

Conservation of ‘moving’ energy in nonholonomic systems with affine constraints and integrability of spheres on rotating surfaces*

FRANCESCO FASSÒ[†] AND NICOLA SANSONETTO^{‡§}

(April 25, 2018)

Abstract

Energy is in general not conserved for mechanical nonholonomic systems with affine constraints. In this article we point out that, nevertheless, in certain cases, there is a modification of the energy that is conserved. Such a function coincides with the energy of the system relative to a different reference frame, in which the constraint is linear. After giving sufficient conditions for this to happen, we point out the role of symmetry in this mechanism. Lastly, we apply these ideas to prove that the motions of a heavy homogeneous solid sphere that rolls inside a convex surface of revolution in uniform rotation about its vertical figure axis, are (at least for certain parameter values and in open regions of the phase space) quasi-periodic on tori of dimension up to three.

Keywords: Nonholonomic mechanical systems, Conservation of energy, Rolling rigid bodies, Symmetries and momentum maps, Integrability.

MSC: 70F25, 37J60, 37J15, 70E18

1. Introduction

This paper is ultimately addressed to the class of mechanical systems formed by a rigid body subject to the nonholonomic constraint of rolling without sliding on a surface which moves in a preassigned way. This type of nonholonomic constraints are affine, not linear, in the velocities. Consequently, even if the system is time-independent (which may easily be the case in presence of symmetries of body and surface, and if the latter moves at uniform speed) energy need not be conserved. In fact, while energy is conserved in time-independent nonholonomic systems with constraints that are linear in the velocities (see e.g. [23, 22]), the same is not always true if the constraints are affine in the velocities (see e.g. [19, 17] and Section 3.5 below). A simple example of this situation is the classical system formed by a sphere that rolls without sliding on a table that rotates uniformly, studied by Pars [23], Neimark and Fufaev [22] and others, in which the energy is not conserved.

*This work is part of the research projects *Symmetries and integrability of nonholonomic mechanical systems* of the University of Padova and PRIN *Teorie geometriche e analitiche dei sistemi Hamiltoniani in dimensioni finite e infinite*.

[†]Università di Padova, Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy. Email: fasso@math.unipd.it

[‡]Università di Padova, Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy. Email: sanson@math.unipd.it

[§]Supported by the Research Project *Symmetries and integrability of nonholonomic mechanical systems* of the University of Padova.

The starting point of this paper is the observation that for a nonholonomic system with affine constraints, even if the energy is not conserved, there might exist a modification of the energy—that may be interpreted as the energy of the system relative to a different reference frame and for this reason will be called *moving energy*—that is conserved. The reason underneath this fact is simply that, in a moving reference frame in which the surface is at rest, the constraint is linear. Therefore, if the system happens to be time-independent relatively to such a moving frame, its energy relative to that frame is conserved. And the condition of time-independence is easily verified in presence of symmetries.

We study the existence of conserved moving energies in Section 3. For simplicity, instead of changing the reference frame with respect to which the system is described, we pass to a moving system of coordinates. After illustrating the mechanism on the well known example of a sphere on a turntable, we give sufficient conditions for the existence of a conserved moving energy (Theorem 1). Even though these conditions might appear very special, we relate them to the existence of symmetries: we assume that a group acts in configuration space and show that certain invariance properties of the system lead to a conserved moving energy (Theorem 2). Interestingly, this conserved function is the sum of two functions that, at variance from what would happen in a holonomic system, are not conserved: the energy of the system and a component of the momentum map of the lifted action. There is here a connection with the failure of Noether theorem in nonholonomic mechanics, that we discuss.

Lastly, in Section 4 we apply the results of Section 3 to the study of the system formed by a heavy homogeneous solid sphere that rolls without sliding inside a convex surface of revolution, which rotates around its (vertical) figure axis with constant speed Ω . When $\Omega = 0$ this system is known to be integrable, with dynamics quasi-periodic on tori of dimension up to three [16, 27]. If $\Omega \neq 0$ the system is $\text{SO}(3) \times S^1$ -invariant and, so far, it was only known that its four-dimensional reduced system admits two first integrals and a conserved measure, and thus that it is integrable by quadratures [6]. By exploiting the existence of an additional first integral given by a moving energy, much stronger integrability results can be obtained. Here we give a first, general result in this direction. Specifically, using essentially a continuity argument from the case $\Omega = 0$, we prove that, for small Ω , there is an open nonempty subset of the reduced phase space in which the reduced dynamics is periodic, and correspondingly an open nonempty subset of the unreduced phase space in which the unreduced dynamics is quasi-periodic on tori of dimension up to three (Theorem 3).

Even though our primary interest is toward time-independent nonholonomic systems, the need of considering time-dependent coordinates forces us to work in the time-dependent context. This somewhat complicates the notation. In the hope of keeping the complexity to a minimum, we adopt a Lagrangian description on the extended phase space of time-dependent mechanical nonholonomic systems, which is quickly described in Section 2.

For general introductions to nonholonomic mechanics see e.g. [23, 22, 7, 4, 19, 3, 8]; the time-dependent case is treated, using the formalism of jet bundles, in [20, 25, 26, 21]. Throughout the paper all manifolds and maps are assumed to be smooth, and all vector fields are assumed to be complete.

2. Nonholonomic systems with affine constraints

2.1 Time-independent nonholonomic systems with affine constraints. First we briefly recall the time-independent case, mainly to fix some notation. The starting point is a Lagrangian system with n -dimensional configuration manifold Q and Lagrangian $L : TQ \rightarrow \mathbb{R}$, that we assume to be regular; hence, in each set of local bundle coordinates (q, \dot{q}) the matrix $\frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}}(q)$ is everywhere invertible.

An affine nonholonomic constraint consists in the prescription that the kinematic states of the system belong to the fibers $(\mathcal{M}_0)_q$, $q \in Q$, of an affine distribution \mathcal{M}_0 on Q , that we assume to have constant rank $r > 1$ and to be nonintegrable. Thus, there are a vector field ξ_0 on Q and a nonintegrable distribution \mathcal{D}_0 of constant rank r on Q such that

$$(\mathcal{M}_0)_q = \{v_q \in T_q Q : v_q - \xi_0(q) \in (\mathcal{D}_0)_q\} \quad \forall q \in Q.$$

Of course, given \mathcal{M}_0 and \mathcal{D}_0 , ξ_0 is defined up to a section of \mathcal{D}_0 . \mathcal{D}_0 will be said to be the distribution associated to \mathcal{M}_0 . The affine distribution \mathcal{M}_0 may also be regarded as a submanifold M_0 of TQ , which is in fact an affine subbundle of TQ . We call M_0 the *constraint submanifold*. The case of linear constraints is recovered when $\xi_0 = 0$; the constraint manifold is thus a linear subbundle of TQ .

We assume that the nonholonomic constraint is ‘ideal’, that is, that it satisfies d’Alembert principle (see e.g. [2, 20, 19]): at each $q \in Q$, the set of reaction forces that the constraint can exert is (an appropriate jet extension of) the annihilator $(\mathcal{D}_0)_q^\circ \subset T_q^* Q$ of the fiber $(\mathcal{D}_0)_q \subset T_q Q$ of the distribution \mathcal{D}_0 associated to the constraint submanifold. It is well known that, under this assumption, there is a unique choice of the reaction force as a function

$$R_{L, M_0} : M_0 \rightarrow \mathcal{D}_0^\circ$$

such that the restriction to M_0 of Lagrange equations with the reaction force defines a vector field on M_0 (see e.g. [1, 3]). We denote this vector field on M_0 as X_{L, Q, M_0} and call it *time-independent nonholonomic system, with affine constraints* if M_0 is an affine subbundle of TQ and *with linear constraint* if M_0 is a linear subbundle of TQ .

2.2 Definitions and notation for the time-dependent case. In order to consider time-dependent nonholonomic systems we pass to the extended phase space. In doing so, we need a number of definitions that we collect in this section. Let Q be an n -dimensional manifold.

An *m -dimensional extended submanifold M of TQ* is an $(m + 1)$ -dimensional submanifold of the extended phase space $TQ \times \mathbb{R}$ of the form

$$M = \{M_t \times \{t\} : t \in \mathbb{R}\}. \quad (1)$$

Thus, for each t , M_t is a submanifold of TQ of dimension m . The reason for the use of the term ‘extended’, instead of the perhaps more natural ‘time-dependent’, is that we need to treat both time-dependent and time-independent cases within the same context. We say that the extended submanifold M is *time-independent* if

$$M = M_0 \times \mathbb{R}$$

for a given submanifold M_0 of TQ , or equivalently if $M_t = M_0$ for all t , and that it is *time-dependent* otherwise.

If all the M_t ’s in (1) are linear subbundles of TQ , then the extended submanifold M is an *extended linear subbundle* of TQ . If they are all affine subbundles, then M is an

extended affine subbundle of TQ . Obviously, we regard extended subbundles as special cases of extended affine subbundles.

An *extended distribution* \mathcal{D} on Q is a distribution on $Q \times \mathbb{R}$ with fibers

$$\mathcal{D}_{(q,t)} = (\mathcal{D}_t)_q \oplus \{0\}$$

where, for each t , the $(\mathcal{D}_t)_q$ are the fibers of a distribution \mathcal{D}_t on Q . If all the distributions \mathcal{D}_t have rank r , then we say that \mathcal{D} has rank r . We say that \mathcal{D} is nonintegrable if (some at least of) the distributions \mathcal{D}_t are nonintegrable. An extended distribution \mathcal{D} on Q of rank r generates an extended linear subbundle $D = \{D_t \times \{t\} : t \in \mathbb{R}\}$ of TQ of dimension $n + r$, with $D_t = \{v_q : q \in Q, v_q \in (\mathcal{D}_t)_q\}$, and vice versa. \mathcal{D} is *time-independent* if $\mathcal{D}_t = \mathcal{D}_0$ for all t .

An *extended vector field* on Q is a vector field ξ on $Q \times \mathbb{R}$ whose \mathbb{R} -component is identically equal to 0, namely

$$\xi(q, t) = \xi_t(q) + 0 \partial_t$$

with each ξ_t a vector field on Q . ξ is *time-independent* if $\xi_t = \xi_0$ for all t .

If \mathcal{D} is an extended distribution on Q of rank r and ξ is an extended vector field on Q , then

$$\mathcal{M} = \mathcal{D} + \xi$$

is an *affine extended distribution* of rank r on Q . Thus \mathcal{M} has fibers $\mathcal{M}_{(q,t)} = (\mathcal{D}_t)_q + \xi_t(q)$, or $\mathcal{M}_{(q,t)} = (\mathcal{M}_t)_q \oplus \{0\}$ with $\mathcal{M}_t = \mathcal{D}_t + \xi_t$. \mathcal{D} is called the *extended distribution associated to* \mathcal{M} . \mathcal{M} can be regarded in an obvious way as an extended affine subbundle M of TQ of dimension $n + r$. \mathcal{M} is *time-independent* if so are ξ and \mathcal{D} .

Finally, a *dynamical system* on an extended submanifold M of TQ is a vector field on M whose \mathbb{R} -component is identically equal to 1, that is

$$X(v_q, t) = X_t(v_q) + \partial_t$$

with each X_t a vector field on M_t . (The difference with respect an extended vector field is that now time does not stay constant, which is necessary for the dynamics). X is time-independent if $X_t = X_0$ for all t .

2.3 Time-dependent nonholonomic systems. We start now from a Lagrangian system with n -dimensional configuration manifold Q and time-dependent regular Lagrangian $L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$. The time dependency of the Lagrangian accounts, for instance, for the presence of time-dependent holonomic constraints.¹

We add now the nonholonomic constraint that its kinematical states belong to an extended affine subbundle $M = \{M_t \times \{t\} : t \in \mathbb{R}\}$ of TQ of dimension $n + r$, for some $1 < r < n$, that we call the *extended constraint submanifold*. This extended affine subbundle corresponds to an extended affine distribution $\mathcal{M} = \mathcal{D} + \xi$ of rank r on Q . We assume that the associated extended distribution \mathcal{D} is nonintegrable.

The condition of ‘ideality’ of the constraint now means that, at each t and q , the set of reaction forces that the constraint can exert is (a jet extension of) the annihilator

¹A large class of time-dependent holonomic constraints for systems of N material points can be modelled in this way. After the choice of a reference frame, that provides a (time-dependent) identification of physical 3-space with \mathbb{R}^3 , a time-dependent holonomic constraint is given by a time-dependent embedding of a manifold Q into the configuration space $(\mathbb{R}^3)^N$ of the unconstrained system, and the Lagrangian is the restriction of the Lagrangian of the unconstrained system to the resulting time-dependent, extended submanifold.

$(\mathcal{D}_t)_q^\circ \subset T_q^*Q$ of the fiber $(\mathcal{D}_t)_q$ [20, 19], and implies that there is a unique choice of the reaction force as a function

$$R_{L,M} : M \rightarrow \mathcal{D}^\circ$$

such that the restriction to M of Lagrange equations with the reaction force defines a dynamical system $X_{L,Q,M}$ on M . Here, \mathcal{D}° is the extended codistribution on $Q \times \mathbb{R}$ with fibers $(\mathcal{D}_t)_q^\circ \oplus \{0\}$.

Definition. Let $L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$ a regular Lagrangian and M an extended affine subbundle of TQ .

- i. The dynamical system $X_{L,Q,M}$ on M is called the nonholonomic system with affine constraints (or, shortly, the nonholonomic system) with Lagrangian L and extended constraint manifold M .
- ii. If M is an extended linear subbundle of TQ then we say that $X_{L,Q,M}$ has linear constraints.
- iii. If L and M are time-independent, then we say that $X_{L,Q,M}$ is time-independent.

In the time-independent case we will routinely identify X_{L,Q,M_0} and $X_{L,Q,M_0 \times \mathbb{R}}$, and, depending on the context, we will regard the Lagrangian L as defined either on TQ or on $TQ \times \mathbb{R}$.

In (possibly time-dependent) bundle coordinates (q, \dot{q}) in TQ , the fibers of the distributions \mathcal{D}_t on Q are the kernels of a (q, t) -dependent $k \times n$ matrix $S(q, t)$ that has everywhere rank k , with $k = n - r$:

$$(\mathcal{D}_t)_q = \{\dot{q} \in T_qQ : S(q, t)\dot{q} = 0\}.$$

Thus, $\dot{q} \in (\mathcal{M}_t)_q$ if and only if $\dot{q} = \xi_t(q) + u$ with $u \in \ker S(q, t)$, that is, if and only if $S(q, t)[\dot{q} - \xi_t(q)] = 0$. It follows that, for each t and q , the affine subspace $(\mathcal{M}_t)_q$ of T_qQ is described by

$$S(q, t)\dot{q} + s(q, t) = 0 \tag{2}$$

with $s(q, t) = -S(q, t)\xi_t(q) \in \mathbb{R}^k$. Of course, only $\ker S$ is uniquely defined, not S , ξ and s . In coordinates, the annihilator of the fiber $(\mathcal{D}_t)_q$ is the range of the matrix $S(q, t)^T$, and the reaction force $R_{L,M}(q, \dot{q}, t) \in \text{range}[S(q, t)^T]$.

2.4 Time-dependent diffeomorphisms and conjugation of nonholonomic systems. In order to implement time-dependent coordinate changes, we need to consider (lifted) diffeomorphisms of the configuration space that depend on time, and use them to transform nonholonomic systems. (We may use now the expression ‘time-dependent’, instead of ‘extended’, because we will never need to consider ‘time-independent’ time-dependent change of coordinates and there will be no ambiguities).

By a *time-dependent diffeomorphism* of a manifold U onto a manifold Q we mean a diffeomorphism $\mathcal{C} = (\mathcal{C}_Q, \mathcal{C}_\mathbb{R}) : U \times \mathbb{R} \rightarrow Q \times \mathbb{R}$ whose second component $\mathcal{C}_\mathbb{R} : U \times \mathbb{R} \rightarrow \mathbb{R}$ is the identity between the \mathbb{R} -factors. The first component $\mathcal{C}_Q : U \times \mathbb{R}$ is a differentiable map, that in the sequel we denote \mathcal{Q} . Thus,

$$\mathcal{C} = (\mathcal{Q}, \text{id}_\mathbb{R}) : U \times \mathbb{R} \rightarrow Q \times \mathbb{R}, \quad \mathcal{C}(u, t) = (\mathcal{Q}(u, t), t)$$

and, for each t , the map

$$\mathcal{Q}_t := \mathcal{Q}(\cdot, t) : U \rightarrow Q$$

is a diffeomorphism (and “smoothly depends on t ”).

With the identifications $T(U \times \mathbb{R}) \simeq TU \times T\mathbb{R}$ and $T(Q \times \mathbb{R}) \simeq TQ \times T\mathbb{R}$, the tangent map $TC : T(U \times \mathbb{R}) \rightarrow T(Q \times \mathbb{R})$ can be seen as a diffeomorphism from $TU \times T\mathbb{R}$ to $TQ \times T\mathbb{R}$ whose second component is the identity on the factor $T\mathbb{R}$. Restricting TC to the unit tangent vector in the $T\mathbb{R}$ -factor gives the diffeomorphism $TC|_{i=1} : TU \times \mathbb{R} \times \{1\} \rightarrow TQ \times \mathbb{R} \times \{1\}$, that we regard as a diffeomorphism

$$DC : TU \times \mathbb{R} \rightarrow TQ \times \mathbb{R}.$$

Explicitly, if for all $u \in U$ and $t \in \mathbb{R}$ we write

$$\dot{Q}_t(u) := \frac{\partial}{\partial t} Q(u, t) \in T_{Q_t(u)} Q,$$

then

$$DC(v_u, t) = (T_u Q_t \cdot v_u + \dot{Q}_t(u), t) \in T_{Q_t(u)} Q \times \mathbb{R}$$

for all $u \in U$, $t \in \mathbb{R}$ and $v_u \in T_u U$. Clearly, DC is a time-dependent diffeomorphism of TU onto TQ .

In coordinates ($u \in U$, $q \in Q$) we will write \mathcal{Q}' for $\frac{\partial Q}{\partial u}$ and $\dot{\mathcal{Q}}$ for $\frac{\partial Q}{\partial t}$ and, for given t , $\mathcal{Q}'_t = \frac{\partial Q}{\partial u}(\cdot, t)$ and $\dot{\mathcal{Q}}_t = \frac{\partial Q}{\partial t}(\cdot, t)$. Thus, \mathcal{Q}' and $\dot{\mathcal{Q}}$ are defined on $U \times \mathbb{R}$ while, for each t , \mathcal{Q}'_t and $\dot{\mathcal{Q}}_t$ are defined on U . With this notation, $TC(u, \dot{u}, t, \dot{t}) = (Q(u, t), \mathcal{Q}'_t(u)\dot{u} + \dot{\mathcal{Q}}_t(u)\dot{t}, t, \dot{t})$ and

$$DC(u, \dot{u}, t) = (Q_t(u), \mathcal{Q}'_t(u)\dot{u} + \dot{\mathcal{Q}}_t(u), t). \quad (3)$$

If \mathcal{C} is a time-dependent diffeomorphism from U onto Q , then the pull back $\tilde{M} := DC^{-1}(M)$ of an extended affine subbundle $M = \{M_t \times \{t\} : t \in \mathbb{R}\}$ of TQ is an extended affine subbundle of TU . In coordinates, if M is described by $S(q, t)\dot{q} + s(q, t) = 0$ then \tilde{M} is described by $\tilde{S}(u, t)\dot{u} + \tilde{s}(u, t) = 0$ with

$$\tilde{S} = (S \circ \mathcal{C})\mathcal{Q}', \quad \tilde{s} = s \circ \mathcal{C} + (S \circ \mathcal{C})\dot{\mathcal{Q}}, \quad (4)$$

as is verified requiring that, for each t , $(u, \dot{u}) \in \tilde{M}_t$ if and only if $(Q_t(u), \mathcal{Q}'_t(u)\dot{u} + \dot{\mathcal{Q}}_t(u)) \in M_t$.

The following fact is proven, in the time-independent case, in [12]; the generalization to the time-dependent case is straightforward and we omit it.

Proposition 1. *Consider a nonholonomic system $X_{L,Q,M}$ and a time-dependent diffeomorphism \mathcal{C} from a manifold U onto Q . Then, the pull-back of $X_{L,Q,M}$ under the restriction to M of DC coincides with the nonholonomic system $X_{\tilde{L},U,\tilde{M}}$ with $\tilde{L} = L \circ DC$ and $\tilde{M} = DC^{-1}(M)$.*

3. Conservation of moving energy

3.1 Example. We begin by illustrating the mechanism we have in mind on a sample system—the well known sphere on a turntable considered by Pars [23], Neimark and Fufaev [22] and several others, see e.g. [5, 7].

This system is formed by a homogeneous solid sphere constrained to roll without sliding on a table which, relatively to an inertial reference frame, rotates with constant rate Ω around an axis orthogonal to it. In the mentioned references, and in all the other works we could find, the system is described with respect to the inertial reference frame,

but is nevertheless time-independent. Let $\{O; x, y, z\}$ be such a frame, and assume that the table lies in the xy -plane, and rotates about the z -axis.

The configuration manifold Q is $\mathbb{R}^2 \times \text{SO}(3) \ni (q, R)$ where $q = (x, y)$ are the coordinates of the point of contact between sphere and table and R is the attitude matrix of the sphere. We identify $T\text{SO}(3)$ and $\text{SO}(3) \times \mathbb{R}^3$ via right trivialization. The constraint manifold M_0 is 8-dimensional and is diffeomorphic to $\mathbb{R}^2 \times \text{SO}(3) \times \mathbb{R}^3 \ni (q, R, \omega)$, where $\omega = (\omega_x, \omega_y, \omega_z)$ is the angular velocity in space of the sphere. Up to an inessential factor m , the mass of the sphere, the Lagrangian is

$$L = \frac{1}{2} \|\dot{q}\|^2 + \frac{1}{2} ca^2 \|\omega\|^2 \quad (5)$$

where a is the radius of the sphere and ca^2 , with $c > 0$, its moment of inertia (divided by m). The condition of rolling without sliding is given by the affine constraint

$$\dot{x} = a\omega_y - \Omega y, \quad \dot{y} = -a\omega_x + \Omega x. \quad (6)$$

The Lagrangian and the constraints are $\text{SO}(3)$ -invariant.² Reduction under this action consists merely in cutting away the $\text{SO}(3)$ factor and produces a 5-dimensional reduced system on $\mathbb{R}^2 \times \mathbb{R}^3 \ni (q, \omega)$. The equations of motion of the reduced system are the two equations (6) and the three equations

$$\dot{\omega}_x = \frac{\nu}{a} (a\omega_y - \Omega y), \quad \dot{\omega}_y = \frac{\nu}{a} (-a\omega_x + \Omega x), \quad \dot{\omega}_z = 0$$

where $\nu = \frac{\Omega}{1+c}$, while the reconstruction equation is $\dot{\omega} = \dot{R}R^T$. It is elementary to show that the solutions of the reduced systems are periodic, with frequency ν .

In some of the quoted references, e.g. in [22], it is remarked that the reduced equations have the three independent first integrals

$$\omega_z, \quad a\omega_x - \nu x, \quad a\omega_y - \nu y.$$

However, the periodicity of a flow in a 5-dimensional phase space implies the existence of four, not just three, independent first integrals. To our knowledge (and surprise), this fact, and the existence of a fourth independent first integral, do not seem to have been noticed before.

The obvious candidate for the missing first integral would seem to be the (projection to the reduced phase space of the) energy E_{L, M_0} of the nonholonomic system, which is the restriction to the constraint manifold M_0 of the energy E_L of the Lagrangian L , that in this case coincides with L . However, energy is not conserved: the equations of motions give

$$\frac{d}{dt} E_{L, M_0} = ac\nu\Omega(x\omega_y - y\omega_x).$$

Nevertheless, a simple computation shows that the function

$$E_{L, M_0} - \Omega^2(x^2 + y^2) + \Omega a(x\omega_x + y\omega_y) \quad (7)$$

is a first integral of the nonholonomic system. Being $\text{SO}(3)$ -invariant, this function is also a first integral of the reduced system, and it is independent of the previous three (except where $\dot{x} = \dot{y} = 0$).

²They are in fact invariant under an action of $\text{SO}(3) \times S^1$, but this is not used in the quoted references.

This additional first integral has a simple interpretation. In a system of time-dependent, rotating coordinates in which the table is at rest, the constraint of rolling without sliding on the table is linear, and the Lagrangian, which is the pull back \tilde{L} of L , is still time-independent. Therefore, the energy is now conserved and its push forward to the original coordinates is a first integral, that turns out to coincide with (7).

The reason why the push forward of the energy in the rotating coordinates is different from the original energy (and may thus be conserved) is due to the time-dependency of the coordinate change. The Lagrangian in the rotating coordinates has the form $\tilde{L} = \tilde{L}_2 + \tilde{L}_1 + \tilde{L}_0$, where the dependence of each \tilde{L}_i on the velocities is homogeneous of degree i .³ The function \tilde{L}_1 does not contribute to the energy $E_{\tilde{L}} = \tilde{L}_2 - \tilde{L}_0$ of \tilde{L} and the push forward of $E_{\tilde{L}}$ to the original coordinates differs from E_L by the push forward of \tilde{L}_1 .

Remark. A completely similar situation is met in the system formed by a vertical disk constrained to roll without sliding on a uniformly rotating plane, considered in [13]. In that reference, the system is actually studied in a rotating frame, where the constraint is linear and the energy is conserved. However, the authors directly integrate the reduced equations of motion without noticing the conservation of energy.

3.2 Moving energy and its conservation. First recall that the energy (or ‘Jacobi integral’) of a Lagrangian $L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$ is the function $E_L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$E_L(v_q, t) := \langle p(v_q, t), v_q \rangle_q - L(v_q, t) \quad \forall q \in Q, v_q \in T_q Q, t \in \mathbb{R},$$

where p is the momentum covector and $\langle \cdot, \cdot \rangle_q$ denotes the pairing between $T_q^* Q$ and $T_q Q$. In coordinates, $E_L = \dot{q} \cdot \frac{\partial L}{\partial \dot{q}} - L$, where the dot denotes the scalar product in \mathbb{R}^n .

Definition. Let $X_{L,Q,M}$ be a (either time-dependent or time-independent) nonholonomic system with affine constraints.

i. The energy $E_{L,M} : M \rightarrow \mathbb{R}$ of $X_{L,Q,M}$ is the restriction of E_L to M :

$$E_{L,M} := E_L|_M.$$

ii. If $\mathcal{C} : U \times \mathbb{R} \rightarrow Q \times \mathbb{R}$ is a time-dependent diffeomorphism, then the moving energy of $X_{L,Q,M}$ induced by \mathcal{C} is the restriction $E_{L,\mathcal{C},M}^*$ to M of the function

$$E_{L,\mathcal{C}}^* := E_{L \circ DC} \circ DC^{-1} : TQ \times \mathbb{R} \rightarrow \mathbb{R}. \quad (8)$$

Proposition 2. In the hypotheses of the above definition,

$$E_{L,\mathcal{C}}^* = E_L - \langle p, \mathring{Q} \circ \mathcal{C}^{-1} \rangle. \quad (9)$$

Proof. The proof can be done in coordinates. If $\tilde{L} = L \circ DC$, then, from (3), $\tilde{L}(u, \dot{u}, t) = L(\mathcal{Q}_t(u), \mathcal{Q}'_t(u)\dot{u} + \mathring{Q}_t(u), t)$. Thus $E_{\tilde{L}}(u, \dot{u}, t) = \dot{u} \cdot \frac{\partial \tilde{L}}{\partial \dot{u}}(u, \dot{u}, t) - \tilde{L}(u, \dot{u}, t) = \mathcal{Q}'_t(u)\dot{u} \cdot \frac{\partial L}{\partial \dot{q}}(DC(u, \dot{u}, t)) - L(DC(u, \dot{u}, t)) = E_L(DC(u, \dot{u}, t)) - \mathring{Q}_t(u) \cdot \frac{\partial L}{\partial \dot{q}}(DC(u, \dot{u}, t))$. In (9) we have written $\mathring{Q} \circ \mathcal{C}^{-1}$ instead of $\mathring{Q} \circ DC^{-1}$ because \mathcal{Q} is independent of the velocities. \square

³Clearly, \tilde{L}_2 may be interpreted as the kinetic energy, $-\tilde{L}_0$ as the potential energy of the centrifugal force and $-\tilde{L}_1$ as the generalized potential of the Coriolis force in a rotating, non-inertial reference frame in which the table is at rest. We prefer changing coordinates, instead of reference frames, since this exempts us from embedding the dependence on the choice of a reference frame into the theory, as e.g. in [20].

The interest of considering a moving energy $E_{L,C,M}^*$ resides in the fact that the function $E_{L,C}^*$ differs from the energy E_L of the Lagrangian L by a term which is produced by the time-dependence of the diffeomorphism.⁴ It is therefore possible that the function $E_{L,C,M}^*$ is a first integral even if $E_{L,M}$ is not. We now formalize this possibility in the case of time-independent nonholonomic systems:

Theorem 1. *Consider a time-independent nonholonomic system with affine constraints $X_{L,Q,M_0 \times \mathbb{R}}$ and a time-dependent diffeomorphism \mathcal{C} from a manifold U to Q . Assume that:*

- i. $L \circ DC$ is independent of t .*
- ii. $E_{L,C}^*$ is independent of t .*
- iii. $DC^{-1}(M_0 \times \mathbb{R})$ is a time-independent extended linear subbundle of TU .*

Then, the moving energy $E_{L,C,M_0 \times \mathbb{R}}^$ is a time-independent first integral of $X_{L,Q,M_0 \times \mathbb{R}}$.*

Proof. Hypothesis iii. means that $DC^{-1}(M_0 \times \mathbb{R}) = \tilde{M}_0 \times \mathbb{R}$ for a fixed linear subbundle \tilde{M}_0 of TU . By Proposition 1, DC conjugates $X_{L,Q,M_0 \times \mathbb{R}}$ to the nonholonomic system $X_{\tilde{L},U,\tilde{M}_0 \times \mathbb{R}}$ with $\tilde{L} = L \circ DC$. By hypotheses i. and iii., $X_{\tilde{L},U,\tilde{M}_0 \times \mathbb{R}}$ has linear constraints and is time-independent. Therefore, the energy $E_{\tilde{L},\tilde{M}_0 \times \mathbb{R}}$ is a (time-independent) first integral of $X_{\tilde{L},U,\tilde{M}_0 \times \mathbb{R}}$. It follows that its push-forward $E_{\tilde{L},\tilde{M}_0 \times \mathbb{R}} \circ (DC^{-1}|_{M_0 \times \mathbb{R}})$ is a first integral of $X_{L,Q,M_0 \times \mathbb{R}}$. Since DC maps $\tilde{M}_0 \times \mathbb{R}$ diffeomorphically onto $M_0 \times \mathbb{R}$,

$$E_{\tilde{L},\tilde{M}_0 \times \mathbb{R}} \circ (DC^{-1}|_{M_0 \times \mathbb{R}}) = (E_{\tilde{L},\tilde{M}_0 \times \mathbb{R}} \circ DC^{-1})|_{M_0 \times \mathbb{R}} = E_{L,C,M_0 \times \mathbb{R}}^*.$$

This proves that $E_{L,C,M_0 \times \mathbb{R}}^*$ is a first integral of $X_{L,Q,M_0 \times \mathbb{R}}$. Hypothesis ii. ensures that it is a time-independent function. \square

We have stated Theorem 1 in terms of time-independent nonholonomic systems on the extended phase space so as to properly regard, in hypotheses i. and ii., the functions L and $E_{L,C}^*$ as defined on $TQ \times \mathbb{R}$, even though constant on \mathbb{R} . But if we identify functions on TQ and functions on $TQ \times \mathbb{R}$ that are constant on \mathbb{R} , then Theorem 1 states that, *under hypotheses i., ii. and iii., a time-independent nonholonomic system with affine constraints X_{L,Q,M_0} has the time-independent first integral E_{L,C,M_0}^** (in fact, since $E_{L,C}^*$ is time-independent, $E_{L,C,M_0 \times \mathbb{R}}^*$ and E_{L,C,M_0}^* may be identified). From now on, we will adopt this point of view.

Remarks. (i) In time-dependent nonholonomic systems, either with linear or with affine constraints, the energy $E_{L,M}$ is ordinarily time-dependent; even though it is not impossible that it is a (time-dependent) first integral, we do not consider this case because, in our opinion, from a dynamical point of view only time-independent first integrals are of interest.

(ii) One might weaken the hypotheses of Theorem 1 in various ways, e.g. by requiring that only the restriction of $E_{L,C}^*$ to $M_0 \times \mathbb{R}$ be time-independent. However, the setting of Theorem 1 is sufficient for our application in Section 4.

3.3 On the conditions of Theorem 1. The situation of Theorem 1 might appear very special. Our next goal, in Section 3.4, is to show that such a situation is instead easily verified in presence of symmetries. In order to gain some insight on this possibility, we begin by establishing conditions under which $E_{L,C}^*$ is time-independent and conditions

⁴A fact which is well known in the theory of time-dependent canonical transformations.

under which the extended constraint submanifold $DC^{-1}(M_0 \times \mathbb{R})$ in the new coordinates is linear (even though possibly time-dependent).

Proposition 3. *Consider a time-independent nonholonomic system X_{L,Q,M_0} with affine constraints. Denote by $\xi_0 + \mathcal{D}_0$ the affine distribution on Q that corresponds to M_0 , with ξ_0 a vector field on Q and \mathcal{D}_0 a distribution on Q . Consider a time-dependent diffeomorphism $\mathcal{C} = (\mathcal{Q}, \text{id}_{\mathbb{R}})$ of Q to itself.*

i. $E_{L,\mathcal{C}}^$ is time-independent if and only if \mathcal{Q} is the flow of a vector field Y on Q , and in that case*

$$E_{L,\mathcal{C}}^* = E_L - \langle p, Y \rangle.$$

ii. Assume that \mathcal{Q} is the flow of a vector field Y on Q . Then, $DC^{-1}(M_0 \times \mathbb{R})$ is an extended linear subbundle of TQ if and only if the vector field $Y - \xi_0$ is a section of \mathcal{D}_0 .

Proof. (i) First note that $\mathring{\mathcal{Q}} \circ \mathcal{C}^{-1}(q, t) = \mathring{\mathcal{Q}}_t \circ \mathcal{Q}_t^{-1}(q)$ for all q and t . Let us write $Y(q, t) = \mathring{\mathcal{Q}}_t \circ \mathcal{Q}_t^{-1}(q)$ and note that $Y(q, t) \in T_q Q$. Since L and E_L are time-independent, it follows from (9) that $E_{L,\mathcal{C}}^*$ is time-independent if and only if $\langle Y, p \rangle$ is time-independent, that is, given that the momentum covector p does not depend on time, if and only if

$$\frac{\partial}{\partial t} \langle Y, p \rangle = \left\langle \frac{\partial Y}{\partial t}, p \right\rangle : TQ \times \mathbb{R} \rightarrow \mathbb{R}$$

vanishes. Since the Lagrangian L is regular, the map $v_q \mapsto \langle p, v_q \rangle_q$ is a local diffeomorphism for each q . Therefore, $\langle \frac{\partial Y}{\partial t}, p \rangle$ vanishes identically in $TQ \times \mathbb{R}$ if and only if $\frac{\partial Y}{\partial t} = 0$. This shows that the time-independence of $E_{L,\mathcal{C}}^*$ is equivalent to $\mathring{\mathcal{Q}}_t \circ \mathcal{Q}_t^{-1} = Y$ with Y independent of t . But then Y is a vector field on Q and, since $\mathring{\mathcal{Q}} = Y \circ \mathcal{Q}$, \mathcal{Q} is the flow of Y .

(ii) The proof can be done in coordinates. Let M_0 be given by $S(q)\dot{q} + s(q) = 0$. Then $DC^{-1}(M_0 \times \mathbb{R})$ is an affine subbundle of TQ that is described by $\mathring{S}(u, t)\dot{u} + \mathring{s}(u, t) = 0$ with \mathring{S} and \mathring{s} as in (4). Its linearity is equivalent to the vanishing of $\mathring{s} = s \circ \mathcal{C} + (S \circ \mathcal{C})\mathring{\mathcal{Q}}$, that is, given that S and s are time-independent, $s \circ \mathcal{C} = s \circ \mathcal{Q}$ and $S \circ \mathcal{C} = S \circ \mathcal{Q}$, to the vanishing of $s + S(\mathring{\mathcal{Q}}_t \circ \mathcal{Q}_t^{-1}) = S(Y - \xi_0)$. \square

This proposition suggests that, in order to obtain a time-independent conserved moving energy, the time-dependent diffeomorphism \mathcal{C} should be constructed as the flow of a vector field on Q that differs from the vector field ξ_0 by a section of the distribution \mathcal{D}_0 . The freedom in the choice of this section might then be used to try to make $L \circ DC$ and $DC^{-1}(M_0 \times \mathbb{R})$ time-independent. In the next section we will show that this is always possible if the system admits a symmetry group, with suitable properties, by choosing Y as an infinitesimal generator of the group action, that is, by choosing \mathcal{Q} as the action of a one-parameter subgroup.

3.4 Symmetry and conservation of moving energy. We consider now a time-independent nonholonomic system X_{L,Q,M_0} with affine constraints whose Lagrangian and constraint distribution are invariant—in a sense made precise in Hypotheses (H1) and (H2) below—under an action $\Psi : G \times Q \rightarrow Q$ of a Lie group G on Q . As in Proposition 3, we denote by $\mathcal{M}_0 = \xi_0 + \mathcal{D}_0$ the affine distribution on Q corresponding to M_0 .

For each $q \in Q$, we write as usual $\Psi_g(q)$ for $\Psi(g, q)$. We denote by $\Psi^{TQ} : G \times TQ \rightarrow TQ$ the tangent lift of the action Ψ , which is the action of G on TQ given by

$$\Psi_g^{TQ}(v_q) = T_q \Psi_g \cdot v_q$$

(in coordinates, $\Psi_g^{TQ}(q, \dot{q}) = (\Psi_g(q), \Psi'_g(q)\dot{q})$ with $\Psi'_g = \frac{\partial \Psi_g}{\partial q}$). We make the following two hypotheses:

(H1) L is invariant under Ψ^{TQ} , namely

$$L \circ \Psi_g^{TQ} = L \quad \forall g \in G$$

(in coordinates, $L(\Psi_g(q), \Psi'_g(q)\dot{q}) = L(q, \dot{q}) \forall g, q, \dot{q}$).

(H2) The distribution \mathcal{D}_0 is invariant under Ψ , in the sense that

$$(\mathcal{D}_0)_{\Psi_g(q)} = T_q \Psi_g \cdot (\mathcal{D}_0)_q \quad \forall g \in G, q \in Q$$

(we need not make any hypothesis on the nonhomogeneous term ξ_0 and on the invariance of M_0 under the group action).

Under these hypotheses, it is rather natural to try to build the time-dependent diffeomorphism $\mathcal{C} = (\mathcal{Q}, \text{id}_{\mathbb{R}})$ that leads to a conserved moving energy by choosing \mathcal{Q} as a one-parameter subgroup of the action Ψ .

For $\eta \in \mathfrak{g}$, the Lie algebra of G , denote by

$$Y_\eta := \frac{d}{dt} \Psi_{\exp(t\eta)}|_{t=0}$$

the infinitesimal generator of the action of the one-parameter subgroup generated by η and by

$$J_\eta := \langle p, Y_\eta \rangle$$

the momentum map of the lifted action of the same one-parameter subgroup. The moving energy of X_{L, \mathcal{Q}, M_0} relative of the time-dependent diffeomorphism $\mathcal{C}_\eta = (\Phi^{Y_\eta}, \text{id}_{\mathbb{R}})$, where $\Phi^{Y_\eta} : Q \times \mathbb{R} \rightarrow Q$ is the flow of Y_η , is thus the restriction to M_0 of the function

$$E_{L, \mathcal{C}_\eta}^* = E_L - J_\eta. \quad (10)$$

Theorem 2. *Consider a time-independent nonholonomic system X_{L, \mathcal{Q}, M_0} with affine constraints and an action Ψ of a Lie group G on Q . Assume (H1), (H2) and*

(H3) $\eta \in \mathfrak{g}$ is such that $Y_\eta - \xi_0$ is a section of \mathcal{D}_0 .

Then, the moving energy $E_{L, \mathcal{C}_\eta, M_0}^$ is a time-independent first integral of X_{L, \mathcal{Q}, M_0} .*

Proof. Let \mathcal{Q} be the flow of Y_η . The conclusion follows if we show that the three hypotheses of Theorem 1 are satisfied with $\mathcal{C} = (\mathcal{Q}, \text{id}_{\mathbb{R}})$. We may check them in coordinates.

Hypothesis i. With this choice of \mathcal{Q} , $\mathcal{Q}_t = \Psi_{\exp(t\eta)}$ for all t . Hence, by hypothesis (H1), $L(\mathcal{Q}_t(q), \mathcal{Q}'_t(q)\dot{q}) = L(q, \dot{q})$ for all q, \dot{q}, t . Define $\tilde{L} = L \circ D\mathcal{C}$. Since a vector field is invariant under its own flow, $\mathcal{Q}_t = Y_\eta \circ \mathcal{Q}_t = \mathcal{Q}'_t Y_\eta$. Thus

$$\tilde{L}(q, \dot{q}, t) = L(\mathcal{Q}_t(q), \mathcal{Q}'_t(q)\dot{q} + \dot{\mathcal{Q}}_t(q)) = L(\mathcal{Q}_t(q), \mathcal{Q}'_t(q)[\dot{q} + Y_\eta(q)]) = L(q, \dot{q} + Y_\eta(q))$$

that shows that \tilde{L} is time-independent.

Hypothesis ii. Under hypothesis (H3), by item ii. of Proposition 3, $\tilde{M} = DC^{-1}(M_0 \times \mathbb{R})$ is an extended linear subbundle of TQ . We prove that this subbundle is time-independent. If M_0 is described by $S(q)\dot{q} + s(q) = 0$, then its associated distribution \mathcal{D}_0 has fibers $\ker S(q)$ and hypothesis (H2) is $\ker S(\Psi_g(q)) = \Psi'_g(q) \ker S(q)$ for all g and q , or

$$\ker [S(\mathcal{Q}_t(q))] = \mathcal{Q}'_t(q) \ker [S(q)] \quad \forall t, q. \quad (11)$$

In turn, the extended distribution $\tilde{\mathcal{M}}$ associated to \tilde{M} has fibers $(\tilde{\mathcal{M}}_t)_q = \ker[\tilde{S}(q, t)]$ with

$$\tilde{S}(q, t) = S(\mathcal{Q}_t(q))\mathcal{Q}'_t(q),$$

see (4). Hence, using (11) and the fact that, if B is an invertible $n \times n$ matrix and S is a $k \times n$ matrix, then $\ker(SB) = B^{-1} \ker S$, we have

$$\ker \tilde{S}(q, t) = \ker [S(\mathcal{Q}_t(q))\mathcal{Q}'_t(q)] = \mathcal{Q}'_t(q)^{-1} \ker S(\mathcal{Q}_t(q)) = \ker S(q).$$

Thus $\tilde{\mathcal{M}}_t = \mathcal{D}_0$ for all t and \tilde{M} is time-independent.

Hypothesis iii. This follows from item i. of Proposition 3. \square

Remark. The condition that $Y_\eta - \xi_0$ is a section of \mathcal{D}_0 implies that the orbits of the group action Ψ^{TQ} must be transversal to the constraint manifold.

Example. The sphere on the turntable of Section 3.1 is an instance of the situation of Theorem 2. As in that section, we identify the tangent spaces to Q with \mathbb{R}^5 via right-trivialization of $T\text{SO}(3)$. The natural symmetry group of the problem is $S^1 \times \text{SO}(3)$ (that acts as in (12) below), but for the sake of applying Theorem 2 we may consider only its subgroup $G = S^1 \times \{\mathbb{I}\}$, that acts on $Q = \mathbb{R}^2 \times \text{SO}(3)$ as

$$\Psi_\theta(q, R) = (H_\theta q, H_\theta R).$$

Here H_θ is the matrix of the anticlockwise rotation of angle θ around the z -axis and, with a small abuse, we identify vectors $(x, y) \in \mathbb{R}^2$ with vectors $(x, y, 0) \in \mathbb{R}^3$. The (right-trivialized) infinitesimal generator of the action that corresponds to the Lie algebra element $\eta \in \mathbb{R}$ is $Y_\eta = (-\eta y, \eta x, 0, 0, \eta)$, and the corresponding momentum is $J_\eta = \eta(x\dot{y} - y\dot{x} + ca^2\omega_z)$. The (right-trivialized) tangent lift of this action is

$$\Psi_\theta^{TQ}(q, R, \dot{q}, \omega) = (H_\theta q, H_\theta R, H_\theta \dot{q}, H_\theta \omega)$$

and leaves the Lagrangian (5) invariant, as in (H1). The distribution \mathcal{D}_0 associated to the affine constraint (6) is given by $\dot{x} = a\omega_y$, $\dot{y} = -a\omega_x$ (that is, $\dot{q} = a\omega \times e_z$ with \times the vector product in \mathbb{R}^3) and is invariant under the action Ψ_θ , as in (H2). Finally, the nonhomogeneous part of the constraint (6) is the vector field $\xi_0 = (-\Omega y, \Omega x, 0, 0, 0)$ and $Y_\Omega - \xi_0 = (0, 0, 0, 0, \Omega)$ lies in \mathcal{D}_0 , as in (H3). By Theorem 2, the moving energy $E_{L, \mathcal{C}_\Omega, M_0}^* = E_{L, M_0} - J_\Omega|_{M_0}$ is conserved. Using (6) one verifies that this moving energy coincides with (7), up to a constant term $ca^2\Omega\omega_z$.

3.5 Connection to (the nonholonomic failure of) Noether theorem. In the setting of Theorem 2 it is natural to view the conservation of the moving energy

$$E_{L, \mathcal{C}_\eta, M_0}^* = E_{L, M_0} - J_\eta|_{M_0},$$

where

$$J_\eta|_{M_0} = J_\eta|_{M_0},$$

as related to the invariance of the system under the action Ψ . If the energy E_{L,M_0} is not conserved, then the conservation of the moving energy E_{L,C_η,M_0}^* is only possible if J_{η,M_0} is not conserved. Thus, Theorem 2 produces a conserved quantity from the sum of two quantities—the energy and a component of the momentum map—that, at variance from what would happen if the system was holonomic, are not conserved. In a way, this first integral seems to be produced notwithstanding—or perhaps thanks to—the failure in nonholonomic mechanics of two cornerstones of Lagrangian mechanics: conservation of energy and Noether theorem.

It has some interest to understand why, at least in the symmetric case considered here, the mechanism of Theorem 1, that obviously has no interest in the Lagrangian case, is instead of interest in the nonholonomic case. In the absence of nonholonomic constraints ($M_0 = \tilde{M}_0 = TQ$), Theorem 1 is obviously true: the time-independence of the two Lagrangians L and \tilde{L} implies that both functions E_L and E_{L,C_η}^* are first integrals of the Lagrangian system described by the Lagrangian L . However, since in Lagrangian mechanics the momentum map of a lifted action that leaves the Lagrangian invariant is conserved, this mechanism can hardly be seen as disclosing a ‘new’ first integral E_{L,C_η}^* : the difference $E_{L,C_\eta}^* - E_L$ is a component of the momentum map, and it is thus a first integral for the very same reason of symmetry that underpins the possibility of passing to moving coordinates without introducing a time-dependence in the Lagrangian.

Explaining why things are different in the nonholonomic case—and how they are different—requires exploiting the role of the reaction forces, along the lines of [11, 12]. The hypothesis of ideality of the constraints assumes that the constraint can—a priori—exert all reaction forces that lie in $(\mathcal{D}_0)_q^\circ$. However, the set of reaction forces that is actually exerted when the system X_{L,Q,M_0} is in a configuration $q \in Q$ with any possible velocity $\dot{q} \in (\mathcal{M}_0)_q$ is given by

$$\mathcal{R}_q := \bigcup_{\dot{q} \in (\mathcal{M}_0)_q} R_{L,M}(q, \dot{q})$$

and can be (and typically is) smaller than $(\mathcal{D}_0)_q^\circ$. Therefore, the annihilators of \mathcal{R}_q can be (and typically are) larger than the fibers of $(\mathcal{D}_t)_q$. These annihilators are the fibers of a distribution \mathcal{R}° on Q , which was introduced in [11] (in the case of time-independent linear constraints, but the generalization to the case of time-independent affine constraints is straightforward [12]) and was called the reaction-annihilator distribution. We refer to these works for further details and we limit ourselves to note that

$$(\mathcal{D}_0)_q \subseteq \mathcal{R}_q^\circ \quad \forall q, t.$$

Proposition 4. [12] *Consider a time-independent nonholonomic system with affine constraints X_{L,Q,M_0} , and let $\mathcal{M}_0 = \mathcal{D}_0 + \xi_0$.*

- i. *The energy E_{L,M_0} is conserved if and only if ξ_0 is a section of \mathcal{R}° .*
- ii. *Assume that the Lagrangian $L : TQ \rightarrow \mathbb{R}$ is invariant under the tangent lift of an action of a Lie group G on Q , namely $L \circ \Psi_g^{TG} = L$ for all $g \in G$. Then, for any $\eta \in \mathfrak{g}$, J_{η,M_0} is a first integral of X_{L,Q,M_0} if and only if Y_η is a section of \mathcal{R}° .*

Assume, thus, that the energy of a time-independent nonholonomic system X_{L,Q,M_0} with affine constraints is not conserved. By Proposition 4, this happens if and only if ξ_0 is not a section of \mathcal{R}° . By Theorem 2, under Hypotheses (H1) and (H2), the existence of a conserved moving energy E_{L,C_η,M_0}^* depends on the existence of an infinitesimal generator Y_η such that the difference $\xi_0 - Y_\eta$ is a section of \mathcal{D}_0 . Since $\mathcal{D}_0 \subseteq \mathcal{R}^\circ$ and ξ_0 is not a

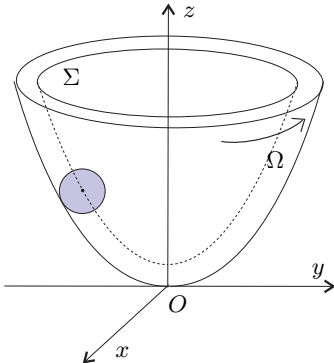


Figure 1: The sphere in a rotating convex surface of revolution.

section of \mathcal{R}° , this necessarily requires that Y_η has a nonzero component off \mathcal{R}° and, still by Proposition 4, the momentum J_{η, M_0} is not conserved. Thus, at the basis of the non-conservation of the energy and of the component of the momentum map—that makes it possible for them to add up to give a conserved function—there is the same reason: ξ_0 is not a section of \mathcal{R}° .

Remark. One might take as well a different point of view, and see the function E_{L, c_η}^* as the momentum map of the action, in the extended phase space, given by a combination of time-translation and of the lift of a one-parameter subgroup of Ψ .

4. Integrability of a sphere rolling on a rotating surface of revolution

We outline now an application of the existence of a conserved moving energy to the class of systems, considered by Borisov, Mamaev and Kilin in [6], that are formed by a heavy homogeneous solid sphere constrained to rotate without sliding on a moving surface of revolution; specifically, the surface rotates—relatively to an inertial frame—with uniform angular velocity Ω around its figure axis, which is assumed to be vertical (that is, directed like gravity). Describing the system in an inertial frame $\{O; x, y, z\}$ and using time-independent coordinates, as done in [6], leads to a time-independent Lagrangian; we will assume that the z -axis coincides with the figure axis of the surface, see Figure 1. The case $\Omega = 0$ is classical. Its study goes back to Routh [24], who also considered special cases with $\Omega \neq 0$ (see also [22]). When $\Omega \neq 0$ and the surface is a horizontal plane, the system reduces to the sphere on the turntable of section 3.1.

In all cases, either with $\Omega = 0$ or with $\Omega \neq 0$, the (time-independent) constraint manifold M_0 has dimension 8 and the system has an obvious $S^1 \times \text{SO}(3)$ symmetry. Reduction leads to a four-dimensional system which has two independent first integrals, that we will denote J_1 and J_2 . We will use the same symbols J_1 and J_2 also for the lifts of these functions to the unreduced phase space, which are first integrals of the unreduced system. The existence of the two first integrals J_1 and J_2 , when $\Omega = 0$, was known already to Routh [24, 22]. Their existence when $\Omega \neq 0$ has been proven in [6].

The integrability of the case $\Omega = 0$ has been studied by Hermans [16] and Zenkov [27]. When $\Omega = 0$ the constraint is linear and the energy E_{L, M_0} is conserved. As a result, the unreduced system has the three first integrals J_1 , J_2 and E_{L, M_0} , which are independent in

an open subset M_0^* of the phase space M_0 (specifically, they are independent everywhere except on motions in which the center of the sphere either moves horizontally or is at rest). Since the function E_{L,M_0} is $S^1 \times \text{SO}(3)$ -invariant, the reduced system has three independent integrals as well. Moreover, if the surface is upward convex and the sphere rolls inside it, as in Figure 1, then the common level sets of these three integrals in the four-dimensional reduced phase space are compact, and hence are closed curves, and the reduced dynamics is periodic [16, 27] (reference [16] uses a different argument; for details on the properties of the first integrals see [10]). Since the symmetry group is compact and acts freely on M_0^* , this in turn implies that the unreduced dynamics in M_0^* is quasi-periodic on tori of dimension up to three. This was proven in [16] using a reconstruction result from periodic dynamics, originally due to Field and Krupa (see particularly [18, 14, 16, 9, 8, 15]). Reference [27] reaches the same conclusion, but restricted to the motion of the center of mass, that undergoes quasi-periodic motions on tori of dimension up to two.

When $\Omega \neq 0$ the constraint is affine, not linear, and even if the constraint and the Lagrangian are time-independent the energy is not conserved. Therefore, the argument used for the case $\Omega = 0$ is not directly applicable. A different approach has been taken by Borisov, Mamaev and Kilin, who proved that the reduced system has an invariant measure and, using Jacobi theorem [2], deduced from this and from the existence of the two first integrals J_1 and J_2 that the reduced system is integrable by quadratures [6].

But as we now prove, a conserved ‘moving energy’ exists in this problem, and much stronger results can be obtained. Leaving for a future work a detailed study of the problem, we limit here ourselves to some conclusions that can be obtained combining Theorem 1 with some general arguments (essentially, continuity from the case $\Omega = 0$):

Theorem 3. *Consider the system formed by a heavy homogeneous solid sphere that rolls without sliding on a surface of revolution, which rotates with constant angular velocity Ω around its figure axis, aligned with gravity. Then, at least for Ω not too large:*

1. *The reduced system has three first integrals, which are independent in some open nonempty subset of the four-dimensional reduced phase space.*

If, moreover, the surface is upward convex, and the sphere rolls inside it, then:

2. *There is a nonempty open subset of the reduced phase space where the reduced dynamics is periodic.*
3. *There is a nonempty open subset of the phase space of the unreduced system in which motions are quasi-periodic, on tori of dimension up to three.*

Proof. Let $r \in \mathbb{R}^3$ be the vector of the coordinates of the center of the sphere relative to the considered inertial frame $\{O; x, y, z\}$, $R \in \text{SO}(3)$ be the matrix that fixes the attitude of the sphere relative to that frame, and $\omega \in \mathbb{R}^3$ be the angular velocity in space of the sphere relative to that frame.

The holonomic constraint that the sphere is in contact with the surface of revolution can be modelled by imposing that the vector r is constrained to a (fixed, time independent) surface of revolution Σ , that we embed in $\mathbb{R}^3 \ni r$. The configuration space of the holonomic system is thus $Q = \Sigma \times \text{SO}(3) \ni (r, R)$ and the phase space TQ can be identified with $\Sigma \times \mathbb{R}^2 \times \text{SO}(3) \times \mathbb{R}^3 \ni (r, \dot{r}, R, \omega)$. The Lagrangian is the restriction to $\Sigma \times \mathbb{R}^2 \times \text{SO}(3) \times \mathbb{R}^3$ of the function $L : \mathbb{R}^3 \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$L(r, \dot{r}, R, \omega) = \frac{1}{2} \|\dot{r}\|^2 + \frac{1}{2} ca^2 \|\omega\|^2 - gr_3,$$

where the constants have obvious meanings, see (5), and is time-independent.

The constraint of rolling without sliding leads to a time-independent nonholonomic system with affine constraints, with constraint submanifold an 8-dimensional affine subbundle M_0 of $\Sigma \times \mathbb{R}^2 \times \text{SO}(3) \times \mathbb{R}^3$ and Lagrangian the restriction of L to M_0 . The Lagrangian and the constraint manifold are invariant under the tangent lift of the action of $S^1 \times \text{SO}(3)$ on Q given by

$$\Psi_{(\theta, P)}(r, R) = (H_\theta r, H_\theta R P) \quad (12)$$

where H_θ is the 3×3 matrix of rotation of angle θ around the third axis. Once the kinematical states with the sphere sitting at the point $r = 0$ and spinning about the z -axis have been removed from phase space, to prevent the need for singular reduction, the (regularly) reduced phase space is a 4-dimensional submanifold of $\Sigma \times \mathbb{R}^2 \times \mathbb{R}^3 \ni (r, \dot{r}, \omega)$.

We pass now to time-dependent coordinates (s, S) in Q with

$$\mathcal{Q}(s, S, t) = (H_{\Omega t} s, H_{\Omega t} S) \quad (13)$$

and lift them to a time-dependent coordinate change $DC : (s, \dot{s}, S, \nu, t) \mapsto (r, \dot{r}, R, \omega, t)$ in TQ . In this coordinate system the surface is at rest; therefore, the constraint of rolling without sliding is linear and time-independent and defines a linear subbundle \tilde{M}_0 of $\Sigma \times \mathbb{R}^2 \times \text{SO}(3) \times \mathbb{R}^3$. Due to the symmetry of the system, the Lagrangian $\tilde{L} = L \circ DC$ is time-independent as well. (Incidentally, since \mathcal{Q} is the restriction of the action Ψ to a one-parameter subgroup, the time-independence of \tilde{L} follows as well from the argument used in the proof of Theorem 2; we mention also that

$$\tilde{L}(s, \dot{s}, S, \nu) = \frac{1}{2} \|\dot{s} + \Omega e_z \times s\|^2 + \frac{1}{2} ca^2 \|\nu + \Omega e_z\|^2 - gs_3).$$

Thus, $X_{\tilde{L}, \mathcal{Q}, \tilde{M}_0}$ is a time-independent nonholonomic system with linear constraints. Moreover, by item i. of Proposition 3 the function $E_{L, \mathcal{C}}^*$ is time-independent. By Theorem 1, we conclude that the original system X_{L, \mathcal{Q}, M_0} has the conserved moving energy $E_{L, \mathcal{C}, M_0}^*$.

We may now prove the three statements.

1. Since the constraint manifolds M_0 and \tilde{M}_0 , the Lagrangians L and \tilde{L} and the change of coordinates are $S^1 \times \text{SO}(3)$ -invariant, the function $E_{L, \mathcal{C}, M_0}^*$ has this very same invariance property and descends to a first integral of the reduced system. The reduced system has therefore the three first integrals J_1 , J_2 and $E_{L, \mathcal{C}, M_0}^*$. From the expressions for J_1 and J_2 given in [6] and from the expression above of \tilde{L} , one sees that these integrals depend continuously on Ω (for $\Omega = 0$, $E_{L, \mathcal{C}, M_0}^*$ reduces to the energy E_{L, M_0}), and the same is obviously true for the constraint manifold M_0 . It is known that, when $\Omega = 0$, J_1 , J_2 and $E_{L, \mathcal{C}, M_0}^*$ are the components of a submersion from an open nonempty set M_{reg} of M_0 to \mathbb{R}^3 [27, 10]. Continuity implies that, at least for Ω sufficiently close to zero, the map $(J_1, J_2, E_{L, \mathcal{C}, M_0}^*)$ is a submersion from an open nonempty subset of M_{reg} to \mathbb{R}^3 .

2. When $\Omega = 0$, the level curves of $(J_1, J_2, E_{L, \mathcal{C}, M_0}^*)$ in M_{reg} are compact [27, 10], hence bounded. It follows that, for each Ω sufficiently close to zero, there is an open nonempty subset of M_{reg} where the submersion $(J_1, J_2, E_{L, \mathcal{C}, M_0}^*)$ has bounded (hence compact, being a submersion) level sets.

3. According to the mentioned reconstruction results from reduced periodic dynamics by Field and Krupa, if the group is compact and acts freely, then each ‘relative periodic orbit’ (that is, the group orbit in the phase space that projects over a periodic orbit of the reduced system) is fibered by tori of dimension up to $1 + \rho$, where ρ is the rank of the group, on which motions are quasi-periodic. In our case, $\rho = 2$. \square

These are clearly partial results, that should be completed and extended under several aspects. We shortly indicate some of them.

First of all, the regions of the phase space where the dynamics is quasi-periodic should be identified, and it should be understood how they depend on Ω and on the shape of the surface.

A complementary question concerns the behaviour of motions that are not quasi-periodic, if present.

There are also interesting questions about the geometry of the (singular) foliation by invariant tori. The proof given above shows that each relative periodic orbit is fibered by tori of some dimension between 1 and 3, but it does not ensure that this dimension is the same across different relative periodic orbits and that the invariant tori are the fibers of a fibration of (an open subset of) the phase space. This property is important, because it implies the existence of the appropriate number of first integrals that are usually associated to integrability (for some results on this point in the case $\Omega = 0$ see [9]).

Finally, even if the reconstruction procedure gives generically a fibration by invariant tori of dimension 3, it might happen that for certain shapes of the surface there are resonance conditions among the frequencies of *all* the quasi-periodic motions—or equivalently, there exist additional first integrals—and there is a fibration by invariant tori of smaller dimension, either 2 (all motions have two frequencies) or 1 (all motions are periodic). An instance of this possibility is met in the limiting case of the sphere on the turntable: since the $SO(3)$ -reduced system has periodic dynamics, and $SO(3)$ has rank 1, the unreduced motions are quasi-periodic on tori of dimension at most 2.

This analysis (which has not yet been performed completely for the case $\Omega = 0$, either) requires manifestly an approach different from the general one used in this section, and will be done elsewhere.

Remark. The integrability result for the reduced system given in [6], based on Jacobi theorem, could be strengthen if the common level sets of the two first integrals J_1 and J_2 were compact. Under such a hypothesis, Jacobi theorem implies that these level sets are two-dimensional tori and motions on them are linear *after a time reparameterization*. However, not only the level sets of J_1 and J_2 are unlikely to be compact (these two functions are linear in some of the coordinates on the reduced phase space), but because of the time reparameterization, this result would be much weaker than those in Theorem 3.

Acknowledgements. We thank Enrico Pagani for a useful conversation and Larry Bates for suggesting the term ‘moving energy’.

References

- [1] C. Agostinelli, *Nuova forma sintetica delle equazioni del moto di un sistema anolonomo ed esistenza di un integrale lineare nelle velocità*. Boll. Un. Mat. Ital. **11** (1956), 1-9.
- [2] V.I. Arnold, V.V. Kozlov and A. Neishtadt *Mathematical Aspects of Classical and Celestial Mechanics. Dynamical Systems, III. Third Edition. Encyclopaedia Math. Sci.* **3**. (Springer, Berlin, 2006).
- [3] S. Benenti, *A ‘user-friendly’ approach to the dynamical equations of non-holonomic systems*. SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007), Paper 036, 33 pp.
- [4] A.M. Bloch, *Nonholonomic Mechanics and Control*. Interdisciplinary Applied Mathematics **24** (Springer-Verlag, New York, 2003).

- [5] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden and R.M. Murray, *Nonholonomic mechanical systems with symmetry*. Arch. Rational Mech. Anal. **136** (1996), 21-99.
- [6] A.V. Borisov, I.S. Mamaev and A.A. Kilin, *The rolling motion of a ball on a surface. New integrals and hierarchy of dynamics*. Regul. Chaotic Dyn. **7** (2002), 201-219.
- [7] J. Cortés Monforte, *Geometric, Control and Numerical Aspects of Nonholonomic Systems*. Lecture Notes in Math. **1793** (Springer-Verlag, Berlin, 2002).
- [8] R. Cushman, J.J. Duistermaat and J. Śnyatnicki, *Geometry of Nonholonomically Constrained Systems*. Advanced Series in Nonlinear Dynamics **26** (World Scientific, Singapore, 2010)
- [9] F. Fassò and A. Giacobbe, *Geometry of invariant tori of certain integrable systems with symmetry and an application to a nonholonomic system*. SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007), Paper 051, 12 pp.
- [10] F. Fassò, A. Giacobbe and N. Sansonetto, *Periodic flows, rank-two Poisson structures, and nonholonomic mechanics*. Regul. Chaotic Dyn. **10** (2005), 267-284.
- [11] F. Fassò, A. Ramos and N. Sansonetto, *The reaction-annihilator distribution and the nonholonomic Noether theorem for lifted actions*. Regul. Chaotic Dyn. **12** (2007), 449-458.
- [12] F. Fassò and N. Sansonetto, *In preparation*.
- [13] C. Ferrario and A. Passerini, *Rolling Rigid Bodies and Forces of Constraint: an Application to Affine Nonholonomic Systems*. Meccanica **35** (2000), 433-442.
- [14] M. Field, *Local structure of equivariant dynamics*, in *Singularity Theory and Its Applications, Part II*. Lecture Notes in Mathematics **1463**, 142-166 (Springer, Berlin, 1991)
- [15] M.J. Field, *Dynamics and symmetry*. ICP Advanced Texts in Mathematics **3** (Imperial College Press, London, 2007).
- [16] J. Hermans, *A symmetric sphere rolling on a surface*. Nonlinearity **8** (1995), 493-515.
- [17] M.H. Kobayashi and W.O. Oliva, *A note on the conservation of energy and volume in the setting of nonholonomic mechanical systems*. Qual. Theory Dyn. Syst. **5**, 4 (2004) 247-259.
- [18] M. Krupa, *Bifurcations of relative equilibria*. SIAM J. Math. Anal. **21** (1990), 1453-1486.
- [19] C.-M. Marle, *On symmetries and constants of motion in Hamiltonian systems with nonholonomic constraints*. In *Classical and Quantum Integrability (Warsaw, 2001)*, 223-242, Banach Center Publ. **59** (Polish Acad. Sci. Warsaw, 2003).
- [20] E. Massa and E. Pagani, *Classical dynamics of nonholonomic systems: a geometric approach*. Ann. Inst. H. Poincaré Phys. Théor. **55** (1991), 511-544.
- [21] E. Massa and E. Pagani, *A new look at classical mechanics of constrained systems*. Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), 1-36.
- [22] J.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems*. Translations of Mathematical Monographs **33** (AMS, Providence, 1972).
- [23] L.A. Pars, *A Treatise on Analytical Dynamics*. (Heinemann, London, 1965).
- [24] E.J. Routh, *Treatise on the Dynamics of a System of Rigid Bodies (Advanced Part)*. (Dover, New York, 1955).
- [25] W. Sarlet, F. Cantrijn and D.J. Saunders, *A geometrical framework for the study of nonholonomic Lagrangian systems*. J. Phys. A **28** (1995), 3253-3268.
- [26] W. Sarlet, F. Cantrijn and D.J. Saunders, *A geometrical framework for the study of nonholonomic Lagrangian systems: II*. J. Phys. A **29** (1996), 4265-4274.
- [27] D.V. Zenkov, *The geometry of the Routh problem*. J. Nonlinear Sci. **5** (1995), 503-519.