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OPTIMAL EXECUTION STRATEGY UNDER ARITHMETIC BROWNIAN MOTION WITH VAR AND ES AS RISK PARAMETERS

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Abstract: We explicitly give the optimal trade execution strategy in the Almgren-Chriss framework, see [1, 2], when the publicly available price process follows an arithmetic Brownian motion with zero drift. The financial setting is completed by choosing the risk parameters to be the *Value at Risk* and the *Expected Shortfall* associated with the *Profit and Loss* distribution of the strategy's position.

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1. Introduction

We consider a non-liquid market for a risky asset, hence allowing an active agent to influence the price process of the asset itself. In this financial setting great attention is given to the study of the difference between *publicly available price* representing the price per share of the asset in a market impact-free world, and the actual price. Such a a difference is called market impact. Our main

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aim is to understand how a large order may be divided into smaller orders to minimize the resulting market impact. We assume that the publicly available prices process follows an arithmetic Brownian motion (ABM) with zero drift. The market impact is split into two component: the temporary market impact and the permanent one. Both impact components are assumed to be linear in the rate of trading and in the number of shares sold respectively. With previous assumptions, in [1, 2] an optimal execution strategy is explicitly computed when the variance of the strategy's cost is used as risk parameter. We solve a slightly different optimal trade execution problem taking the *Value at Risk* (VAR) and the *Expected Shortfall* (ES) as risk parameters. In [4], resp. in [3], the same problem is solved under the assumption that the unaffected price process follows a geometric Brownian motion, resp. a displaced diffusion process. Moreover in [5] a robustness property for the optimal strategies is found. Indeed, under a specified cost criterion, the form of the solution is independent on the unaffected price process, provided that it is a square integrable martingale. This paper is organized as as follows: in Sect. 2 the model is presented, we state the conditions which characterize the set of admissible strategies and we specify the price processes. Moreover we compute the *Implementation Shortfall* (IS) of each admissible strategy. In Sect. 3 we introduce the chosen risk parameters, namely the VAR and the ES, deriving an explicit computation for them. In Sect. 4 we define the criterium that we want to minimize. For such a criterion, which involves the expected cost and risk parameter associated to a strategy, we are able to exhibit the related optimal strategy.

2. The Model

The general framework is based on a trader which has a position x_0 in the risky asset at time $t = 0$. If such a position is positive then the trader's goal is to sell all of the x_0 shares within a fixed deadline $T > 0$ minimizing, at the same time, a function involving the expected cost and some risk parameters. Otherwise, if $x_0 < 0$, then the trader has the objective to buy x_0 shares of the risky asset within a fixed time $T > 0$, maximizing a given revenue function which may depend on some parameters.

Let S_t denote the price per share at time $t \in [0, T]$, of the asset that is publicly available, i.e. the unaffected stock price level. This is the price per share of the asset which will occur in a market impact-free world or, similarly, the price will occur if the trader will not participate in the market.

We would like to underline that S_t is not the amount per share received by the trader. Indeed we assume that liquidity effects are present in the market. In particular the paper value of the asset and the value it will be sold for, may be significantly different. The realized price, that is the price the trader actually receives on each trade per share, is called *actual price* and it will be denoted by $\tilde{S}_t$. Note that $\tilde{S}_t$ depends both on the unaffected price and the behaviour of the trader in the market.

We assume that the publicly available price process S_t follows an ABM with zero drift, therefore S_t satisfies the following stochastic differential equation $dS_t = \sigma dW_t$, where W_t is a Brownian motion and σ is a positive constant representing the volatility of the price process. We would like to underline that, following [1, 2], the volatility term does not depend on the particular strategy, since it results as an average over all the market's endogenous inputs. Assuming that the initial value of the unaffected price is a fixed and known positive constant S_0, we have that $S_t = S_0 + \sigma W_t$, and the price process S_t is a martingale with respect to its natural filtration. Concerning the actual price process $\tilde{S}_t$, we have that it is defined by

$$\tilde{S}_t = S_t + \eta \dot{X}_t + \gamma(X_t - x_0) , \tag{1}$$

where X is the trade execution strategy adopted by the trader, this means X_t represents the number of shares that the trader still has to sell at time t within the deadline T, and where η and γ are given positive constants.

Exploiting equation (1) we can split the market impact, i.e. the difference between the actual price $\tilde{S}_t$ and the publicly available price S_t, into two components

- $\eta \dot{X}_t$ outlines the temporary (or instantaneous) market impact of trading ,
- $\gamma(X_t - x_0)$ describes the permanent market impact.

Note that while the permanent impact is accumulated by all transactions from the initial time up to time t, the temporary impact only affects the trading in the *infinitesimal* interval $[t, t + dt)$. We point out that both the temporary and the permanent market impact are assumed to be linear in the rate of trading and in the sold/purchased shares respectively, so that we are able to find explicitly the related optimal strategy, see below Sect. (4).

Since at the initial time, the units of the asset held by the trader are fixed and equal to x_0, while at the final time $T > 0$, all the shares are sold, then the financial transactions we are interested in happen in the time interval $[0, T]$. Consequently we define the set of admissible strategies $\mathcal{A}$ as the class of all the absolutely continuous stochastic processes $\{X_t\}_{t\in[0,T]}$ which are adapted to the natural filtration generated by the Brownian motion $\{W_t\}_{t\in[0,T]}$, fulfilling the boundary conditions $X_0 = x_0$ and $X_T = 0$.

2.1. Cost of a Trading Strategy

To understand how to optimally trade in the market, we have to compute the costs arising from each admissible strategy. The marked-to-market value of trader's initial position, i.e the value under the classical price taking condition, equals $x_0 S_0$, and we will use such a value as benchmark. If we fix a certain time $t \in [0, T)$ and we consider a fixed admissible strategy $X \in \mathcal{A}$, we have that, in the infinitesimal time interval $[t, t+dt)$, the trader sells $-dX(t) = -\dot{X}_t dt$ shares of the assets at the price $\tilde{S}_t$, earning $-\tilde{S}_t\,\dot{X}_t dt$. By integrating the earning over all the strategy's lifetime, we have that the *total capture* $G(X)$ associated to the strategy $X \in \mathcal{A}$, reads as follows

$$G(X) = \int_0^T -\tilde{S}_t\,\dot{X}_t\,dt = \int_0^T -(S_t + \eta\dot{X}_t + \gamma(X_t - x_0))\dot{X}_t\,dt =$$
$$= -\int_0^T S_t\,\dot{X}_t\,dt - \eta\int_0^T \dot{X}_t^2\,dt - \gamma\int_0^T X_t\dot{X}_t\,dt + \gamma x_0\int_0^T \dot{X}_t\,dt\,.$$

Using the boundary conditions, we have $\int_0^T \dot{X}_t\,dt = X_t - X_0 = -x_0$ and by the stochastic Itô version of the integration by parts formula, it follows

$$\int_0^T X_t\dot{X}_t\,dt = X_t^2|_0^T - \int_0^T \dot{X}_t X_t\,dt = -X_0^2 - \int_0^T \dot{X}_t X_t\,dt\,,$$

which implies $\int_0^T X_t\dot{X}_t\,dt = -\frac{1}{2}x_0^2$. Exploiting again the integration by parts formula and the maturity condition $S_T\,X_T = 0$, we obtain

$$\int_0^T S_t\,\dot{X}_t\,dt = -S_0 X_0 - \int_0^T \sigma X_t\,dW_t\,. \tag{2}$$

Notice that the stochastic integral in (2) is well defined since $X \in \mathcal{A}$. Summing up, the total capture of a strategy X is given by

$$G(X) = S_0 x_0 - \frac{\gamma}{2}x_0^2 - \eta\int_0^T \dot{X}_t^2\,dt + \int_0^T \sigma X_t\,dW_t.$$

We define the cost $C(X)$ of a trading strategy $X \in \mathcal{A}$ as the difference between the marked-to-market of the initial position, i.e. the quantity $x_0 S_0$, and the strategy's capture, therefore

$$C(X) = S_0 x_0 - G(X) = \frac{\gamma}{2}x_0^2 + \eta\int_0^T \dot{X}_t^2\,dt - \int_0^T \sigma X_t\,dW_t. \tag{3}$$

3. Risk Parameters

The optimal execution strategy is defined as the strategy that, over all admissible strategies, minimizes a given criterion. Along with the expected costs, our criterion takes into account as risk parameter the VaR or the ES.

3.1. Risk Parameter: Value at Risk

At time $t \in [0, T]$, the trader is currently holding X_t shares of the asset, so the marked-to-market value of his position is $H_t = X_t S_t$. After a time h if the trader does not enter in the market the new marked-to-market value of his position is $H_{t+h} = X_t S_{t+h}$. Therefore, the loss in position due to the change in price is equal to

$$L_{[t,t+h]}(X) = X_t(S_t - S_{t+h})\,.$$

The VaR associated with the P&L of the position $x(t)$ with $X \in \mathcal{A}$ over a time horizon h at the confidence level α is given by

$$\mathrm{VaR}_{\alpha,t,h}[X_t(S_t - S_{t+h})] = \sigma X_t \mathrm{VaR}_{\alpha,t,h}[W_t - W_{t+h}]$$

due to the homogeneity property. The term $\mathrm{VaR}_{\alpha,t,h}[W_t - W_{t+h}]$ does not depend on time t and, since $W_t - W_{t+h} \sim N(0, h)$, it represents the α quantile of the random variable $H \sim N(0, h)$. Therefore if ϕ is the density of standard Gaussian random variable and Φ being its related cumulative distribution, then the quantile we are looking for is given by $\sqrt{h}\Phi^{-1}(\alpha)$. Summing up the VaR measure for the loss of the instantaneous strategy's position at time t turns out to be

$$\mathrm{VaR}_{\alpha,t,h}[X_t(S_t - S_{t+h})] = \sigma\sqrt{h}\Phi^{-1}(\alpha)X_t\,.$$

When we want to use the VaR as risk parameter we have to taking into account the whole liquidation time. To this end, we integrate the VaR over the strategy lifetime $[0, T]$, obtaining that the risk function becomes

$$R_{\mathrm{VaR}_\alpha}(X) = \sigma\sqrt{h}\Phi^{-1}(\alpha)\int_0^T X_t\,dt. \tag{4}$$

3.2. Risk Parameter: Expected Shortfall

We may also choose the Expected Shortfall (ES) as measure of risk. The ES of the marked-to-market losses, given t, h and α is equal to

$$\begin{aligned}\mathrm{ES}_{\alpha,t,h}[X_t(S_t - S_{t+h})] :=& \mathbb{E}[\sigma X_t(W_t - W_{t+h})|\sigma X_t(W_t - W_{t-h}) \geq \mathrm{VaR}_{\alpha,t,h}]\\ =& \sigma X_t \mathbb{E}\left[W_t - W_{t+h}\middle| W_t - W_{t+h} \geq \frac{\mathrm{VaR}_{\alpha,t,h}}{\sigma X_t}\right].\end{aligned}$$

Recalling that $\mathrm{VaR}_{\alpha,t,h} = \sigma\sqrt{h}\Phi^{-1}(\alpha)X_t$, we have

$$\mathbb{E}\left[W_t - W_{t+h}\middle| W_t - W_{t+h} \geq \frac{\mathrm{VaR}_{\alpha,t,h}}{\sigma X_t}\right] = \sqrt{h}\mathbb{E}[Z|Z \geq \Phi^{-1}(\alpha)]\,,$$

hence we are left with the computation of

$$\mathbb{E}[Z|Z \geq \Phi^{-1}(\alpha)] = \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} z\phi(z)\,dz = -\frac{\phi(z)|_{\Phi^{-1}(\alpha)}^{\infty}}{1-\alpha} = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\,.$$

Therefore, for a given confidence level α, we have that the ES of the instantaneous position at time t equals

$$\mathrm{ES}_{\alpha,t,h}[X_t(S_t - S_{t+h})] = \frac{\sigma\sqrt{h}\phi(\Phi^{-1}(\alpha))X_t}{1-\alpha}\,,$$

and the risk function associated with the ES for the whole period of liquidation, is given by

$$R_{\mathrm{ES}_\alpha}(x) = \frac{\sigma\sqrt{h}\phi(\Phi^{-1}(\alpha))}{1-\alpha}\int_0^T X_t\,dt\,. \tag{5}$$

4. Optimal Strategy based on VaR or ES

We would like to underline that the two risk functions, given by (4) and by (5), are of the same form. Indeed, they are both proportional to the integral of the trading strategy over its lifetime but with different coefficients. Let $R(X) = R(X) = \lambda\int_0^T X_t\,dt$, then setting $\lambda = \sigma\sqrt{h}\Phi^{-1}(\alpha)$ or $\lambda = \frac{\sigma\sqrt{h}\phi(\Phi^{-1}(\alpha))}{1-\alpha}$, we will choose as risk function the VaR, resp. the ES. Thus, our aim is minimize the function

$$\mathbb{E}[C(X) + R(X)] = \frac{\gamma}{2}x_0^2 + \mathbb{E}\left[\eta\int_0^T \dot{X}_t^2\,dt + \lambda\int_0^T X_t\,dt - \int_0^T \sigma X_t\,dW_t\right] \tag{6}$$

over all the admissible strategies $\mathcal{A}$. Since the constant value $\frac{\gamma}{2}x_0^2$ is additive, it does not affect the minimum point even if it affects the minimum value, therefore we do not have to take it into consideration in our minimization procedure. Since by Itô integral property, we have that $\mathbb{E}\left[\int_0^T X_t\,dS_t\right] = 0$, minimizing the function in (6) is equivalent to minimize

$$\mathbb{E}\left[\int_0^T \dot{X}_t^2\,dt + \Lambda\int_0^T X_t\,dt\right], \tag{7}$$

where $\Lambda := \frac{\lambda}{\eta}$.

Theorem 1. *The unique optimal trading strategy which solves the following problem*

$$\min_{X\in\mathcal{A}} \mathbb{E}\left[\int_0^T \dot{X}_t^2\,dt + \Lambda\int_0^T X_t\,dt\right] \tag{8}$$

is given by

$$X_t^* = \frac{T-t}{T}\left(x_0 - \frac{\Lambda T}{4}t\right), \tag{9}$$

for which (8) *returns the value*

$$\mathbb{E}\left[\int_0^T (\dot{X}_t^*)^2\,dt + \Lambda\int_0^T X_t^*\,dt\right] = \frac{x_0^2}{T} + \frac{1}{2}x_0\Lambda T - \frac{1}{48}x_0\Lambda T^3.$$

Proof. Consider the following perturbation of X_t

$$X_t^\varepsilon = X_t + \varepsilon h(t), \tag{10}$$

where $h(t)$ is an arbitrary function satisfying $h(0) = h(T) = 0$ and ε is a real constant. In this way, X^ε still satisfies the boundary condition $X^\varepsilon(0) = x_0$ and $X^\varepsilon(T) = 0$. Moreover we require h be differentiable, so that $\dot{X}_t^\varepsilon = \dot{X}_t + \varepsilon\dot{h}(t)$.

Substituting (10) and (4) into (7), we obtain a functional in ε, i.e. $H(\varepsilon) = \mathbb{E}\left[\int_0^T (\dot{X}_t^\varepsilon)^2 + \Lambda X_t^\varepsilon\,dt\right]$, hence, differentiating with respect to ε and evaluating the resulting expression at $\varepsilon = 0$, we have $\dot{H}(0) = \mathbb{E}\left[\int_0^T 2\dot{X}_t\dot{h}(t) + \Lambda h(t)\,dt\right]$. Since, using integration by parts formula, we have

$$\int_0^T \dot{X}_t\,\dot{h}(t)\,dt = h(t)\dot{X}_t|_0^T - \int_0^T h(t)\ddot{X}_t\,dt = -\int_0^T h(t)\ddot{X}_t\,dt,$$

where we have used the boundary condition on h, $h(0) = h(T) = 0$, it follows

$$\dot{H}(0) = \mathbb{E}\left[\int_0^T -2\ddot{X}_t\,h(t) + \Lambda h(t)\,dt\right] = \mathbb{E}\left[\int_0^T -(2\ddot{X}_t - \Lambda)h(t)\,dt\right].$$

The optimal trading strategy X_t^* is obtained by setting $\dot{H}(0) = 0$. Since the function $h(t)$ is arbitrarily, X^* has to satisfy the following equality $2\ddot{X}_t - \Lambda = 0$, for all $t \in [0, T]$. Then, in order to find the optimal solution, we have to solve the following Cauchy problem

$$\begin{cases} \ddot{X}_t = \dfrac{\Lambda}{2} & (11)\\ X_0 = x_0 & (12)\\ X_T = 0 \quad . & (13) \end{cases}$$

The family of solutions of the differential equation (11) are given by $X_t = \frac{\Lambda}{4}t^2 + At + B$, where A and B are real constants. Constraint (12) implies $B = x_0$ and then, by (13), $A = -\frac{x_0}{T} - \frac{\Lambda T}{4}$.

Therefore the unique solution which attain the minimum in (8) is $X^*_t = \frac{T-t}{T}\big(x_0 - \frac{\Lambda T}{4}t\big)$ and the associated minimum value is given by $\mathbb{E}\big[\int_0^T (\dot{X}^*_t)^2\,dt + \Lambda \int_0^T X^*_t\,dt\big]$. $\square$

5. Conclusions

In the present paper we give the explicit execution strategy that minimizes a criterion involving expected cost and the *value at risk* or the *expected shortfall* as risk parameters, see Th. 1, when the publicly available price process follows an arithmetic Brownian motion with zero drift. Our result is linked to the ones obtained in [5], but, see (9), we give the optimal execution strategy with respect to different minimization criteria.

We would also like to underline that in the optimal execution strategy X^* exhibited in (9), the price process S_t does not appear explicitly, indeed X^* turns out to be deterministic.

References

[1] R. Almgren, N. Chriss, Optimal execution of portfolio transactions, *Journal of Risk*, **3** (2011), 5-39.

[2] R. Almgren, N. Chriss, Value under liquidation, *Risk*, **12** (1999), 61-63.

[3] D. Brigo, G. Di Graziano, Optimal execution comparison across risks and dynamics, with solutions for displaced diffusions, *Unpublished* (2013).

[4] J. Gatheral, A. Schied, Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework, *International Journal of Theoretical and Applied Finance*, **14**, No 3 (2011), 353-368.

[5] A. Schied, Robust strategies for optimal order execution in the Almgren-Chriss framework, *Applied Mathematical Finance*, **20**, No 3 (2013), 264-286.