P}_{\nu}^{x_0}$. For simplicity, we denote by $\bar{\nu}$ the total jump rate $\sum_{j=1}^A \nu^j$. Moreover, we denote by $j(k)$ the unique element of $\{1, \dots, A\}$ such that $X_{T_k} - X_{T_{k-1}} = a^{j(k)}$. With an elementary computation based on

the explicit distribution of $\mathbb{P}_\nu^{x_0}$:

$$\begin{aligned} \mathbb{P}_\nu^x(L_\varepsilon^\gamma) &= \mathbb{P}_\nu^x\left(\{|N|_\varepsilon = |\mathbf{c}|\} \cap \bigcap_{k=1}^{|\mathbf{c}|} \{X_{T_k} - X_{T_{k-1}} = a^{j(k)}\}\right) \\ &= \exp(-\varepsilon\bar{\nu}) \frac{(\varepsilon\bar{\nu})^{|\mathbf{c}|}}{|\mathbf{c}|!} \prod_{k=1}^{|\mathbf{c}|} \frac{\nu^{j(k)}}{\bar{\nu}} = \exp(-\varepsilon\bar{\nu}) \varepsilon^{|\mathbf{c}|} \prod_{j=1}^A (\nu^j)^{\sharp\{k:j(k)=j\}} \\ &= \exp(-\varepsilon\bar{\nu}) \frac{\varepsilon^{|\mathbf{c}|}}{|\mathbf{c}|!} \prod_{j=1}^A (\nu^j)^{c_j} = \exp(-\varepsilon\bar{\nu}) \frac{1}{\Phi_\nu^{\mathbf{c}} |\mathbf{c}|!} \varepsilon^{|\mathbf{c}|} \end{aligned}$$

from which the conclusion follows. \square

Even more interesting, the same time-asymptotics holds under any $\mathbb{Q} \in \mathfrak{R}(\nu)$ and in particular under any bridge $\mathbb{P}_\nu^{x,y}$.

Proposition 3.12. *Let $\nu \in \mathcal{J}$ be a time-homogeneous jump measure and $\mathbb{Q} \in \mathfrak{R}(\nu)$. Then for any time $t \geq 0$, $\mathbf{c} \in \ker_{\mathbb{Z}}^*(\mathbf{A})$ and any cycle γ with $\mathbf{N}(\gamma) = \mathbf{c}$, we have:*

$$\mathbb{Q}\text{-a.s.} \quad \mathbb{Q}\left(\tau_t(X) \in L_\varepsilon^\gamma \mid X_t\right) = \frac{1}{\Phi_\nu^{\mathbf{c}} |\mathbf{c}|!} \varepsilon^{|\mathbf{c}|} + o(\varepsilon^{|\mathbf{c}|}) \quad \text{as } \varepsilon \rightarrow 0$$

Proof. Assume that $\mathbb{Q} \in \mathfrak{R}(\nu)$. Observe that w.l.o.g we can assume that $\mathbb{Q}_0 = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^d$, the general result following by mixing over the initial condition. Then by Proposition 1.5, $d\mathbb{Q}/d\mathbb{P}_\nu^{x_0} = h(X_1)$. We first show the identity:

$$\mathbb{P}_\nu^{x_0}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} h(X_1) \mid X_t\right) = \mathbb{Q}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \mathbb{P}_\nu^{X_t}\left(h(X_{1-t})\right). \quad (3.10)$$

Indeed, let us take any test function of the form $\mathbf{1}_{\{X_t \in A\}}$. We have:

$$\begin{aligned} \mathbb{P}_\nu^{x_0}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} h(X_1) \mathbf{1}_{\{X_t \in A\}}\right) &= \mathbb{Q}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mathbf{1}_{\{X_t \in A\}}\right) \\ &= \mathbb{Q}\left(\mathbb{Q}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \mathbf{1}_{\{X_t \in A\}}\right) \\ &= \mathbb{P}_\nu^{x_0}\left(\mathbb{Q}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) h(X_1) \mathbf{1}_{\{X_t \in A\}}\right) \\ &= \mathbb{P}_\nu^{x_0}\left(\mathbb{Q}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \mathbb{P}_\nu^{x_0}(h(X_1) \mid X_t) \mathbf{1}_{\{X_t \in A\}}\right) \end{aligned}$$

from which (3.10) follows. Consider now the left hand side of (3.10). We have, by applying the Markov property and the fact that γ is a cycle:

$$\begin{aligned} \mathbb{P}_\nu^{x_0}\left(h(X_1) \mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) &= \mathbb{P}_\nu^{x_0}\left(\mathbb{P}_\nu^{x_0}(h(X_1) \mid \mathcal{F}_{[t, t+\varepsilon]}) \mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \\ &= \mathbb{P}_\nu^{x_0}\left(\mathbb{P}_\nu^{X_{t+\varepsilon}}(h(X_{1-(t+\varepsilon)})) \mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \\ &= \mathbb{P}_\nu^{x_0}\left(\mathbb{P}_\nu^{X_t}(h(X_{1-(t+\varepsilon)})) \mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \\ &= \mathbb{P}_\nu^{x_0}\left(\mathbf{1}_{\{\tau_t(X) \in L_\varepsilon^\gamma\}} \mid X_t\right) \mathbb{P}_\nu^{X_t}\left(h(X_{1-(t+\varepsilon)})\right). \end{aligned}$$

Applying (3.10) and Proposition 3.11 and the continuity of

$$(\omega, t, \cdot) \mapsto \mathbb{P}_\nu^{X_t}(h(X_{1-\cdot}))$$

we obtain:

$$\frac{1}{\Phi_\nu^c |\mathbf{c}|!} \mathbb{P}_\nu^{X_t}(h(X_{1-t})) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\mathbf{c}|} \mathbb{Q}(\mathbb{1}_{\{\tau_t \in L_\varepsilon^\gamma\}} | X_t) \mathbb{P}_\nu^{X_t}(h(X_{1-t})) \quad (3.11)$$

We observe that $\mathbb{P}_\nu^{X_t}(h(X_{1-t})) = d\mathbb{Q}_t/d(\mathbb{P}_\nu^{x_0})_t$ and therefore it is strictly positive \mathbb{Q} -a.s. . This allows us to divide on both sides by $\mathbb{P}_\nu^{X_t}(h(X_{1-t}))$ and the conclusion follows. \square

We have thus shown that each element of the reciprocal class has the same probability to *spin around* its current state in a very short time interval.

Remark 3.13. In the statement of Proposition [3.12](#) we could have replaced X_t with \mathcal{F}_t , i.e. the following asymptotics holds true:

$$\mathbb{Q}(\tau_t(X) \in L_\varepsilon^\gamma | \mathcal{F}_t) = \frac{1}{\Phi_\nu^c |\mathbf{c}|!} \varepsilon^{|\mathbf{c}|} + o(\varepsilon^{|\mathbf{c}|}) \quad \text{as } \varepsilon \rightarrow 0.$$

4. APPENDIX

Proof. (Step 1 in Theorem [3.3](#))

We first observe that it is sufficient to prove that

$$\mathbb{Q}(\cdot | \mathbf{N}_1 = \mathbf{n}) \ll \mathbb{P}_\nu(\cdot | \mathbf{N}_1 = \mathbf{n}) \quad \text{for all } \mathbf{n} \text{ such that } \mathbb{Q}(\mathbf{N}_1 = \mathbf{n}) > 0.$$

To this aim, we use an approximation argument.

Let us fix \mathbf{n} and construct a discrete (dyadic) approximation of the jump times. For $m \geq \max_{j=1, \dots, A} \log_2(n^j) + 1 := \bar{m}$, \mathcal{D}^m is composed by A ordered sequences of dyadic numbers, the j -th sequence having length n^j :

$$\mathcal{D}^m := \left\{ \mathbf{k} = (k_i^j)_{j \leq A, i \leq n^j} : k_i^j \in 2^{-m} \mathbb{N}, 0 < k_{i-1}^j < k_i^j \leq 1, \forall j \leq A, \forall i \leq n^j \right\}$$

For $\mathbf{k} \in \mathcal{D}^m$ we define the subset of trajectories whose jump times are localized around \mathbf{k} :

$$O_{\mathbf{k}}^m = \{\mathbf{N}_1 = \mathbf{n}\} \cap \bigcap_{\substack{j \leq A \\ i \leq n^j}} \left\{ 0 \leq k_i^j - T_i^j < 2^{-m} \right\} \quad (4.1)$$

Moreover, as a final preparatory step, we observe for every $m \geq \bar{m}$, $\mathbf{k}, \mathbf{k}' \in \mathcal{D}^m$ one can easily construct $u \in \mathcal{U}$ such that:

$$u(j, t) = t + k_i^{j'} - k_i^j, \quad \forall j \leq A, i \leq n^j \text{ and } t \text{ s.t. } 0 \leq k_i^j - t < 2^{-m} \quad (4.2)$$

We can observe that [\(4.2\)](#) ensures $\dot{u}(j, T_i^j) = 1$ over $O_{\mathbf{k}}^m$, and that $O_{\mathbf{k}'}^m = \pi_u^{-1}(O_{\mathbf{k}}^m)$. We choose $F = \mathbb{1}_{O_{\mathbf{k}'}^m} \mathbb{1}_{\{\mathbf{N}_1 = \mathbf{n}\}} / \mathbb{Q}(\mathbf{N}_1 = \mathbf{n})$ and u as in [\(4.2\)](#) and apply [\(3.1\)](#) to obtain :

$$\begin{aligned} \mathbb{Q}\left(O_{\mathbf{k}'}^m | \mathbf{N}_1 = \mathbf{n}\right) &= \mathbb{Q}\left(\{\omega : \pi_u(\omega) \in O_{\mathbf{k}}^m\} | \mathbf{N}_1 = \mathbf{n}\right) \\ &= \mathbb{Q}\left(\mathbb{1}_{O_{\mathbf{k}}^m} \exp\left(\sum_{j=1}^A \int_0^1 \log \Xi^\nu(j, t, u(j, t)) \dot{u}(j, t) dN_t^j\right) \Big| \mathbf{N}_1 = \mathbf{n}\right) \\ &\geq C \mathbb{Q}\left(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}\right), \end{aligned}$$

where

$$C := \left(\inf_{s,t \in [0,1], j \leq A} \Xi^\nu(j, s, t) \right)^{\sum_j n_j} > 0 \quad (4.3)$$

since $\nu \in \mathcal{J}$. With a simple covering argument we obtain, for all $m \geq \bar{m}$ and $\mathbf{k} \in \mathcal{D}^m$,

$$\begin{aligned} \#\mathcal{D}^m \min\{1, \frac{1}{C}\} \mathbb{Q}(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}) \\ \leq \mathbb{Q}(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}) + \sum_{\substack{\mathbf{k}' \in \mathcal{D}^m \\ \mathbf{k}' \neq \mathbf{k}}} \mathbb{Q}(O_{\mathbf{k}'}^m | \mathbf{N}_1 = \mathbf{n}) \leq 1. \end{aligned}$$

It can be shown with a direct computation that $\frac{1}{\#\mathcal{D}^m} \leq C' \mathbb{P}_\nu(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n})$ for some $C' > 0$ uniformly in $m, \mathbf{k} \in \mathcal{D}^m$ (the proof is given in Lemma [4.1](#)). Therefore there exists a constant $C'' > 0$ such that:

$$\mathbb{Q}(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}) \leq C'' \mathbb{P}_\nu(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}), \quad \forall m \geq \bar{m}, \mathbf{k} \in \mathcal{D}^m.$$

With a standard approximation argument, using the fact that $\mathbb{Q}(\Omega) = 1$, we can extend the last bound to any measurable set. This completes the proof of the absolute continuity. \square

Lemma 4.1. *Let \mathcal{D}^m and \mathbb{P}_ν as before. Then there exists a constant C' such that for m large enough,*

$$C' \mathbb{P}_\nu(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}) \geq \frac{1}{\#\mathcal{D}^m}$$

Proof. We want to prove that, for $\mathbf{n} \in \mathbb{N}^A$:

$$\frac{1}{\#\mathcal{D}^m} \leq C' \mathbb{P}_\nu(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}), \quad \forall m \geq \max_{j \leq A} \log(n^j) + 1, \mathbf{k} \in \mathcal{D}^m \quad (4.4)$$

We can first compute explicitly $\#\mathcal{D}^m$ with a simple combinatorial argument: each $\mathbf{k} \in \mathcal{D}^m$ is constructed by choosing n^j dyadic intervals, $j \leq A$, and ordering them. Therefore

$$\#\mathcal{D}^m = \prod_{j=1}^A \binom{2^m}{n^j}. \quad (4.5)$$

On the other hand, we observe that defining $\tilde{\nu}(dxdt) = \sum_{j=1}^A \delta_{a_j}(dx) \otimes dt$, $\mathbb{P}_{\tilde{\nu}}$ is equivalent to \mathbb{P}_ν , and therefore, we can prove [\(4.4\)](#) replacing \mathbb{P}_ν with $\mathbb{P}_{\tilde{\nu}}$. To do this, for each $\mathbf{k} \in \mathcal{D}^m$ we define the function:

$$\begin{aligned} \delta : \{1, \dots, 2^m\} \times \{1, \dots, A\} &\longrightarrow \{0, 1\} \\ \delta(i, j) &:= \begin{cases} 1, & \text{if } i \in \{2^m \mathbf{k}_1^j, \dots, 2^m \mathbf{k}_{n^j}^j\} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then, using the explicit distribution of $\mathbb{P}_{\tilde{\nu}}$,

$$\begin{aligned} & \mathbb{P}_{\tilde{\nu}}(O_{\mathbf{k}}^m | \mathbf{N}_1 = \mathbf{n}) \\ &= \mathbb{P}_{\tilde{\nu}}\left(\bigcap_{(i,j) \in \{1, \dots, 2^m\} \times \{1, \dots, A\}} \{N_{\frac{i}{2^m}}^j - N_{\frac{i}{2^m}}^j = \delta(i, j)\} | \mathbf{N}_1 = \mathbf{n}\right) \\ &= \exp(A) \exp(-2^{-m})^{2^m A} (2^{-m})^{(\sum_j n^j)} \prod_{j=1}^A n^{j!} = \prod_{j=1}^A 2^{-mn^j} n^{j!} \end{aligned}$$

It is now easy to see that there exists a constant $C_0 > 0$ such that:

$$\binom{2^m}{n^j} \geq C_0 \frac{2^{mn^j}}{n^{j!}}, \quad \forall j \leq A, \quad m \geq \max_{j=1, \dots, A} \log(n^j) + 1, \quad \mathbf{k} \in \mathcal{D}^m$$

from which the conclusion follows. \square

Proof. (of Proposition [2.15](#))

- i) Let $\mathbf{n} \in \mathbb{N}^A$, $\mathbf{m} \in \mathfrak{F}_{\mathbf{A}, \mathbf{n}}$. Since $\ker_{\mathbb{Z}}^*(\mathbf{A})$ is a lattice basis there exists $\mathbf{c}_1, \dots, \mathbf{c}_K \subseteq (\ker_{\mathbb{Z}}^*(\mathbf{A}) \cup -\ker_{\mathbb{Z}}^*(\mathbf{A}))^K$ such that, if we define recursively

$$w_0 = \mathbf{n}, \quad w_k = \theta_{\mathbf{c}_k} w_{k-1}$$

then we have that $w_K = \mathbf{m}$. Let us consider l large enough such that

$$l \min_{j=1, \dots, A} \bar{c}^j \geq \left| \min_{\substack{j=1, \dots, A \\ k=1, \dots, K}} w_k^j \right|. \quad (4.6)$$

We then consider the sequence w'_k , $k = 0, \dots, K+2l$ defined as follows:

$$w'_k = \begin{cases} \theta_{\bar{c}} w'_{k-1}, & \text{if } 1 \leq k \leq l \\ \theta_{\mathbf{c}_{k-l}} w'_{k-1}, & \text{if } l+1 \leq k \leq K+l \\ \theta_{-\bar{c}} w'_{k-1} & \text{if } K+l+1 \leq k \leq K+2l. \end{cases}$$

It is now easy to check, thanks to condition [\(4.6\)](#) that

$$w'_k \in \mathfrak{F}_{\mathbf{A}, \mathbf{n}} \quad \forall k \leq K+2l.$$

Since all the shifts involved in the definition of w'_k are associated to vectors in $\ker_{\mathbb{Z}}^*(\mathbf{A}) \cup -\ker_{\mathbb{Z}}^*(\mathbf{A})$ we also have that $w'_k \in \mathfrak{F}_{\mathbf{A}, \mathbf{n}}$ and (w'_{k-1}, w'_k) is an edge of $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \ker_{\mathbb{Z}}^*(\mathbf{A}))$, $k \leq K+2l$.

Moreover we can check that

$$w'_{K+2l} = \mathbf{n} + l\bar{c} + \sum_{k \leq K} c_k - l\bar{c} = \mathbf{m}$$

Therefore \mathbf{n} and \mathbf{m} are connected in $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \ker_{\mathbb{Z}}^*(\mathbf{A}))$ and the conclusion follows since the choice of \mathbf{m} is arbitrarily in $\mathfrak{F}_{\mathbf{A}, \mathbf{n}}$ and \mathbf{n} any point in \mathbb{N}^A .

- ii) Let $\mathbf{n} \in \mathbb{N}^A$, $\mathbf{m} \in \mathfrak{F}_{\mathbf{A}, \mathbf{n}}$. Since $\ker_{\mathbb{Z}}^*(\mathbf{A})$ is a lattice basis there exists $K < \infty$ and $\mathbf{c}_1, \dots, \mathbf{c}_K \subseteq (\ker_{\mathbb{Z}}^*(\mathbf{A}) \cup -\ker_{\mathbb{Z}}^*(\mathbf{A}))^K$ such that if we define recursively :

$$w_0 = \mathbf{n}, \quad w_k = \theta_{\mathbf{c}_k} w_{k-1} \quad (4.7)$$

then we have that $w_K = \mathbf{m}$

Observe that w.l.o.g there exists K^+ s.t. $\mathbf{c}_k \in \ker_{\mathbb{Z}}^*(\mathbf{A})$ for all $k \leq K^+$ and $\mathbf{c}_k \in -\ker_{\mathbb{Z}}^*(\mathbf{A})$, $k \in \{K^+ + 1, \dots, A\}$. Applying the hypothesis one can check directly that $\{w_k\}_{0 \leq k \leq K}$ is a path which connects \mathbf{n} to \mathbf{m} in $\mathcal{G}(\mathfrak{F}_{\mathbf{A}, \mathbf{n}}, \ker_{\mathbb{Z}}^*(\mathbf{A}))$. \square

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