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SMALL NOISE EXPANSION FOR
THE LÉVY PERTURBED VASICEK MODEL

Francesco Cordoni[1], Luca Di Persio[2] [§]

[1]Department of Mathematics
University of Trento
Via Sommarive, 14, Trento, ITALY

[2]Department of Computer Science
University of Verona
Strada le Grazie, 15, Verona, ITALY

Abstract: We present rigorous small noise expansion results for a Lévy perturbed Vasicek model. Estimates for the remainders as well as an application to ZCB pricing are also provided.

1. Introduction

In the present paper we shall provide rigorous small noise expansion results for the Lévy perturbed Vasicek model. Our analysis is based on [3], in the setting proposed in [9, Sec.6.2]. Let us underline that during recent years a wide range of small noise expansion techniques have been developed, particularly with respect to the *so called* Loval Volatility Models (LVMs), see, e.g., [4, 8, 11, 13]. LVMs are commonly used to analyse options markets where the underlying volatility strongly depends on the level of the underlying itself, therefore LVMs are also widely accepted as tools to model interest-rate derivatives as is the case for the Vasicek model. The paper is organized as follows: in Sect. 2 the we

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[§]Correspondence author

present the approach developed in [3] is presented and then applied, in Sect. 3, to provide order corrections to both the Vasicek model and its zero coupon bond price.

2. The Asymptotic Expansion

2.1. The General Setting

Let us consider the following stochastic differential equation (SDE), indexed by a parameter $\epsilon > 0$

$$\begin{cases} dX_t^\epsilon = \mu^\epsilon\left(X_t^\epsilon\right)dt + \sigma^\epsilon\left(X_t^\epsilon\right)dL_t, \\ X^\epsilon\left(0\right) = x_0^\epsilon \in \mathbb{R}, \quad t \in [0, \infty)\,, \end{cases} \tag{1}$$

where L_t, $t \in [0, \infty)$, is a Lévy noise of jump diffusion type and $\mu^\epsilon : \mathbb{R}^d \to \mathbb{R}$, $\sigma^\epsilon : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are Borel measurable functions for any $\epsilon \geq 0$. In order to guarantee existence and uniqueness of strong solutions of (1) we assume μ^ϵ and σ^ϵ to be locally Lipschitz and with sublinear growth at infinity, see, e.g., [10]. We note that, if the Lévy process L_t has a jump component, then X_t^ϵ in eq. (1) has to be understood as $X_{t-}^\epsilon := \lim_{s\uparrow t} X_s^\epsilon$, see, e.g., [12] for details. In what follows we denote by $stlim_{n\to\infty}X_n := X$, the limit in probability, namely,

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \delta\right) = 0\,.$$

Let us state the following

Hypothesis 2.1. Let us assume that:

(i) $\mu^\epsilon, \sigma^\epsilon \in C^{k+1}(\mathbb{R})$ in the space variable, for any fixed value $\epsilon > 0$ and for all $k \in \mathbb{N}_+$

(ii) the maps $\epsilon \mapsto \alpha^\epsilon(x)$, where $\alpha = \mu, \sigma$, are of $C^M(I)$ in ϵ, for some $M \in \mathbb{N}$, for every fixed $x \in \mathbb{R}$ and $I := [0, \epsilon_0]$, $\epsilon_0 = \epsilon_0\left(\mu^\epsilon, \sigma^\epsilon\right) > 0$.

It has been shown in [1, 3] that under hypothesis 2.1 on μ^ϵ and σ^ϵ, a solution X_t^ϵ of equation (1) can be represented as a power series with respect to the parameter ϵ, namely

$$X_t^\epsilon = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + \cdots + \epsilon^N X_t^N + R_N(t, \epsilon)\,, \tag{2}$$

where $X^i : [0, \infty) \to \mathbb{R}$, $i = 0, \ldots, N$, are continuous functions, while $|R_N(t, \epsilon)| \leq C_N(t)\epsilon^{N+1}$, $\forall N \in \mathbb{N}$ and $\epsilon \geq 0$, for some $C_N(t)$ independent of ϵ, but in general dependent of randomness, through $X_t^0, X_t^1, \ldots, X_t^N$. The functions X_t^i are

determined recursively as solutions of random differential equations in terms of the X_t^j, $j \leq i - 1$, $\forall i \in \{1, \ldots, N\}$. If we assume that $x = x(\epsilon)$, with $\epsilon \mapsto x(\epsilon)$ in C^{N+1}, $0 \leq \epsilon < \epsilon_0$, $0 < \epsilon_0 < 1$ and $x(0) = x_0$ independent from ϵ, then the following Taylor type expansion result holds

Proposition 2.2. *Let assume* $x(\epsilon)$ *can be written as a power series*

$$x(\epsilon) = \sum_{j=0}^{N} \epsilon^j x_j + R_N^x(\epsilon), \quad N \in \mathbb{N}_0, \quad x_j \in \mathbb{R}, j = 0, 1, \ldots, N , \tag{3}$$

and let $f_\epsilon \in C^{k+1}$, $k \in \mathbb{N}^+$, *be of the following form*

$$f_\epsilon(x) = \sum_{j=0}^{K} f_j(x)\epsilon^j + R_K^f(\epsilon, x) , \tag{4}$$

then we have $f_\epsilon(x(\epsilon)) = \sum_{k=0}^{K+M} \epsilon^k \left[f_\epsilon(x(\epsilon)) \right]_k + R_{K+M}(\epsilon)$, *with* $|R_{K+M}(\epsilon)| \leq C_{K+M}\epsilon^{K+M+1}$, *for some constant* $C_{K+M} \geq 0$, *independent of* ϵ, $0 \leq \epsilon \leq \epsilon_0$, *and coefficients* $\left[f_\epsilon(x(\epsilon)) \right]_k$ *defined by*

$$[f_\epsilon(x(\epsilon))]_0 = f_0(x_0);$$
$$[f_\epsilon(x(\epsilon))]_1 = Df_0(x_0)x_1 + f_1(x_0);$$
$$[f_\epsilon(x(\epsilon))]_2 = Df_0(x_0)x_2 + \frac{1}{2}D^2 f_0(x_0)x_1^2 + Df_1(x_0)x_1 + f_2(x_0);$$
$$[f_\epsilon(x(\epsilon))]_3 = Df_0(x_0)x_3 + \frac{1}{6}D^3 f_0(x_0)x_1^3 + Df_1(x_0)x_2$$
$$+ Df_2(x_0)x_1 + D^2 f_1(x_0)x_1^2 + f_3(x_0).$$

The general case reads as

$$[f_\epsilon(x(\epsilon))]_k = Df_0(x_0)x_k + \frac{1}{k!}D^k f_0(x_0)x_1^k + f_k(x_0) \\ + B_k^f(x_0, x_1, \ldots, x_{k-1}) , \tag{5}$$

where B_k^f *is a real function depending on* $(x_0, x_1, \ldots, x_{k-1})$ *only.*

Proof. See, e.g. [1]. □

The following theorem establish our main expansion result with respect to the solutions to the SDE in eq. (1)

Theorem 2.3. *Let us assume that the coefficients α^ϵ, $\alpha = \mu, \sigma$, of the stochastic differential equation (1) are in $C^{K_\alpha}(I)$ as function of ϵ, $\epsilon \in [0, \epsilon_0]$, and in $C^{M_\alpha}(\mathbb{R})$ as function of x. Let us also assume that α^ϵ are such that there exists a solution X_t^ϵ in the probabilistic strong, resp. weak sense of (1). Let us also assume that the recursive system of random differential equations*

$$dX_t^j = [\mu^\epsilon(X_t^\epsilon)]_j \, dt + [\sigma^\epsilon(X_t^\epsilon)]_j \, dL_t, \quad j = 0, 1, \ldots, N, \ t \geq 0,$$

has a unique solution.

Then there exists a decreasing sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$, $\epsilon_n \in (0, \epsilon_0]$, with $\epsilon_0 > 0$ as in hyp. 2.1 (ii), and $\epsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$, such that $X_t^{\epsilon_n}$ has an asymptotic expansion in powers of ϵ_n, up to order N, in the following sense

$$X_t^{\epsilon_n} = X_t^0 + \epsilon_n X_t^1 + \cdots + \epsilon_n^N X_t^N + R_N(\epsilon_n, t),$$

with

$$\text{st-lim}_{\epsilon_n \downarrow 0} \frac{\sup_{s\in[0,t]} |R_N(\epsilon_n, s)|}{\epsilon_n^{N+1}} \leq C_{N+1},$$

for some deterministic $C_N > 0$.

Remark 2.4. It can be seen that in general the k−th equation for X_t^k in Th. 2.3 is a nonhomogeneous linear equation in X_t^k, but with random coefficients depending on $X_t^0, \ldots, X_t^{k-1}$ and with a random inhomogeneity depending on X_t^k. Thus it has the general form

$$\begin{aligned}
dX_t^k &= f_k\left(X_t^0, \ldots, X_t^{k-1}\right) X_t^k dt + g_k\left(X_t^0, \ldots, X_t^{k-1}\right) dt \\
&\quad + \tilde{g}_k\left(X_t^0\right) dL_t + h_k\left(X_t^0, \ldots, X_t^{k-1}\right) X_t^k dL_t
\end{aligned} \tag{6}$$

for some function $f_k, g_k, \tilde{g}_h$ and h_k.

Example 2.1. Let $\mu^\epsilon = ax + \epsilon bx$ and $\sigma^\epsilon = \sigma_0 x + \epsilon \sigma_1 x$ with a, b, σ_0 and σ_1 some real constants. Applying Proposition 2.2 we get

$$\begin{aligned}
X_t^0 &= x_0 + \int_0^t a X_s^0 ds + \int_0^t \sigma_0 X_s^0 dL_s, \\
X_t^1 &= \int_0^t a X_s^1 ds + \int_0^t b X_s^0 ds + \int_0^t \sigma_1 X_s^0 dL_t + \int_0^t \sigma_0 X_s^1 dL_t, \\
X_t^k &= \int_0^t a X_s^k ds + \int_0^t b X_s^{k-1} ds + \int_0^t \sigma_1 X_s^{k-1} dL_s + \int_0^t \sigma_0 X_t^k dL_t, k \geq 2.
\end{aligned} \tag{7}$$

In particular, if we consider the special case of $\mu^\epsilon(x) = ax + b$ independent of ϵ, $\sigma^\epsilon(x) = cx + \epsilon \tilde{d}x$, for some constants a, b, c and $\tilde{d}$, and where the Lévy

process is taken to be a standard Brownian motion, $L_t = W_t$, then by eq. (5) we have that X_t^k satisfies a linear equation with constant coefficients for any $k \in \mathbb{N}$, thus applying standard results, see, e.g., [6, 9], an explicit solution for X_t^k can be retrieved. Moreover, if $L_t = W_t$ is a Brownian motion and we consider a set of K coupled linear stochastic equations with random coefficients, we have that the $k - th$ equation is of the form

$$\begin{cases} dX_t^k = \left[A^k(t)X_t^k + f^k(t)\right] dt + \left[B^k(t)X_t^k + g(t)\right] dW_t, \\ X_0^k = x_0^k \in \mathbb{R}, \quad t \geq 0 \end{cases} \tag{8}$$

where for any k with all the functions A^k, B^k, f^k and g are assumed to be Lipschitz and with linear growth. A solution of equation (8) is given by

$$X_t^k = \sum_{k=0}^{K} \Phi_k(t) \left[\int_0^t \Phi_k^{-1}(s) \left(f^k(s) - B^k(s)g^k(s)\right) ds + \int_0^t \Phi_k^{-1}(s)g^k(s)dW_s \right] \tag{9}$$

where Φ_t is the fundamental $K \times K$ matrix solution of the corresponding homogeneous equation, i.e. it is the solution of the problem

$$\begin{cases} d\Phi_k(t) = A^k(t)\Phi_k(t)dt + B^k(t)\Phi_k(t)dW_t, \\ \Phi_k(0) = I \end{cases} . \tag{10}$$

Remark 2.5. In the trivial case of $K = 1$ then Φ reduces to a scalar and we have

$$\Phi(t) = \exp \left\{ \int_0^t \left(A(s) - \frac{1}{2}B^2(s) \right) ds + \int_0^t B(s)dW_s \right\} .$$

In the more general case where L_t is a Lévy process composed by a Brownian motion plus a jump component, i.e.

$$\begin{cases} d\Phi(t) = A(t)\Phi(t)dt + B(t)\Phi(t)dW_t + \int_{\mathbb{R}_0} \Phi(t)x\tilde{N}(dt, dx) = \Phi(t)dX_t, \\ \Phi(0) = I, \quad t \geq 0 \end{cases} . \tag{11}$$

where $\tilde{N}(dt, dx)$ is a Poisson compensated measure to be understood in the following sense. We have $\tilde{N}(t, A) := N(t, A) - t\nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}, 0)$, $0 \notin \bar{A}$, N being a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ and $\nu(A) = \mathbb{E}(N(1, A)$, while $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $\int_{\mathbb{R}_0}(|x|^2 \wedge 1)\nu(dx) < \infty$. Then we have that a solution to eq. (11) is explicitly given by

$$\Phi(t) = \exp \left\{ \int_0^t \left(A(s) - \frac{1}{2}B^2(s) \right) ds + \int_0^t B(s)dW_s \right\} \prod_{0 < s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$$

where $\Delta \eta_J(s) := X_s - X_{s_-}$ being the jump at time $s \in (0, t)$. This object is called *Doóleans-Dade exponential* (or stochastic exponential) and it is denoted

by $\Phi(t) = \mathcal{E}(X_t)$. The *Doóleans-Dade exponential* has a wide use in finance since it is the natural extension to the Lévy case of the standard geometric Brownian motion, see, e.g., [6, 9] for a more extensive treatment on fundamental solution of homogeneous equation and [5] for more details on the *Doóleans-Dade exponential*.

3. Application to Financial Mathematics

3.1. Approximation of an Option Price

The Vasicek model (together with the CIR model) is one of the most used short rate modes. It assumes that the interest rate under the the risk neutral measure $\mathbb{Q}$ evolves according to a mean reverting Ornstein-Uhlenbeck process with constant coefficients, see, e.g. [7] for details. In particular the interest rate r_t is the solution of the following linear stochastic equation

$$\begin{cases} dr_t = \kappa \left[\theta - r_t \right] dt + \sigma dW_t, \\ r_0 = r_0, \end{cases} \tag{12}$$

with κ, θ, σ and r_0 some positive constants. The price of a pure discounted bond, better known as *zero-coupon bond* (ZCB), in the Vasicek model can be explicitly computed, see, e.g. [7] as

$$ZCB(t;T) = \mathbb{E}_t \left[e^{- \int_t^T r_s ds} \right] = A(t;T) e^{-B(t;T) r_t}, \tag{13}$$

$$A(t;T) := \exp \left\{ \left(\theta - \frac{\sigma^2}{2\kappa^2} \right) (B(t;T) - T + t) - \frac{\sigma^2}{4\kappa} B(t;T)^2 \right\},$$

$$B(t;T) := \frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)} \right).$$

The price of an option with payoff $\Phi(r_T)$ written on the interest rate r_t is given by

$$ZBO(t;T) = \mathbb{E}_t \left[e^{- \int_t^T r_s ds} \Phi(r_T) \right].$$

In the particular case of an European call/put option, as the one introduced in the previous BS model, the formula can be explicitly computed, see, e.g. [7] Sec. 3.2.1.

From Theorem 2.3 we deduce that $\Phi(r_t^\epsilon)$ has an asymptotic expansion in powers of ϵ of the form

$$\Phi(r_t^\epsilon) = \sum_{k=0}^{H} \epsilon^k \left[\Phi(r_t^\epsilon)\right]_k + R_H(\epsilon, t), \tag{14}$$

with

$$\sup_{s \in [0,t]} |R_H(\epsilon, s)| \leq C_{H+1}(t)\epsilon^{H+1},$$

and the coefficients can be computed from the expansions coefficients of r_t^ϵ, as discussed in section 2, where also the Taylor coefficients of Φ are treated.

3.1.1. The Vasicek Model: A First Order Correction

Applying the results in Sec. 2, let us then consider the following perturbed Vasicek model

$$\begin{cases} dr_t^\epsilon = \kappa \left[\theta - r_t^\epsilon\right] dt + \left(\sigma_0 + \epsilon \sigma_1 f(r_t^\epsilon)\right) dW_t, \\ r_0^\epsilon = r_0, \end{cases} \tag{15}$$

with σ_0 and σ_1 some positive constants, f a smooth real valued function, $0 \leq \epsilon \leq \epsilon_0$.

Let us now consider the particular case $f(r) = e^{\alpha r}$, for some $\alpha \in \mathbb{R}$, then we get the following proposition.

Proposition 3.1. *For the particular case where $f(r) = e^{\alpha r}$, for some $\alpha \in \mathbb{R}_0$, we have that r_t^ϵ can be written a power series, namely*

$$r_t^\epsilon = r_t^0 + \epsilon r_t^1 + R_1(\epsilon, t),$$

where the expansion coefficients read as

$$r_t^0 = r_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t}\right) + \sigma_0 \int_0^t e^{\kappa(t-s)} dW_s, \quad \text{with law} \quad \mathcal{N}(\mu_t, Q_t),$$

$$r_t^1 = \frac{\sigma_1}{\alpha \sigma_0} \left(e^{\alpha r_t^0} - e^{\alpha r_0}\right) + \int_0^t C_\alpha^1 e^{-\kappa(t-s)} e^{\alpha r_s^0} ds + \int_0^t C_\alpha^2 e^{-\kappa(t-s)} r_s^0 e^{\alpha r_s^0}, \tag{16}$$

with

$$\mu_t = r_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t}\right), \quad Q_t = \frac{\sigma_0^2}{2\kappa} \left(1 - e^{-2\kappa t}\right),$$

$$C_1^\alpha = -\frac{\sigma_1}{\alpha \sigma_0} \left(\kappa + \kappa \theta \alpha + \frac{1}{2}\alpha^2 \sigma_0^2\right), \quad C_2^\alpha = \frac{\sigma_1}{\alpha \sigma_0} \kappa \alpha.$$

Proof. Applying Th. 2.3 we have that expanding eq. (15) up to the first order we get

$$
\begin{aligned}
r_t^0 &= r_0 + \int_0^t \kappa \left[\theta - r_s^0 \right] ds + \int_0^t \sigma_0 dW_s \,, \\
r_t^1 &= - \int_0^t \kappa r_s^1 ds + \int_0^t \sigma_1 e^{\alpha r_s^0} dW_s \,,
\end{aligned}
\tag{17}
$$

An application of Itô's lemma to $g(s,r) = e^{\kappa t} r_t^0$ gives us that

$$
r_t^0 = r_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t} \right) + \sigma_0 \int_0^t e^{\kappa(t-s)} W_s, \text{ with law } \mathcal{N} \left(\mu_t, Q_t \right) \,.
$$

Computing r_t^1, in the same manner, we have that applying Itô's lemma to $g(s,r) = e^{\kappa t} r_t^1$ it follows

$$
r_t^1 = \sigma_1 \int_0^t e^{-\kappa(t-s)} e^{\alpha r_s^0} dW_s,
$$

Applying again Itô's lemma to the function $h(r) = e^{\alpha r_t^0} e^{\kappa s}$ we get

$$
e^{\alpha r_t^0} - e^{\alpha r_0^0} = \int_0^t \left(\gamma - \kappa \alpha r_s^0 \right) e^{\alpha r_s^0} ds + \int_0^t \alpha \sigma_0 e^{\alpha r_s^0} dW_s \,,
\tag{18}
$$

with $\gamma := \kappa + \alpha \kappa \theta + \frac{\alpha^2 \sigma_0^2}{2}$. The expression for r_t^1 thus follows applying eq. (18) and solving for integral w.r.t. the Brownian motion. $\qquad\square$

Remark 3.2. With the same argument we can derive also the second correction term r_t^2. In fact applying Th. 2.3 we have that

$$
r_t^2 = - \int_0^t \kappa r_s^2 ds + \int_0^t \sigma_1 \alpha e^{\alpha r_s^0} dW_s = \sigma_1 \alpha \int_0^t e^{-\kappa(t-s)} e^{\alpha r_s^0} dW_s \,.
$$

The particular choice of $f(r) = e^{\alpha r}$ can easily be extended to any real function which can be written as a Fourier transform, resp. Laplace transform, $f(r) = \int_{\mathbb{R}_0} e^{ir\alpha} \lambda(d\alpha)$, resp. $f(r) = \int_{\mathbb{R}_0} e^{\alpha r} \lambda(d\alpha)$ of some positive measure λ on $\mathbb{R}_0$ (e.g. a probability measure) resp. which has finite Laplace transform. Formulae (20) holds with $K_\alpha e^{\alpha r_\tau^0}$ replaced by $\int_{\mathbb{R}_0} K_\alpha e^{i\alpha r_\tau^0} \lambda(d\alpha)$, resp. $\int_{\mathbb{R}_0} K_\alpha e^{\alpha r_\tau^0} \lambda(d\alpha)$, which are finite if, e.g. $\int_{\mathbb{R}_0} |K_\alpha| \lambda(d\alpha) < \infty$, resp. λ has compact support. In fact eq. (18) gets replaced by

$$
\int_{\mathbb{R}_0} e^{\alpha r_t^0} \lambda(d\alpha) = 1 + \int_{\mathbb{R}} \left[\int_0^t \left(e^{\alpha r_s^0} \alpha \mu + \frac{\alpha^2}{2} \sigma_0^2 e^{\alpha r_s^0} \right) ds \right] \lambda(d\alpha)
$$

$$
+ \int_{\mathbb{R}} \left[\int_0^t e^{\alpha r_s^0} \alpha \sigma_0 dW_s \right] \lambda(d\alpha) \,.
$$

By repeating the steps used before we get the statements in Prop. 3.3 extended to these more general cases.

3.1.2. The Vasicek Model: A First Order Correction with Jumps

In the present section we will deal with the previous model with an addition of a small perturbed Poisson compensated measure $\mathcal{N}$. In particular we will assume $\tilde{N}(t, A) := N(t, A) - t\nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}, 0)$, $0 \notin \bar{A}$, N being a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ and $\nu(A) = \mathbb{E}(N(1, A))$, while $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $\int_{\mathbb{R}_0} (|x|^2 \wedge 1)\nu(dx) < \infty$. Eventually the Poisson random measure is assumed to be independent of the Brownian motion W_t. We refer to [5] for details on Levy processes.

Under previous conditions let us assume we are given an interest rate r_t^ϵ evolving according to the SDE

$$
\begin{cases}
dr_t = \kappa\left[\theta - r_t\right] dt + \left(\sigma_0 + \epsilon\sigma_1 f(r_t^\epsilon)\right) dW_t + \epsilon \int_0^t \int_{\mathbb{R}_0} x\tilde{N}(ds, dx)\,, \\
r_0 = r_0,
\end{cases}
\tag{19}
$$

with the notation as previously introduced.

Let us again consider the particular case $f(r) = e^{\alpha r}$, for some $\alpha \in \mathbb{R}$, then we get the following proposition.

Proposition 3.3. *For the particular case where $f(r) = e^{\alpha r}$, for some $\alpha \in \mathbb{R}_0$, we have that r_t^ϵ can be written a power series, namely*

$$
r_t^\epsilon = r_t^0 + \epsilon r_t^1 + R_1(\epsilon, t)\,,
$$

where the expansion coefficients read as

$$
r_t^0 = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma_0 \int_0^t e^{\kappa(t-s)} dW_s, \quad \text{with law} \quad \mathcal{N}\left(\mu_t, Q_t\right),
$$

$$
r_t^1 = \frac{\sigma_1}{\alpha\sigma_0}\left(e^{\alpha r_t^0} - e^{\alpha r_0}\right) + \int_0^t C_\alpha^1 e^{-\kappa(t-s)} e^{\alpha r_s^0} ds + \int_0^t C_\alpha^2 e^{-\kappa(t-s)} r_s^0 e^{\alpha r_s^0}
$$

$$
+ \int_0^t \int_{\mathsf{R}_0} x\tilde{N}(ds, dx)\,,
\tag{20}
$$

with constants as in Prop. 3.4.

Proof. The proof is completely analogous to the one of Prop. 3.4 just taking into account the Poisson random measure. $\square$

3.1.3. Application to Pricing

Expanding the payoff function Φ, assumed to be smooth, according to eq. (14) we have that the first order correction to the fair price of an option written on the underlying r^ϵ is given by

$$ZBO^1(0;T) = \mathbb{E}\left[e^{-\int_0^T r_s^\epsilon ds}\Phi(r_T^\epsilon)\right] = \mathbb{E}\left[e^{-\int_0^t r_s^0 ds}\left(1 - \epsilon\int_0^t r_s^1 ds\right)\Phi\left(r_T^0\right)\right]$$
$$+ \mathbb{E}\left[e^{-\int_0^t r_s^0 ds}\left(1 - \epsilon\int_0^t r_s^1 ds\right)\Phi'\left(r_T^0\right)\right]. \tag{21}$$

Proposition 3.4. *The first order corrected fair price of an option written on the underlying r_t^ϵ reads as*

$$ZBO^1(0;T) = \mathbb{E}\left[e^{-\int_0^T r_s^\epsilon ds}\Phi(r_T^\epsilon)\right] = ZBO + \epsilon\mathbb{E}\left[e^{-\int_0^t r_s^0 ds}\int_0^T r_s^1 ds\Phi\left(r_T^0\right)\right]$$
$$+ \epsilon\mathbb{E}\left[e^{-\int_0^t r_s^0 ds}\Phi'\left(r_T^0\right)\right]. \tag{22}$$

Proof. Expanding the r_s^ϵ in a converging power series we have that

$$e^{-\int_0^t r_s^\epsilon ds} = e^{-\int_0^t r_s^0 ds - \epsilon\int_0^t r_s^1 ds + R_1(\epsilon,t)} = e^{-\int_0^t r_s^0 ds}\left(1 - \epsilon\int_0^t r_s^1 ds + \tilde{R}_1(\epsilon,t)\right),$$

where r_t^0, r_t^1 are given in Prop. 3.3 for the particular case of $f(x) = e^{\alpha x}$.

The expansion in eq. (22) follows applying Th. 2.3 to the payoff function Φ.   □

If we consider the particular case of pricing a *zero coupon bond* (ZCB), namely we consider a terminal payoff $\Phi = 1$, we have the following.

Proposition 3.5. *The first order corrected fair price of an option written on the underlying r_t^ϵ reads as*

$$ZCB^1(0;T) = ZCB + \epsilon\mathbb{E}\left[e^{-\int_0^t r_s^0 ds}\int_0^T r_s^1 ds\right], \tag{23}$$

where ZCB is the price given in eq. (13).

Proof. The claim follows by Prop. 3.4, with $\Phi = 1$.   □

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