SYNCHRONIZATION AND FUNCTIONAL CENTRAL LIMIT THEOREMS FOR INTERACTING REINFORCED RANDOM WALKS

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Abstract. We obtain Central Limit Theorems in Functional form for a class of time-inhomogeneous interacting random walks on the simplex of probability measures over a finite set. Due to a reinforcement mechanism, the increments of the walks are correlated, forcing their convergence to the same, possibly random, limit. Random walks of this form have been introduced in the context of urn models and in stochastic algorithms. We also propose an application to opinion dynamics in a random network evolving via preferential attachment. We study, in particular, random walks interacting through a mean-field rule and compare the rate they converge to their limit with the rate of *synchronization*, i.e. the rate at which their mutual distances converge to zero. Under certain conditions, synchronization is faster than convergence.

Keywords. interacting random systems; synchronization; functional central limit theorems; urn models; reinforced processes; dynamics on random graphs

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1. Introduction

Let S be a finite set and denote by $\mathcal{P}(S)$ the simplex of probabilities on S:

$$\mathcal{P}(S) := \left\{ \mu : S \to [0,1] : \sum_{x \in S} \mu(x) = 1 \right\}.$$

In this paper we consider stochastic evolutions on $\mathcal{P}(S)$ of the form

$$\mathcal{Z}_{n+1} = (1 - r_n)\mathcal{Z}_n + r_n K_n(I_{n+1}), \tag{1}$$

where $0 \leq r_n < 1$ are given numbers, $K_n : S \to \mathcal{P}(S)$ are given functions, and $(I_n)_{n \geq 1}$ is a sequence of S-valued random variables such that, for $\mathcal{F}_n := \sigma(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_n)$,

$$\mathbb{P}(I_{n+1} = x | \mathcal{F}_n) = \mathcal{Z}_n(x). \tag{2}$$

We think of this as a *generalized reinforcement mechanism*: note indeed that, in the particular case $K_n(x) = \delta_x$, with δ_x denoting the Dirac measure at $x \in S$, the larger $\mathcal{Z}_n(x)$, the higher the probability of increasing it at the next step.

Models of type (1) can be viewed as time-inhomogeneous random walks on $\mathcal{P}(S)$, and arise naturally in at least two distinct contexts.

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1.1. Urn Models. Let S be the set of the colors of the balls in a urn. Consider the following scheme. A ball is randomly drawn, uniformly among all balls. Suppose this is the $(n+1)$-st draw. If its color is y then we reinsert it in the urn and, for each color $x \in S$, we add $A_n(y,x)$ balls of color x, where

$$A_n : S \times S \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

is a given function, called *reinforcement matrix*, satisfying the *balance* condition: the sum $\sum_{x \in S} A_n(y,x) = \overline{A}_n$ does not depend on y. As a consequence, the total number

$$N(n) = N(0) + \sum_{k=0}^{n-1} \overline{A}_k$$

of balls in the urn after n steps does not depend on the sequence of colors drawn. Denote by $N(n,x)$ the number of balls in the urn of color x after n steps, and

$$\mathcal{Z}_n(x) := \frac{N(n,x)}{N(n)}.$$

Then $\mathcal{Z}_n \in \mathcal{P}(S)$, and it evolves as in (1) with

$$r_n := \frac{\overline{A}_n}{N(n+1)} \quad K_n(y)(\cdot) = \frac{A_n(y,\cdot)}{\overline{A}_n}.$$

This model includes the Pólya and the Friedman scheme as special cases, as well as many generalizations with time dependent reinforcement scheme (see e.g. [33] for an introduction to the subject). Note that in the most classical schemes (Pólya, Friedman) $\overline{A}_n$ is constant in n. More generally, in all cases in which $\overline{A}_n$ grows at most polynomially in n, we have that r_n is of order $\frac{1}{n}$ as $n \to +\infty$.

1.2. Opinion dynamics on preferential attachment graphs. Consider a sequence of random non-oriented graphs $G_n = (V_n, E_n)$, evolving through a preferential attachment rule (see e.g. [4, 28]). More specifically, for a given $\delta > -1$, the graph evolves according to the following rules:

- at time $n = 2$ the graph consists of the two vertices $\{1, 2\}$ connected by one edge;
- at time $n+1$ the new vertex $n+1$ is added and it is linked with an edge to vertex $i \in V_n = \{1, 2, \ldots, n\}$ with probability $\frac{d_i(n)+\delta}{2(n-1)+n\delta}$, where $d_i(n)$ is the *degree* of the vertex i at time n, i.e. the number of edges having i as endpoint.

Note that G_n is a connected graph.

We now define a stochastic dynamics, whose evolution depends on the realization of the graph sequence $(G_n)_{n \geq 2}$, which therefore plays a role analogous of that of a dynamic *random environment*. We adopt here the standard "quenched" point of view: we assume a realization of the sequence $(G_n)_{n \geq 2}$ is given, and we aim at proving results that hold for almost every realization of the graph sequence.

We consider the following random evolution, indexed by the same time variable $n \geq 2$. Let S be a finite set, representing possible choices made by "individuals" $i \in V_n$. To each vertex $i \in V_n$ is associated a probability $p_{n,i} \in \mathcal{P}(S)$. The quantity $p_{n,i}(x)$ $(x \in S)$ represents the inclination of individual i to adopt the choice x at time n or, in different terms, the *relative opinion* of individual i about x: the higher this value, the better the opinion of i on x compared with that on the other alternatives $y \neq x$ $(y \in S)$. The following two-steps dynamics occurs before the arrival of the $(n+1)^{\text{st}}$ vertex.

Step 1: (*consensus*) Through a *fast* consensus dynamics on the graph G_n, the $p_{n,i}$ are homogenized: every vertex ends up with the same inclination

$$\mathcal{Z}_{n,i} = \mathcal{Z}_n := \frac{1}{n}\sum_{i=1}^{n} p_{n,i}.$$

Step 2: (*hub's influence*) Let j_n be a vertex chosen arbitrarily among those of maximal degree. This vertex exhibits a choice $I_{n+1} = x$ with probability $\mathcal{Z}_n(x)$, i.e. according to his (and everyone else's) inclination. The exhibition of the choice has influence on the inclination of the vertex j_n's neighbors, so that, given $I_{n+1} = x$:

$$\begin{array}{ll} p_{n+1,j} = \lambda\delta_x + (1-\lambda)\mathcal{Z}_n & \text{if } j \text{ is a neighbor of } j_n \\ p_{n+1,j} = \mathcal{Z}_n & \text{otherwise,} \end{array}$$

where $\lambda \in (0,1)$ is a given constant and δ_x denotes the Dirac measure at $x \in S$.

After these two steps, the vertex $n+1$ is added; its inclination $p_{n+1,n+1}$ right after arrival could be taken arbitrarily; just for simplicity in next formulas, we set $p_{n+1,n+1} = \mathcal{Z}_n$.
This dynamics allows to obtain a recursive formula for $\mathcal{Z}_n$:

$$\begin{aligned} \mathcal{Z}_{n+1} &= \frac{1}{n+1}\left[\left(n+1-d_{j_n}(n)\right)\mathcal{Z}_n + d_{j_n}(n)(\lambda\delta_{I_{n+1}} + (1-\lambda)\mathcal{Z}_n)\right] \\ &= \frac{1}{n+1}\left[(n+1-\lambda\, d_{j_n}(n))\mathcal{Z}_n + \lambda d_{j_n}(n)\delta_{I_{n+1}}\right] \\ &= (1-r_n)\mathcal{Z}_n + r_n\delta_{I_{n+1}}, \end{aligned}$$

where

$$r_n := \frac{\lambda d_{j_n}(n)}{n+1}. \tag{3}$$

This has the form (1) with $K_n(y) = \delta_y$.

It should be stressed that many variants of this consensus-influence dynamics could be considered as well; for instance, influence could be exercised by vertices other than those with maximal degree, e.g. with a degree dependent probability. Our specific choice makes particularly easy to verify the conditions of some of the results below, see Remark 2.4 for details.

Finally, we remark that the dynamics (1) are special cases of *stochastic algorithms*, that are treated with stochastic approximation methods and are used in many different contexts and applications (see [6] for an overview, [30] for a general reference and applications and [19] for classical results in the spirit of this paper). In particular, (1) admits the following algorithmic interpretation. Let $K : S \to \mathcal{P}(S)$ be given. It can be viewed as a stochastic kernel that induces a map $T_K : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$T_K\mu := \sum_{y\in S}\mu(y)K(y).$$

Then, (1) is a version of the Robbins-Monro algorithm (see [30]) to obtain a fixed point of T_K, i.e. a stationary distribution of the S-valued Markov chain with transition kernel K.

This paper is concerned with systems of N interacting random walks in which, to N evolutions as in (1), we add an *interaction* term of *mean-field* type. We are particularly interested in the phenomenon of *synchronization*, that could be roughly defined as the tendency of different components to adopt a common long-time behavior. This phenomenon has been subject to recent investigation in systems of many interacting particles, where synchronization emerges in the large-scale limit [11, 22, 37]. More recently, interacting urn models have attracted attention as prototypical dynamics subject to reinforcement [2, 7, 34, 36, 38]. For some of these dynamics, synchronization is induced by reinforcement, so it does not require a large-scale limit [13, 16, 31, 32]. Another context in which synchronization emerges naturally is that of opinion dynamics in a population [20]. We have proposed here in Example 1.2 a version of opinion dynamics in an evolving population: the interacting version could be interpreted as related to different homogeneous groups within a given population, in the same spirit as in [10, 12].

In this work, to avoid complications, we focus on the case $S = \{0,1\}$, so that there is only one relevant variable, $Z_n := \mathcal{Z}_n(1)$; moreover we assume $K_n(y) = K(y)$ to be independent of time. Concerning the examples considered above, this time-independence property holds for the opinion models in preferential attachment graphs; in urn models a sufficient condition is that the reinforcement matrix A_n is of the form $A_n = c_n A$ for some $c_n > 0$ and a given matrix A independent of n. This includes generalizations of Pólya and Friedman models, where the reinforcement matrix is allowed to depend on n. The most general function $K : S \to \mathcal{P}(S)$ can be written in the form

$$K(y) = \rho\delta_y + (1-\rho)q, \tag{4}$$

for some $\rho \in [0,1]$ and a given $q \in \mathcal{P}(S)$. After identifying q with $q(1)$, the evolution of the i-th walk is therefore given by

$$Z_{n+1}(i) = (1-r_n)Z_n(i) + r_n(\rho I_{n+1}(i) + (1-\rho)q). \tag{5}$$

The interaction enters in the conditional law of $I_{n+1}(i)$: Setting

$$\mathcal{F}_n := \sigma(Z_k(i) : \, i = 1,2,\dots,N \, ; \; 0 \le k \le n) \,,$$

we assume that the random variables $\{I_{n+1}(i),\, i = 1,2,\dots,N\}$ are conditionally independent given $\mathcal{F}_n$ with

$$P(I_{n+1}(i) = 1|\mathcal{F}_n) = (1-\alpha)Z_n(i) + \alpha Z_n \tag{6}$$

where $\alpha \in [0,1]$ is the interaction parameter and

$$Z_n := \frac{1}{N}\sum_{i=1}^{N} Z_n(i). \tag{7}$$

Under suitable conditions on r_n, but actually no conditions if $\rho = 1$, the sequence (Z_n) converges almost surely to a limit Z. In the spirit of similar results for urn models, we study the corresponding rate of convergence and, in some cases, we obtain a fluctuation Theorem in functional form. We compare this rate of convergence with the *rate of synchronization*, which we define as the rate at which $Z_n(i) - Z_n$ converges to zero. As observed in [40] for interacting Friedman urns, synchronization may be *faster* than convergence. In our model we show that this occur when $\rho = 1$ and $r_n \sim \frac{c}{n^\gamma}$ with $c > 0$ and $\frac{1}{2} < \gamma < 1$. We stress the fact that this is not a large-scale phenomenon, in the sense that it holds *for any* value of N.

The paper is organized as follows. In section 2 we present our main results. Section 3 contains some basic identities often used in the proofs. Sections 4-7 are then devoted to proofs.

2. Main results

From now on we study interacting dynamics of the form defined by (5), (6) and (7). We assume that the initial configuration $[Z_0(i)]_{i=1}^N$ has a permutation invariant distribution with $\mathrm{E}[Z_0(i)] = \frac{1}{2}$ and $\mathrm{E}[Z_0(1-Z_0)] > 0$ where $Z_0 := N^{-1}\sum_{i=1}^N Z_0(i)$. The value $\frac{1}{2}$ could be replaced by $z_0 \in (0,1)$ at the only cost of longer formulas. It is worthwhile also to note that these assumptions will be used only in some of our proofs and they could be weakened.

2.1. **Convergence and synchronization.** The following theorem describes the convergence of the sequence (Z_n).

Theorem 2.1.

(i) *If $\rho = 1$, then Z_n converges almost surely to a random variable Z. Moreover:*
  a) *If $\alpha > 0$, then*

$$\mathbb{P}(Z \in \{0,1\}) = 1 \Leftrightarrow \sum_n r_n^2 = +\infty; \tag{8}$$

  b) *If $\alpha = 0$, each $Z_n(i)$ converges almost surely to a random variable $Z(i)$ such that*

$$\mathbb{P}(Z(i) \in \{0,1\}) = 1 \Leftrightarrow \sum_n r_n^2 = +\infty. \tag{9}$$

(ii) *If $\rho < 1$ and*

$$\sum_n r_n = +\infty \quad \textit{and} \quad \sum_n r_n^2 < +\infty\,, \tag{10}$$

then $Z_n \to q$ almost surely.

The following result particularly points out that in the case of a single walk, strengthening condition (8) one gets the phenomenon of *fixation*. Note that this phenomenon has been observed in various urn models, see e.g. [17, 31, 32].

Proposition 2.2. *Assume $N = 1$. If*

$$\rho = 1 \quad \textit{and} \quad \sum_{n\geq 1} \exp\left[-\sum_{k=0}^{n} r_k^2\right] < +\infty \tag{11}$$

or

$$\rho < 1, \quad q \in \{0,1\} \quad \textit{and} \quad \sum_n r_n = +\infty\,, \tag{12}$$

then there exists a random index M such that with probability one the indicator functions $\{I_n : n \geq M\}$ have all the same value.

Note that, if $r_n = O(n^{-\gamma})$, then (8) holds for $\gamma \leq \frac{1}{2}$, while (11) for $\gamma < \frac{1}{2}$. Moreover, (10) holds for $\frac{1}{2} < \gamma \leq 1$.

Next theorem establishes the fact that *synchronization* indeed takes place as soon as either interaction is present ($\alpha > 0$) or the limit of Z_n is deterministic ($\rho < 1$).

Theorem 2.3. *Suppose that* (10) *holds and* $\rho(1-\alpha)<1$. *Then, for all* $i \in \{1,2,\dots,N\}$, *we have*

$$Z_n(i) - Z_n \longrightarrow 0 \quad a.s.$$

In particular, if Z *is the almost sure limit of* $Z_n = \frac{1}{N}\sum_{i=1}^N Z_n(i)$ *(note that, for* $\rho<1$, *by Theorem* 2.1, $Z=q$*), we have* $Z_n(i)\to Z$ *almost surely.*

As we will see, the proof of this result does not require the assumptions on the initial configuration.

Remark 2.4. At this point it is worth discussing the assumptions in the previous results, in the context of the applications 1.1 and 1.2 proposed in the introduction.

Urn models. Consider an urn model with reinforcement matrix A_n of the form $A_n = c_n A$. So, after the $(n+1)$-st drawing, the number of balls added into the urn is $\overline{A}_n = c_n\overline{A}$ and therefore

$$r_n = \frac{c_n\overline{A}}{N(0)+\overline{A}\sum_{k=0}^n c_k}.$$

As observed in the introduction, r_n is of order $\frac{1}{n}$ whenever c_n grows polynomially, so (10) holds. For a different behavior one has to consider a faster growing reinforcement, e.g $c_n = \exp\left(n^\beta\right)$. In this case, for $0<\beta<1$, it is easily shown that r_n is of order $\frac{1}{n^{1-\beta}}$, thus (8), (11) or (10) may hold depending on the value of β.

Opinion dynamics. Note that in this model $\rho=1$. By Theorem 8.8 in [28], the maximal degree $d_{j_n}(n)$ at time n is such that the limit

$$\lim_{n\to+\infty}\frac{d_{j_n}(n)}{n^{\frac{1}{2+\delta}}} =: l$$

exists for almost every realization of the graph sequence $(G_n)_{n\geq 2}$. Thus, given the definition (3) of r_n, we have $r_n \sim \frac{c}{n^\gamma}$, with $\gamma = \frac{1+\delta}{2+\delta}$ and $c=\lambda l$. It follows that the conditions (10) hold for every $\delta>0$. For $-1<\delta\leq 0$, condition (8) holds and so the population's inclination "polarizes", i.e. it converges to the Dirac measure concentrated on one choice.

2.2. **Fluctuation theorems.** Assume $\rho(1-\alpha)<1$ and $r_n \sim \frac{c}{n^\gamma}$, where $\frac{1}{2}<\gamma\leq 1$, with the meaning

$$\lim_{n\to+\infty} n^\gamma r_n = c > 0.$$

Note that the above assumptions imply that Theorem 2.3 holds. In all the following theorems, the notation $\xrightarrow{d}$ denotes convergence in distribution with respect to the classical Skorohod's topology (see e.g. [8]).

Next result describes the fluctuations of Z_n around its limit Z in terms of a functional Central Limit Theorem in the case $\rho=1$.

Theorem 2.5. *Suppose* $\rho=1$ *(and so* $\alpha>0$*). Then the random limit* Z *of* Z_n *is such that* $\mathbb{P}(Z\in\{0,1\})<1$ *and* $\mathbb{P}(Z=z)=0$ *for all* $z\in(0,1)$. *Moreover the following holds:*

$$\left(t^{2\gamma-1}n^{\gamma-\frac{1}{2}}(Z_{\lfloor nt\rfloor}-Z)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0} \tag{13}$$

where

$$V_t = \frac{c^2}{N(2\gamma-1)}Z(1-Z)t^{2\gamma-1}$$

and $W=(W_t)_{t\geq 0}$ *is a Wiener process independent of* $V=(V_t)_{t\geq 0}$.

Next theorem characterizes the rate of synchronization, i.e. the rate of convergence to zero of $Z_n(i) - Z_n$, in terms of a functional Central Limit Theorem in the case $\rho = 1$.

Theorem 2.6. *Suppose $\rho = 1$ (and so $\alpha > 0$).*

(i) *If $\frac{1}{2} < \gamma < 1$ then*

$$\left(n^{\gamma/2} e^{c\alpha t} \left(Z_{\lfloor n+n^\gamma t \rfloor}(i) - Z_{\lfloor n+n^\gamma t\rfloor}\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where

$$V_t = \left(1 - \frac{1}{N}\right) \frac{cZ(1-Z)}{2\alpha} e^{2c\alpha t}.$$

and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion independent of $V = (V_t)_{t\geq 0}$.

(ii) *If $\gamma = 1$ and $2c\alpha > 1$, then*

$$\left(n^{1/2}(1+t)^{c\alpha} \left(Z_{\lfloor n+nt \rfloor}(i) - Z_{\lfloor n+nt\rfloor}\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where

$$V_t = \left(1 - \frac{1}{N}\right) \frac{c^2 Z(1-Z)}{2c\alpha - 1} (1+t)^{2c\alpha-1}.$$

and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion independent of $V = (V_t)_{t\geq 0}$

Remark 2.7. When $\rho = 1$, since $\gamma/2 > \gamma - 1/2$ for $\frac{1}{2} < \gamma < 1$, by Theorems 2.5 and 2.6, we have that in this regime synchronization is faster than convergence. More precisely, the proof of Theorem 2.6 implicitly contains the fact that, for $1/2 < \gamma < 1$,

$$\mathrm{E}\left[(Z_n(i) - Z_n)^2\right] \sim \left(1 - \frac{1}{N}\right) C_1\, n^{-\gamma} \tag{14}$$

with a suitable constant $C_1 = c(2\alpha)^{-1}\mathrm{E}[Z(1-Z)] > 0$ by Theorem 2.1. This fact, by permutation invariance, implies that we have for $i \neq j$

$$\mathrm{E}\left[(Z_n(i) - Z_n(j))^2\right] = 2\frac{N}{N-1}\mathrm{E}\left[(Z_n(i) - Z_n)^2\right] \sim 2C_1\, n^{-\gamma}.$$

On the other hand, the proof of Theorem 2.5 (see also the proof of Proposition 5.1) implicitly contains the fact that, for $1/2 < \gamma < 1$,

$$\mathrm{E}\left[(Z_n - Z)^2\right] \sim \frac{1}{N} C_2\, n^{-(2\gamma-1)}$$

with a suitable constant $C_2 = c^2(2\gamma - 1)^{-1}\mathrm{E}[Z(1-Z)] > 0$ by Theorem 2.1. This fact, together with (14), implies that, for each i, we have

$$\mathrm{E}\left[(Z_n(i) - Z)^2\right] \sim \mathrm{E}\left[(Z_n - Z)^2\right] \sim \frac{1}{N} C_2\, n^{-(2\gamma-1)}.$$

Therefore, for $\rho = 1$ and $1/2 < \gamma < 1$, the velocity of convergence to zero of $\|Z_n(i) - Z_n(j)\|_{L^2}$ is greater than the one of $\|Z_n(i) - Z\|_{L^2}$.

In the case of $\rho < 1$ and $q \notin \{0, 1\}$ the rate of convergence of Z_n to its limit q is, for $\frac{1}{2} < \gamma < 1$, different from the scaling in Theorem 2.5, and matches that in Theorem 2.6, that is $\gamma/2$. The following two results, in particular, show that convergence and synchronization occur at the same rate.

Theorem 2.8. *Suppose $\rho < 1$ and $q \notin \{0, 1\}$.*

(i) *If* $\frac{1}{2} < \gamma < 1$, *then*

$$\left(n^{\gamma/2} e^{c(1-\rho)t}\left(Z_{\lfloor n+n^{\gamma}t\rfloor}-q\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where $W=(W_t)_{t\geq 0}$ *is a standard Brownian motion and*

$$V_t = \frac{cq(1-q)\rho^2}{2N(1-\rho)} e^{2c(1-\rho)t}.$$

(ii) *If* $\gamma = 1$ *and* $2c(1-\rho) > 1$, *then*

$$\left(n^{1/2}(1+t)^{c(1-\rho)}\left(Z_{\lfloor n+nt\rfloor}-q\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where $W=(W_t)_{t\geq 0}$ *is a standard Brownian motion and*

$$V_t = \frac{c^2 q(1-q)}{N(2c(1-\rho)-1)}(1+t)^{2c(1-\rho)-1}.$$

Theorem 2.9. *Suppose* $\rho < 1$ *and* $q \notin \{0,1\}$.

(i) *If* $\frac{1}{2} < \gamma < 1$, *then*

$$\left(n^{\gamma/2} e^{c(1-\rho(1-\alpha))t}\left(Z_{\lfloor n+n^{\gamma}t\rfloor}(i)-Z_{\lfloor n+n^{\gamma}t\rfloor}\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where $W=(W_t)_{t\geq 0}$ *is a standard Brownian motion and*

$$V_t = \left(1-\frac{1}{N}\right)\frac{c\rho^2 q(1-q)}{2(1-\rho(1-\alpha))} e^{2c(1-\rho(1-\alpha))t}.$$

(ii) *If* $\gamma = 1$ *and* $2c(1-\rho(1-\alpha)) > 1$, *then*

$$\left(n^{1/2}(1+t)^{c(1-\rho(1-\alpha))}\left(Z_{\lfloor n+nt\rfloor}(i)-Z_{\lfloor n+nt\rfloor}\right)\right)_{t\geq 0} \xrightarrow{d} (W_{V_t})_{t\geq 0},$$

where $W=(W_t)_{t\geq 0}$ *is a standard Brownian motion and*

$$V_t = \left(1-\frac{1}{N}\right)\frac{c^2 q(1-q)}{2c(1-\rho(1-\alpha))-1}(1+t)^{2c(1-\rho(1-\alpha))-1}.$$

Remark 2.10. Functional Central Limit Theorems in the spirit of those above have been proved for various urn models (e.g. [3, 5, 24, 29, 42]). In particular [3, 24] and [42] contain results for Friedman urn models and Pólya urn models respectively, that in our model correspond to the case $N = 1$ and $\gamma = 1$. The results for the fluctuations of the Friedman urn, in particular, show that the condition $2c(1-\rho(1-\alpha)) > 1$ in Theorem 2.9 is essential: the Friedman urn that corresponds to $2c(1-\rho(1-\alpha)) < 1$ is known to have non-Gaussian fluctuations (see [21]), so no convergence to a Gaussian process is possible. The case of one Friedman urn with $2c(1-\rho) = 1$ is considered in [24]: a functional central limit theorem holds with a logarithmic correction in the scaling. We do not consider this case here.
For interacting Pólya urns ($\gamma = 1$) a non functional version of Theorem 2.6 is proved in [13], under the same condition $2c\alpha > 1$. For interacting Friedman urns ($\gamma = 1$) a non functional version of Theorem 2.9 is proved in [40].
We finally remark that the non-functional version of Theorem 2.8 could be alternatively derived by following stochastic approximation methods (see e.g. [6, 19, 30]).

Remark 2.11. Regarding the assumption on q of Theorems 2.8 and 2.9, we note that, if $q \in \{0,1\}$, when $\gamma < 1$, the behaviors of Z_n and $Z_n(i)$ are "eventually deterministic" (as can be easily derived by the same argument used for Proposition 2.2). Therefore the only case to be considered is when $\gamma = 1$, but we will not deal with it in this paper. For $N = 1$, a functional central limit Theorem could be obtained as in Proposition 2.2 of [24].

3. Basic properties

In this section we derive some simple recursions on the random walks $Z_n(i)$ and on $Z_n = \frac{1}{N}\sum_{i=1}^N Z_n(i)$, that will be used several times.

By averaging over i in (5), we have

$$Z_{n+1} - Z_n = r_n \left[\rho \left(N^{-1} \sum_{i=1}^N I_{n+1}(i) - Z_n \right) - (1-\rho)(Z_n - q) \right]$$

with

$$\mathrm{E}\left[N^{-1} \sum_{i=1}^N I_{n+1}(i) \Big| \mathcal{F}_n \right] = Z_n.$$

Therefore we can write

$$Z_{n+1} = Z_n - r_n(1-\rho)(Z_n - q) + \rho r_n \Delta M_{n+1}, \tag{15}$$

where

$$\Delta M_{n+1} = N^{-1}\sum_{i=1}^N I_{n+1}(i) - \mathrm{E}\left[N^{-1} \sum_{i=1}^N I_{n+1}(i) \Big| \mathcal{F}_n \right] = N^{-1}\sum_{i=1}^N I_{n+1}(i) - Z_n. \tag{16}$$

Then, subtracting (15) to (5), we obtain

$$Z_{n+1}(i) - Z_{n+1} = [1 - r_n(1-\rho(1-\alpha))](Z_n(i) - Z_n) + r_n\rho[\Delta M_{n+1}(i) - \Delta M_{n+1}], \tag{17}$$

where

$$\Delta M_{n+1}(i) = I_{n+1}(i) - \mathrm{E}[I_{n+1}(i)|\mathcal{F}_n]. \tag{18}$$

In particular, from the relations above, we have

$$\mathrm{E}[Z_{n+1} - q|\mathcal{F}_n] = [1-(1-\rho)r_n](Z_n - q) \tag{19}$$

$$\mathrm{E}[Z_{n+1}(i) - Z_{n+1}|\mathcal{F}_n] = [1-(1-\rho(1-\alpha))r_n]\,(Z_n(i) - Z_n) \tag{20}$$

and

$$\begin{aligned}\mathrm{Var}[Z_{n+1}(i) - Z_{n+1}|\mathcal{F}_n] &= r_n^2\rho^2 \mathrm{E}\left[(\Delta M_{n+1}(i) - \Delta M_{n+1})^2|\mathcal{F}_n\right] \\ &= r_n^2\rho^2\left(1-\frac{1}{N}\right)^2 \mathrm{Var}[I_{n+1}(i)|\mathcal{F}_n] + \frac{r_n^2\rho^2}{N^2}\sum_{j\neq i}\mathrm{Var}[I_{n+1}(j)|\mathcal{F}_n] \end{aligned} \tag{21}$$

$$\mathrm{Var}[Z_{n+1}|\mathcal{F}_n] = \frac{r_n^2\rho^2}{N^2}\sum_{j=1}^N \mathrm{Var}[I_{n+1}(j)|\mathcal{F}_n]. \tag{22}$$

4. Proofs: convergence and synchronization

4.1. **Proof of Theorem 2.1.**

Part (i)(a) Here we assume $\rho = 1$ and $\alpha > 0$. By (15), we immediately get that (Z_n) is a bounded martingale. Therefore, it converges a.s. (and in L^p) to a random variable Z, with values in $[0, 1]$.

Since by assumption $\mathrm{E}[Z_0(i)] = \frac{1}{2}$ for every i, we have $\mathrm{E}(Z) = \frac{1}{2}$. Moreover $P(Z \in \{0, 1\}) = 1$ if and only if

$$\mathrm{Var}(Z) = \lim_{n\to+\infty} \mathrm{Var}(Z_n) = \frac{1}{4}.$$

By using (22), we have

$$\begin{aligned}
\mathrm{Var}[Z_{n+1}] &= \mathrm{E}[\mathrm{Var}(Z_{n+1}|\mathcal{F}_n)] + \mathrm{Var}[\mathrm{E}(Z_{n+1}|\mathcal{F}_n)] \\
&= r_n^2 N^{-2} \sum_{i=1}^{N} \mathrm{E}\left[((1-\alpha)Z_n(i) + \alpha Z_n)\left(1 - (1-\alpha)Z_n(i) - \alpha Z_n\right)\right] + \mathrm{Var}[Z_n] \\
&= r_n^2 \mathrm{E}\left[Z_n/N - (\alpha^2 + 2(1-\alpha)\alpha)Z_n^2/N - (1-\alpha)^2 \sum_{i=1}^{N} Z_n^2(i)/N^2 \right] + \mathrm{Var}[Z_n] \\
&= r_n^2 \mathrm{E}\left[Z_n/N - (1-(1-\alpha)^2)Z_n^2/N - (1-\alpha)^2 Z_n^2 + (1-\alpha)^2 \sum_{i\neq j} Z_n(i)Z_n(j)/N^2 \right] + \mathrm{Var}[Z_n] \\
&= [1/N - (1-\alpha)^2(1-1/N)]r_n^2/4 + (1-\alpha)^2 r_n^2 \sum_{i\neq j} \mathrm{E}[Z_n(i)Z_n(j)]/N^2 \\
&+ \left\{1 - [1/N + (1-\alpha)^2(1-1/N)]r_n^2\right\} \mathrm{Var}[Z_n] \\
&= [1/N + (1-\alpha)^2(1-1/N)]r_n^2/4 - (1-\alpha)^2(1-1/N)r_n^2/2 + (1-\alpha)^2 r_n^2 \sum_{i\neq j} \mathrm{E}[Z_n(i)Z_n(j)]/N^2 \\
&+ \left\{1 - [1/N + (1-\alpha)^2(1-1/N)]r_n^2\right\} \mathrm{Var}[Z_n].
\end{aligned}$$

Now we observe that, using the permutation invariance, we have $\mathrm{E}[Z_n^2(i)] \leq \mathrm{E}[Z_n(i)] = \frac{1}{2}$ for all i and

$$\sum_{i\neq j} \mathrm{E}[Z_n(i)Z_n(j)]/N^2 = \mathrm{E}[Z_n^2] - \frac{1}{N}\mathrm{E}[Z_n^2(1)] \geq \mathrm{Var}[Z_n] + \frac{1}{4} - \frac{1}{2N}.$$

Using this fact in the formula above, we get

$$\mathrm{Var}[Z_{n+1}] \geq \left[1 - \frac{1}{N}(1-(1-\alpha)^2)r_n^2\right] \mathrm{Var}[Z_n] + \frac{1}{N}(1-(1-\alpha)^2)\frac{r_n^2}{4},$$

which, by letting $x_n := \frac{1}{4} - \mathrm{Var}[Z_n] \geq 0$, is equivalent to

$$x_{n+1} \leq \left(1 - C r_n^2\right) x_n \tag{23}$$

with $C := \frac{1-(1-\alpha)^2}{N} \in]0, 1]$ for $0 \leq (1-\alpha) < 1$. Therefore

$$x_n \leq x_0 \prod_{k=0}^{n-1} \left(1 - C r_k^2\right)$$

which implies $x_n \to 0$ if $\sum_n r_n^2 = +\infty$.

We are left to prove that if $\sum_n r_n^2 < +\infty$ then $x_n \not\to 0$. From the above equalities, we have

$$\begin{aligned}
\mathrm{Var}[Z_{n+1}] &= r_n^2 N^{-2} \sum_{i=1}^N \mathrm{E}\left[(1-\alpha)Z_n(i) + \alpha Z_n\right] - \mathrm{E}\left[\left((1-\alpha)Z_n(i) + \alpha Z_n\right)^2\right] + \mathrm{Var}[Z_n] \\
&= \frac{r_n^2}{N}\mathrm{E}[Z_n] - \frac{r_n^2}{N^2}\sum_{i=1}^N \mathrm{E}\left[\left(Z_n + (1-\alpha)(Z_n(i) - Z_n)\right)^2\right] + \mathrm{Var}[Z_n] \\
&= \frac{r_n^2}{N}\mathrm{E}[Z_n] - \frac{r_n^2}{N}\mathrm{E}[Z_n^2] - \frac{r_n^2}{N^2}(1-\alpha)^2 \sum_{i=1}^N \mathrm{E}\left[(Z_n(i) - Z_n)^2\right] + \mathrm{Var}[Z_n] \\
&\leq \frac{r_n^2}{N}\mathrm{E}[Z_n] - \frac{r_n^2}{N}\mathrm{E}[Z_n^2] + \mathrm{Var}[Z_n] = \frac{r_n^2}{4N} + \left(1 - \frac{r_n^2}{N}\right)\mathrm{Var}[Z_n],
\end{aligned}$$

where we have used the identities $\mathrm{E}(Z_n) = \frac{1}{2}$ and $\mathrm{E}[Z_n^2] = \frac{1}{4} + \mathrm{Var}[Z_n]$. Thus we have

$$x_{n+1} \geq \left(1 - \frac{r_n^2}{N}\right) x_n,$$

from which it follows

$$x_n \geq x_0 \prod_{k=0}^{n-1} \left(1 - \frac{r_k^2}{N}\right),$$

where $x_0 > 0$ by assumption. Since, $\sum_n r_n^2 < +\infty$ by assumption, we obtain $\lim_{n\to+\infty} x_n > 0$.

Part(i)(b). In the case when $\rho = 1$ and $\alpha = 0$, each $(Z_n(i))$ is a bounded martingale and so we have the almost sure (and in L^p) convergence of $Z_n(i)$ to a random variable $Z(i)$ with values in $[0,1]$. Moreover, with similar computation as above, we have

$$\mathrm{Var}[Z_{n+1}(i)] = \frac{r_n^2}{4} + (1 - r_n^2)\mathrm{Var}[Z_n(i)]$$

that is

$$x_{n+1}(i) = (1 - r_n^2) x_n(i) \tag{24}$$

where $x_n(i) := \frac{1}{4} - \mathrm{Var}[Z_n(i)]$. Therefore the conclusion immediately follows.

Part (ii). We are now assuming $\rho < 1$ and (10), that is $\sum_n r_n = +\infty$ and $\sum_n r_n^2 < +\infty$.

Using (15), we have

$$\mathrm{E}\left[(Z_{n+1} - q)^2 \,|\mathcal{F}_n\right] = (Z_n - q)^2[1 - 2(1-\rho)r_n] + r_n^2\left\{(1-\rho)^2(Z_n - q)^2 + \rho^2 \mathrm{E}\left[(\Delta M_{n+1})^2|\mathcal{F}_n\right]\right\} \tag{25}$$

from which we obtain

$$\mathrm{E}\left[(Z_{n+1} - q)^2 \,|\mathcal{F}_n\right] \leq (Z_n - q)^2 + r_n^2 \xi_n$$

where $\xi_n := \left\{(1-\rho)^2(Z_n - q)^2 + \rho^2 \mathrm{E}\left[(\Delta M_{n+1})^2|\mathcal{F}_n\right]\right\}$ is bounded. Therefore, since $\sum_n r_n^2 < +\infty$, we can conclude that $\left((Z_n - q)^2\right)_n$ is a positive almost supermartingale (see [39]) and so it converges almost surely (and in L^p). In order to show that this limit is 0, we are left to show that

$$\lim_{n\to+\infty} \mathrm{E}\left[(Z_n - q)^2\right] = 0. \tag{26}$$

Averaging in (25) and letting $x_n := \mathrm{E}\left[(Z_n - q)^2\right]$, we have

$$x_{n+1} = [1 - 2(1-\rho)r_n]x_n + K_n r_n^2,$$

with

$$0 \le K_n := \left\{(1-\rho)^2 \mathrm{E}[(Z_n - q)^2] + \rho^2 \mathrm{E}\left[(\Delta M_{n+1})^2\right]\right\} \le 1.$$

Recalling the assumptions (10) and $\rho < 1$, the conclusion follows from Lemma A.1. ∎

4.2. **Proof of Proposition 2.2.** First suppose that (11) holds. Setting $x_n := \frac{1}{4} - \mathrm{Var}[Z_n]$ and using (24), we have, for a suitable positive constant C,

$$x_n = x_0 \prod_{k=0}^{n-1} (1 - r_k^2) \sim C \exp\left[-\sum_{k=0}^{n} r_k^2\right]. \tag{27}$$

Now

$$\begin{aligned}
\sum_{n=1}^{+\infty} P(I_n = 0, I_{n+1} = 1) &= \sum_{n=1}^{+\infty} \mathrm{E}[P(I_n = 0, I_{n+1} = 1 | \mathcal{F}_n)] = \sum_{n=1}^{+\infty} \mathrm{E}[(1 - I_n) Z_n] \\
&= \sum_{n=1}^{+\infty} \mathrm{E}[(1 - I_n)(Z_{n-1}(1 - r_{n-1}) + r_{n-1} I_n)] \\
&= \sum_{n=1}^{+\infty} \mathrm{E}[(1 - I_n) Z_{n-1}(1 - r_{n-1})] \ \text{ then, conditioning on } \mathcal{F}_{n-1}, \\
&= \sum_{n=1}^{+\infty} (1 - r_{n-1}) \mathrm{E}[Z_{n-1}(1 - Z_{n-1})] \\
&\le \sum_{n=1}^{+\infty} \mathrm{E}[Z_{n-1}(1 - Z_{n-1})] = \sum_{n=1}^{+\infty} x_{n-1} < +\infty
\end{aligned}$$

by (27) and (11). Then by Borel-Cantelli lemma $P(\limsup_n \{I_n = 0, I_{n+1} = 1\}) = 0$ and the conclusion follows.

Now, assume (12) and $q = 0$ (the case $q = 1$ is specular), by a similar argument as above, we get

$$\sum_n P(I_{n+1} = 1) = \sum_n \mathrm{E}(Z_n) < +\infty$$

since $E[Z_n] \sim C \exp[-(1-\rho) \sum_{k=0}^{n} r_k]$. ∎

4.3. **Proof of Theorem 2.3.** We aim at showing that

$$Z_n(i) - Z_n \longrightarrow 0 \ \ \text{a.s} \tag{28}$$

Set $x_n := \mathrm{E}\left[(Z_n(i) - Z_n)^2\right]$. The proof is essentially the same as that of Theorem 2.1, (ii): we first show that $\left((Z_n(i) - Z_n)^2\right)$ is a positive almost supermartingale, which implies almost sure (and in L^p) convergence to a limit, and then we show that

$$\lim_{n \to +\infty} x_n = 0, \tag{29}$$

so that the limit of $Z_n(i) - Z_n$ is a.s. zero.

By (17), we obtain

$$\begin{aligned}
\mathrm{E}[(Z_{n+1}(i) - Z_n)^2 | \mathcal{F}_n] &= [1 - 2r_n(1 - \rho(1-\alpha))](Z_n(i) - Z_n)^2 \\
&\quad + r_n^2 \left\{(1 - \rho(1-\alpha))^2 (Z_n(i) - Z_n)^2 + \rho^2 \mathrm{E}\left[(\Delta M_{n+1}(i) - \Delta M_{n+1})^2 | \mathcal{F}_n\right]\right\}
\end{aligned} \tag{30}$$

and so

$$\mathrm{E}[(Z_{n+1}(i) - Z_n)^2 | \mathcal{F}_n] \leq (Z_n(i) - Z_n)^2 + r_n^2 \xi_n,$$

where $\xi_n := (1 - \rho(1-\alpha))^2 (Z_n(i) - Z_n)^2 + \rho^2 \mathrm{E}\left[(\Delta M_{n+1}(i) - \Delta M_{n+1})^2 | \mathcal{F}_n\right]$ is bounded. Since $\sum_n r_n^2 < +\infty$, this implies that $((Z_{n+1}(i) - Z_n)^2)$ is a positive almost supermartingale.

It remains to prove (29). Taking the expected value in (30), we obtain

$$x_{n+1} = [1 - 2(1 - \rho(1-\alpha)) r_n] x_n + K_n r_n^2$$

for a bounded sequence $(K_n)_n$ of positive numbers. Since we assume (10) and $\rho(1-\alpha) < 1$, the conclusion follows by applying Lemma A.1.

5. Fluctuation theorems I: proof of Theorem 2.5

The synchronization result in Theorem 2.3 gives, for $\rho(1-\alpha) < 1$,

$$\mathrm{Var}[I_{n+1}(i) | \mathcal{F}_n] = ((1-\alpha) Z_n(i) + \alpha Z_n)\left(1 - (1-\alpha) Z_n(i) - \alpha Z_n\right) \longrightarrow Z(1-Z) \text{ a.s.} \tag{31}$$

for all i. This, together with (21) and (22), implies the following useful relations:

$$\mathrm{Var}[Z_{n+1}(i) - Z_{n+1} | \mathcal{F}_n] \sim r_n^2 \rho^2 \left(1 - \frac{1}{N}\right) Z(1-Z) \tag{32}$$

$$\mathrm{Var}[Z_{n+1} | \mathcal{F}_n] \sim \frac{r_n^2 \rho^2}{N} Z(1-Z). \tag{33}$$

Before proving fluctuation theorems in the functional form, we prove a fluctuation theorem in non-functional form, but with a stronger form of convergence, the almost sure conditional convergence (see the appendix for details). This result, which has independent interest, is useful here to prove that the limit Z has no point mass in $(0,1)$. We recall that, in this section, we assume $\rho = 1$ and $(1-\alpha) < 1$.

Proposition 5.1. *Under the assumptions of Theorem 2.5,*

$$n^{\gamma - \frac{1}{2}} (Z_n - Z) \stackrel{stably}{\longrightarrow} \mathcal{N}\left(0, \frac{c^2}{N(2\gamma - 1)} Z(1-Z)\right).$$

Moreover, the above convergence is in the sense of the almost sure conditional convergence w.r.t. $\mathcal{F} = (\mathcal{F}_n)$.

Proof. We want to apply Theorem B.1. Let us consider, for each $n \geq 1$, the filtration $(\mathcal{F}_{n,h})_{h \in \mathbb{N}}$ and the process $(M_{n,h})_{h \in \mathbb{N}}$ defined by

$$\mathcal{F}_{n,0} = \mathcal{F}_{n,1} = \mathcal{F}_n, \qquad M_{n,0} = M_{n,1} = 0$$

and, for $h \geq 2$,

$$\mathcal{F}_{n,h} = \mathcal{F}_{n+h-1}, \qquad M_{n,h} = n^{\gamma - \frac{1}{2}} (Z_n - Z_{n+h-1}).$$

By (15) (with $\rho = 1$), it is easy to verify that, with respect to $(\mathcal{F}_{n,h})_{h \geq 0}$, the process $(M_{n,h})_{h \geq 0}$ is a martingale which converges in L^1 (for $h \to +\infty$) to the random variable $M_{n,\infty} := n^{\gamma - \frac{1}{2}} (Z_n - Z)$. In addition, the increment $X_{n,j} := M_{n,j} - M_{n,j-1}$ is equal to zero for $j = 1$ and, for $j \geq 2$, it coincides with a random variable of the form $n^{\gamma - \frac{1}{2}} (Z_k - Z_{k+1})$ with $k \geq n$. Therefore, we have

$$\begin{aligned}\sum_{j\geq 1} X_{n,j}^2 &= n^{2\gamma-1}\sum_{k\geq n}(Z_k - Z_{k+1})^2 = n^{2\gamma-1}\sum_{k\geq n} r_k^2\left(N^{-1}\sum_{i=1}^N I_{k+1}(i) - Z_k\right)^2\\ &\overset{a.s.}{\sim} c^2 n^{2\gamma-1}\sum_{k\geq n} k^{-2\gamma}\left(N^{-1}\sum_{i=1}^N I_{k+1}(i) - Z_k\right)^2\\ &\xrightarrow{a.s.} \frac{c^2}{N(2\gamma-1)} Z(1-Z)\end{aligned}$$

where the last convergence follows from Lemma 4.1 in [13] (where, using the notation of such lemma, $a_k = k^{1-(2\gamma-1)}$, $b_k = k^{2\gamma-1}$, $Y_k = \left(N^{-1}\sum_{i=1}^N I_{k+1}(i) - Z_k\right)^2$ and $\mathcal{G}_k = \mathcal{F}_{k+1}$) and the fact that, by (31), we have

$$\begin{aligned}\mathrm{E}\left[\left(N^{-1}\sum_{i=1}^N I_{n+1}(i) - Z_n\right)^2 \Big| \mathcal{F}_n\right] &= \mathrm{Var}\left[N^{-1}\sum_{i=1}^N I_{n+1}(i)\Big|\mathcal{F}_n\right]\\ &= N^{-2}\sum_{i=1}^N \mathrm{Var}\left[I_{n+1}(i)|\mathcal{F}_n\right]\\ &\xrightarrow{a.s.} N^{-1}Z(1-Z).\end{aligned} \tag{34}$$

Moreover, again by (15) (with $\rho = 1$), we have

$$\begin{aligned}X_n^* = \sup_{j\geq 1}|X_{n,j}| &= n^{\gamma-\frac{1}{2}}\sup_{k\geq n}|Z_k - Z_{k+1}| \leq \sup_{k\geq n} k^{\gamma-\frac{1}{2}}|Z_k - Z_{k+1}|\\ &\leq \sup_{k\geq n} k^{\gamma-\frac{1}{2}} r_k \xrightarrow{a.s.} 0.\end{aligned} \tag{35}$$

Hence, if in Theorem B.1 we take $k_n = 1$ for each n and $\mathcal{U}$ equal to the σ-field $\bigvee_n \mathcal{F}_n$, then the conditioning system $(\mathcal{F}_{n,k_n})_n$ coincides with the filtration $\mathcal{F}$ and the assumptions are satisfied. The proof is thus complete. ∎

We are now ready for the proof of Theorem 2.5. We split it into two steps.

Step 1: The fact that $\mathbb{P}(Z \in \{0,1\}) < 1$ follows from Theorem 2.1. The proof that $\mathbb{P}(Z = z) = 0$ for all $z \in (0,1)$ is now a consequence of the almost sure conditional convergence in Proposition 5.1, exactly as in Theorem 3.2 in [13]. Indeed, if we denote by K_n a version of the conditional distribution of $n^{\gamma-\frac{1}{2}}(Z_n - Z)$ given $\mathcal{F}_n$, then there exists an event A such that $P(A) = 1$ and, for each $\omega \in A$,

$$\lim_n Z_n(\omega) = Z(\omega) \quad \text{and} \quad K_n(\omega) \overset{weakly}{\longrightarrow} \mathcal{N}\left(0, \frac{c^2}{N(2\gamma-1)}(Z(\omega) - Z^2(\omega))\right).$$

Assume now, by contradiction, that there exists $z \in (0,1)$ with $P(Z = z) > 0$, and set $A' = A \cap \{Z = z\}$ and define B_n as the $\mathcal{F}_n$-measurable random set $\{n^{\gamma-\frac{1}{2}}(Z_n - z)\}$. Then $P(A') > 0$ and, since $E\left[I_{\{Z=z\}}\,|\,\mathcal{F}_n\right]$ converges almost surely to $I_{\{Z=z\}}$, there exists an event A'' such that $P(A'') > 0$, $A'' \subseteq A'$ and, for each $\omega \in A''$,

$$K_n(\omega)(B_n(\omega)) = \mathrm{E}\left[I_{\{n^{\gamma-\frac{1}{2}}(Z_n - z)\}}\left(n^{\gamma-\frac{1}{2}}(Z_n - Z)\right)\,|\,\mathcal{F}_n\right](\omega) = \mathrm{E}\left[I_{\{Z=z\}}\,|\,\mathcal{F}_n\right](\omega) \longrightarrow I_{\{Z=z\}}(\omega) = 1.$$

On the other hand, we observe that $Z(\omega)-Z^2(\omega)\neq 0$ when $\omega\in A'$. Hence, if D is the discrepancy metric defined by

$$D[\mu,\nu]=\sup_{\{B\in\{\text{closed balls of }\mathbb{R}\}\}}|\mu(B)-\nu(B)|,$$

which metrizes the weak convergence of a sequence of probability distributions on $\mathbb{R}$ in the case when the limit distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ (see [23]), then, for each $\omega\in A'$, we have

$$\begin{aligned}K_n(\omega)(B_n(\omega)) &= \left|K_n(\omega)(B_n(\omega))-\mathcal{N}\left(0,\frac{c^2}{N(2\gamma-1)}(Z(\omega)-Z^2(\omega))\right)(B_n(\omega))\right|\\ &\leq D\left[K_n(\omega),\mathcal{N}\left(0,\frac{c^2}{N(2\gamma-1)}(Z(\omega)-Z^2(\omega))\right)\right]\longrightarrow 0.\end{aligned}$$

This contradicts the previous fact and the proof of the first step is concluded.

Step 2: We now prove the functional fluctuation result in (13). First of all, we want to verify the three conditions $(a2)$, $(b2)$, $(c2)$ in Theorem B.5 for the stochastic processes

$$S_t^{(n)}=\frac{1}{n^{(2\gamma-1)/2}}\sum_{k=1}^{\lfloor nt^{1/(2\gamma-1)}\rfloor}k^{2\gamma-1}(Z_k-Z_{k-1})$$

in order to obtain a convergence result for $S^{(n)}$ on the space (T,m) defined in the appendix. Finally, by applying a suitable continuous transformation, we will arrive to (13).

Proof of condition (a2): We want to use Theorem B.2. Let us set $\mathcal{F}_{n,k}=\mathcal{F}_k$,

$$X_{n,k}=\frac{k^{2\gamma-1}(Z_k-Z_{k-1})}{n^{(2\gamma-1)/2}}\quad\text{and}\quad k_n(t)=\lfloor n\,t^{1/(2\gamma-1)}\rfloor$$

so that $S_t^{(n)}=\sum_{k=1}^{k_n(t)}X_{n,k}$. We observe that

$$\begin{aligned}\sum_{k=1}^{k_n(t)}\mathrm{E}[X_{n,k}^2\,|\,\mathcal{F}_{n,k-1}] &= \frac{1}{n^{2\gamma-1}}\sum_{k=1}^{k_n(t)}k^{4\gamma-2}\mathrm{E}[(Z_k-Z_{k-1})^2|\mathcal{F}_{k-1}]\\ &\overset{a.s.}{\sim}\left(\frac{k_n(t)}{n}\right)^{2\gamma-1}\frac{c^2}{(k_n(t))^{2\gamma-1}}\sum_{k=1}^{k_n(t)}\frac{1}{k^{1-(2\gamma-1)}}\mathrm{E}\left[\left(N^{-1}\sum_{i=1}^{N}I_k(i)-Z_{k-1}\right)^2\Big|\mathcal{F}_{k-1}\right]\\ &\xrightarrow{a.s.}\widetilde{V}_t=\frac{c^2}{N(2\gamma-1)}Z(1-Z)t\end{aligned}\tag{36}$$

(where, in the last step, we have used (34) and $\lim_n k_n(t)/n=t^{1/(2\gamma-1)}$) and so condition (a1) of Theorem B.2 is verified. Furthermore, for any $u>2$, we have

$$\begin{aligned}\sum_{k=1}^{k_n(1)}\mathrm{E}[\,|X_{n,k}|^u\,] &= \frac{1}{n^{u(2\gamma-1)/2}}\sum_{k=1}^{n}k^{u(2\gamma-1)}\mathrm{E}[\,|Z_k-Z_{k-1}|^u\,]=\frac{1}{n^{u(2\gamma-1)/2}}\sum_{k=1}^{n}k^{u(2\gamma-1)}O(k^{-u\gamma})\\ &= \frac{1}{n^{u(2\gamma-1)/2}}\sum_{k=1}^{n}O\left(1/k^{1-[1-u(1-\gamma)]}\right)=\begin{cases}O(n^{-(u/2-1)}) & \text{if } 1-u(1-\gamma)>0\\ O(n^{-u(2\gamma-1)/2}) & \text{if } 1-u(1-\gamma)<0\\ O\left(\frac{\ln(n)}{n^{(2\gamma-1)/2(1-\gamma)}}\right) & \text{if } 1-u(1-\gamma)=0.\end{cases}\end{aligned}$$

Hence also condition (b1) of Theorem B.2 holds true (by Remark B.3 with $u > 2$) and we conclude that

$$S_t^{(n)} = \sum_{k=1}^{k_n(t)} X_{n,k} = \frac{1}{n^{(2\gamma-1)/2}} \sum_{k=1}^{\lfloor nt^{1/(2\gamma-1)} \rfloor} k^{2\gamma-1}(Z_k - Z_{k-1}) \xrightarrow{d} \widetilde{W} = \left(W_{\widetilde{V}_t}\right)_{t\geq 0}$$

(w.r.t. Skorohod's topology).

Proof of condition (b2): For each $\epsilon > 0$, we observe that we have

$$\begin{aligned} \mathrm{E}[n^{2(1/2+\epsilon)(1-2\gamma)} n^{4\gamma-2}(Z_n - Z_{n-1})^2] \leq n^{2(1/2+\epsilon)(1-2\gamma)+4\gamma-2} r_{n-1}^2 &\sim c^2 n^{2(1/2+\epsilon)(1-2\gamma)+4\gamma-2-2\gamma} \\ &= \frac{c^2}{n^{1+2\epsilon(2\gamma-1)}}. \end{aligned}$$

Therefore the martingale

$$\left(\sum_{k=1}^{n} k^{(1/2+\epsilon)(1-2\gamma)} k^{(2\gamma-1)}(Z_k - Z_{k-1})\right)$$

is bounded in L^2 and so $\sum_{k=1}^{+\infty} k^{(1/2+\epsilon)(1-2\gamma)} k^{(2\gamma-1)}(Z_k - Z_{k-1})$ is a.s. convergent. By Kronecker's lemma, we get $\sum_{k=1}^{n} k^{(2\gamma-1)}(Z_k - Z_{k-1}) = o(n^{(1/2+\epsilon)(2\gamma-1)})$ a.s. This fact implies that, for each fixed n, the process $S^{(n)}$ is such that $S_t^{(n)} = o(t^{(1/2+\epsilon)})$ a.s. as $t \to +\infty$.

Proof of condition (c2): Fix $\theta > 1/2$, $\epsilon > 0$ and $\eta > 0$. We want to verify that there exists t_0 such that

$$P\left\{\sup_{t\geq t_0} \frac{|S_t^{(n)}|}{t^\theta} > \epsilon\right\} \leq \eta. \tag{37}$$

To this purpose, we observe that

$$S_t^{(n)} = \frac{1}{n^{(2\gamma-1)/2}} L_{\lfloor nt^{1/(2\gamma-1)} \rfloor} \quad \text{where } L_k = \sum_{j=1}^{k} j^{2\gamma-1}(Z_j - Z_{j-1}) = \sum_{j=1}^{k} \xi_j.$$

Denoting by C a suitable positive constant (which may vary at each step), we have

$$\begin{aligned} P\left\{\sup_{t\geq (n_0/n)^{2\gamma-1}} \frac{|S_t^{(n)}|}{t^\theta} > \epsilon\right\} &\leq P\left\{|L_k| > \frac{\epsilon k^{(2\gamma-1)\theta}}{2n^{(2\gamma-1)\theta-(2\gamma-1)/2}} \text{ for some } k \geq n_0\right\} \\ &\leq \sum_{i=1}^{+\infty} P\left\{\max_{2^{i-1}n_0 \leq k \leq 2^i n_0} |L_k| > \frac{\epsilon\left(2^{i-1}n_0\right)^{(2\gamma-1)\theta}}{2n^{(2\gamma-1)\theta-(2\gamma-1)/2}}\right\} \\ &\leq \frac{16 n^{2(2\gamma-1)\theta-(2\gamma-1)}}{\epsilon^2 n_0^{2(2\gamma-1)\theta}} \sum_{i=1}^{+\infty} 2^{-2i} \sum_{j=1}^{2^i n_0} \mathrm{E}[\xi_j^2] \\ &\leq C \frac{n^{2(2\gamma-1)\theta-(2\gamma-1)}}{\epsilon^2 n_0^{2(2\gamma-1)\theta}} \sum_{i=1}^{+\infty} 2^{-2i}(2^i n_0)^{2\gamma-1} \\ &= \frac{C}{\epsilon^2}\left(\frac{n}{n_0}\right)^{(2\gamma-1)(2\theta-1)} \sum_{i=1}^{+\infty}\left(\frac{1}{2^{3-2\gamma}}\right)^i = \frac{C}{\epsilon^2}\left(\frac{n}{n_0}\right)^{(2\gamma-1)(2\theta-1)} \end{aligned}$$

where the third inequality is the Hájek-Rényi inequality for martingales (see the appendix) and we used the fact that $E[\xi_j^2] \sim C/j^{1-(2\gamma-1)}$ with $C > 0$. Therefore, in order to obtain (37), it is enough to set $t_0^{2\theta-1} = \frac{C}{\epsilon^2\eta}$.

Conclusion: By Theorem B.5, we have

$$S^{(n)} \xrightarrow{d} \widetilde{W} = \left(W_{\widetilde{V}_t}\right)_{t\geq 0} \qquad \text{on } (T, m).$$

Now, let $g : T \to T_1^*$ be the Barbour's transform (defined in the appendix) and observe that, since

$$(\Delta S^{(n)})_s = S_s^{(n)} - S_{s-}^{(n)} = \begin{cases} \frac{k^{2\gamma-1}(Z_k - Z_{k-1})}{n^{(2\gamma-1)/2}} & \text{if } k = ns^{1/(2\gamma-1)} \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} g(S^{(n)})(t) &= \sum_{s\geq 1/t} \frac{(\Delta S^{(n)})_s}{s} = \sum_{k > \lfloor nt^{-1/(2\gamma-1)} \rfloor} \left(\frac{n}{k}\right)^{2\gamma-1} \frac{k^{2\gamma-1}(Z_k - Z_{k-1})}{n^{(2\gamma-1)/2}} \\ &= n^{(2\gamma-1)/2} \sum_{k > \lfloor nt^{-1/(2\gamma-1)} \rfloor} (Z_k - Z_{k-1}) = n^{(2\gamma-1)/2} \left(Z - Z_{\lfloor nt^{-1/(2\gamma-1)} \rfloor}\right). \end{aligned}$$

Therefore, by the properties of g (see the appendix), we get

$$g(S^{(n)}) = \left(n^{(2\gamma-1)/2} \left(Z - Z_{\lfloor nt^{-1/(2\gamma-1)} \rfloor}\right)\right)_{t\geq 0} \xrightarrow{d} g(\widetilde{W}) \stackrel{d}{=} \widetilde{W} \qquad \text{on } (T_1^*, m_1^*).$$

Immediately, by symmetry of W, we get the convergence result of $\left(n^{(2\gamma-1)/2}(Z_{\lfloor nt^{-1/(2\gamma-1)} \rfloor} - Z)\right)_{t\geq 0}$ to the stochastic process $\widetilde{W}$. Then, we can apply the continuous map $h : T_1^* \to D$ defined as

$$h(f)(0) = 0 \quad \text{and} \quad h(f)(t) = tf(t^{-1})$$

(see [27]) and obtain

$$\left(tn^{\gamma-\frac{1}{2}}(Z_{\lfloor nt^{1/(2\gamma-1)} \rfloor} - Z)\right)_{t\geq 0} \xrightarrow{d} \left(h(\widetilde{W})\right)_{t\geq 0} \stackrel{d}{=} \widetilde{W} = \left(W_{\widetilde{V}_t}\right)_{t\geq 0} \qquad \text{(w.r.t. Skorohod's topology).} \tag{38}$$

Finally, we can set $t = s^{2\gamma-1}$ and obtain

$$\left(s^{2\gamma-1}n^{\gamma-\frac{1}{2}}(Z_{\lfloor ns \rfloor} - Z)\right)_{s\geq 0} \xrightarrow{d} (W_{V_s})_{s\geq 0} \qquad \text{(w.r.t. Skorohod's topology),}$$

which coincides with (13).

6. Fluctuation theorems II: proof of Theorem 2.6

In all the sequel, we denote by $C > 0$ a suitable constant (which may vary at each step).

By (17), recalling that we are assuming $\rho = 1$, we have

$$E[Z_{n+1}(i) - Z_{n+1}|\mathcal{F}_n] = (1 - \alpha r_n)\left(Z_n(i) - Z_n\right).$$

Thus, setting

$$l_n = \prod_{k=0}^{n-1} \frac{1}{1 - \alpha r_k},$$

so that $l_n = (1-\alpha r_n)l_{n+1}$, we have that $L_n = l_n\left(Z_n(i) - Z_n\right)$ forms a martingale with

$$\xi_n = \Delta L_n = L_n - L_{n-1} = l_n\left(Z_n(i) - Z_n - \mathrm{E}[Z_n(i) - Z_n|\mathcal{F}_{n-1}]\right).$$

Observe that, if we fix $\epsilon \in (0,1)$, for all $n \geq \bar{n}$ and $\bar{n}$ sufficiently large

$$l_{\bar{n}} \exp\left[-\sum_{k=\bar{n}}^{n} \ln\left(1 - \frac{\alpha c_1}{k^\gamma}\right)\right] \leq l_n \leq l_{\bar{n}} \exp\left[-\sum_{k=\bar{n}}^{n} \ln\left(1 - \frac{\alpha c_2}{k^\gamma}\right)\right]$$

where $c_1 = c(1-\epsilon)$ and $c_2 = c(1+\epsilon)$.

In particular, recalling that the function $\varphi(x) = -x - \ln(1-x)$ is such that $0 \leq \varphi(x) \leq Cx^2$ for $0 \leq x \leq \bar{x} < 1$, and assuming $\bar{n}$ large enough such that $\alpha c_2/k^\gamma \leq \bar{x}$ for $k \geq \bar{n}$, we have for $n \geq \bar{n}$

$$l_{\bar{n}} \exp\left[\sum_{k=\bar{n}}^{n} \frac{\alpha c_1}{k^\gamma}\right] \leq l_n \leq l_{\bar{n}} \exp\left[\sum_{k=\bar{n}}^{n} \left(\frac{\alpha c_2}{k^\gamma} + C\frac{(\alpha c_2)^2}{k^{2\gamma}}\right)\right]. \tag{39}$$

As a simple consequence, we have for n large enough

$$l_n \geq l_{\bar{n}} \exp\left[\sum_{k=\bar{n}}^{n} \frac{\alpha c_1}{k^\gamma}\right] \geq \begin{cases} C\exp\left(\frac{c_1\alpha}{1-\gamma} n^{1-\gamma}\right) & \text{for } 1/2 < \gamma < 1 \\ Cn^{c_1\alpha} & \text{for } \gamma = 1. \end{cases} \tag{40}$$

This, in particular, implies

$$\lim_n \frac{n^\gamma}{l_n^2} = 0 \qquad \text{and} \qquad \sum_{n=1}^{+\infty} l_n^2 r_{n-1}^2 = +\infty. \tag{41}$$

Indeed, these facts follow from (40) immediately for $1/2 < \gamma < 1$; while, for $\gamma = 1$ one has to note that, since we assume $2c\alpha > 1$, we can choose ε small enough so that $c_1\alpha > \frac{1}{2}$.

We now use Theorem B.2 to obtain a functional central limit theorem for (L_n), from which the corresponding result for $(Z_n(i) - Z_n)$ will follow. Set

$$X_{n,k} = \frac{n^{\gamma/2}\xi_k}{l_n},$$

$\mathcal{F}_{n,k} = \mathcal{F}_k$ and $k_n(t) = \lfloor n + n^\gamma t \rfloor$. We start with showing condition (a1) of Theorem B.2. Note that

$$\begin{aligned}\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2 \,|\, \mathcal{F}_{n,k-1}] &= \frac{n^\gamma}{l_n^2} \sum_{k=1}^{k_n(t)} \mathrm{E}\left[\xi_k^2|\mathcal{F}_{k-1}\right] = \frac{n^\gamma}{l_n^2} \sum_{k=1}^{k_n(t)} l_k^2 \mathrm{Var}[Z_k(i) - Z_k|\mathcal{F}_{k-1}] \\ &\overset{a.s.}{\sim} (1-1/N)Z(1-Z)\frac{n^\gamma}{l_n^2} \sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2,\end{aligned} \tag{42}$$

where, in the last step, we have used the fact that $\lim_n n^\gamma/l_n^2 = 0$ and (32) with $\rho = 1$. In order to estimate the sum $\sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2$, we observe that, for $k \to +\infty$, we have

$$\begin{aligned}
\frac{1}{k^\gamma} l_k^2 - \frac{1}{(k-1)^\gamma} l_{k-1}^2 &= \left[\frac{1}{k^\gamma} - \frac{1}{(k-1)^\gamma}\right] l_{k-1}^2 + \frac{1}{k^\gamma}\left(l_k^2 - l_{k-1}^2\right) \\
&= \left[-\frac{\gamma}{k^{\gamma+1}} + o(1/k^{\gamma+1})\right] l_k^2 \frac{l_{k-1}^2}{l_k^2} + \frac{1}{k^\gamma} l_k^2 \left(1 - \frac{l_{k-1}^2}{l_k^2}\right) \\
&= -\frac{\gamma}{k^{\gamma+1}} l_k^2 (1-\alpha r_{k-1})^2 + o(l_k^2/k^{\gamma+1}) + \frac{1}{k^\gamma} l_k^2 \left[1 - (1-\alpha r_{k-1})^2\right] \\
&= -\frac{\gamma}{k^{\gamma+1}} l_k^2 + \frac{1}{k^\gamma} l_k^2 \left(-\alpha^2 r_{k-1}^2 + 2\alpha r_{k-1}\right) + o(l_k^2/k^{\gamma+1}) \\
&= \begin{cases} \left(\frac{2\alpha r_{k-1}}{k} - \frac{1}{k^2}\right) l_k^2 + o(l_k^2 r_{k-1}^2) & \text{if } \gamma = 1,\ 2c\alpha > 1 \\ \frac{2\alpha r_{k-1}}{k^\gamma} l_k^2 + o(l_k^2 r_{k-1}^2) & \text{if } 1/2 < \gamma < 1 \end{cases} \\
&\sim \begin{cases} \frac{2c\alpha - 1}{c^2} l_k^2 r_{k-1}^2 & \text{if } \gamma = 1,\ 2c\alpha > 1 \\ \frac{2\alpha}{c} l_k^2 r_{k-1}^2 & \text{if } 1/2 < \gamma < 1. \end{cases}
\end{aligned} \tag{43}$$

Therefore, recalling (41), we have

$$\frac{n^\gamma}{l_n^2} \sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2 \sim \begin{cases} \frac{c^2}{2c\alpha - 1} \frac{n}{l_n^2} \sum_{k=2}^{k_n(t)} \left(\frac{1}{k} l_k^2 - \frac{1}{(k-1)} l_{k-1}^2\right) \sim \frac{c^2}{2c\alpha-1} \frac{n}{k_n(t)} \frac{l_{k_n(t)}^2}{l_n^2} & \text{if } \gamma = 1,\ 2c\alpha > 1 \\ \frac{c}{2\alpha} \frac{n^\gamma}{l_n^2} \sum_{k=2}^{k_n(t)} \left(\frac{1}{k^\gamma} l_k^2 - \frac{1}{(k-1)^\gamma} l_{k-1}^2\right) \sim \frac{c}{2\alpha} \frac{n^\gamma}{k_n(t)^\gamma} \frac{l_{k_n(t)}^2}{l_n^2} & \text{if } 1/2 < \gamma < 1. \end{cases} \tag{44}$$

Observing that

$$\lim_n \left(\frac{n}{k_n(t)}\right)^\gamma = \begin{cases} 1 & \text{if } 1/2 < \gamma < 1 \\ \frac{1}{1+t} & \text{if } \gamma = 1, \end{cases} \tag{45}$$

we are left to compute the limit

$$\lim_n \frac{l_{k_n(t)}}{l_n}.$$

This is done in the following Lemma.

Lemma 6.1. *The following convergence holds uniformly over compact subsets of* $[0, +\infty)$*:*

$$\lim_n \frac{l_{k_n(t)}}{l_n} = \begin{cases} e^{c\alpha t} & \text{for } 1/2 < \gamma < 1 \\ (1+t)^{c\alpha} & \text{for } \gamma = 1,\ 2c\alpha > 1. \end{cases} \tag{46}$$

We postpone the proof of Lemma 6.1 and complete the proof of Theorem 2.6. Inserting (45) and (46) in (44) and (42) we obtain

$$\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2 \,|\, \mathcal{F}_{n,k-1}] \xrightarrow{a.s.} \begin{cases} (1-1/N)\, \frac{c}{2\alpha} e^{2c\alpha t} Z(1-Z) & \text{if } 1/2 < \gamma < 1 \\ (1-1/N)\, \frac{c^2}{2c\alpha - 1} (1+t)^{2c\alpha - 1} Z(1-Z) & \text{if } \gamma = 1,\ 2c\alpha > 1. \end{cases}$$

and condition (a1) of Theorem B.2 is verified.

We now prove condition (b1) of Theorem B.2, via Remark B.3 (with $u > 2$) and the sufficient condition in (63). Let $u > 2$. Note that, using (17) and (20) with $\rho = 1$, we have $|\xi_{n+1}| \le 2 l_{n+1} r_n$ for all n. Then

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[\,|X_{n,k}|^u\,] \le 2^u \frac{n^{\frac{\gamma u}{2}}}{l_n^u} \sum_{k=1}^{k_n(1)} l_k^u r_{k-1}^u. \tag{47}$$

This last sum can be estimated as we have done in (44). A computation analogous to the one made in (43) gives

$$\frac{1}{k^{\gamma(u-1)}}l_k^u - \frac{1}{(k-1)^{\gamma(u-1)}}l_{k-1}^u \sim \begin{cases} \frac{uc\alpha-(u-1)}{c^u} l_k^u r_{k-1}^u & \text{if } \gamma=1,\ uc\alpha > (u-1) \\ \frac{u\alpha}{c^{u-1}} l_k^u r_{k-1}^u & \text{if } 1/2<\gamma<1. \end{cases} \tag{48}$$

Then, for $1/2 < \gamma \le 1$ and suitable $u > 2$ (note that, when $\gamma = 1$, we have to choose $u > 2$ such that $uc\alpha - (u-1) = u(c\alpha - 1) + 1 > 0$ and this choice is possible since in this case $2c\alpha > 1$ by assumption), we have

$$\frac{n^{\frac{\gamma u}{2}}}{l_n^u} \sum_{k=1}^{k_n(1)} l_k^u r_{k-1}^u \sim C\left(\frac{l_{k_n(1)}}{l_n}\right)^u \frac{n^{\frac{\gamma u}{2}}}{n^{\gamma(u-1)}} \longrightarrow 0.$$

Thus, by applying Theorem B.2, we get

$$\left(\frac{n^{\gamma/2}}{l_n} L_{k_n(t)}\right)_{t\ge 0} \xrightarrow{d} (W_{V_t})_{t\ge 0},$$

where $(V_t)_{t\ge0}$ and $(W_t)_{t\ge0}$ are defined in the statement of the theorem. Finally, recalling that $\frac{n^{\gamma/2}}{l_n}L_{k_n(t)} = n^{\gamma/2}\frac{l_{k_n(t)}}{l_n}\left(Z_{k_n(t)}(i) - Z_{k_n(t)}\right)$ and using Lemma 6.1 the proof is complete. ∎

Proof of Lemma 6.1. We begin by proving the claim for $1/2 < \gamma < 1$. Note first that $\frac{l_{k_n(t)}}{l_n} = 1$ if $t < n^{-\gamma}$, then for any $T \ge 1$ we have

$$\sup_{t\in[0,T]}\left|\frac{l_{k_n(t)}}{l_n} - e^{c\alpha t}\right| = \max\left\{\sup_{t\in[0,n^{-\gamma})}|1-e^{c\alpha t}|,\ \sup_{t\in[n^{-\gamma},T]} e^{c\alpha t}\left|\frac{l_{k_n(t)}}{l_n}e^{-c\alpha t} - 1\right|\right\}. \tag{49}$$

Since $\sup_{t\in[0,n^{-\gamma})}|1-e^{c\alpha t}| = e^{c\alpha n^{-\gamma}} - 1$ converges to 0, it is enough to show that

$$\lim_n \sup_{t\in[n^{-\gamma},T]}\left|\frac{l_{k_n(t)}}{l_n}e^{-c\alpha t} - 1\right| = 0.$$

We fix $\sigma > 0$ and take $0 < \epsilon < \min\{\frac{\ln(1+\sigma)}{2c\alpha T}, 1\}$. Then, using (39), we have, for all $n \ge \bar{n}$ and $\bar{n}$ sufficiently large,

$$\exp\left[\sum_{k=n}^{k_n(t)} \frac{\alpha c_1}{k^\gamma}\right] \le \frac{l_{k_n(t)}}{l_n} \le \exp\left[\sum_{k=n}^{k_n(t)}\left(\frac{\alpha c_2}{k^\gamma} + C\frac{(\alpha c_2)^2}{k^{2\gamma}}\right)\right] \qquad \forall t \in [n^{-\gamma}, T] \tag{50}$$

where $c_1 = c(1-\epsilon)$ and $c_2 = c(1+\epsilon)$.

Now we use the following asymptotics, which hold for all $p > 0$ and $n, m \ge 1$:

$$\sum_{k=n}^{n+m-1}\frac{1}{k^p} = \begin{cases} \varepsilon_p(n,m) & \text{if } p>1 \\ \ln\left(1+\frac{m}{n}\right) + \varepsilon_1(n,m) & \text{if } p=1 \\ \frac{n^{1-p}}{1-p}\left[\left(1+\frac{m}{n}\right)^{1-p} - 1\right] + \varepsilon_p(n,m) & \text{if } p<1 \end{cases} \tag{51}$$

where $\varepsilon_p(n,m)$ denotes a positive function such that $\lim_n \sup_{m\ge1}\varepsilon_p(n,m) = 0$. Thus, recalling that $1/2 < \gamma < 1$,

$$\exp\left\{\frac{\alpha c_1 n^{1-\gamma}}{1-\gamma}\left[\left(1+\frac{\lfloor n^\gamma t\rfloor}{n}\right)^{1-\gamma} - 1\right]\right\} \le \frac{l_{k_n(t)}}{l_n} \le \exp\left\{\frac{\alpha c_2 n^{1-\gamma}}{1-\gamma}\left[\left(1+\frac{\lfloor n^\gamma t\rfloor}{n}\right)^{1-\gamma} - 1\right] + A_{n,t}\right\} \tag{52}$$

where $A_{n,t} = \alpha c_2 \varepsilon_\gamma(n, \lfloor n^\gamma t\rfloor) + C(\alpha c_2)^2 \varepsilon_{2\gamma}(n, \lfloor n^\gamma t\rfloor)$. Moreover, using the relation $1 + (1-\gamma)x - \frac{(1-\gamma)\gamma}{2}x^2 \le (1+x)^{1-\gamma} \le 1 + (1-\gamma)x$ for all $x \ge 0$ and $1/2 < \gamma < 1$, we have that

$$t - \frac{1}{n^\gamma} - \frac{\gamma t^2}{2n^{1-\gamma}} \le \frac{n^{1-\gamma}}{1-\gamma}\left[\left(1 + \frac{\lfloor n^\gamma t\rfloor}{n}\right)^{1-\gamma} - 1\right] \le t \tag{53}$$

Therefore, using (52) and (53) we obtain

$$e^{-c\alpha\epsilon t - \frac{c\alpha(1-\epsilon)}{n^\gamma} - \frac{c\alpha(1-\epsilon)\gamma t^2}{2n^{1-\gamma}}} \le \frac{l_{k_n(t)}}{l_n} e^{-c\alpha t} \le e^{c\alpha\epsilon t + A_{n,t}} \qquad \forall t \in [n^{-\gamma}, T]$$

and, since $\lim_n \sup_t A_{n,t} = 0$, for all n sufficiently large we have

$$e^{-2c\alpha\epsilon T} \le \frac{l_{k_n(t)}}{l_n} e^{-c\alpha t} \le e^{2c\alpha\epsilon T} \qquad \forall t \in [n^\gamma, T]$$

from which it follows

$$\sup_{t\in[n^{-\gamma},T]} \left| \frac{l_{k_n(t)}}{l_n} e^{-c\alpha t} - 1\right| \le e^{2c\alpha\epsilon T} - 1 < \sigma$$

for all n sufficiently large and the proof is complete, since σ can be taken arbitrarily small.

We now turn to the case $\gamma = 1$ and $2c\alpha > 1$. As for $1/2 < \gamma < 1$, it is enough to show that $\sup_{t\in[n^{-1},T]} \left|\frac{l_{k_n(t)}}{l_n}(1+t)^{-c\alpha} - 1\right|$ converges to 0. We fix $\sigma > 0$ and take $0 < \epsilon < \min\{\frac{\ln(1+\sigma)}{2c\alpha\ln(1+T)}, 1\}$. Using (50) and (51) for $\gamma = 1$, we obtain, for all $n \ge \bar{n}$ and $\bar{n}$ sufficiently large,

$$\exp\left[\alpha c_1 \ln\left(1 + \frac{\lfloor nt\rfloor}{n}\right)\right] \le \frac{l_{k_n(t)}}{l_n} \le \exp\left[\alpha c_2 \ln\left(1 + \frac{\lfloor nt\rfloor}{n}\right) + A_{n,t}\right] \quad \forall t \in [n^{-1}, T] \tag{54}$$

where $c_1 = c(1-\epsilon)$, $c_2 = c(1+\epsilon)$ and $A_{n,t} = \alpha c_2 \varepsilon_1(n, \lfloor nt\rfloor) + C(\alpha c_2)^2 \varepsilon_2(n, \lfloor nt\rfloor)$. Then, for all n sufficiently large

$$\left(1 + \frac{\lfloor nt\rfloor}{n}\right)^{c(1-\epsilon)\alpha}(1+t)^{-c\alpha} \le \frac{l_{k_n(t)}}{l_n}(1+t)^{-c\alpha} \le \left(1 + \frac{\lfloor nt\rfloor}{n}\right)^{c(1+\epsilon)\alpha} e^{A_{n,t}}(1+t)^{-c\alpha} \quad \forall t \in [n^{-1}, T]$$

and, using the fact that $\lim_n \sup_t A_{n,t} = 0$, for all n sufficiently large we can write

$$(1+T)^{-2c\alpha\epsilon} \le \frac{l_{k_n(t)}}{l_n}(1+t)^{-c\alpha} \le (1+T)^{2c\alpha\epsilon} \qquad \forall t \in [n^{-1}, T]$$

from which it follows

$$\sup_{t\in[n^{-1},T]} \left|\frac{l_{k_n(t)}}{l_n}(1+t)^{-c\alpha} - 1\right| \le (T+1)^{2c\alpha\epsilon} - 1 < \sigma$$

for all n sufficiently large and the proof is complete, since σ can be taken arbitrarily small. ∎

7. Fluctuation theorems III: proof of Theorems 2.8 and 2.9

7.1. **Proof of Theorem 2.8.** *Proof of case (i).* The proof is the same as for Theorem 2.6, part (i). Indeed, using (19) we can define $l_n = \prod_{k=0}^{n-1} \frac{1}{1-(1-\rho)r_k}$ and $X_{n,k} = \frac{n^{\gamma/2}\xi_k}{l_n}$, where $\xi_n = l_n(Z_n - \mathrm{E}[Z_n|\mathcal{F}_{n-1}])$. Using the estimate (41) and (33) we obtain

$$\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2 \,|\, \mathcal{F}_{n,k-1}] \overset{a.s.}{\sim} \frac{1}{N} q(1-q)\rho^2 \frac{n}{l_n^2} \sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2. \tag{55}$$

Then, using estimate (44) and Lemma 6.1 with $(1-\rho)$ instead of α, we can conclude that $\frac{n}{l_n^2}\sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2$ converges to $\frac{c}{2(1-\rho)}e^{2c(1-\rho)t}$ and so condition (a1) of Theorem B.2 is satisfied.

Using (15) and (19) we have $|\xi_{n+1}| \leq l_{n+1}r_n$ and so, fixing $u>2$, we can repeat the same argument used in the proof of Theorem 2.6 in order to prove

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[\,|X_{n,k}|^u\,] \longrightarrow 0 \tag{56}$$

and the conclusion follows from Theorem B.2, Remark B.3 (with $u>2$) and Lemma 6.1.

Proof of case (ii). Following proof and notations of Theorem 2.6, part (ii), with $1-\rho$ in place of α, since $2c(1-\rho)>1$, relation (42) becomes

$$\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2\,|\,\mathcal{F}_{n,k-1}] \overset{a.s.}{\sim} \frac{1}{N}q(1-q)\rho^2\frac{n}{l_n^2}\sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2.$$

and, by estimate (44) and Lemma 6.1 with $(1-\rho)$ instead of α, the last sum converges to $\frac{c^2}{2c(1-\rho)-1}(1+t)^{2c(1-\rho)-1}$. Analogously, condition (b1) of Theorem B.2 and then the conclusion can be easily derived as seen in Theorem 2.6.

7.2. **Proof of Theorem 2.9.** *Proof of case (i).* We again follow the proof of Theorem 2.6 by posing $\beta = 1-(1-\alpha)\rho$, using (20) and defining $l_n = \prod_{k=0}^{n-1}\frac{1}{1-\beta r_k}$. Then, given $X_{n,k} = \frac{n^{\gamma/2}\xi_k}{l_n}$, where $\xi_n = l_n(Z_n(i) - Z_n - \mathrm{E}[Z_n(i)-Z_n|\mathcal{F}_{n-1}])$, we use the estimate (41) and (32) and we have

$$\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2\,|\,\mathcal{F}_{n,k-1}] \overset{a.s.}{\sim} (1-\frac{1}{N})q(1-q)\rho^2\frac{n}{l_n^2}\sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2.$$

Then, by estimate (44) and Lemma 6.1 with β instead of α, it follows $\frac{n}{l_n^2}\sum_{k=1}^{k_n(t)} l_k^2 r_{k-1}^2 \longrightarrow \frac{c}{2(1-(1-\alpha)\rho)}e^{2c(1-(1-\alpha)\rho)t}$ and condition (a1) of Theorem B.2 is verified.
For condition (b1), by (17) and (20), we obtain $|\xi_{n+1}| \leq 2l_{n+1}r_n$ and so, for a fixed $u>2$

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[\,|X_{n,k}|^u\,] \longrightarrow 0$$

and the conclusion follows from Theorem B.2, Remark B.3 (with $u>2$) and Lemma 6.1.

Proof of case (ii). The proof is analogous to the one of case (ii) of Theorem 2.8 above.

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Appendix A. A technical lemma

Lemma A.1. *Let* (x_n) *be a sequence of positive numbers that satisfies the following equation:*

$$x_{n+1} = (1 - ar_n)x_n + K_n r_n^2 \tag{57}$$

where $a > 0$*,* $r_n \geq 0$ *and* $0 \leq K_n \leq K$*. Suppose that*

$$\sum_n r_n = +\infty \quad \text{and} \quad \sum_n r_n^2 < +\infty. \tag{58}$$

Then

$$\lim_{n\to+\infty} x_n = 0.$$

Proof. The case $K = 0$ is well-known and so we will prove the statement with $K > 0$. Let l be such that $ar_n < 1$ for all $n \geq l$. Then for $n \geq l$ we have $x_n \leq y_n$, where

$$\begin{cases} y_{n+1} &= (1 - ar_n)y_n + Kr_n^2 \\ y_l &= x_l \end{cases}$$

Set $\varepsilon_n = ar_n$ and $\delta_n = Kr_n^2$. It holds

$$y_n = y_l \prod_{i=l}^{n-1}(1 - \varepsilon_i) + \sum_{i=l}^{n-1} \delta_i \prod_{j=i+1}^{n-1}(1 - \varepsilon_j).$$

Using the fact that $\sum_n \varepsilon_n = +\infty$, it follows that

$$\prod_{i=l}^{n-1}(1 - \varepsilon_i) \longrightarrow 0.$$

Moreover, for every $m \geq l$,

$$\begin{aligned} \sum_{i=l}^{n-1} \delta_i \prod_{j=i+1}^{n-1}(1 - \varepsilon_j) &= \sum_{i=l}^{m-1} \delta_i \prod_{j=i+1}^{n-1}(1 - \varepsilon_j) + \sum_{i=m}^{n-1} \delta_i \prod_{j=i+1}^{n-1}(1 - \varepsilon_j) \\ &\leq \prod_{j=m}^{n-1}(1 - \varepsilon_j) \sum_{i=l}^{m-1} \delta_i + \sum_{i=m}^{+\infty} \delta_i. \end{aligned} \tag{59}$$

Using the fact that $\prod_{j=m}^{n-1}(1 - \varepsilon_j) \longrightarrow 0$ and that $\sum_n \delta_n < +\infty$, letting first $n \to +\infty$ and then $m \to +\infty$ in (59) the conclusion follows.

∎

Appendix B. Some definitions and known results

B.1. **Stable convergence and its variants.** We recall here some basic definitions and results. For more details, we refer the reader to [15, 25] and the references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let S be a Polish space, endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto Kf(\omega) = \int f(x)\, K(\omega)(dx)$$

is $\mathcal{A}$-measurable. Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel K is said $\mathcal{H}$-measurable if all the above random variables Kf are $\mathcal{H}$-measurable.

On $(\Omega, \mathcal{A}, P)$, let (Y_n) be a sequence of S-valued random variables, let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$, and let K be a $\mathcal{H}$-measurable kernel on S. Then we say that Y_n converges $\mathcal{H}$-*stably* to K, and we write $Y_n \overset{\mathcal{H}-stably}{\longrightarrow} K$, if

$$P(Y_n \in \cdot \,|\, H) \overset{weakly}{\longrightarrow} E\left[K(\cdot)\,|\,H\right] \qquad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0.$$

In the case when $\mathcal{H} = \mathcal{A}$, we simply say that Y_n converges *stably* to K and we write $Y_n \overset{stably}{\longrightarrow} K$. Clearly, if $Y_n \overset{\mathcal{H}-stably}{\longrightarrow} K$, then Y_n converges in distribution to the probability distribution $E[K(\cdot)]$. Moreover, the $\mathcal{H}$-stable convergence of Y_n to K can be stated in terms of the following convergence of conditional expectations:

$$E[f(Y_n)\,|\,\mathcal{H}] \overset{\sigma(L^1, L^\infty)}{\longrightarrow} Kf \tag{60}$$

for each bounded continuous real function f on S.

In [15] the notion of $\mathcal{H}$-stable convergence is firstly generalized in a natural way replacing in (60) the single sub-σ-field $\mathcal{H}$ by a collection $\mathcal{G} = (\mathcal{G}_n)$ (called conditioning system) of sub-σ-fields of $\mathcal{A}$ and then it is strengthened by substituting the convergence in $\sigma(L^1, L^\infty)$ by the one in probability (i.e. in L^1, since f is bounded). Hence, according to [15], we say that Y_n converges to K *stably in the strong sense*, with respect to $\mathcal{G} = (\mathcal{G}_n)$, if

$$E\left[f(Y_n)\,|\,\mathcal{G}_n\right] \overset{P}{\longrightarrow} Kf \tag{61}$$

for each bounded continuous real function f on S.

Finally, a strengthening of the stable convergence in the strong sense can be naturally obtained if in (61) we replace the convergence in probability by the almost sure convergence: given a conditioning system $\mathcal{G} = (\mathcal{G}_n)$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence*, with respect to $\mathcal{G}$, if

$$E\left[f(Y_n)\,|\,\mathcal{G}_n\right] \overset{a.s.}{\longrightarrow} Kf \tag{62}$$

for each bounded continuous real function f on S. Evidently, this last type of convergence can be reformulated using the conditional distributions. Indeed, if K_n denotes a version of the conditional distribution of Y_n given $\mathcal{G}_n$, then the random variable $K_n f$ is a version of the conditional expectation $E\left[f(Y_n)|\mathcal{G}_n\right]$ and so we can say that Y_n converges to K in the sense of the almost sure conditional convergence, with respect to $\mathcal{F}$, if, for almost every ω in Ω, the probability measure $K_n(\omega)$ converges weakly to $K(\omega)$. The almost sure conditional convergence has been introduced in [14] and, subsequently, employed by others in the urn model literature (e.g. [1, 42]).

We now conclude this section with a convergence result for martingale difference arrays.
Given a conditioning system $\mathcal{G} = (\mathcal{G}_n)_n$, if $\mathcal{U}$ is a sub-σ-field of $\mathcal{A}$ such that, for each real integrable random variable Y, the conditional expectation $\mathrm{E}[Y\,|\,\mathcal{G}_n]$ converges almost surely to the conditional expectation $\mathrm{E}[Y\,|\,\mathcal{U}]$, then we shall briefly say that $\mathcal{U}$ is an *asymptotic σ-field* for $\mathcal{G}$. In order that there exists an asymptotic σ-field $\mathcal{U}$ for a given conditioning system $\mathcal{G}$, it is obviously sufficient that the sequence $(\mathcal{G}_n)_n$ is increasing or decreasing. (Indeed we can take $\mathcal{U} = \bigvee_n \mathcal{G}_n$ in the first case and $\mathcal{U} = \bigcap_n \mathcal{G}_n$ in the second one.)

Theorem B.1. *(Theorem A.1 in [14])*
On $(\Omega, \mathcal{A}, P)$, for each $n \geq 1$, let $(\mathcal{F}_{n,h})_{h\in\mathbb{N}}$ be a filtration and $(M_{n,h})_{h\in\mathbb{N}}$ a real martingale with respect to $(\mathcal{F}_{n,h})_{h\in\mathbb{N}}$, with $M_{n,0} = 0$, which converges in L^1 to a random variable $M_{n,\infty}$. Set

$$X_{n,j} := M_{n,j} - M_{n,j-1} \quad \text{for } j \geq 1, \quad U_n := \sum_{j\geq 1} X_{n,j}^2, \quad X_n^* := \sup_{j\geq 1} |X_{n,j}|.$$

Further, let $(k_n)_{n\geq 1}$ be a sequence of strictly positive integers such that $k_n X_n^* \overset{a.s.}{\to} 0$ and let $\mathcal{U}$ be a sub-σ-field which is asymptotic for the conditioning system $\mathcal{G}$ defined by $\mathcal{G}_n = \mathcal{F}_{n,k_n}$. Assume that the sequence $(X_n^*)_n$ is dominated in L^1 and that the sequence $(U_n)_n$ converges almost surely to a positive real random variable U which is measurable with respect to $\mathcal{U}$.

Then, with respect to the conditioning system $\mathcal{G}$, the sequence $(M_{n,\infty})_n$ converges to the Gaussian kernel $\mathcal{N}(0, U)$ in the sense of the almost sure conditional convergence.

B.2. **Durrett-Resnick result.** We recall the following convergence result for martingale difference arrays:

Theorem B.2. *(Th. 2.5 in [18])*
Let $(X_{n,k})$ be a square-integrable martingale difference array with respect to $(\mathcal{F}_{n,k})$. Suppose that $(\mathcal{F}_{n,k})$ increases as n increases and let $k_n(t)$ a non-decreasing right continuous function with values in $\mathbb{N}$ such that the following conditions hold true:
(a1) for each $t > 0$,

$$\sum_{k=1}^{k_n(t)} \mathrm{E}[X_{n,k}^2 \,|\, \mathcal{F}_{n,k-1}] \overset{P}{\longrightarrow} V_t$$

where $P(t \mapsto V_t \ \text{ is continuous}) = 1$;
(b1) for each $\epsilon > 0$,

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[X_{n,k}^2 I_{\{|X_{n,k}|>\epsilon\}} \,|\, \mathcal{F}_{n,k-1}] \overset{P}{\longrightarrow} 0.$$

Then, if we set $S_{n,k_n(t)} = \sum_{k=1}^{k_n(t)} X_{n,k}$, we have

$$S^{(n)} = (S_{n,k_n(t)})_{t\geq 0} \overset{d}{\longrightarrow} \widetilde{W} = \big(W_{V_t}\big)_{t\geq 0} \qquad \text{(w.r.t. Skorohod's topology)},$$

where $W = (W_t)_{t\geq 0}$ is a Wiener process independent of $V = (V_t)_{t\geq 0}$.

Remark B.3. If there exists a number $u \geq 2$ such that

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[\,|X_{n,k}|^u\,] \longrightarrow 0, \tag{63}$$

then condition (b1) holds true with convergence in L^1. Indeed, it is enough to observe that

$$\sum_{k=1}^{k_n(1)} \mathrm{E}[X_{n,k}^2 I_{\{|X_{n,k}|>\epsilon\}}] = \sum_{k=1}^{k_n(1)} \mathrm{E}[|X_{n,k}|^u |X_{n,k}|^{-(u-2)} I_{\{|X_{n,k}|>\epsilon\}}] \leq \frac{1}{\epsilon^{u-2}} \sum_{k=1}^{k_n(1)} \mathrm{E}[\,|X_{n,k}|^u\,] \longrightarrow 0.$$

B.3. **Hájek-Rényi inequality.** We recall the following martingale inequality (e.g. [9]):

Theorem B.4. *If $M_n = \sum_{j=1}^n \xi_j$ is a square integrable martingale and (a_n) is a positive, nondecreasing sequence of numbers, then for each $\lambda > 0$, we have*

$$P\left(\max_{1\leq j\leq n} \frac{|M_j|}{a_j} \geq \lambda\right) \leq \frac{1}{\lambda^2}\sum_{j=1}^{n} \frac{\mathrm{E}[\xi_j^2]}{a_j^2}\,.$$

B.4. **Barbour's transform.** Let $D = D[0,+\infty)$ be the space of right-continuous functions with left limits on $[0,+\infty)$, endowed with the classical Skorohod's topology (e.g. [8]).

Let T be the subspace of functions $f(t)$ in the space D such that

$$\limsup_{t\to+\infty} \frac{|f(t)|}{t} = 0 \tag{64}$$

$$\int_1^{+\infty} \frac{|f(t)|}{t^2}\,\mathrm{d}t < +\infty \tag{65}$$

$$\int_0^{1} \frac{|f(t)|}{t}\,\mathrm{d}t < +\infty. \tag{66}$$

Let m be the metric on T such that $m(f_1, f_2)$ is the infimum of those $\epsilon > 0$ for which there exists some continuous strictly increasing function $\lambda : [0,+\infty) \mapsto [0,+\infty)$ with $\lambda(0) = 0$, such that

$$\sup_{t\geq 0} \frac{f_1(t) - f_2(\lambda(t))}{t+1} < \epsilon \tag{67}$$

$$\int_1^{+\infty} \frac{|f_1(t) - f_2(\lambda(t))|}{t^2}\,\mathrm{d}t < \epsilon \tag{68}$$

$$\int_0^{1} \frac{|f_1(t) - f_2(\lambda(t))|}{t}\,\mathrm{d}t < \epsilon \tag{69}$$

$$\sup_{t\neq s} \left|\log \frac{\lambda(t) - \lambda(s)}{t-s}\right| < \epsilon. \tag{70}$$

Let T_1 and m_1 be defined similarly, without the restrictions (66) and (69). We shall denote by T^* and T_1^* the corresponding subspaces of the space $D^* = D^*[0,+\infty)$ of left-continuous functions with right limits on $[0,+\infty)$ (endowed with the corresponding Skorohod's topology).

The topology induced by m on T is stronger than the Skorohod's topology. Moreover, the *Barbour's transform* $g : T \to T_1^*$ defined as

$$g(f)(0) = 0 \quad \text{and} \quad g(f)(t) := \int_{1/t}^{+\infty} s^{-1}\mathrm{d}f(s) = -tf(t^{-1}) + \int_{1/t}^{+\infty} s^{-2} f(s)\,\mathrm{d}s \qquad \text{for } t \in (0,+\infty).$$

is continuous and, if W is a Wiener process, then also $g(W)$ is a Wiener process. Finally, the following result holds (for more details, see [24, 26, 35, 41]).

Theorem B.5. *Let $(Y^{(n)})_n$ be a sequence of stochastic processes satisfying the following conditions:*

- (a2) *$Y^{(n)} \xrightarrow{d} \widetilde{W}$ (w.r.t. Skorohod's topology), where $\widetilde{W}$ is a stochastic process of the form $\widetilde{W}_t = W_{V_t}$ where W is a Wiener process and V is a stochastic process, independent of W and with $P(t \mapsto V_t$ is continuous$) = 1$;*
- (b2) *for each n and $\epsilon > 0$, $Y_n(t) = o(t^{1/2+\epsilon})$ a.s. as $t \to +\infty$;*

(c2) *for each* $\theta > 1/2$, $\epsilon > 0$ *and* $\eta > 0$, *there exists* t_0 *such that*

$$P\left\{\sup_{t\geq t_0}\frac{|Y_t^{(n)}|}{t^\theta} > \epsilon\right\} \leq \eta. \tag{71}$$

Then each $Y^{(n)}$ *takes values in the space* T *and* $Y^{(n)} \xrightarrow{d} \widetilde{W}$ *on* (T, m).

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