Moving energies as first integrals of nonholonomic systems with affine constraints

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Abstract

In nonholonomic mechanical systems with constraints that are affine (linear nonhomogeneous) functions of the velocities, the energy is typically not a first integral. It was shown in [17] that, nevertheless, there exist modifications of the energy, called there moving energies, which under suitable conditions are first integrals. The first goal of this paper is to study the properties of these functions and the conditions that lead to their conservation. In particular, we enlarge the class of moving energies considered in [17]. The second goal of the paper is to demonstrate the relevance of moving energies in nonholonomic mechanics. We show that certain first integrals of some well known systems (the affine Veselova and LR systems), which had been detected on a case-by-case way, are instances of moving energies. Moreover, we determine conserved moving energies for a class of affine systems on Lie groups that include the LR systems, for a heavy convex rigid body that rolls without slipping on a uniformly rotating plane, and for an n-dimensional generalization of the Chaplygin sphere problem to a uniformly rotating hyperplane.

Keywords: Moving energies · Nonholonomic mechanical systems · Affine constraints · Nonhomogeneous constraints · Conservation of energy · LR systems · Rolling rigid bodies · Veselova system

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1 Introduction

1.1 Moving energies. Conservation of energy in time-independent mechanical systems with nonholonomic constraints is an important feature that has received extended consideration. It is well known that the energy is conserved if the nonholonomic constraints are linear (or more generally homogeneous) functions of the velocities [33, 30] and that this typically does not happen if the constraints are arbitrary nonlinear functions of the velocities (see e.g. [3, 29, 28]). The situation is better understood in the case of systems with nonholonomic constraints that are *affine* (namely, linear non-homogeneous) functions of the velocities. This case is important in mechanics because it is encountered in systems formed by rigid bodies that roll without sliding on surfaces that move in a preassigned way; instances of these systems have been considered, e.g., by Routh [34] and more recently in [5]. This is the case that we consider in this paper.

The conditions under which the energy is conserved in nonholonomic systems with affine constraints have been clarified in [16], and are very special. However, it was noticed and proved

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in [17] that, in such systems, when the energy is not conserved, there may exist modifications of it which are conserved. Such functions were called *moving energies* in [17] because they were there constructed by means of time-dependent changes of coordinates that transform the nonholonomic system with affine constraints into a nonholonomic system with linear constraints. If time-independent, the transformed system has a conserved energy, and the moving energy is the pull-back of this function to the original coordinates—namely, the energy of the transformed system written in the original coordinates. This moving energy is always a conserved function for the original system, but the interesting case is when it is time-independent. In [17], the time-independence of the moving energy was linked to the presence of symmetries of the system. In such a case the conserved moving energy is the sum of two non-conserved functions: one is the energy, and the other is the momentum of an infinitesimal generator of the symmetry group.

This mechanism has an elementary mechanical interpretation for systems that consist of rigid bodies constrained to roll without sliding on moving surfaces: here the moving energy is the energy of the system relative to a moving reference frame in which the surface is at rest—so that the no-sliding constraint is linear—written however in the original coordinates. Most of the examples of moving energies produced so far [17, 6] are indeed of this type.[1]

The existence of this energy-like first integral may play an important role. Reference [17] proved its existence (without however determining it) in the system formed by a heavy homogeneous sphere that rolls without sliding inside an upward convex surface of revolution that rotates uniformly. The existence of this integral, together with that of two other first integrals that had been previously determined in [5], implies the integrability of this system, in the sense of quasi-periodicity of the dynamics [17, 9]. The expression of the moving energy for a homogeneous sphere that rolls on an arbitrary rotating surface was given in [6], and there used to show integrability by the Euler-Jacobi theorem (which is weaker than quasi-periodicity) in the case of axisymmetric surfaces. Other systems, whose conserved moving energy is given in [6], include the Chaplygin sleigh on a rotating plane and the special case of the affine Suslov problem considered in [24] where the axis of forbidden rotations is also an axis of symmetry of the body. The authors of [6] also remark that a known first integral of the Veselova system [37, 35] is an instance of a moving energy.

It is therefore important to understand how general and effective this mechanism can be, and the aim of the present paper is to investigate this question.

1.2 Aim of the paper. First, we will extend in a natural way the notion of moving energy, going beyond the relation to a time-dependent change of coordinates. Specifically, we define here a moving energy as the difference between the energy of the system and the momentum of a vector field defined on the configuration manifold of the system. This will allow a clearer, simpler and more general treatment.

We will investigate which vector fields produce a conserved moving energy (Proposition 4) and how this relates to the existence of symmetries of the Lagrangian (Corollary 5).[2] Then, we will investigate some properties of moving energies, including their nonuniqueness (Propositions 6 and 7).

We will also compare this extended notion of moving energy to the one considered so far and described above. We will say that a moving energy is *kinematically interepretable* if it arises

[1] Reference [6], that refers to the moving energy as Jacobi integral (see also Section 2.3), claims that this mechanism can be extended to less symmetrical situations. It is unclear to us to which extent this goal can be achieved: without symmetry, the moving energy exists but is usually time-dependent. For more precise comments see footnote nr. 2 of [16]. We note that while in [17] and in the present article the moving energy is regarded as a first integral of the original system, and is therefore written in the original coordinates, in [6] it is written in the new, time-dependent coordinates.

[2] While the writing of this article was almost completed, we were informed of the existence of the very recent article [27]. Following the approach in [26], this article considers moving energies from the point of view of Noether symmetries for time-dependent systems—calling them Noether integrals—and generalizes to this context some of the results of [17]; in particular, it proves a statement analogous to Corollary 5.

from a time-dependent change of coordinates, as in the mechanism described above, and we will characterize the moving energies which are kinematically interpretable (Proposition 8).

Next, we will show that certain known first integrals of some important nonholonomic systems with affine constraints are instances of moving energies. Specifically, we will consider a class of affine nonholonomic systems on Lie groups which includes the affine Veselova system [37, 35] and the more general affine LR systems introduced in [36].

Finally, we will determine the explicit form of a conserved moving energy for two other important nonholonomic systems, for which the existence of an energy-like first integral was so far unknown: a convex body that rolls on a rotating plane and the n-dimensional Chaplygin sphere that rolls without slipping on a rotating hyperplane.

We will also indicate some dynamical consequences of their existence (Corollary 16) and remark their usage for the Hamiltonization of reduced systems (see the Remark at the end of section 5.2).

The resulting picture is that the notion of moving energy is a unifying concept in the study of nonholonomic systems with affine constraints. Our view is that this class of functions—rather than the energy itself—should be considered the primary 'energy-like' first integrals to be considered in these systems.

1.3 Outline of the paper. In section 2 we recall the general framework for mechanical systems with affine nonholonomic constraints and review some of their properties, that are needed in the subsequent study. In particular, given the role played by symmetry in the conservation of moving energies, we give there a 'Noether theorem' for nonholonomic systems with affine constraints that extends previous formulations (Proposition 2). In section 3 we introduce the moving energies, give conditions for their conservation, and analyze some of their properties. In section 4 we investigate the relation between the notion of moving energy introduced in section 3 and the original definition of [17].

Sections 5, 6 and 7 are devoted to show that the aforementioned examples possess a moving energy and to compute it explicitly. We point out that this analysis only requires the definitions presented in section 2 and the results of subsection 3.1. In fact, the reader who is interested in applying the methods of this paper to specific examples of nonholonomic systems with affine constraints need only concentrate on these sections.

The Appendix is devoted to some additional material relative to the two systems studied in Sections 6 and 7. In both cases we identify a symmetry group and obtain the reduced equations of motion.

Throughout the work, we assume that all objects (functions, manifolds, distributions, etc.) are smooth and that all vector fields are complete. If $\mathcal{E}$ is a distribution over a manifold Q, then $\Gamma(\mathcal{E})$ denotes the space of sections of $\mathcal{E}$.

2 Nonholonomic systems with affine constraints

2.1 The setting. We start with a Lagrangian system with n-dimensional configuration manifold Q and Lagrangian $L : TQ \to \mathbb{R}$, that describes a holonomic mechanical system. We assume that the Lagrangian has the mechanical form

$$L = T + b - V \circ \pi \,, \tag{1}$$

where T is a Riemannian metric on Q, b is a 1-form on Q regarded as a function on TQ, V is a function on Q and $\pi : TQ \to Q$ is the tangent bundle projection. We interpret T as the kinetic energy, V as the potential energy of the positional forces that act on the system, and the 1-form b as the generalized potential of the gyrostatic forces that act on the system.

We add now the nonholonomic constraint that, at each point $q \in Q$, the velocities of the system belong to an affine subspace $\mathcal{M}_q$ of the tangent space T_qQ. Specifically, we assume that there are a nonintegrable distribution $\mathcal{D}$ on Q of constant rank r, with $1 < r < n$, and a vector field Z on Q such that, at each point $q \in Q$,

$$\mathcal{M}_q = Z(q) + \mathcal{D}_q \,. \tag{2}$$

Note that the vector field Z is defined up to a section of $\mathcal{D}$. The affine distribution $\mathcal{M}$ with fibers $\mathcal{M}_q$ may also be regarded as a submanifold $M \subset TQ$ of dimension $n + r$. This submanifold is an affine subbundle of TQ of rank r and will be called the *constraint manifold*. The case of linear constraints is recovered when the vector field Z is *horizontal* (namely, it is a section of the distribution $\mathcal{D}$), since then $\mathcal{M} = \mathcal{D}$.

We assume that the nonholonomic constraint is ideal, namely, that it satisfies d'Alembert principle: when the system is in a configuration $q \in Q$, then the set of reaction forces that the nonholonomic constraint is capable of exerting coincides with the annihilator $\mathcal{D}_q^\circ$ of $\mathcal{D}_q$ (see e.g. [32, 29]). Under this hypothesis there is a unique function $R : M \to \mathcal{D}^\circ$, which is interpreted as associating an ideal reaction force $R(v_q)$ to each constrained kinematic state $v_q \in M$, with the property that the equations of motion of the system are given by the restriction to M of Lagrange equations with reaction forces R; for a detailed proof, see [16]. We will denote $(L, Q, \mathcal{M})$ the nonholonomic system determined by these data.

In bundle coordinates $(q, \dot q)$ on TQ the Lagrangian L has the form

$$L(q, \dot q) = \frac{1}{2} \dot q \cdot A(q) \dot q + b(q) \cdot \dot q - V(q) \tag{3}$$

with $A(q)$ an $n \times n$ symmetric nonsingular matrix and $b(q) \in \mathbb{R}^n$. (In order to keep the notation to a minimum we do not distinguish between global objects and their coordinate representatives). Here, and in all expressions written in coordinates, the dot denotes the standard scalar product in $\mathbb{R}^n$. In bundle coordinates, the fibers of the distribution $\mathcal{D}$ can be described as the null spaces of a q-dependent $k \times n$ matrix $S(q)$ that has everywhere rank k, with $k = n - r$:

$$\mathcal{D}_q = \{\dot q \in T_q Q \,:\, S(q)\dot q = 0\} \,.$$

In turn $\mathcal{M}_q = \{\dot q \in T_q Q \,:\, S(q)(\dot q - Z(q)) = 0\}$ and

$$M = \big\{(q, \dot q) \,:\, S(q)\dot q + s(q) = 0\big\}$$

with

$$s(q) = -S(q) Z(q) \in \mathbb{R}^n \,.$$

In coordinates, the equations of motion of the nonholonomic mechanical system $(L, Q, \mathcal{M})$ are

$$\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}\Big)\Big|_M \;=\; R|_M \tag{4}$$

with

$$R = S^T (S A^{-1} S^T)^{-1} (S A^{-1} \ell - \sigma) \tag{5}$$

where $\ell \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^k$ have components

$$\ell_i = \sum_{j=1}^n \frac{\partial^2 L}{\partial \dot q_i \partial q_j} \dot q_j - \frac{\partial L}{\partial q_i} \,, \qquad \sigma_a = \sum_{i,j=1}^n \frac{\partial S_{ai}}{\partial q_j} \dot q_i \dot q_j + \sum_{j=1}^n \frac{\partial s_a}{\partial q_j} \dot q_j \tag{6}$$

$(i = 1, \ldots, n,\ a = 1, \ldots, k)$. For details see [16]; in the case of linear constraints, these or analogue expressions are given in [1, 2, 14]. We note that the restriction of R to M is independent of the arbitrariness that affects the choices of the vector field Z, of the matrix S and of the vector s, see [16].

2.2 The reaction-annihilator distribution. We need to introduce now the so-called reaction-annihilator distribution $\mathcal{R}^\circ$, from [14, 16]. This object plays a central role in the conservation of energy and of moving energies of nonholonomic systems with affine constraints [16, 17] (as well as in the conservation of momenta in nonholonomic systems with either linear or affine constraints [14, 16]).

The observation underlying the consideration of this object is that, while the condition of ideality assumes that, at each point $q \in Q$, the constraint is capable of exerting all reaction forces that lie in $\mathcal{D}_q^\circ$, expression (5) shows that, ordinarily, only a subset of these possible reaction forces is actually exerted in the motions of the system. Specifically, in bundle coordinates, $\mathcal{D}_q^\circ$ is the orthogonal complement to $\ker S(q)$, namely the range of $S(q)^T$, but the map

$$S^T(SA^{-1}S^T)^{-1}(SA^{-1}\ell - \sigma)\big|_{M_q} \;:\; M_q \to \mathrm{range}\big[S(q)^T\big]$$

need not be surjective. Instead, the reaction forces that the constraint exerts, when the system $(L, Q, \mathcal{M})$ is in a configuration $q \in Q$ with any possible velocity in $\mathcal{M}_q$, are the elements of the set

$$\mathcal{R}_q := \bigcup_{v_q \in \mathcal{M}_q} R(v_q)\,,$$

which is a subset of $\mathcal{D}_q^\circ$—and typically a proper subset of it. For instance, an extreme case is that of a heavy homogeneous sphere that rolls without sliding on a steady horizontal plane: all motions consist of the ball rolling with constant linear and angular velocitiy, as in the system without the nonholonomic constraint [33, 30], and hence $\mathcal{R}_q = \{0\}$ at all points q.

The *reaction-annihilator distribution* $\mathcal{R}^\circ$ of the nonholonomic system $(L, Q, \mathcal{M})$ is the distribution on Q whose fiber $\mathcal{R}_q^\circ$ at $q \in Q$ is the annihilator of $\mathcal{R}_q$. In other words, a vector field Y on Q is a section of $\mathcal{R}^\circ$ if and only if, *in all constrained kinematic states* of the the system, the reaction force does no work on it, namely[3]

$$\langle R(v_q), Y(q)\rangle = 0 \qquad \forall v_q \in T_qM,\ q \in Q\,.$$

This is a system-dependent condition, which is weaker than being a section of $\mathcal{D}$ because

$$\mathcal{D}_q \subseteq \mathcal{R}_q^\circ \qquad \forall\, q \in Q\,.$$

For further details and examples on the reaction-annihilator distribution see [14, 11, 12, 25, 16, 27] and for a discussion of its relation to d'Alembert principle see [15].

2.3 Conservation of energy. The *energy* of the nonholonomic system $(L, Q, \mathcal{M})$ is the restriction $E_L|_M$ to the constraint manifold M of the energy

$$E_L := \langle p, \cdot\rangle - L$$

of the Lagrangian system (L, Q). Here $p : TQ \to \mathbb{R}$ is the momentum 1-form generated by the Lagrangian L, regarded as a function on TQ. If the Lagrangian is of the form (1), then in coordinates $p = \frac{\partial L}{\partial \dot q} = A(q)\dot q + b(q)$, and

$$E_L = T + V \circ \pi\,.$$

We note that, in Lagrangian mechanics, the function E_L is variously called energy, generalized energy, Jacobi integral, Jacobi-Painlevé integral, sometimes with slightly different meanings attached to each of these terms (see the discussion in a remark in section 1.1 of [16]). We simply call it energy.

As we have already mentioned, it is well known that the energy is always conserved if the constraints are linear in the velocities. For affine constraints, with constraint distribution as in (2), the situation is as follows:

[3] Here and in the sequel, except in Section 7.2, $\langle\ ,\ \rangle$ denotes the cotangent-tangent pairing.

Proposition 1. [16] *The energy of* $(L, Q, \mathcal{M} = Z + \mathcal{D})$ *is conserved if and only if* $Z \in \Gamma(\mathcal{R}^\circ)$.

Thus, energy conservation is not a universal property of nonholonomic systems with affine constraints. Instead, it is a system-dependent property. In particular, note that $\mathcal{R}^\circ$ depends on the potentials b and V that enter the Lagrangian, see (6). Therefore, changing the (active) forces that act on the system—even within the class of gyrostatic and conservative forces—may destroy or restore the conservation of energy. For some examples of this phenomenon, which includes e.g. a sphere that rolls inside a rotating cylinder, see [16]. An extension of Proposition 1 to a time-dependent setting is given in Corollary 4.2 of [27].

2.4 Conservation of momenta of vector fields. We conclude this short panoramic of nonholonomic systems with affine constraints with some results on the conservation of momenta of vector fields and of lifted actions. Even though we will not strictly need these results in the sequel, they will be useful for appreciating certain aspects of the conservation of moving energies. Moreover, we will introduce here some notation and terminology which will be used throughout this work.

Given a nonholonomic system $(L, Q, \mathcal{M})$, we define the *momentum* of a vector field Y on Q as the restriction to M of the function

$$J_Y := \langle p, Y \rangle : TQ \to \mathbb{R}$$

(in coordinates, $J_Y(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \cdot Y(q)$). A geometric characterization of the vector fields whose momenta are first integrals of a nonholonomic system $(L, Q, \mathcal{M})$ with affine constraints does not exist. However, just as in the case of systems with linear constraints, see Proposition 2 of [14], we may characterize those among them which have another property as well.

Here and everywhere in the sequel we denote by Y^{TQ} the tangent lift of a vector field Y on a manifold Q, namely the vector field on TQ which, in bundle coordinates, is given by $Y^{TQ} = \sum_i Y_i \partial_{q_i} + \sum_{ij} \dot{q}_j \frac{\partial Y_i}{\partial q_j} \partial_{\dot{q}_i}$.

Proposition 2. *Any two of the following three conditions imply the third:*

i. $Y \in \Gamma(\mathcal{R}^\circ)$.

ii. $\hat{Y}(L)|_M = 0$.

iii. $J_Y|_M$ *is a first integral of* $(L, Q, \mathcal{M})$.

Proof. We may work in coordinates. It is understood that all functions are evaluated in M, and time derivatives are along the flow of the equations of motion (4). Compute

$$\frac{dJ_Y}{dt} = \sum_i \frac{dp_i}{dt} Y_i + \sum_{ij} p_i \dot{q}_j \frac{\partial Y_i}{\partial q_j} = \sum_i \big(\frac{\partial L}{\partial q_i} + R_i\big) Y_i + \sum_{ij} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_j \frac{\partial Y_i}{\partial q_j} = \hat{Y}(L) + R \cdot Y\,. \qquad (7)$$

From this it follows that, at each point $q \in Q$, the vanishing in all of $\mathcal{M}_q$ of any two among $\frac{dJ_Y}{dt}$, $R \cdot Y$ and $\hat{Y}(L)$ implies the vanishing in all of $\mathcal{M}_q$ of the third. But the vanishing in all of $\mathcal{M}_q$ of $R \cdot Y$ is equivalent to the fact that Y belongs to the fiber at q of $\mathcal{R}^\circ$. □

Let now Ψ be an action of a Lie group G on Q. For each $g \in G$ we write as usual $\Psi_g(q)$ for $\Psi(g, q)$. The infinitesimal generator relative to an element $\xi \in \mathfrak{g}$, the Lie algebra of G, is the vector field

$$Y_\xi := \frac{d}{dt} \Psi_{\exp(t\xi)}\big|_{t=0}$$

on Q. The tangent lift Ψ^{TQ} of Ψ is the action of G on TQ given by $\Psi_g^{TQ}(v_q) = T_q \Psi_g \cdot v_q$ for all $v_q \in TQ$ (in coordinates, $\Psi_g^{TQ}(q, \dot{q}) = \big(\Psi_g(q), \Psi'_g(q)\dot{q}\big)$ with $\Psi'_g = \frac{\partial \Psi_g}{\partial q}$). For any $\xi \in \mathfrak{g}$, the ξ-component of the momentum map of Ψ^{TQ} is the momentum J_{Y_ξ} of Y_ξ.

The following consequence of Proposition 2 extends a result in [16] and is a possible statement of a 'Nonholonomic Noether theorem' for nonholonomic systems with affine constraints:

Corollary 3. *Assume that L is invariant under Ψ^{TQ}, namely $L \circ \Psi^{TQ}_g = L$ for all $g \in G$. Given $\xi \in \mathfrak{g}$, $J_{Y_\xi}|_M$ is a first integral of $(L, Q, \mathcal{M})$ if and only if $Y_\xi \in \Gamma(\mathcal{R}^\circ)$.*

3 Moving energies

3.1 Definition and conservation. In all of this section, $(L, Q, \mathcal{M} = Z+\mathcal{D})$ is a nonholonomic system with affine constraints and we freely use the notation and the terminology introduced in the previous section. For any vector field Y on Q define

$$E_{L,Y} := E_L - \langle p, Y \rangle$$

(in coordinates, $E_{L,Y} = E_L - p \cdot Y = p \cdot (\dot{q} - Y) - L$ with $p = \frac{\partial L}{\partial \dot{q}}$).

Definition 1. *A function $f : M \longrightarrow \mathbb{R}$ is called a* moving energy *of $(L, Q, \mathcal{M})$ if there exists a vector field Y on Q, called a* generator *of f, such that f equals the restriction of $E_{L,Y}$ to the constraint manifold,*

$$f = E_{L,Y}|_M .$$

We note that, because of the restriction to the constraint manifold, the generator of a moving energy is never unique. Different vector fields may lead to the same moving energy (see Proposition 7 below).

As we have mentioned in the Introduction, the notion of moving energy given here is an extension of that originally given in [17], which has a kinematical interpretation. A comparison between the two is done in section 4 below.

Obviously, the consideration of moving energies has interest only when the energy is not conserved, namely if Z is not a section of $\mathcal{R}^\circ$. The central question, then, is which vector fields Y produce conserved moving energies for a given nonholonomic system $(L, Q, \mathcal{M})$. The situation is very similar to that of which vector fields produce conserved momenta, see Proposition 2:

Proposition 4. *Any two of the following three conditions imply the third:*
i. $Y - Z \in \Gamma(\mathcal{R}^\circ)$.
ii. $Y^{TQ}(L) = 0$ *in* M.
iii. $E_{L,Y}|_M$ *is a first integral of* $(L, Q, \mathcal{M} = Z + \mathcal{D})$.

Proof. We work in coordinates. All functions are evaluated in M. We have

$$\frac{d}{dt}E_L = \frac{d}{dt}(p \cdot \dot{q} - L) = \Big(\dot{p} - \frac{\partial L}{\partial q}\Big) \cdot \dot{q} + \Big(p - \frac{\partial L}{\partial \dot{q}}\Big) \cdot \ddot{q} = R \cdot \dot{q} = R \cdot Z$$

given that $\dot{q} - Z \in \mathcal{D}$ and R annihilates $\mathcal{D}$. Therefore, by (7),

$$\frac{dE_{L,Y}}{dt} = R \cdot (Y - Z) + Y^{TQ}(L)$$

and the proof goes as that of Proposition 2. □

Proposition 4 does not characterize all vector fields that generate conserved moving energies, but only those which satisfy either one (and hence the other) of the two conditions i. and ii. It has some immediate consequences:

Corollary 5.

i. *If $Y^{TQ}(L)|_M = 0$ then $E_{L,Y}|_M$ is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$ if and only if $Y - Z \in \Gamma(\mathcal{R}^\circ)$.*

ii. *If $Z^{TQ}(L)|_M = 0$ then*

$$E_{L,Z}|_M = E_L - \langle p, Z\rangle|_M$$

is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$.

iii. *Assume that L is invariant under the tangent lift Ψ^{TQ} of an action Ψ on Q, namely $L \circ \Psi_g^{TQ} = L$ for all $g \in G$. Then for any $\xi \in \mathfrak{g}$, $E_{L,Y_\xi}|_M$ is a first integral of $(L, Q, \mathcal{M} = Z + \mathcal{D})$ if and only if $Y_\xi - Z \in \Gamma(\mathcal{R}^\circ)$.*

Statement ii. is a particular case of statement i., but we have made it explicit because—as special as it may appear—it is precisely the case of all the affine LR systems and of their generalizations considered in section 4. Statement iii. formalizes the idea that, in presence of a symmetry group of the Lagrangian, the natural candidates to generate conserved moving energies are the infinitesimal generators of the group action that are sections of $\mathcal{R}^\circ$.

Remarks. i. Since the fibers of $\mathcal{D}$ are contained in those of $\mathcal{R}^\circ$, the condition in item i. of Proposition 4 is independent of the arbitrariness in the choice of the component along $\mathcal{D}$ of the vector field Z.

ii. Statement iii. of Corollary 5 generalizes Theorem 2 of [17] in two respects: it drops the assumption of the invariance of the distribution $\mathcal{D}$ under the group action and requires $Z - Y_\xi$ to be a section of $\mathcal{R}^\circ$, not of the smaller distribution $\mathcal{D}$. As is clarified in section 4, these hypotheses were present in [17] because they are related to the possibility of interpreting the moving energy as the energy in a different system of coordinates, which is the case considered there.

3.2 Nonuniqueness of moving energies and their generators. A system may have different conserved moving energies and, on the other hand, different vector fields may produce the same moving energy. The following Proposition is a direct consequence of the definitions:

Proposition 6. *Consider a nonholonomic system with affine constraints $(L, Q, \mathcal{M})$ which has a conserved moving energy $E_{L,Y_1}|_M$. Then, for any vector field Y_2 on Q, the moving energy $E_{L,Y_2}|_M$ is conserved if and only if $J_{Y_1-Y_2}|_M$ is a first integral of $(L, Q, \mathcal{M})$.*

We analyze the second question only in the special case of a Lagrangian $L = T - V \circ \pi$ without terms that are linear in the velocities; the general case can be easily worked out. We denote here by $\perp$ the orthogonality with respect to the Riemannian metric defined by the kinetic energy T and by $\langle Z\rangle$ the distribution on Q generated by Z.

Proposition 7. *Assume that the Lagrangian does not contain gyrostatic terms. Let Y_1 and Y_2 be two vector fields on Q. Then $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ if and only if $Y_1 - Y_2$ is a section of $(\mathcal{D} \oplus \langle Z\rangle)^\perp$.*

Proof. The equality $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ is equivalent to the condition $\langle p, Y_1 - Y_2\rangle|_M = 0$. Since $p = A\dot{q}$, this is in turn equivalent to the condition that, at each point $q \in Q$, $Y_1(q) - Y_2(q)$ is T-orthogonal to the fiber $\mathcal{M}_q$, namely, to all tangent vectors $Z(q) + u$ with $u \in \mathcal{D}_q$. Since $0 \in \mathcal{D}_q$, this is equivalent to the two conditions $Y_1(q) - Y_2(q) \perp Z(q)$ and $Y_1(q) - Y_2(q) \perp \mathcal{D}_q$. It follows that $E_{L,Y_1}|_M = E_{L,Y_2}|_M$ if and only if, for all $q \in Q$, $(Y_1 - Y_2)(q) \perp \mathcal{D}_q \oplus \langle Z\rangle_q$. □

4 Kinematically interpretable moving energies.

We now compare the definition of moving energy given above with the one originally given in [17]. This is not strictly needed to apply the results of section 3 to specific nonholonomic systems, and the reader may safely decide to skip to the treatment of examples in sections 5, 6 and 7 if he or she desires. The discussion is however important to understand how Definition 1 for a moving energy enlarges the class of these energy-type integrals compared to the cases considered so far [17, 6].

4.1 Kinematically interpretable and horizontal moving energies. As already explained in the Introduction, the construction of moving energies in [17] is as follows. One looks for a time-dependent change of coordinates that transforms the given nonholonomic system with affine constraints into a nonholonomic system with *linear* constraints. If time-independent, the transformed system has a conserved energy. The moving energy of ref. [17] is the energy of the transformed system written in the original coordinates.

Let us be more precise. Let $\mathcal{C}$ be a time-dependent diffeomorphism from a manifold U onto Q, namely a smooth map $\mathcal{C} : \mathbb{R} \times U \to Q$ such that, for each $t \in \mathbb{R}$, the map $\mathcal{C}_t := \mathcal{C}(t,\ ) : U \to Q$ is a diffeomorphism. As proven in [17] (Proposition 1; see also Proposition 4 of [16]), any nonholonomic system with affine constraints $(L, Q, \mathcal{M} = Z + \mathcal{D})$ on the configuration space Q pull-backs, under the tangent bundle lift of $\mathcal{C}$, to a nonholonomic system with affine constraints $(\tilde{L}, U, \tilde{\mathcal{M}})$ on the configuration manifold U. Specifically, in coordinates, the Lagrangian of the transformed system is

$$\tilde{L}(u, \dot{u}, t) = L\big(\mathcal{C}_t(u), \mathcal{C}_t'(u)\dot{u} + \dot{\mathcal{C}}_t(u)\big) \tag{8}$$

where $\mathcal{C}_t'$ is the Jacobian matrix of $\mathcal{C}_t$ and $\dot{\mathcal{C}}_t = \frac{\partial \mathcal{C}_t}{\partial t}$, and the transformed, time-dependent, constraint distribution $\tilde{\mathcal{M}}$ has fibers

$$\begin{aligned} \tilde{\mathcal{M}}_{t,u} &= \mathcal{C}_t'(u)^{-1}\big[\mathcal{M}_{t,\mathcal{C}_t(u)} - \dot{\mathcal{C}}_t(u)\big] \\ &= \mathcal{C}_t'(u)^{-1}\big[\mathcal{D}_{\mathcal{C}_t(u)} + Z(\mathcal{C}_t(u)) - \dot{\mathcal{C}}_t(u)\big] . \end{aligned} \tag{9}$$

Let now $E^*_{L,\mathcal{C}} : \mathbb{R} \times TQ \to \mathbb{R}$ be the push-forward of the energy $E_{\tilde{L}} : \mathbb{R} \times TU \to \mathbb{R}$ of the Lagrangian $\tilde{L}$ under the tangent bundle lift of $\mathcal{C}$. The restriction of $E^*_{L,\mathcal{C}}$ to $\mathbb{R} \times M$ is the "moving energy of (L, Q, M) induced by $\mathcal{C}$" of reference [17]; in the sequel, however, 'moving energy' will always have the meaning of Definition 1. A simple computation gives

$$E^*_{L,\mathcal{C}} = E_L - \langle p, \dot{\mathcal{C}}_t \circ \mathcal{C}_t^{-1} \rangle . \tag{10}$$

Without further hypotheses, this function may be time-dependent.

Definition 2. *A moving energy of a nonholonomic system with affine constraints* $(L, Q, \mathcal{M})$ *is* kinematically interpretable *if it has a generator* Y *and there exists a time-dependent diffeomorphism* $\mathcal{C} : \mathbb{R} \times U \to Q$ *which are such that*

$$E_{L,Y} = E^*_{L,\mathcal{C}} \tag{11}$$

and, with the notation just introduced,

D1. $\tilde{\mathcal{M}}$ *is a linear distribution.*

D2. $\tilde{\mathcal{M}}$ *is time-independent.*

D3. $\tilde{L}$ *is time-independent.*

(Note that (11) implies that $E^*_{L,\mathcal{C}}$ *is time-independent).*

In formulating this definition we have taken into account the fact that, because of the restriction to the constraint manifold M, the generator of a moving energy is not unique (see Proposition 7 above). We also note that this definition might be weakened by requiring only, instead of (11), that the moving energy equals $E^*_{L,\mathcal{C}}|_M$, namely $E_{L,Y}|_M = E^*_{L,\mathcal{C}}|_M$ for any generator Y. The stronger requirement (11) allow us to characterize the moving energies which are kinematically interpretable, instead of giving only a sufficient condition for their kinematical interpretability. Recall that Y^{TQ} denotes the tangent lift of a vector field Y on Q.

Proposition 8. *A moving energy of a nonhonolonomic system with affine constraints* $(L, Q, \mathcal{M} = Z + \mathcal{D})$ *is kinematically interpretable if and only if it has a generator* Y *that satisfies the following three conditions:*

P1. $Y - Z \in \Gamma(\mathcal{D})$.

P2. $\mathcal{D}$ *is* Y*-invariant (namely,* $\mathcal{D}_{\Phi_t^Y(q)} = (\Phi_t^Y)'(q)\mathcal{D}_q$ *for all* t, q*, where* $\Phi^Y : \mathbb{R} \times Q \to Q$ *is the flow of* Y*).*

P3. $Y^{TQ}(L) = 0$.

The proof of this Proposition is given in the next subsection.

In view of Proposition 4, items P1 and P3 of Proposition 8 imply that any kinematically interpretable moving energy is a conserved quantity. However, the comparison with Proposition 4 shows that the class of kinematically interpretable moving energies is (in general) a subclass of that of all conserved moving energies. One reason is that the kinematic interpretability requires the vector field $Y - Z$ to be a section of the distribution $\mathcal{D}$ and not of the (generally) larger distribution $\mathcal{R}^\circ$.

Definition 3. *A moving energy of* $(L, Q, \mathcal{M} = \mathcal{D} + Z)$ *is said to be* horizontal *if it has a generator* Y *such that* $Y - Z \in \Gamma(\mathcal{D})$.

Hence, a necessary condition for a moving energy to be kinematically interpretable is that it is horizontal. All examples of moving energies given in [17, 6] and in the following sections of this work are horizontal, and all, except for possibly a subclass of the ones treated in section 5.1, are kinematically interpretable. It would be interesting to find systems that do not have conserved horizontal moving energies but do have conserved moving energies.

Horizontal moving energies have other reasons of interest. Changing the Lagrangian changes $\mathcal{R}^\circ$ and—as it happens for the energy—may destroy or restore the conservation of a moving energy. Since the distribution $\mathcal{D}$ is independent of the Lagrangian, horizontal moving energies are in this respect special. Specifically, Proposition 4 has the following consequence:

Corollary 9. *If* $Y - Z \in \Gamma(\mathcal{D})$*, then* Y *generates a conserved moving energy for any nonholonomic system* $(L, Q, \mathcal{M} = Z + \mathcal{D})$ *whose Lagrangian satisfies* $Y^{TQ}(L)|_M = 0$.

In particular, if we have an action Ψ of a Lie group G on Q and, for some $\xi \in \mathfrak{g}$, the infinitesimal generator Y_ξ is such that $Y_\xi - Z \in \Gamma(\mathcal{D})$, then Y_ξ generates a conserved moving energy for all nonholonomic systems $(L, Q, \mathcal{M} = Z + \mathcal{D})$ with Ψ^{TQ}-invariant Lagrangian L. This may be viewed as a 'Noetherian' property (in the sense of [31, 12, 13]) of horizontal moving energies.

An instance of this property is encountered in the affine Suslov problem. If the axis of forbidden rotations is also an axis of symmetry of the body, the system is invariant under rotations of the body frame about this axis. Associated to this S^1-symmetry there is a preserved moving energy that persists in the presence of invariant potentials (see [6]).

4.2 Proof of Proposition 8. The proof rests on a few results that are refinements of results from [17] (and, in one case, correct a minor error of [17]).

Lemma 10. *Let* $\mathcal{C} : \mathbb{R} \times U \to Q$ *be a time-dependent diffeomorphism. Then* $E^*_{L,\mathcal{C}}$ *is time-independent if and only if* $\mathcal{C}$ *is, up to a diffeomorphism* $\mathcal{C}_0 : U \to Q$*, the flow* Φ^Y *of a vector field* Y *on* Q*,*[4] *namely*

$$\mathcal{C}_t = \Phi_t^Y \circ \mathcal{C}_0 \qquad \forall t \in \mathbb{R}\,.$$

[4] In [17], Proposition 3, it is erroneously stated that $\mathcal{C}$ is a flow, without contemplating the possibility of the presence of a diffeomorphism $\mathcal{C}_0$. We correct this here. This error does not impair the other results of [17].

Proof. We may work in coordinates. Let $Y(q,t) := \dot{\mathcal{C}}_t \circ \mathcal{C}_t^{-1} \in T_qQ$. From (10) and from the time-independency of E_L and of $p = \frac{\partial L}{\partial \dot q}$ it follows that $E^*_{L,\mathcal{C}}$ is time-independent if and only if $\langle p(q,\dot q), \frac{\partial Y}{\partial t}(q,t)\rangle = 0$ for all $q, \dot q, t$. Since, by the assumptions made on the Lagrangian (3), $\dot q \mapsto p(q,\dot q)$ is, for each q, a linear invertible map $T_qQ \to T_q^*Q$, this is equivalent to $\frac{\partial Y}{\partial t} = 0$. Hence $E^*_{L,\mathcal{C}}$ is time-independent if and only if there is a vector field Y on Q such that $\dot{\mathcal{C}}_t = Y \circ \mathcal{C}_t$ for all t. Define a map $\Phi : \mathbb{R} \times Q \to Q$ through $\Phi_t := \mathcal{C}_t \circ \mathcal{C}_0^{-1}$ $\forall t$. Since $\Phi_0 = \mathrm{id}$ and $\frac{\partial \Phi_t}{\partial t} = \dot{\mathcal{C}}_t \circ \mathcal{C}_0^{-1} = Y \circ \Phi_t$ $\forall t$, Φ is the flow of Y. □

If $\mathcal{C}_t = \Phi_t^Y \circ \mathcal{C}_0$, then (10) gives

$$E^*_{L,\mathcal{C}} = E_L - \langle p, Y\rangle\,. \tag{12}$$

Lemma 11. *Let $\mathcal{C} : \mathbb{R} \times U \to Q$ be a time-dependent diffeomorphism. Assume $\mathcal{C}_t = \Phi_t^Y \circ \mathcal{C}_0$ for all t. Then:*

L1. $\tilde{\mathcal{M}}$ is a linear distribution if and only if $Y - Z \in \Gamma(\mathcal{D})$.

L2. $\tilde{\mathcal{M}}$ is time-independent if and only if $\mathcal{M}$ is Y-invariant (namely, $(\Phi_t^Y)'(q)\mathcal{M}_q = \mathcal{M}_{\Phi_t^Y(q)}$ $\forall t, q$).

L3. $\tilde L$ is time-independent if and only if $Y^{TQ}(L) = 0$.

Proof. (L1) Expression (9) shows that $\tilde{\mathcal{M}}_{t,u}$ is a linear subspace of T_uU if and only if, at each t and u, the vector $Z(\mathcal{C}_t(u)) - \dot{\mathcal{C}}_t(u) = (Z - Y)(\mathcal{C}_t(u))$ belongs to $\mathcal{D}_{\mathcal{C}_t(u)}$.

(L2) Write Φ for Φ^Y. First note that if $\mathcal{C}_t = \Phi_t \circ \mathcal{C}_0$ then $\mathcal{C}_t' = (\Phi_t' \circ \mathcal{C}_0)\mathcal{C}_0'$ and $\dot{\mathcal{C}}_t = Y \circ \Phi_t \circ \mathcal{C}_0 = (\Phi_t' Y) \circ \mathcal{C}_0$, where the last equality uses the invariance of a vector field under its own flow. Therefore, (9) gives

$$\tilde{\mathcal{M}}_{t,u} = \mathcal{C}_0'(u)^{-1}\Phi_t'(\mathcal{C}_0(u))^{-1}\big[\mathcal{D}_{\Phi_t(\mathcal{C}_0(u))} + Z(\Phi_t(\mathcal{C}_0(u))) - \Phi_t'(\mathcal{C}_0(u)Y(\mathcal{C}_0(u))\big] \qquad \forall u, t\,.$$

Thus, using the fact that $\mathcal{C}_0$ is a diffeomorphism and $\Phi_0 = \mathrm{id}$, the condition of time-independence $\tilde{\mathcal{M}}_{t,u} = \tilde{\mathcal{M}}_{0,u}$ $\forall t, u$ is

$$\Phi_t'(q)^{-1}\big[\mathcal{D}_{\Phi_t(q)} + Z(\Phi_t(q))\big] = \mathcal{D}_q + Z(q)\,,$$

namely, $\Phi_t'(q)^{-1}\mathcal{M}_{\Phi_t(q)} = \mathcal{M}_q$, for all t, q.

(L3) Proceeding as in the proof of (L2) we see that (8) gives

$$\tilde L(u, \dot u, t) = L\Big(\Phi_t(\mathcal{C}_0(u)), \Phi_t'(\mathcal{C}_0(u))\big[\mathcal{C}_0'(u)\dot u + Y(\mathcal{C}_0(u))\big]\Big) \quad \forall u, \dot u, t\,.$$

It follows that $\tilde L$ is time-independent if and only if

$$L\Big(\Phi_t(\mathcal{C}_0(u)), \Phi_t'(\mathcal{C}_0(u))\big[\mathcal{C}_0'(u)\dot u + Y(\mathcal{C}_0(u))\big]\Big) = L\big(\mathcal{C}_0(u), \mathcal{C}_0'(u)\dot u + Y(\mathcal{C}_0(u))\big) \quad \forall u, \dot u, t\,.$$

Given that $\mathcal{C}_0 : U \to Q$ is a diffeomorphism and so is, for each u, the map $T_uU \ni \dot u \mapsto \mathcal{C}_0'(u)\dot u + Y(\mathcal{C}_0(u)) \in T_qQ$, the latter condition is equivalent to

$$L(\Phi_t(q), \Phi_t'(q)\dot q) = L(q, \dot q) \qquad \forall q, \dot q, t$$

namely, to the Y^{TQ}-invariance of L. □

We may now prove Proposition 8. Consider first a moving energy $E_{L,Y}|_M$ of a nonholonomic system $(L, Q, \mathcal{M} = Z + \mathcal{D})$ whose generator Y satisfies the three conditions P1, P2 and P3 of Proposition 8. Let $\mathcal{C} = \Phi^Y$, the flow of Y. By (12), $E_{L,Y} = E^*_{L,\mathcal{C}}$. Therefore, on account of item L1 of Lemma 11, P1 implies that $\mathcal{C}$ satisfies D1 and, on account of item L3 of that Lemma, P3 implies that $\mathcal{C}$ satisfies D3. To show that $E_{L,Y}$ is kinematically interpretable it remains to show that $\mathcal{C}$ satisfies D2. On account of L2, this is equivalent to the Y-invariance of $\mathcal{M}$. By itself P2 gives only the Y-invariance of $\mathcal{D}$ but, as we now show, together with P1 this implies the Y-invariance of $\mathcal{M}$. Let us write Φ for Φ^Y. Since $\Phi_t'Y = Y \circ \Phi_t$ for all t, and since $\mathcal{D} + Z - Y = \mathcal{D}$ given that $Z - Y$ is a section of $\mathcal{D}$, we have

$$\begin{aligned}
\Phi_t'(\mathcal{M}_q) &= \Phi_t'\big(\mathcal{D}_q + Z(q)\big) \\
&= \Phi_t'\big(\mathcal{D}_q + Z(q) - Y(q)\big) + Y(\Phi_t(q)) \\
&= \Phi_t'(\mathcal{D}_q) + Y(\Phi_t(q)) \\
&= \mathcal{D}_{\Phi_t(q)} + Y(\Phi_t(q)) \\
&= \mathcal{D}_{\Phi_t(q)} + (Y - Z)(\Phi_t(q)) + Z(\Phi_t(q)) \\
&= \mathcal{D}_{\Phi_t(q)} + Z \circ \Phi_t(q) \\
&= \mathcal{M}_{\Phi_t(q)}
\end{aligned}$$

for all q and t. This proves that $E_{L,Y}$ is kinematically interpretable.

Conversely, assume that $E_{L,Y}|_M$ is kinematically intrepretable. Then there exist an n-dimensional manifold U and a time-dependent diffeomorphism $\mathcal{C} : \mathbb{R} \times U \to Q$ such that $E_{L,Y} = E^*_{L,\mathcal{C}}$ and $\mathcal{C}$ satisfies properties D1, D2 and D3. Therefore, by L1 and L3, Y satisfies P1 and P3. Thus $Y - Z$ is a section of $\mathcal{D}$ and an argument similar to the one just used shows that D2, namely the time-independence of $\tilde{\mathcal{M}}$, implies that not only $\mathcal{M}$ (as in L2) but also $\mathcal{D}$ are Y-independent. Hence Y satisfies P2 as well.

5 Moving energies for LR systems

5.1 A moving energy for a class of affine nonholonomic systems on Lie groups. As a first example, we consider here a class of nonholonomic systems $(L, Q, \mathcal{M})$ with affine constraints whose configuration manifold Q is a Lie group G. This class includes the so-called (affine) LR systems, that we will consider in the next subsection.

As usual, we denote by L and R, respectively, the actions of G on itself by left and right translations.[5] A function $f : TG \to \mathbb{R}$ is left-invariant if it is invariant under the lifted action L^{TG} on TG, namely $f \circ L_g^{TG} = f$ for all $g \in G$. A vector field Y on G is right-invariant if $(R_g)_*Y = Y$ for all $g \in G$.

Corollary 5 has the following immediate consequence (which we formulate in a way that takes into account the fact that the vector field Z that determines the inhomogeneous part of the affine constraint distribution is nonunique):

Proposition 12. *Consider a nonholonomic system with affine constraints $(L, G, \mathcal{M})$, where G is a Lie group. Assume that the Lagrangian L is left-invariant and that the affine distribution $\mathcal{M}$ can be written as $\mathcal{M} = Z + \mathcal{D}$ with a right-invariant vector field Z. Then, the (horizontal) moving energy*

$$E_{L,Z}|_M = E_L - \langle p, Z\rangle\big|_M$$

is a first integral of $(L, G, \mathcal{M})$.

Proof. Since the flow of a right-invariant vector field consists of left-translations, under the stated hypotheses we have $Z^{TG}(L) = 0$ in all of TG. Hence, the conclusion follows either from item ii. of Corollary 5 or from Corollary 9, given that $Z - Z = 0 \in \Gamma(\mathcal{D})$. □

If the Lagrangian has the form (1), then the condition of left-invariance implies that the kinetic energy T is a left-invariant Riemannian metric on G, that the gyrostatic term b is a left-invariant 1-form on G, and that the potential energy V is a constant. We give here the expression of $E_{L,Z}$ in the case in which the Lagrangian does not contain gyrostatic terms, so that $L = T$.

[5] The symbols L and R are also used for other objects, but there will be no risk of confusion from the context.

We employ the left-trivialization of TG, namely the identification $\Lambda : TG \to G \times \mathfrak{g}$ given by the maps $T_gL_{g^{-1}} : T_gG \to T_eG \equiv \mathfrak{g}$. We write Ω for $T_gL_{g^{-1}}\dot{g}$ (the 'body angular velocity'; in a matrix group, $\Omega = g^{-1}\dot{g}$). We denote $\langle\ ,\ \rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$ the $\mathfrak{g}^*$-$\mathfrak{g}$ pairing. On account of the left-invariance of the Lagrangian $L = T$, its left-trivialization $T \circ \Lambda^{-1}$, which we keep denoting T, is given by

$$T(g,\Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$$

where $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is the positive definite symmetric tensor determined by the kinetic energy at the group identity e. Furthermore, the left-trivialization of the Legendre transformation is the map $\mathfrak{g} \to \mathfrak{g}^*$ given by $\Omega \mapsto \mathbb{I}\Omega$. Given that the vector field Z is right-invariant, there exists a Lie algebra vector $\zeta \in \mathfrak{g}$ such that

$$Z(g) = T_eR_g \cdot \zeta \quad \forall g \in G\,,$$

and its left-trivialization is $\mathrm{Ad}_{g^{-1}}\zeta$. Hence, the left-trivialization of the momentum J_Z is $\langle \mathbb{I}\Omega, \mathrm{Ad}_{g^{-1}}\zeta\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}$ and that of $E_{T,Z}$ is

$$E_{T,Z}(g,\Omega) \;=\; \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}} - \langle \mathbb{I}\Omega, \mathrm{Ad}_{g^{-1}}\zeta\rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}}\,. \tag{13}$$

We conclude this subsection with two remarks. First, even though the left-invariance of the Lagrangian requires the potential energy to be a constant, and hence $L = T + b$, on account of the remark at the end of section 4.1 any Z-invariant potential energy V can be included into the Lagrangian without destroying the existence of a conserved moving energy, which is now the restriction to M of $T + V - J_Z$. Hence, in the absence of gyrostatic terms,

$$E_{T-V,Z} = E_{T,Z} + V\,. \tag{14}$$

In the special case of the LR systems, a particular class of Z-invariant potentials has been identified in [22] (section 4.6).

In all these systems, the moving energy $E_{L,Z}|_M$ is horizontal and the generator Z preserves the Lagrangian. Therefore, by Proposition 8, if $\mathcal{D}$ is Z-invariant then $E_{L,Z}|_M$ is kinematically interpretable, in the sense of Definition 2. A particular class of systems with this property are the LR systems of the next subsection.

5.2 The case of LR systems. LR systems are the subclass of the class of systems on Lie groups considered in the previous subsection in which the affine constraint distribution $\mathcal{M} = Z + \mathcal{D}$ is right-invariant, namely, not only Z but also $\mathcal{D}$ is right-invariant. The right-invariance of $\mathcal{D}$ means that $\mathcal{D}_g = T_eR_g \cdot \mathcal{D}_e$ for all $g \in G$, or equivalently, that $\mathcal{D}$ is the null space of a set of right-invariant 1-forms on G.

LR systems were introduced by Veselov and Veselova in [36], who focussed mostly on the case of linear constraints, with $\mathcal{M} = \mathcal{D}$, and of purely kinetic Lagrangian, namely $L = T$. The prototype of these systems is the renowned Veselova system [37, 35], which describes the motion by inertia of a rigid body with a fixed point under the constraint that the angular velocity remains orthogonal to a direction fixed in space (linear case) or, more generally, that the component of the angular velocity in a direction fixed in space is constant (affine case).

As proven in [36], LR systems have remarkable properties: the existence of an invariant measure and the conservation of the (restriction to the constraint manifold of the) momentum covector. A difference between the linear and the affine cases concerns of course the energy, which is conserved in the former case but not in the latter case. However, it was shown in [18, 19, 20, 23] that the affine Veselova system, and an n-dimensional generalization of it, possess a first integral which was there regarded as an "analog of the Jacobi-Painlevé integral" or as a "modified Hamiltonian". As remarked in [6], this function turns out to be a moving energy. In fact, in full generality, Proposition 12 implies the following

Corollary 13. *Any affine LR system* $(L, G, \mathcal{M} = Z + \mathcal{D})$ *has the conserved moving energy* $E_{L,Z}|_M$*.*

In order to compare the moving energy $E_{L,Z}|_M$ of Corollary 13 with the first integral found in [18, 19, 23], we give the expression of $E_{L,Z}$ in the special case of a Lie group G for which there is an Ad-invariant inner product in $\mathfrak{g}$, which includes the case of $\mathrm{SO}(n)$. Since [18, 19, 23] did not consider gyrostatic terms, on account of (14) we limit ourselves to $L = T$. In general, the right-invariant affine distribution $\mathcal{M} = Z + \mathcal{D}$ can be specified, via right-translations, by a set of independent covectors $a^1, \dots, a^k \in \mathfrak{g}^*$ that span the annihilator $\mathcal{D}_e^\circ$ and by the vector $\zeta \in \mathfrak{g}$ that specifies the vector field Z. Hence, the constraint can be written

$$\langle a^j, \omega - \zeta \rangle_{\mathfrak{g}^*\text{-}\mathfrak{g}} = 0, \qquad j = 1, \dots, k \tag{15}$$

where $\omega = \mathrm{Ad}_g \Omega$ is the right-trivialization of the velocity vector $\dot{g} \in T_g G$. By means of an Ad-invariant inner product $\langle \ , \ \rangle_{\mathfrak{g}}$, the covectors $a^1, \dots, a^k$ are interpreted as elements of $\mathfrak{g}$ and can be chosen to be orthonormal. Also, Z can be chosen so that ζ belongs to the span of $a^1, \dots, a^k$. Hence $\zeta = \sum_{j=1}^k \langle a^j, \zeta \rangle_{\mathfrak{g}} a^j$ and the left-trivialization of the momentum J_Z takes the form $\sum_{j=1}^k \langle a^j, \zeta \rangle_{\mathfrak{g}} \langle \mathbb{I}\Omega, \gamma^j \rangle_{\mathfrak{g}}$, where $\gamma^j = \mathrm{Ad}_g^{-1} a^j$ are the so-called Poisson vectors. In conclusion,

$$E_{T,V}(g, \Omega) = \frac{1}{2} \langle \mathbb{I}\Omega, \Omega \rangle_{\mathfrak{g}} - \sum_{j=1}^{k} \langle a^j, \zeta \rangle_{\mathfrak{g}} \langle \mathbb{I}\Omega, \gamma^j \rangle_{\mathfrak{g}} \,.$$

When $G = \mathrm{SO}(n)$, this coincides with the first integral of the n-dimensional affine Veselova system given in [18, 19, 23], except that the constants $\langle a^j, \zeta \rangle_{\mathfrak{g}}$ are there written, using the constraint (15), as $\langle \gamma^j, \Omega \rangle_{\mathfrak{g}}$.

The right-invariance of the distribution $\mathcal{D}$ makes the affine LR systems very special among those considered in the previous subsection. In particular, as mentioned above, the restriction to the constraint manifold of the momentum covector is conserved [36]. This fact is accounted for, without any computation, by Proposition 2:

Proposition 14. [36] *Consider an affine LR system* $(L, G, \mathcal{M} = Z + \mathcal{D})$*. Denote* Y_ξ *the infinitesimal generator of the left-action of* G *on itself by left-translations associated to* $\xi \in \mathfrak{g}$*. Then, for any* $\xi \in \mathcal{D}_e \subset \mathfrak{g}$*,* $J_{Y_\xi}|_M = \langle p, Y_\xi \rangle|_M$ *is a first integral of the system.*

Proof. $Y_\xi(g) = T_e R_g \cdot \xi$. By left-invariance of L, $Y_\xi^{TG}(L) = 0$. By right-invariance of $\mathcal{D}$, Y_ξ is a section of $\mathcal{D}$. The statement now follows from Proposition 2. □

Note that if $\mathcal{D}$ is not right-invariant then the infinitesimal generators Y_ξ might be not sections of $\mathcal{D}$: this is the reason why, notwithstanding the fact that the Lagrangian has the appropriate invariance property, the momenta $J_{Y_\xi}|_M$ are in general not conserved for the systems considered in the previous subsection.

Proposition 14 implies that the affine LR systems have a multitude of conserved moving energies: for any $\xi \in \mathcal{D}_e$, $E_{L,Z-Y_\xi}|_M = E_{L,Z}|_M - J_{Y_\xi}|_M$ is a conserved moving energy. On account of Proposition 8, they are all kinematically interpretable and hence associated to time-dependent changes of coordinates.

Remark. The 3-dimensional affine Veselova system allows a Hamiltonization after reduction, in terms of a rank-four Poisson structure, with the above moving energy playing the role of the Hamiltonian [23].

6 A convex body that rolls on a steadily rotating plane

6.1 The system. We consider now the system formed by a heavy convex rigid body constrained to roll without slipping on a horizontal plane Π, which rotates uniformly around a vertical

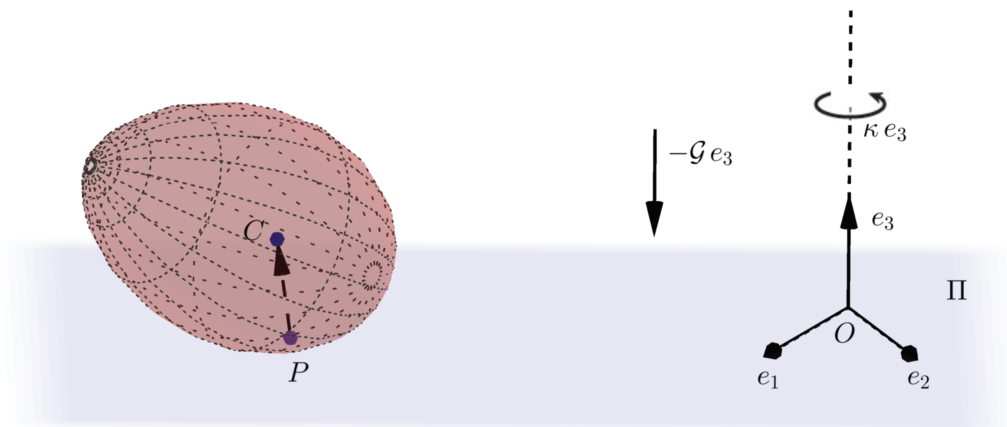

Figure 1: The heavy convex rigid body with smooth surface that rolls without slipping on a plane that rotates with constant angular velocity κe_3.

axis, see Figure 1. The case in which the plane is at rest is classical and was studied for specific geometries of the body already by Routh [34] and Chaplygin [7] (see [4, 8] for recent treatments) while, to our knowledge, the case in which the plane Π is rotating has been studied only in two particular cases—that of a homogeneous sphere [10, 33, 30, 3, 17] and that of a disk [21] (which however describes the disk in a frame that co-rotates with the plane, in which the nonholonomic constraint is linear). Here, we exclude the latter case because we assume that the body has a smooth (i.e. C^∞) surface.

We describe the system relatively to a spatial inertial frame $\Sigma_s = \{O; e_1, e_2, e_3\}$. We assume that the plane Π rotates around the axis e_3 of this frame, with constant angular velocity κe_3, $\kappa \in \mathbb{R}$, and that it is superposed to the subspace spanned by e_1 and e_2.

Let $x = (x_1, x_2, x_3)$ be the coordinates in Σ_s of the center of mass C of the body. The system is subject to the holonomic constraint that the body has a point in contact with the plane. The configuration manifold is thus $Q = \mathbb{R}^2 \times \mathrm{SO}(3) \ni (q, g)$, where $q = (x_1, x_2) \in \mathbb{R}^2$ and $g \in \mathrm{SO}(3)$ is the attitude matrix that relates the inertial frame Σ_s to a frame $\Sigma_b = \{C; E_1, E_2, E_3\}$ attached to the body, and with the origin in C. We assume that g is chosen so that the representatives u^s and u^b of a same vector in the two frames Σ_s and Σ_b are related by $u^s = gu^b$.

We denote by ω the angular velocity of the body relative to the inertial frame Σ_s, and by Ω its representative ω^b in the body frame Σ_b.

As in the previous section we will left-trivialize the factor $T\mathrm{SO}(3)$ of TQ, but now we identify the Lie algebra so(3) with $\mathbb{R}^3$ via the hat map $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$, with $\hat{a} = a\times$ for any $a \in \mathbb{R}^3$. Thus, $TQ \equiv \mathbb{R}^2 \times \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (q, \dot{q}, g, \Omega)$.

The holonomic constraint that the body touches the plane is $e_3 \cdot OC = e_3 \cdot PC$, where P is the point of the body in contact with the plane, or $x_3 = e_3 \cdot PC$. Hence, $\dot{x}_3 = e_3 \cdot (\omega \times PC)$. If we denote by $\rho(g)$ the representative in Σ_b of the vector PC and by $\gamma(g)$ the representative of e_3 in Σ_b, which is the so-called Poisson vector and equals $g^{-1}e_3^s = g^{-1}(0,0,1)^T$, then the holonomic constraint is $x_3 = \gamma(g) \cdot \rho(g)$ and

$$\dot{x}_3 = \gamma(g) \cdot [\Omega \times \rho(g)]\,. \tag{16}$$

The potential energy of the weight force is $V(g) = m\mathcal{G}e_3 \cdot OC = m\mathcal{G}\gamma(g) \cdot \rho(g)$, with the obvious meaning of the constants m and $\mathcal{G}$. Here, γ and ρ are known functions of the attitude $g \in \mathrm{SO}(3)$, but they are related by the Gauss map $G : \mathcal{S} \to S^2$ of the surface $\mathcal{S}$ of the body. Specifically, given that γ is the inward unit normal vector to $\mathcal{S}$ at the point of $\mathcal{S}$ of coordinates $-\rho$, $\gamma(g) = -G(-\rho(g))$. Since $\mathcal{S}$ is assumed to be smooth and convex, G is a diffeomorphism and we may also write $\rho = F\circ\gamma$ with $F(\gamma) = -G^{-1}(-\gamma)$. Hence $V = v\circ\gamma$ with $v(\gamma) = m\mathcal{G}\gamma\cdot F(\gamma)$. In the sequel, we shall routinely write γ for $\gamma(g)$ and ρ for $F(\gamma(g))$. With these conventions, the left-trivialized Lagrangian of the system is

$$L(q,g,\dot q,\Omega) = \frac{1}{2}\mathbb{I}\Omega\cdot\Omega + \frac{m}{2}\Big(\dot q_1^2 + \dot q_2^2 + \big(\gamma\cdot[\Omega\times\rho]\big)^2\Big) - v(\gamma) \tag{17}$$

and depends on the attitude g only through γ. Here, $\mathbb{I}$ is the inertia tensor of the body relative to its center of mass.

The condition of no-slipping of the body on the plane is obtained by equating the velocities (relative to Σ_s) of the point P of the body, which is $\frac{d}{dt}(OC) + \omega\times CP$, and of the point of the plane which is in contact with P, which is $\kappa e_3\times OP = \kappa e_3\times(OC+CP)$. Using representatives, the condition of no slipping is thus

$$\dot x = g(\Omega\times\rho) + \kappa e_3^s\times(x - g\rho)\,. \tag{18}$$

The first two components of this condition define an 8-dimensional affine subbundle M of TQ which can be identified with $\mathbb{R}^2\times\mathrm{SO}(3)\times\mathbb{R}^3\ni(q,g,\Omega)$ (the third component of (18) is nothing but (16)).

In order to simplify the notation we identify $Q = \mathbb{R}^2\times\mathrm{SO}(3)\ni(q,g)$ with its embedding in $\mathbb{R}^3\times\mathrm{SO}(3)\ni(x,g)$ and $M = \mathbb{R}^2\times\mathrm{SO}(3)\times\mathbb{R}^3\ni(q,g,\Omega)$ with its embedding in $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3\ni(x,g,\Omega)$ given, in both cases, by $x(q,g) = (q_1,q_2,\gamma\cdot\rho)$. Correspondingly, we identify $TQ\equiv\mathbb{R}^2\times\mathrm{SO}(3)\times\mathbb{R}^2\times\mathbb{R}^3\ni(q,g,\dot q,\Omega)$ with its embedding in $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3\ni(x,g,\dot x,\Omega)$, with x as above and $\dot x = (\dot q_1,\dot q_2,\dot x_3)$ with $\dot x_3$ as in (16).

The affine subbundle M corresponds to an affine distribution $\mathcal{M} = Z + \mathcal{D}$ on $Q = \mathbb{R}^2\times\mathrm{SO}(3)$ which, once left-trivialized and embedded in $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3$, is given by

$$\begin{aligned}\mathcal{D}_{(q,g)} &= \big\{\big(g(\Omega\times\rho)\,,\,\Omega\big)\;:\;\Omega\in\mathbb{R}^3\big\}\\ Z(q,g) &= \big(\kappa e_3^s\times(x-g\rho)\,,\,0\big)\,.\end{aligned} \tag{19}$$

6.2 The conserved moving energy. The energy is not conserved in the nonholonomic system $(L,Q,\mathcal{M})$ just constructed. However, being independent of q and depending on g only through the Poisson vector γ, the Lagrangian (17) is invariant under the lift of an action of SE(2). We may thus try to construct a moving energy using the infinitesimal generator of the action of a subgroup.

Specifically, we consider the S^1-action which (after the aforementioned embedding of Q in $\mathbb{R}^3\times\mathrm{SO}(3)$) is given by

$$\theta.(x,g) = (R_\theta x, R_\theta g) \tag{20}$$

where R_θ is the rotation matrix in SO(3) that rotates an angle $\theta\in S^1$ about the third axis $(0,0,1)^T = e_3^s$. This action leaves the Poisson vector γ invariant. Its lift to M (as embedded in $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3$) is

$$\theta.(x,g,\Omega) = (R_\theta x, R_\theta g,\Omega) \tag{21}$$

and clearly leaves the Lagrangian (17) invariant.

The infinitesimal generator of the action (20) that corresponds to $\xi\in\mathbb{R}\cong\mathfrak{s}^1$, once left-trivialized, is the vector field with components

$$Y_\xi(q,g) = \big(\xi e_3^s\times x\,,\,\xi\gamma\big)\in\mathbb{R}^3\times\mathbb{R}^3\,. \tag{22}$$

Therefore

$$Y_k - Z = \big(\kappa e_3^s\times g\rho\,,\,\kappa\gamma\big) = \big(g(\kappa\gamma\times\rho)\,,\,\kappa\gamma\big)\in\mathcal{D}_{(q,g)}\,.$$

Hence, the hypotheses of Corollary 5 are satisfied and the moving energy $E_{L,Y_\kappa}|_M$ is a first integral of the system.

In order to give an expression for this moving energy we find convenient to introduce the vector function

$$K(g,\Omega) = \mathbb{I}\Omega + m\rho \times \big(\Omega \times \rho\big)\,. \tag{23}$$

Proposition 15.

$$E_{L,Y_\kappa}|_M = \frac{1}{2}K\cdot\Omega + m\mathcal{G}\rho\cdot\gamma - \kappa K\cdot\gamma + \frac{1}{2}m\kappa^2\big(\|\rho\|^2 - \|x\|^2\big)\,. \tag{24}$$

Proof. Instead of parameterizing the embedding of M in $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3$ with (x,g,Ω), as done so far, we will parameterize it with (X,g,Ω), with $X = g^{-1}x$, the representative of OC in the body frame. Specifically, M is the the submanifold of $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3$ given by the condition $X\cdot\gamma = \rho\cdot\gamma$ (the holonomic constraint). Similarly, we parametrize $TQ \equiv \mathbb{R}^3\times\mathrm{SO}(3)\times\mathbb{R}^3\times\mathbb{R}^3$ with $(X,g,\dot{x},\Omega)$ (notice that we keep the spatial representative $\dot{x}$ of the velocity of the center of mass). Thus, M is the subbundle of TQ defined by the two conditions

$$X\cdot\gamma = \rho\cdot\gamma\,, \qquad g^{-1}\dot{x} = \Omega\times\rho + \kappa\gamma\times(X-\rho)\,. \tag{25}$$

The energy E_L of the system is the sum of the kinetic and potential energies:

$$E_L = \frac{1}{2}\mathbb{I}\Omega\cdot\Omega + \frac{m}{2}\|\dot{x}\|^2 + m\mathcal{G}\gamma\cdot\rho$$

Its restriction to M is given by

$$\begin{aligned} E_L|_M \;&=\; \frac{1}{2}\mathbb{I}\Omega\cdot\Omega + m\mathcal{G}\gamma\cdot\rho + \frac{m}{2}\|\rho\times\Omega\|^2 + m\kappa\,(\Omega\times\rho)\cdot(\gamma\times(X-\rho)) + \frac{m\kappa^2}{2}\|\gamma\times(X-\rho)\|^2 \\ &=\; \frac{1}{2}K\cdot\Omega + m\mathcal{G}\gamma\cdot\rho - m\kappa\gamma\cdot(\rho\times(\Omega\times\rho)) + m\kappa\,(\Omega\times\rho)\cdot(\gamma\times X) + \frac{m\kappa^2}{2}\|\gamma\times X\|^2 \\ &\qquad + \frac{m\kappa^2}{2}\|\gamma\times\rho\|^2 - m\kappa^2(\gamma\times X)\cdot(\gamma\times\rho). \end{aligned}$$

On the other hand, given (22) and the form of the kinetic energy metric defined by (17), we find

$$J_{Y_\kappa} \;=\; m\kappa(q_1\dot{q}_2 - q_2\dot{q}_1) + \kappa\mathbb{I}\Omega\cdot\gamma \;=\; m\kappa\gamma\cdot(X\times(g^{-1}\dot{x})) + \kappa\mathbb{I}\Omega\cdot\gamma.$$

Its restriction to M is computed to be

$$J_{Y_\kappa}|_M \;=\; m\kappa(\Omega\times\rho)\cdot(\gamma\times X) - m\kappa^2(\gamma\times X)\cdot(\gamma\times\rho) + m\kappa^2\|\gamma\times X\|^2 + \kappa\mathbb{I}\Omega\cdot\gamma\,.$$

Hence the moving energy $E_{L,Y_\kappa}|_M = E_L|_M - J_{Y_\kappa}|_M$ is given by

$$E_{L,Y_\kappa}|_M \;=\; \frac{1}{2}K\cdot\Omega + m\mathcal{G}\rho\cdot\gamma - \kappa(K\cdot\gamma) + \frac{m\kappa^2}{2}\left(\|\gamma\times\rho\|^2 - \|\gamma\times X\|^2\right)\,.$$

This is equivalent to (24) because, in M, $X\cdot\gamma = \rho\cdot\gamma$ and hence $\|\gamma\times\rho\|^2 - \|\gamma\times X\|^2 = \|\rho\|^2 - \|X\|^2$, and because $\|X\| = \|x\|$. ☐

We note that the existence of the moving energy $E_{L,Y_\kappa}|_M$ has the following dynamical consequence:

Corollary 16. *If the motion of the rolling body is unbounded, then its angular velocity Ω satisfies $\limsup_{t\to\infty}\|\Omega\| = \infty$.*

Proof. Since ρ and γ are bounded, the only way in which the conserved moving energy (24) remains bounded as $\|x\|$ becomes large is that $\|\Omega\|$ becomes large. ☐

Remark. The distribution $\mathcal{D}$ and the vector field Z are also invariant under the lifted S^1-action (21). Therefore, the system is invariant under this action. In the Appendix, we give for completeness the reduced equations of motion on M/S^1.

7 The n-dimensional Chaplygin sphere that rolls on a steadily rotating hyperplane

7.1 The system. It is natural to expect that the discussion of the previous section admits a multi-dimensional generalization. Here we consider the particular case in which the body is an n-dimensional sphere and the center of mass coincides with its geometric center. If the hyperplane where the rolling takes plane is not rotating we recover the n-dimensional Chaplygin sphere problem introduced in [19].

Let $x \in \mathbb{R}^n$ denote the position of the center of mass of the sphere written with respect to an inertial frame $\Sigma_s = \{O; e_1, \dots, e_n\}$. We assume that the hyperplane where the rolling takes place passes through O and has e_n as its normal vector. Moreover, we assume that the sphere is 'above' this hyperplane so at all times the holonomic constraint $x_n = r$ is satisfied, where r is the radius of the sphere.

The configuration space is $Q = \mathbb{R}^{n-1} \times \mathrm{SO}(n) \ni (q, g)$, where $q = (x_1, \dots, x_{n-1})$. For convenience, in all this section we embed $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ by putting $x_n = r$. We will also work with the induced embedding of tangent bundles $TQ \hookrightarrow T(\mathbb{R}^n \times \mathrm{SO}(n))$ defined by the simultaneous relations $x_n = r$ and $\dot{x}_n = 0$.

The Lagrangian $L : T(\mathbb{R}^n \times \mathrm{SO}(n)) \to \mathbb{R}$ is written in the left-trivialization as

$$L(x, g, \dot{x}, \Omega) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle + m\|\dot{x}\|^2. \tag{26}$$

As usual $\Omega = g^{-1}\dot{g} \in \mathfrak{so}(n)$ is the angular velocity written in body coordinates (the left-trivialization of the velocity). The pairing $\langle \cdot, \cdot \rangle$ in (26) denotes the Killing metric

$$\langle \zeta_1, \zeta_2 \rangle = -\frac{1}{2}\mathrm{Trace}(\zeta_1\zeta_2), \qquad \zeta_1, \zeta_2 \in \mathfrak{so}(n),$$

and the inertia tensor $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is a positive definite symmetric linear operator.

The steady rotation of the hyperplane where the rolling takes place is specified by a fixed element $\eta \in \mathfrak{so}(n)$ that satisfies

$$\eta e_n = 0. \tag{27}$$

The nonholonomic constraints of rolling without slipping are

$$\dot{x} = r\omega e_n + \eta x, \tag{28}$$

where $\omega = \dot{g}g^{-1}$ is the angular velocity written in space coordinates (the right-trivialization of the velocity) that satisfies $\omega = \mathrm{Ad}_g\Omega$. Note that the last component of (28) reads $\dot{x}_n = 0$ so that (28) defines an affine constraint subbundle of $TQ \subset T(\mathbb{R}^n \times \mathrm{SO}(n))$.

The constraint (28) may be rewritten as

$$\dot{x} = r(\mathrm{Ad}_g\Omega)e_n + \eta x, \tag{29}$$

that defines an affine distribution $\mathcal{M} = Z + \mathcal{D}$ on $\mathbb{R}^n \times \mathrm{SO}(n)$ that is given in the left-trivialization by

$$\begin{aligned} \mathcal{D}_{(x,g)} &= \{(\dot{x}, \Omega) \,:\, \dot{x} = r(\mathrm{Ad}_g\Omega)e_n\}\,, \\ Z(x, g) &= \big(\eta x, 0\big)\,. \end{aligned} \tag{30}$$

The constraint manifold M is diffeomorphic to $\mathbb{R}^{n-1} \times \mathrm{SO}(n) \times \mathfrak{so}(n) \subset \mathbb{R}^n \times \mathrm{SO}(n) \times \mathfrak{so}(n) \ni (x, g, \Omega)$.

7.2 The conserved moving energy. The energy is not conserved for the above system. A conserved moving energy may be found by considering the action of $\mathrm{SO}(n-1)$ on $\mathbb{R}^n \times \mathrm{SO}(n)$ defined by $h \cdot (x, g) = (\tilde{h}x, g)$, where for $h \in \mathrm{SO}(n-1)$ we denote

$$\tilde{h} = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(n). \tag{31}$$

This action leaves Q invariant and its tangent lift clearly preserves the Lagrangian (26).

Using (27) and the embedding $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ given by (31), we can naturally think of η as an element in $\mathfrak{so}(n-1)$. Its infinitesimal generator Y_η is readily computed to be the vector field $Y_\eta(x, g) = (\eta x, 0)$. Therefore, $Y_\eta - Z = 0$ and by Corollary 5 the moving energy $E_{L,Y_\eta}|_M$ is preserved.

In order to give an expression for this moving energy we introduce the Poisson vector $\gamma(g) = g^{-1}e_n$ and the matrix[6]

$$K := \mathbb{I}\Omega + mr^2\big(\Gamma(g)\Omega + \Omega\Gamma(g)\big) \in \mathfrak{so}(n), \tag{32}$$

where $\Gamma(g)$ denotes the symmetric, rank one matrix $\Gamma := \gamma \otimes \gamma = \gamma\gamma^T$.

Proposition 17.

$$E_{L,Y_\eta}|_M(x, g, \dot{x}, \Omega) = \frac{1}{2}\langle K, \Omega\rangle - \frac{m}{2}\|\eta x\|^2 \,. \tag{33}$$

Proof. The restriction of the energy E_L to M is obtained by substituting the constraint (28) into the Lagrangian (26). Notice that

$$\frac{m}{2}||r\omega e_n + \eta x||^2 = \frac{mr^2}{2}||\Omega\gamma||^2 + \frac{m}{2}||\eta x||^2 + mr(\omega e_n, \eta x),$$

where $(\cdot, \cdot)$ is the euclidean scalar product in $\mathbb{R}^n$ (so far denoted by a dot). Also,

$$\frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle + \frac{mr^2}{2}||\Omega\gamma||^2 = \frac{1}{2}\langle K, \Omega\rangle.$$

Therefore,

$$E_L|_M\,(x, g, \dot{x}, \Omega) = \frac{1}{2}\langle K, \Omega\rangle + \frac{m}{2}||\eta x||^2 + mr(\omega e_n, \eta x). \tag{34}$$

On the other hand, considering that $Y_\eta(x, g) = (\eta x, 0)$ and the form of the Lagrangian (26), we have

$$J_{Y_\eta}(x, g, \dot{x}, \Omega) \;=\; m(\dot{x}, \eta x).$$

Its restriction to M is given by

$$J_{Y_\eta}\big|_M\,(x, g, \dot{x}, \Omega) \;=\; mr(\omega e_n, \eta x) + m\|\eta x\|^2.$$

Hence the moving energy $E_{L,Y_\eta}|_M = E_L|_M - J_{Y_\eta}\big|_M$ is given by (33). □

Remark. The system is invariant under the action of a certain subgroup of $\mathrm{SO}(n-1)$. The precise form of this subgroup and its action, together with the unreduced and the reduced equations of motion, is given, for completeness, in the Appendix.

[6]This matrix was introduced in [19] as the angular momentum of the sphere about the contact point.

8 Appendix: Equations of motion for the examples

We present here the (reduced) equations of motion of the examples treated in sections 6 and 7.

8.1 The S^1-reduced equations of motion of the convex body that rolls on a rotating plane. In the system studied in section 6, not only the Lagrangian (17) but also the distribution $\mathcal{D}$ and the vector field Z as in (19) are invariant under the lift of the S^1-action (21) to TQ. Therefore, this lifted action can be restricted to the 8-dimensional phase space M, and the dynamics is equivariant. For completeness, we give here the reduced equations of motion on the quotient space M/S^1.

The S^1-action (21) on M is free. The 7-dimensional quotient manifold M/S^1 can be identified with $\mathbb{R}^2 \times S^2 \times \mathbb{R}^3 \ni (q, \gamma, \Omega)$, with projection

$$(q, g, \Omega) \mapsto \big(q, \gamma(g), \Omega\big) .$$

We embed M/S^1 in $\mathbb{R}^9 \ni (X, \gamma, \Omega)$, as the submanifold given by

$$\|\gamma\| = 1 , \qquad (X - \rho) \cdot \gamma = 0 \tag{35}$$

where ρ stands for $\rho = F(\gamma)$ (recall that $X = g^{-1}x$ and see the first equation of (25)).

The definition (23) of K can be inverted to give

$$\Omega(\gamma, K) = AK + \frac{mA\rho \cdot K}{1 - mA\rho \cdot \rho} A\rho, \tag{36}$$

where $A = (\mathbb{I} + m\|\rho\|^2\mathbb{1})^{-1}$ and $\rho = F(\gamma)$. Therefore, as (global) coordinates on $\mathbb{R}^9$ we may use (X, γ, K) instead of (X, γ, Ω).

Proposition 18. *The equations of motion of the S^1-reduced system are the restriction to the submanifold (35) of the equations*

$$\begin{aligned}
\dot K &= K \times \Omega + m\dot\rho \times (\Omega \times \rho) + m\mathcal{G}\gamma \times \rho + m\kappa\rho \times (\kappa X - \dot\rho \times \gamma), \\
\dot X &= (X - \rho) \times (\Omega - \kappa\gamma), \\
\dot\gamma &= \gamma \times \Omega,
\end{aligned} \tag{37}$$

on $\mathbb{R}^9 \ni (X, \gamma, K)$, where $\Omega = \Omega(\gamma, K)$ is as in (36) and $\dot\rho$ is shorthand for $DF(\gamma)(\gamma \times \Omega)$.[7]

The equation for γ is the well-known evolution equation of the Poisson vector γ that can be deduced by direct differentiation of the defining relation $\gamma = g^{-1}e_3^s$. The evolution equation for X follows by differentiating $X = g^{-1}x$ and using the nonholonomic constraint (25). Both of these equations are kinematical. The equation for K is a balance of momentum. The full dynamics of the system on M is obtained by adjoining the reconstruction equation $\dot g = g\hat\Omega$.

Proof. We begin by writing the equations of motion as

$$m\ddot x = -m\mathcal{G}e_3 + R_1, \qquad \frac{d}{dt}(\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega + R_2, \tag{38}$$

where $R = (R_1, R_2)$ is the nonholonomic constraint force/torque. D'Alembert's principle states that (R_1, R_2) should annihilate any vector in the distribution $\mathcal{D}$. In view of (19) one finds that

[7] It is immediate to check that both $\|\gamma\|^2$ and $\gamma \cdot (X - \rho)$ are first integrals of (37). For the latter one should use the kinematic relation $\dot\rho \cdot \gamma = 0$.

$R_2 = (g^{-1}R_1) \times \rho$. On the other hand, differentiating the constraint (25) and combining it with the first of the above equations yields

$$g^{-1}R_1 = m\mathcal{G}\gamma + m\Omega \times (\Omega \times \rho) + m\dot{\Omega} \times \rho + m\Omega \times \dot{\rho} + m\kappa\gamma \times (g^{-1}\dot{x} - \Omega \times \rho - \dot{\rho})$$

Using again (25) and (35) this simplifies to

$$g^{-1}R_1 = m\mathcal{G}\gamma + m\Omega \times (\Omega \times \rho) + m\dot{\Omega} \times \rho + m\Omega \times \dot{\rho} + m\kappa\dot{\rho} \times \gamma - m\kappa^2(X - \rho).$$

Using this expression and substituting $R_2 = (g^{-1}R_1) \times \rho$ in the second equation of (38) gives

$$\begin{aligned}\frac{d}{dt}(\mathbb{I}\Omega) = \mathbb{I}\Omega \times \Omega + m\mathcal{G}\gamma \times \rho + m(\Omega \times (\Omega \times \rho)) \times \rho + m(\dot{\Omega} \times \rho) \times \rho + m(\Omega \times \dot{\rho}) \times \rho \\ + m\kappa\rho \times (\kappa X - \dot{\rho} \times \gamma)\,.\end{aligned}$$

A simple calculation that uses the definition of K shows that the above relation is equivalent to the first equation in (37). □

Remark. For future reference we note that if the body is a sphere whose center of mass coincides with its geometric center (a Chaplygin ball) then $\rho = r\gamma$, where $r > 0$ is the sphere's radius, and (37) simplifies to

$$\begin{aligned}\dot{K} &= K \times \Omega - mr^2\kappa\gamma \times \Omega + mr\kappa^2\gamma \times X,\\ \dot{X} &= (\kappa\gamma - \Omega) \times X + r\Omega \times \gamma,\\ \dot{\gamma} &= \gamma \times \Omega.\end{aligned} \tag{39}$$

As it is checked directly, $K \cdot \gamma$ is a first integral of the system, so conservation of the moving energy (24) implies conservation of

$$\tilde{E} = \frac{1}{2}K \cdot \Omega - \frac{m}{2}\kappa^2\|X\|^2\,, \tag{40}$$

that is also a moving energy (see the remark at the end of the next section).

8.2 The equations of motion for an n-dimensional Chaplygin sphere rolling on a steadily rotating hyperplane. We continue using the notation introduced in section 7. Here too, (32) can be solved for Ω and as global coordinates on M we may use (x, g, K).

Proposition 19. *The equations of motion for an n dimensional Chaplygin ball that rolls without slipping on a hyperplane that steadily rotates with angular velocity $\eta \in \mathfrak{so}(n)$ are given by*

$$\begin{aligned}\dot{K} &= [K, \Omega] - mr(g^{-1}\eta\dot{x}) \wedge \gamma,\\ \dot{x} &= r(\mathrm{Ad}_g\Omega)e_n + \eta x,\\ \dot{g} &= g\Omega,\end{aligned} \tag{41}$$

where $[\cdot,\cdot]$ denotes the matrix commutator in $\mathfrak{so}(n)$ and the wedge product of vectors $a, b \in \mathbb{R}^n$ is defined by $a \wedge b = ab^T - ba^T$.

Proof. The second and third equation of (41) follow, respectively, from the constraint (29) and the definition of Ω. So we only need to prove that the first equation, that is a balance of momentum, holds. Denote by $(R_1, R_2) \in \mathbb{R}^n \times \mathfrak{so}(n)$ the force/torque exerted by the constraint. The equations of motion are

$$m\ddot{x} = R_1, \qquad \mathbb{I}\dot{\Omega} = [\mathbb{I}\Omega, \Omega] + R_2.$$

Differentiating (29) we obtain

$$R_1 = mr(\mathrm{Ad}_g\dot{\Omega})e_n + m\eta\dot{x}. \tag{42}$$

In order to determine R_2, we use the fact that d'Alembert's principle implies that the reaction force (R_1, R_2) annihilates any $(\dot{x}, \Omega)$ belonging to the distribution $\mathcal{D}$. As we now show, this condition gives

$$R_2 = -r\mathrm{Ad}_{g^{-1}}(R_1 \wedge e_n). \tag{43}$$

Let $(\dot{x}, \Omega) \in \mathcal{D}_{(x,g)}$. Then $\dot{x} = r(\mathrm{Ad}_g\Omega)e_n$, and d'Alembert's principle implies that

$$R_1 \cdot (r(\mathrm{Ad}_g\Omega)e_n) + \langle R_2, \Omega\rangle = 0. \tag{44}$$

Note however that

$$\begin{aligned} R_1 \cdot ((\mathrm{Ad}_g\Omega)e_n) &= \frac{1}{2}\mathrm{Trace}\left(R_1((\mathrm{Ad}_g\Omega)e_n)^T\right) + \frac{1}{2}\mathrm{Trace}\left(((\mathrm{Ad}_g\Omega)e_n)R_1^T\right) \\ &= -\frac{1}{2}\mathrm{Trace}\left(R_1e_n^T(\mathrm{Ad}_g\Omega) - e_nR_1^T(\mathrm{Ad}_g\Omega)\right) \\ &= \langle R_1 \wedge e_n, \mathrm{Ad}_g\Omega\rangle \\ &= \langle \mathrm{Ad}_{g^{-1}}(R_1 \wedge e_n), \Omega\rangle, \end{aligned}$$

so (44) may be rewritten as

$$\langle R_2 + r\mathrm{Ad}_{g^{-1}}(R_1 \wedge e_n), \Omega\rangle = 0.$$

Since $\Omega \in \mathfrak{so}(n)$ may be chosen arbitrarily, and the pairing $\langle\cdot,\cdot\rangle$ is non-degenerate, then (43) holds.

Inserting (42) into (43) leads to

$$\begin{aligned} R_2 = -r\mathrm{Ad}_{g^{-1}}(R_1 \wedge e_n) &= -mr^2(\dot{\Omega}\gamma) \wedge \gamma - mr(g^{-1}\eta\dot{x}) \wedge \gamma \\ &= -mr^2(\Gamma\dot{\Omega} + \dot{\Omega}\Gamma) - mr(g^{-1}\eta\dot{x}) \wedge \gamma, \end{aligned}$$

where we have used the identity $\mathrm{Ad}_{g^{-1}}(a \wedge b) = (g^{-1}a) \wedge (g^{-1}b)$, and the definitions of γ, Γ. Therefore,

$$\mathbb{I}\dot{\Omega} = [\mathbb{I}\Omega, \Omega] - mr^2(\Gamma\dot{\Omega} + \Gamma\dot{\Omega}) - mr(g^{-1}\eta\dot{x}) \wedge \gamma,$$

that is seen to be equivalent to the last equation in (41) by using the definition of K. □

A direct calculation shows that both the Lagrangian (26) and the affine constraint distribution determined by (28) are invariant under the tangent lift of the H-action on Q defined by

$$h \cdot (x, g) = (hx, hg)\,,$$

where H is the following closed, Lie subgroup of $\mathrm{SO}(n)$

$$H := \{h \in \mathrm{SO}(n) \,:\, h^{-1}e_n = e_n, \quad \mathrm{Ad}_h\eta = \eta\}.$$

The reduced equations are conveniently written in terms of K, γ, $X := g^{-1}x$, and $\Xi := \mathrm{Ad}_{g^{-1}}\eta$. These variables are not independent but satisfy

$$\Xi \in \mathcal{O}_\eta, \qquad \gamma \cdot X = r, \qquad \Xi\gamma = 0, \qquad \|\gamma\| = 1. \tag{45}$$

Here $\mathcal{O}_\eta$ denotes the adjoint orbit through $\eta \in \mathfrak{so}(n)$, and the other relations follow from our previous assumptions $x_n = r$, $\eta e_n = 0$, and the definition of the Poisson vector γ.

Proposition 20. *The H-reduction of the system* (41) *is given by the restriction to the invariant manifold* (45) *of the following system for* $(K, X, \gamma, \Xi) \in \mathfrak{so}(n) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathfrak{so}(n)$

$$\begin{aligned} \dot{K} &= [K, \Omega] - mr^2[\Xi, (\Omega\gamma) \wedge \gamma] - mr[\Xi, [\Xi, X \wedge \gamma]], \\ \dot{X} &= (\Xi - \Omega)X + r\Omega\gamma, \\ \dot{\gamma} &= -\Omega\gamma, \\ \dot{\Xi} &= [\Xi, \Omega]. \end{aligned} \tag{46}$$

Moreover, the conserved moving energy (33) *is H-invariant and may be written as*

$$E_{L,Y_\eta}|_M(\Omega, X, \gamma, \Xi) = \frac{1}{2}\langle K, \Omega\rangle - \frac{m}{2}\|\Xi X\|^2 \,. \tag{47}$$

Proof. The equations for γ and Ξ are purely kinematical and follow from the definition of $\Omega = g^{-1}\dot{g}$. The equation for X follows by differentiating $X = g^{-1}x$ and using the constraint (29). On the other hand, using again (29), we find

$$g^{-1}\eta\dot{x} = r\Xi\Omega\gamma + \Xi^2 X.$$

Now, given that $\Xi\gamma = 0$, we have $(\Xi a) \wedge \gamma = [\Xi, a \wedge \gamma]$ for all $a \in \mathbb{R}^n$. Therefore

$$(\Xi\Omega\gamma) \wedge \gamma = [\Xi, (\Omega\gamma) \wedge \gamma], \qquad (\Xi^2 X) \wedge \gamma = [\Xi, [\Xi, X \wedge \gamma]].$$

Whence,

$$(g^{-1}\eta\dot{x}) \wedge \gamma = r[\Xi, (\Omega\gamma) \wedge \gamma] + [\Xi, [\Xi, X \wedge \gamma]],$$

and the first equation in (46) follows from the first equation in (41).

Finally, the statement for the conserved moving energy follows by noticing that $\|\eta x\|^2 = \|\Xi X\|^2$. □

Remark. The symmetry group H is a subgroup of the copy of $\mathrm{SO}(n-1)$ inside $\mathrm{SO}(n)$ that fixes e_n. It is in fact a proper subgroup except when $n = 3$ where $H = \mathrm{SO}(2)$. An explanation for this is that in 3 dimensions the vector that is normal to the plane where the rolling takes place is also the axis of rotation of the plane. In dimension $n \geq 4$ this interpretation of the vector that is normal to the hyperplane where the rolling takes place is no longer possible and, consequently, the symmetry group of the problem is smaller.

In the special case $n = 3$, one may identify $\Xi \in \mathfrak{so}(3)$ with $\kappa\gamma \in \mathbb{R}^3$, where $\kappa \in \mathbb{R}$ is the angular speed of rotation of the plane where the rolling takes place. One may then check that (46) is equivalent to (39) via the hat map isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, and that the conserved moving energy (47) differs from (40) by a constant.

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