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LIE SYMMETRY APPROACH TO THE CEV MODEL

Francesco Cordoni[1], Luca Di Persio[2 §]

[1]Mathematics Department
University of Trento
Via Sommarive, 14 Trento, ITALY

[2]Department of Computer Science
University of Verona
Via le Grazie, 14 Verona, ITALY

Abstract: Using a Lie algebraic approach we explicitly provide both the probability density function of the constant elasticity of variance (CEV) process and the fundamental solution for the associated pricing equation. In particular we reduce the CEV stochastic differential equation (SDE) to the SDE characterizing the Cox, Ingersoll and Ross (CIR) model, being the latter easier to treat. The fundamental solution for the CEV pricing equation is then obtained following two methods. We first recover a fundamental solution via the *invariant solution method*, while in the second approach we exploit Lie classical result on classification of linear partial differential equations (PDEs). In particular we find a map which transforms the pricing equation for the CIR model into an equation of the form $v_\tau = v_{yy} - \frac{A}{y^2}v$ whose fundamental solution is known. Then, by inversion, we obtain a fundamental solution for the CEV pricing equation.

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[§]Correspondence author

1. Introduction

Lie's theory of group invariant dates back to the middle of the XIX century, see, e.g., [16], and has been widely studied both from a theoretic and applied point of view, spanning from the theory of partial differential equations, to biological and mathematical physics applications, see, e.g., [21, 22]. In particular Lie groups techniques can be used to solve differential equations by quadratures in connection with groups of continuous transformations. Such an approach provides new solutions of a given differential equation, starting from a known one, via suitable group of symmetries. The classical Lie theory consider continuous transformation group acting on the system's graph space of dependent and independent variables of a given partial differential equation (PDE). If a differential equation is invariant under a point symmetry one can find *similarity solutions* which are invariant under subgroups of the full group. This paper is aimed to both explicitly find the probability density function for the constant elasticity of variance (CEV) model and the fundamental solution for the related pricing equation. We first give a basic introduction to Lie classical result on classification of linear partial differential equations. Then we introduce results later used to retrieve the probability density function of the CEV stochastic process. A concise presentation of the financial background is also provided as well as a method to transform the CEV process into the Cox, Ingersoll and Ross (CIR) process to obtain the CEV pricing equation in an easier way.

2. Lie Classical Symmetry Method

Let us recall that a group $(G, \circ)$ having the structure of an analytic manifold M such that $\mu : (x, y) \to x \circ y^{-1}$, where y^{-1} denotes the inverse of y w.r.t. the group law $\circ$, is analytic is called a *Lie group*. Lie groups naturally arise as *transformation groups*. A Lie group can be seen as the smooth action of a *Lie group* on a manifold M into itslef. Moreover *Lie theorem* allows to reduce the study of Lie group properties into the analysis of the related properties of a pure algebraic object, the *Lie algebra*, by considering the space V, spanned by the *infinitesimal generators* of the Lie group, which is tangent to G in its identity element. We underline that the space V determines the local structure of the Lie group and it is itself a *Lie algebra*, see, e.g., [2, 18] for a detailed treatment of the topic. Let us suppose we are dealing with an $m-$th order partial differential equation with N independent variables $x = (x_1, \dots, x_N)$

and a single dependent variable u of the form

$$F(x, u^{(m)}) = 0 \,, \tag{1}$$

where F is a given differential operator, $u^{(m)}$ is the *vector* of all the derivatives of the real valued function u, which is defined from a subset of $\mathbb{R}^N$, up to order m, then we search for the *Lie group of transformations* leaving invariant the surface $\mathcal{S} := \{(x_1, \dots, x_N, u(x_1, \dots, x_N)) : (x_1, \dots, x_N) \in \Omega \simeq \mathbb{R}^N\}$ determined by the PDE (1).

Definition 2.0.1. Let $F(x, u^{(m)})$ be a partial differential equation. A symmetry group of the PDE $F(x, u^{(m)})$ is a *local Lie group of transformations* G acting on an open subset M of the space of independent and dependent variables for the PDE with the property that whenever $u = f(x)$ is a solution of $F(x, u^{(m)})$ and whenever $\exp(\epsilon\mathbf{v})(f)$ is defined then $u = \exp(\epsilon\mathbf{v})(f(x))$ is also a solution of the PDE. Thus the symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property of mapping solutions to other solutions.

Lie theory allows us to study the admitted transformations locally, namely we can consider the Taylor expansion of the transformation

$$\begin{cases} \tilde{x}_i = X^i(x, u) = x_i + \epsilon\, \xi^i(x, u) + O(\epsilon^2) \quad i = 1, \dots, N, \quad N \in N^+ \\ \tilde{u} = U(x, u) = u + \epsilon\nu(x, u) + O(\epsilon^2) \end{cases} \tag{2}$$

depending continuously on the parameter $\epsilon \in R \subset \mathbb{R}$ s.t. $0 \in R$. Without loss of generality we will always consider *local Lie group* with composition law $+$ and identity element $\epsilon = 0$. Further X^i are U diffeomorphisms which define the *Lie group of transformations* and ξ^i and ν are smooth functions such that

$$\begin{cases} \frac{dX^i_\epsilon(x,u)}{d\epsilon} = \xi^i(x, u), \quad \tilde{x}_i\big|_{\epsilon=0} = x_i, \quad i = 1, \dots, N \\ \frac{dU_\epsilon(x,u)}{d\epsilon} = \nu(x, u), \quad \tilde{u}\big|_{\epsilon=0} = u \end{cases} . \tag{3}$$

In particular the functions $\xi^1, \dots, \xi^N$ and ν define the *tangent vector field*

$$\big(\xi^1(x, u), \dots, \xi^N(x, u), \nu(x, u)\big)$$

of the Lie group G. System (2) is known as *Lie point symmetry group* whereas system (3) are known as *Lie equations*. All of the previous smooth functions have to be evaluated locally at a specific point (x, u). In particular local properties of the transformation characterize the transformation globally. Previous system suggests the standard notation used in literature, namely $\exp(\epsilon\mathbf{v})x$, to indicate the *one-parameter Lie group* with *infinitesimal generator* $\mathbf{v}$ of the form

$$\mathbf{v} = \sum_{i=1}^{N} \xi^i(x,u)\partial_{x_i} + \nu(x,u)\partial_u. \tag{4}$$

The set of all transformations of the form (2) admitted by the PDE (1) forms a continuous group called the *Lie group* G. According to theory developed by Lie the construction of the *r-dimensional Lie group* G is equivalent to the determination of r *infinitesimal generators* of the group G of the form

$$\mathbf{v}_j = \sum_{i=1}^{N} \xi^i_j(x,u)\partial_{x_i} + \nu(x,u)_j\partial_u, \quad j = 1,\dots,r\,. \tag{5}$$

The set of all *infinitesimal generators* of the form (5) span the *Lie algebra* $\mathfrak{g}$ which completely determines the global structure of the *Lie group*. The Lie algebra allows us to reduce the study of a complicated object such as a *Lie group* to the simpler analysis of an algebraic structure, i.e. the *Lie algebra*. Moreover Lie proved that whenever the following invariance condition holds

$$\mathrm{pr}^{(m)}\mathbf{v}[F(x,u^{(m)})]\Big|_{F(x,u^{(m)})=0} = 0\,, \tag{6}$$

where $\mathrm{pr}^{(m)}\mathbf{v}$ denotes the *prolongation* of the vector field, then the given PDE is left invariant under a suitable transformation, see e.g. [2, 18]. Condition (6) leads to a set of linear partial differential equations which are called *determining equations* and whose solution give rises to the most general form of transformations admitted by the PDE (1) i.e. equation (6) allows to obtain a basis that span the *Lie algebra* $\mathfrak{g}$.

It is thus possible to map a solution of a given partial differential equation into another solution of the same PDE. Moreover Lie showed that one can also map a solution of a given PDE into a solution of a different PDE, namely the *target equation*, see, e.g., [2] for a complete treatment of the theory.

The latter result suggests to develop a theoretical framework based on the quest for a transformation which maps a given PDE into a *target one* in such a way that a one-to-one correspondence between solutions of the two problems can be stated. Such an approach implies also a one-to-one correspondence between the infinitesimal generators of both the given and the target PDE. In particular the Lie algebra admitted by the given PDE has to be isomorphic to the Lie algebra admitted by the target PDE, hence the correspondence is also invertible.

With respect to the PDE in (1), let us define G_x and H_x with $H_x \subset G_x$ to be the group and subgroup of all continuous transformations with $\mathfrak{g}_x$, resp. $\mathfrak{h}_x$, the Lie algebra of G_x, resp. H_x. We consider r infinitesimal generators of the form (5) such that $\mathfrak{h}_x := \langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$ where each $\mathbf{v}_j$ generates a one-parameter transformations $\exp(\epsilon \mathbf{v}_j)x \in H_x$.

We further consider the target PDE

$$P(y, v^{(m)}) = 0\,, \tag{7}$$

where $y = (y_1, \ldots, y_N)$, for $N \in \mathbb{N}^+$, is the vector of independent variables, while v is the dependent one. As before, we denote by G_y, resp. H_y, the group, resp. the subgroup of all continuous transformations with $H_y \subset G_y$, while $\mathfrak{g}_y$, resp. $\mathfrak{h}_y$, denotes the Lie algebra of G_y, resp. H_y. We consider again r *infinitesimal generators* of the form

$$\mathbf{w}_j = \sum_{i=1}^{N} \chi_j^i(y, v)\partial_{y_i} + \omega(y, v)_j \partial_v, \quad j = 1, \ldots, r \tag{8}$$

such that $\mathfrak{h}_y := \langle \mathbf{w}_1, \ldots, \mathbf{w}_r \rangle$ where each $\mathbf{w}_j$ generates a one-parameter transformations $\exp(\epsilon \mathbf{w}_j)y \in H_y$.

Let μ denote a mapping which transforms any solution $u = u(x)$ of equation (1) into a solution $v = v(y)$ of equation (7). Of course this transformation need not to exist a priori and even if it does it is unknown. We consider therefore the general case of μ to be a map of the following form

$$\begin{cases} y = \phi(x, u, \ldots, u^{(l)}) \\ v = \psi(x, u, \ldots, u^{(l)}) \end{cases} \tag{9}$$

We will denote by $\mathcal{M}_l$ the class of mapping of the form (9) which depends on at most the l-th derivatives of u. In particular any map $\mu \in \mathcal{M}_m$ induces an action as in following diagram

$$\begin{array}{ccc} F(x, u^{(m)}) & \xrightarrow{\exp(\epsilon \mathbf{v})(x, u^{(m)})} & F(\tilde{x}, \tilde{u}^{(m)}) \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ P(y, v^{(m)}) & \xrightarrow[\exp(\epsilon \mathbf{w})(y, v^{(m)}]{} & P(\tilde{y}, \tilde{v}^{(m)}) \end{array} \tag{10}$$

Since diagram (10) has to commute, namely $\exp(\epsilon \mathbf{w}_j) \circ \mu x = \mu \circ \exp(\epsilon \mathbf{v}_j)x$ we can retrieve the necessary conditions that the mapping μ has to satisfy reads as follows

$$\begin{cases} \mathbf{v}^{(m)}\phi = \mathbf{w}y \\ \mathbf{v}^{(m)}\psi = \mathbf{w}v \end{cases} \tag{11}$$

see, e.g. [2]§6 for more details.

In the case of one-to-one (invertible) mapping from a given PDE to a target PDE the class of possible mappings $\mathcal{M}_l$ is predetermined. In particular for the case on a single dependent variable, as shown in the following theorem, we have that $l = 1$, i.e. $\mu \in \mathcal{M}_1$.

Theorem 2.1 (Bäcklund(1876)). *Let us consider a PDE with a single dependent variable u. Then a mapping μ defines an invertible mapping from $(x, u, \dots, u^{(p)})$-space to $(y, v, \dots, v^{(p)})$-space for any fixed p, iff μ is a one-to-one contact transformation of the form:*

$$\begin{cases} y = \phi(x, u, u^{(1)}) \\ v = \psi(x, u, u^{(1)}) \\ v^{(1)} = \psi^{(1)}(x, u, u^{(1)}) \end{cases} .$$

Let us further notice that if ϕ and ψ are independent of $u^{(1)}$ then they define a point transformation. We now recall Lie's classical results on classification of second order linear partial differential equations. In the case of a parabolic equation we have the following.

Theorem 2.2. *Let us consider the family of linear parabolic equations*

$$P(x,t)u_t + Q(x,t)u_x + R(x,t)u_{xx} + S(x,t)u = 0, \quad P \neq 0, \quad R \neq 0\,, \tag{12}$$

with P, Q, R and S some smooth functions depending on the independent variables (x, t). The principal Lie algebra $\mathfrak{g}$ associated equation (12) is spanned by the generators of trivial symmetries

$$\begin{cases} \mathbf{v}_1 = u\partial_u \\ \mathbf{v}_\beta = \beta(x,t)\partial_u \end{cases} ,$$

with β a generic solution of equation (12) see, e.g [15, 17] for details. The group of dilations generated by the operator $\mathbf{v}_1$ reflects the homogeneity of Equation (12), while the infinite group with the operator $\mathbf{v}_\alpha$ represents the linear superposition principle for Equation (12). Thus any equation of the form (12) can be reduced to an equation of the form $v_\tau = v_{yy} - Z(y,\tau)v$, by the Lie's equivalence transformations $y = \alpha(x,t)$, $\tau = \beta(t)$, $v = \gamma(x,t)u$, $\alpha_x \neq 0$,

$\beta_t \neq 0$. *If equation* (12) *admits an extension of the principal Lie algebra by one additional operator then the original equation reduces to the form*

$$v_\tau = v_{yy} - Z(y)v \tag{13}$$

with an additional infinitesimal generator given by $\mathbf{w}_2 = \partial_\tau$. *If furthermore* $\mathfrak{g}$ *admits an extension by three additional operator then equation* (12) *reduces to the form*

$$v_\tau = v_{yy} - \frac{A}{y^2}v\ , \tag{14}$$

for which the additional infinitesimal generators are

$$\begin{cases} \mathbf{w}_2 = \partial_\tau \\ \mathbf{w}_3 = 2\tau\partial_\tau + y\partial_y \\ \mathbf{w}_4 = \tau^2\partial_\tau + \tau y\partial_y - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\partial_v \end{cases} .$$

If $\mathfrak{g}$ *admits five additional operator then equation* (12) *can be mapped in the heat equation*

$$v_\tau = v_{yy}\ , \tag{15}$$

with infinitesimal generators

$$\begin{cases} \mathbf{w}_2 = \partial_\tau \\ \mathbf{w}_3 = 2\tau\partial_\tau + y\partial_y \\ \mathbf{w}_4 = \tau^2\partial_\tau + \tau y\partial_y - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\partial_v \\ \mathbf{w}_5 = \partial_y \\ \mathbf{w}_6 = 2\tau\partial_y - yv\partial_v \end{cases} .$$

Equations (13), (14) *and* (15) *provide the canonical forms of all linear parabolic second-order equations* (12) *that admit non-trivial symmetries.*

Our aim is to follow previously sketched procedure to characterize the group of symmetries associated to a particular class of PDE which are related to a widely used financial model, namely the Constant Elasticity of Variance model (CEV), see [3], Section 10.2, for details. In particular, following [9, 10, 11], in Section 3 we first characterize the group of symmetries of certain class of PDE, then we use such a group to explicitly compute the transition density function of the CEV process and the fundamental solution of the associated pricing equation.

3. The Constant Elasticity of Variance (CEV) Model

Empirical analysis of observed option prices, and observed phenomena like the volatility smile, led researchers to formulate option pricing models that included non-constant volatility. In response to observations of an inverse relationship between share price and share price volatility, documented by Fischer Black (1975), the Constant Elasticity of Variance (CEV) model was derived by Cox and Ross in [8]. In fact, the CEV model was derived after a direct request from Fischer Black to John Cox, for a share price evolution model that includes an inverse dependence of volatility and the share price.

The CEV process is a generalization of the CIR model but whereas the CIR is widely used in modelling interest rate the CEV model deals with equity markets and option pricing.

Let $S = \{S_t : t \geq 0\}$ be the stock price governed by the CEV stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t^{\beta/2} dW_t \,, \tag{16}$$

with W_t a standard Wiener process, $\beta < 2$, and $\beta - 2$ is called the *elasticity*, while $\sigma > 0$ is a constant.

The return variance $\zeta(S_t, t)$ with respect to prices S_t has the following relationship

$$\frac{d\zeta(S_t,t)/dS_t}{\zeta(S_t,t)/S_t} = \beta - 2$$

it follows then integrating and exponentiating both side that

$$\zeta(S_t, t) = \sigma^2 S_t^{\beta-2} \,,$$

If $\beta = 2$ then the elasticity is 0 and the prices are lognormally distributed, therefore we recover the standard Black and Scholes (BS) model. If $\beta = 1$ we have the well known Cox-Ingersoll-Ross (CIR) model, see [6] for details. We would like to recall the study by Emanuel & MacBeth for the case $\beta > 2$, see e.g. [12], and we refer to [8, 7] for an extensive treatment of the CEV model.

3.1. CEV Transition Density Function

Since a direct approach to obtain the CEV transition density function is not trivial, we instead proceed transforming the related SDE (16) into the CIR SDE via Itô-Doeblin lemma (ID). We will then use the Lie algebraic approach, developed in section 3.2, to study the CIR process and then transform everything back in order to retrieve the desired results in the CEV framework.

Let us define $f(x) = x^{2-\beta}$. Thus by ID lemma we have

$$\begin{aligned} dX_t &= \frac{\partial f}{\partial S} dS_t + \frac{1}{2}\sigma^2 S_t^\beta \frac{\partial^2 f}{\partial S^2} dt \\ &= \left[(2-\beta)\mu S_t S_t^{1-\beta} + \frac{\sigma^2 S_t^\beta}{2}(2-\beta)(1-\beta)S_t^{-\beta} \right] dt + \sigma S_t^{\frac{\beta}{2}}(2-\beta)S_t^{1-\beta} dW_t \\ &= \left[(2-\beta)\mu S_t^{2-\beta} + \frac{\sigma^2}{2}(2-\beta)(1-\beta) \right] dt + \sigma(2-\beta)S_t^{1-\frac{\beta}{2}} dW_t \\ &= \left[(2-\beta)\mu X_t + \frac{\sigma^2}{2}(2-\beta)(1-\beta) \right] dt + \sigma(2-\beta)\sqrt{X_t} dW_t \, . \end{aligned} \tag{17}$$

In order to recover the standard form of the CIR process we set

$$\begin{cases} \kappa = (\beta - 2)\mu \\ \theta = \frac{\sigma^2}{2\mu}(\beta - 1) \\ \tilde{\sigma} = (2-\beta)\sigma \end{cases} \tag{18}$$

Then see that equation (18) is the equation for a process We have therefore retrieved the process $X = \{X_t : t \geq 0\}$, which is the unique solution to the CIR SDE

$$dX_t = \kappa(\theta - X_t)d\tau + \tilde{\sigma}\sqrt{X_t} dW_t \, . \tag{19}$$

Applying now theorem **??** we can recover the transition density function for the CIR process (19) by inverse Laplace transform, see, e.g. [5, 9, 10, 11] for details. In particular we have

$$p_{CIR}(x, \hat{x}, t) = \frac{2\kappa e^{\kappa(\frac{2\kappa\theta}{\tilde{\sigma}^2}+1)t}}{\tilde{\sigma}^2(e^{\kappa t}-1)} \left(\frac{\hat{x}}{x}\right)^{\frac{\nu}{2}} \exp\left(\frac{-2\kappa(x + e^{\kappa t}\hat{x})}{\tilde{\sigma}^2(e^{\kappa t}-1)}\right) I_\nu\left(\frac{2\kappa\sqrt{x\hat{x}}}{\tilde{\sigma}^2 \sinh(\frac{\kappa t}{2})}\right), \tag{20}$$

with $\nu = \frac{2\kappa\theta}{\tilde{\sigma}^2} - 1$ and

$$I_\nu(x) = \frac{x}{2}^\nu \sum_{n \geq 0} \frac{(x/2)^{2n}}{n!\Gamma(\nu + n + 1)} \, ,$$

a modified Bessel function of the first kind of order ν.

We can now easily invert the original transformation, since it is clearly increasing, in order to obtain the transition density function of the CEV process

(16) to be

$$
\begin{aligned}
p_{CEV}(x,\hat{x},t) &= \left|\frac{\partial f(\hat{x})}{\partial \hat{x}}\right| p_{CIR}(f(x),f(\hat{x}),t) \\
&= \frac{2\mu e^{\mu(2\beta-3)t}}{\sigma^2(e^{(\beta-2)\mu t}-1)}\left(x^{1-\beta}\sqrt{\frac{\hat{x}}{x}}\right) \\
&\exp\left\{\frac{-2\mu(x^{2-\beta}+e^{(\beta-2)\mu t}\hat{x}^{2-\beta})}{\sigma^2(\beta-2)(e^{(\beta-2)\mu t}-1)}\right\} I_{\frac{1}{2-\beta}}\left(\frac{2\mu(x\hat{x})^{\frac{2-\beta}{2}}}{\sigma^2(\beta-2)\sinh(\frac{(\beta-2)\mu t}{2})}\right) .
\end{aligned}
\tag{21}
$$

3.2. CEV Option Pricing Model

Let be $S=\{S_t : t\geq 0\}$ the stock price driven by the CEV stochastic differential equation (16). Let Π_t indicates a portfolio containing the option to be priced at time t, while $u(S_t)$ shall be the value of the option in the portfolio Π_t and Δ will stand for the quantity of the stock in Π_t at time t. We thus have

$$\Pi_t = u - \Delta S_t .$$

An infinitesimal change in the portfolio in a time interval dt, setting $\Delta = \frac{\partial u}{\partial S}$ in order to get rid of the stochastic terms, is given by

$$d\Pi_t = \left\{\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}dS + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}dS^2 - \frac{\partial u}{\partial S}dS\right\} = \left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^\beta\right) dt . \tag{22}$$

Since no stochastic terms appear on the right hand side of equation (22), the value of the portfolio is certain. Thus in order to preclude arbitrage the payoff must be equal to $\Pi_t r dt$, then we get the equation

$$\left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^\beta\right) dt = \left(u - \frac{\partial u}{\partial S}S\right) r dt .$$

From now on we will use the more compact notation $f_x := \frac{\partial f}{\partial x}$ for any function f.

We thus have the following PDE

$$u_t + Sru_S + \frac{\sigma^2}{2}S^\beta u_{SS} - ru = 0 \tag{23}$$

If for instance we have to price a call option $u(S_t,t) := C(S_t,t)$, we prescribe the boundary condition $C_T := \max(S_T - K, 0)$, K being the related *strike price*.

We thus have to solve the following PDE

$$\begin{cases} C_t + srC_S + \frac{1}{2}C_{SS}\sigma^2 S^\beta - rC = 0 \\ C_T = \max(S_T - K, 0) \end{cases} .$$

The standard approach in literature is to rewrite everything in terms of the logarithm of the spot price, i.e. setting $x = ln(S)$, so that one has to evaluate the following

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{1}{S}, \quad \frac{\partial^2 C}{\partial S}^2 = \frac{1}{S^2}\frac{\partial^2 C}{\partial x^2} - \frac{1}{S}\frac{\partial C}{\partial x} .$$

Equation (23) can thus be rewritten as

$$\begin{cases} C_t + \left(r - \frac{1}{2}\sigma^2\right) C_x + \frac{1}{2}C_{xx}\sigma^2 e^{x(\beta-2)} - rC = 0 \\ C_T = \max(e^{xT} - K, 0) \end{cases} .$$

Then, in analogy with the BS approach, a solution of the form

$$C_t(x,t) = e^x P_1(x,t) - Ke^{-r(T-t)}P_2(x,t)$$

is guessed, see [4] for details.

We will now follow a different approach. As done previously, we will transform the CEV process (16) into the CIR process (19). Avoiding all the tedious algebraic computation, we will start considering the process $X = \{X_t : t \geq 0\}$ satisfying the CIR stochastic differential equation (19).

We again define a portfolio $\Pi_t := u - \Delta X$, and we change the notation for the time variable setting $t := T - t$, in order to have the backward Cauchy problem, therefore we are left with the following problem

$$u_t = \frac{\sigma^2}{2}xu_{xx} + xru_x - ru \, , \tag{24}$$

and we compute the *infinitesimal generators* spanning the Lie algebra admitted by eq.(24).

Proposition 3.1. *Eq.* (24) *admits a four dimensional Lie-algebra spanned by*

$$\begin{cases} \mathbf{v}_1 = e^{rt}x\partial_x + \frac{e^{rt}}{r}\partial_t + (\frac{2rx+\sigma^2}{\sigma^2})e^{rt}u\partial_u \\ \mathbf{v}_2 = e^{-rt}x\partial_x - \frac{e^{-rt}}{r}\partial_t + e^{-rt}u\partial_u \\ \mathbf{v}_3 = \partial_t \\ \mathbf{v}_4 = u\partial_u \end{cases} \tag{25}$$

Proof. The general *infinitesimal generator* of eq. (24) are of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \tilde{\nu}(x,t,u)\frac{\partial}{\partial u}\,, \tag{26}$$

with ξ, τ and $\tilde{\nu}$ some smooth functions. Applying thus the theory briefly introduced in Sec. 2 and deeply treated in [2, 18] we find seven determining equations, namely

$$\begin{aligned}
\xi_u &= 0\\
\tau_u &= 0\\
\tilde{\nu}_{uu} &= 0\\
\tau_x &= 0\\
2x\sigma^2\tilde{\nu}_{x,u} + 2\xi_t - 2rx\xi_x - x\sigma^2\xi_{x,x} + 2rx\tau_t + 2r\xi &= 0\\
2\tilde{\nu}_t - 2ru\tilde{\nu}_u - 2rx\tilde{\nu}_x - x\sigma^2\tilde{\nu}_{x,x} + 2ru\tau_t + 2r\tilde{\nu} &= 0\\
-2x\xi_x + x\tau_t + \xi &= 0
\end{aligned}$$

Solving previous system for ξ, τ and $\tilde{\nu}$ we retrieve the most general form of the *infinitesimal generator* for eq. (26), namely

$$\begin{cases}
\xi(x,t) = e^{-rt}(e^{2rt}k_1 + k_2)x\\
\tau(x,t) = \frac{e^{rt}k_1 - e^{-rt}k_2 + k_3 r}{r}\\
\nu(x,t) = e^{-rt}k_2 + k_4 - \frac{e^{rt}k_1(2rx+\sigma^2)}{\sigma^2} + \beta(x,t)
\end{cases}, \tag{27}$$

where $k_i \in \mathbb{R}$, $i = 1,2,3,4$, are some constants to be chosen at will, $\tilde{\nu}(x,t,u) = \nu(x,t)u$ and $\beta(x,t)$ is a general solution of equation (24). Choosing thus for a fixed $k_i = 1$ $k_j = 0$ for any $j \neq i$, $i = 1,\dots,4$, we can determine from system (27) a basis for the Lie algebra to be

$$\begin{cases}
\mathbf{v}_1 = e^{rt}x\partial_x + \frac{e^{rt}}{r}\partial_t + (\frac{2rx+\sigma^2}{\sigma^2})e^{rt}u\partial_u\\
\mathbf{v}_2 = e^{-rt}x\partial_x - \frac{e^{-rt}}{r}\partial_t + e^{-rt}u\partial_u\\
\mathbf{v}_3 = \partial_t\\
\mathbf{v}_4 = u\partial_u
\end{cases} \tag{28}$$

Computing now the *Lie brackets* of the basis (25) spanning the *Lie algebra* admitted by equation (24) we can find the commutator table 1.

□

Standard approach is to retrieve the fundamental solution via invariant method, see, e.g. [2] for a detailed introduction and [5, 9, 10, 19, 20] for application with particular attention to finance. We will instead use a different

Table 1: Commutator table of eq. (24)

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	$-\frac{2\mathbf{v}_3}{r} + 2\mathbf{v}_4$	$r\mathbf{v}1$	0
$\mathbf{v}_2$	$\frac{2\mathbf{v}_3}{r} - 2\mathbf{v}_4$	0	$-r\mathbf{v}2$	0
$\mathbf{v}_3$	$-r\mathbf{v}_1$	$r\mathbf{v}_2$	0	0
$\mathbf{v}_4$	0	0	0	0

approach introduced in section 2 in order to retrieve the fundamental solution of equation (23).

In order to achieve such a goal we transform the CEV process (16) into the CIR process (19). Then we recover a fundamental solution of the backward pricing equation (24). We have already proved in Prop. 3.1 that eq. (24) admits a four dimensional Lie algebra spanned by the *infinitesimal generators* (25), with commutator table given in (1).

Recalling theorem 2.2 we can find a one-to-one mapping to the partial differential equation

$$v_y = v_{\tau\tau} - \frac{A}{y^2} v \,, \tag{29}$$

whose Lie algebra is spanned by

$$\begin{cases} \mathbf{w}_1 = \tau y \partial_y + \tau^2 \partial_\tau - \left(\frac{1}{4} y^2 + \frac{1}{2}\tau\right) v \partial_v \\ \mathbf{w}_2 = \partial_\tau \\ \mathbf{w}_3 = y\partial_y + 2\tau\partial_\tau \\ \mathbf{w}_4 = v\partial_v \end{cases} , \tag{30}$$

with commutator table

Table 2: Commutator table of eq. (29)

	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_4$
$\mathbf{w}_1$	0	$-\mathbf{w}_3 + \frac{1}{2}\mathbf{w}_4$	$-2\mathbf{w}_1$	0
$\mathbf{w}_2$	$\mathbf{w}_3 - \frac{1}{2}\mathbf{w}_4$	0	$2\mathbf{w}_2$	0
$\mathbf{w}_3$	$2\mathbf{w}_1$	$-2\mathbf{w}_2$	0	0
$\mathbf{w}_4$	0	0	0	0

Proposition 3.2. *The fundamental solution of the CEV pricing equation* (23)

$$u_t + Sru_S + \frac{\sigma^2}{2} S^\beta u_{SS} - ru = 0 \tag{31}$$

is given by

$$\begin{aligned} p(S,\hat{S},t) =& \frac{2r\hat{S}^{1-\beta}}{\sigma^2(2-\beta)} e^{-\frac{3}{4}rt} \\ & \exp\{-\frac{2r}{\sigma^2(2-\beta)^2}(S^{2-\beta}+\hat{S}^{2-\beta})\} I_1\left(\frac{4r}{\sigma^2(2-\beta)^2}(S\hat{S})^{\frac{2-\beta}{2}}\right) . \end{aligned} \tag{32}$$

Before proving Prop. 3.2 we need a lemma that gives us the explicit transformation that connect eq. (29) to eq. (24).

Lemma 3.3. *The mapping*

$$\begin{cases} x = \frac{y^2\sigma^2}{8r\tau} \,, \\ t = \frac{\log(\frac{8r\tau}{\sigma^2})}{r} \,, \\ u(x,t) = \frac{\sqrt{y}}{\tau} v(y,\tau) \,, \end{cases} . \tag{33}$$

transforms eq. (24) *into the target PDE*

$$v_y = v_{\tau\tau} - \frac{3}{4y^2} v \,, \tag{34}$$

whose Lie algebra is spanned by eq. (30).

Proof. Let us notice that if we consider the Lie subalgebra $\mathfrak{h}^{(23)}$ spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$ and we take the scaling $\mathbf{w}_i = \alpha_i \tilde{\mathbf{w}}_i$, with $\alpha_i \in \mathbb{R}$, $i = 2,3$, the commutators arising from $\{\tilde{\mathbf{w}}_i\}_i$ are the same as the commutators in table (1) arising from $\{\mathbf{v}_i\}_i$, therefore we take

$$\alpha_2 = -\frac{8r}{\sigma^2}, \quad \alpha_3 = \frac{2}{r} \,.$$

The latter fact suggests to search for a one-to-one point transformation mapping of the PDE (24) to the target PDE (29). In particular we look for a transformation of the form

$$\begin{cases} y = \alpha(x,t) \\ \tau = \beta(t) \\ v = \gamma(x,t)u \end{cases} ,$$

such a transformation has to satisfy conditions

$$\mathbf{v}_j\alpha(x,t) = \tilde{\mathbf{w}}_j y|_{(y,\tau,v)=(\alpha,\beta,\gamma)} \;,\; \mathbf{v}_j\beta(t) = \tilde{\mathbf{w}}_j \tau|_{(y,\tau,v)=(\alpha,\beta,\gamma)} \;, \tag{35}$$

with $j = 2, 3$. System (35) reads as follows

$$\begin{aligned} e^{-rt}x\alpha_x + \frac{e^{+rt}}{r}\alpha_t = 0 \\ \alpha_t = \frac{r}{2}\alpha \\ -\frac{e^{-rt}}{r}\beta_t = -\frac{\sigma^2}{8r} \\ \beta_t = r\beta \end{aligned} \;,$$

ant it has the following solutions

$$\begin{cases} y = \alpha(x,t) = \exp\{\frac{rt}{2}\}\sqrt{x} \\ \tau = \beta(t) = \frac{\sigma^2}{8r}\exp\{rt\} \end{cases} . \tag{36}$$

We have to determine the parameter A appearing in equation (29). We first of all apply the change of variables (36) in order to have

$$\begin{cases} x = \frac{y^2\sigma^2}{8r\tau} \\ t = \frac{\log(\frac{8r\tau}{\sigma^2})}{r} \\ u(x,t) = \tilde{v}(y(x,t),\tau(x,t)) = \tilde{v}\left(\exp\{\frac{rt}{2}\}\sqrt{x}, \exp\{rt\}\frac{\sigma^2}{8r}\right) \end{cases} .$$

By the standard chain rule we can express the partial derivatives of u w.r.t. variables (x,t) in term of partial derivative of $\tilde{v}$ w.r.t. variables (y,τ) as

$$\begin{cases} u_t = \sqrt{x}\frac{r}{2}\exp\{\frac{rt}{2}\}\tilde{v}_y + \frac{\sigma^2}{8}\exp\{rt\}\tilde{v}_\tau \\ u_x = \frac{1}{2\sqrt{x}}\exp\{\frac{rt}{2}\}\tilde{v}_y \\ u_{xx} = \frac{1}{4x}\exp\{rt\}\tilde{v}_{yy} - \frac{1}{4x^{\frac{3}{2}}}\exp\{\frac{rt}{2}\}\tilde{v}_y \end{cases} . \tag{37}$$

Substituting now equations (37) into equation (23) we get the following

$$\tilde{v}_\tau = \tilde{v}_{yy} - \frac{1}{y}\tilde{v}_y - \frac{1}{\tau}\tilde{v} \,. \tag{38}$$

Then, making use of the following change of variable

$$\tilde{v}(y,\tau) = \frac{\sqrt{y}}{\tau}v(y,\tau) \,,$$

and computing again the partial derivatives, we get

$$\begin{cases} \tilde{v}_\tau = -\frac{\sqrt{y}}{\tau^2}v + \frac{\sqrt{y}}{\tau}v_\tau \\ \tilde{v}_y = \frac{1}{2\tau\sqrt{y}}v + \frac{\sqrt{y}}{\tau}v_y \\ \tilde{v}_{yy} = \frac{1}{4\tau y^{\frac{3}{4}}}v + \frac{1}{\tau\sqrt{y}}v_y + \frac{\sqrt{y}}{\tau}v_{yy} \end{cases} . \tag{39}$$

Substituting thus system (39) into equation (38) we thus obtain

$$v_\tau = v_{yy} - \frac{3}{4y^2}v \,. \tag{40}$$

□

proof of Prop. (3.1). Exploiting lemma 3.3 we have that eq. (24) can be transformed into eq. (30). In particular equation (40) has a fundamental solution of the form

$$q(y,\hat{y},\tau) = \frac{\sqrt{y\hat{y}}}{2\tau}\exp\{-\frac{y^2+\hat{y}^2}{4\tau}\}I_1\left(\frac{y\hat{y}}{2\tau}\right) , \tag{41}$$

see, e.g. [9, 11, 15] for details. Hence exploiting the inverse transformation of eq. (33) we can retrieve the fundamental solution of equation (24) which reads as follows

$$p(x,\hat{x},t) = \frac{2r}{\sigma^2}e^{-\frac{3}{4}rt}\exp\{-\frac{2r}{\sigma^2}(x+\hat{x})\}I_1\left(\frac{4r}{\sigma^2}\sqrt{x\hat{x}}\right) .$$

Exploiting the transformation that maps the CIR process into the CEV process introduced in Sec. 3.1 we have that the fundamental solution of the CEV pricing equation (23) is given by

$$\begin{aligned} p(S,\hat{S},t) =& \frac{2r\hat{S}^{1-\beta}}{\sigma^2(2-\beta)}e^{-\frac{3}{4}rt} \\ & \exp\{-\frac{2r}{\sigma^2(2-\beta)^2}(S^{2-\beta}+\hat{S}^{2-\beta})\}I_1\left(\frac{4r}{\sigma^2(2-\beta)^2}(S\hat{S})^{\frac{2-\beta}{2}}\right) . \end{aligned} \tag{42}$$

□

4. Conclusion and Future Development

We have performed a Lie classification of some particular linear second order differential equations arising in mathematical finance in connection with the stochastic differential equation defining the CEV model. We were able to retrieve, via Lie symmetry method, the transition density function of the related stochastic process. In particular, we have likewise used two classical approach in Lie's theory in order to explicitly find a fundamental solution for the CEV pricing equation. Much has been done in the study of second order linear differential equation arising in finance, in particular we refer to [9, 10, 11] for integrating symmetries and probability transition density function of particular models. For an application of Lie symmetry method to interest rate models we refer to [19, 20]. Eventually for an extensive treatment of mapping relating differential equations we refer to [17, 15].

Anyway little attention has been given to stochastic volatility models arising in finance, e.g. the *stochastic alpha, beta, rho* (SABR) model which is used to capture the volatility smile in derivatives markets. These models present a different level of mathematical problems, compared to those with non stochastic volatility, nevertheless methods presented in this paper can be still applied and this is one of the topic of our future work on the subject.

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