Integrable almost–symplectic Hamiltonian systems

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Abstract

We extend the notion of Liouville integrability, which is peculiar to Hamiltonian systems on symplectic manifolds, to Hamiltonian systems on almost–symplectic manifolds, namely, manifolds equipped with a nondegenerate (but not closed) 2–form. The key ingredient is to require that the Hamiltonian vector fields of the integrals of motion in involution (or equivalently, the generators of the invariant tori) are symmetries of the almost–symplectic form. We show that, under this hypothesis, essentially all of the structure of the symplectic case (from quasi–periodicity of motions to an analogue of the action–angle coordinates and of the isotropic–coisotropic dual pair structure characteristic of the fibration by the invariant tori) carries over to the almost–symplectic case.

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1 Introduction

A. Motivations. Given the growing interest for integrable non–Hamiltonian systems, e.g. in connection with nonholonomic mechanics [18, 3, 2], it seems appropriate to begin to investigate their structure. As a guiding model, one can at first take the Hamiltonian case, which is well understood.

Even though the common integrability notion for the Hamiltonian case is the so–called complete integrability (maximal number of first integrals in involution, quasi–periodic motions on Lagrangian tori) for a thorough comprehension of this matter it is advisable to refer to the generic case of a system with d degrees of freedom and quasi–periodic motions on tori of a given dimension $n \leq d$, with n ranging from $n = 1$ in the case of periodic flow to $n = d$ in the completely integrable case. For all these cases, which for $n < d$ are sometimes referred to as 'superintegrable', there are:

- A unifying integrability notion—that of 'Liouville integrability', which links quasi–periodicity of motions to the existence of first integrals with certain involutivity properties [24, 23]. (The expressions 'generalized Liouville integrability' and 'noncommutative integrability' are often used if $n < d$, the latter especially when integrability is related to the invariance under a group action).
- A unifying geometric structure—that of an isotropic–coisotropic dual pair [12]. At a semilocal level, this structure is described by the existence of action–angle coordinates.[1] Globally, there is a fibration by isotropic invariant tori which has a coisotropic polar foliation, with the level sets of the actions as leaves. These two foliations together form what in symplectic geometry is called an isotropic–coisotropic "dual pair". Moreover, the base of the fibration by the invariant tori is a Poisson manifold of co–rank n, with the actions as Casimirs; hence, the level sets of the actions project onto the symplectic leaves of the base. Since the frequencies of motion depend on the actions, the polar foliation has a dynamical meaning: its leaves carry motions with given frequencis. This bi–foliated structure plays an important role for the comprehension of the systems, for the study of their perturbations, etc.

Reviews of these topics can be found in [19, 15]. The reason why we insist here on the generic 'superintegrable' case with $n \leq d$, rather than considering only the completely integrable case, is that if the tori are Lagrangian then the two foliations coincide and the dual pair structure is hidden.

It cannot be expected of course that anything comparable to the dual pair structure exists in any non–Hamiltonian integrable system, but one could begin the analysis from cases where there is some remnant of a symplectic structure. In this article we thus consider the almost–symplectic case. Our aim is to understand how much of the structure of symplectic Hamiltonian integrable systems, and under which condi-

[1] Sometimes called 'partial' or 'generalized' action–angle coordinates if $n < d$

tions, carries over to the almost–symplectic case. Almost symplectic manifolds are met e.g. in nonholonomic mechanics [6]. Moreover, they are particular instances of (almost, or not–closed) Dirac manifolds [11, 8] and it is thus conceivable that a comprehension of the almost–symplectic case might be a first step towards the study of the (almost) Dirac case.

B. The almost–symplectic case. An *almost–symplectic structure* on a manifold M is given by a nondegenerate (but possibly not closed) two–form σ. We will say that a vector field X on M is

- *Hamiltonian* with respect to σ if $i_X\sigma$ is exact, that is,
$$i_X\sigma = -dH$$
for some function H. We shall customarily write X_H for X and call H its Hamiltonian.
- *Strongly Hamiltonian* with respect to σ if it is Hamiltonian and, moreover, it is a symmetry of σ, namely
$$L_X\sigma = 0\,.$$

Note that, at variance with the symplectic and pre–symplectic cases, in the almost–symplectic case strong Hamiltonianity does not follow from Hamiltonianity: by Cartan's magic formula $L_X\sigma = d(i_X\sigma) + i_Xd\sigma$, a Hamiltonian vector field X is strongly Hamiltonian if and only if

$$i_Xd\sigma \;=\; 0\,,$$

that is, it annhilates $d\sigma$. (In a way, this amounts to require the closedness of σ only in the 'direction' of X).

Our first goal is to give a notion of 'Liouville integrability' for the almost–symplectic context. The 'involutivity' properties of the first integrals, which characterize the symplectic case, may be defined, rather naturally, in terms of the almost–Poisson bracket induced by the almost–symplectic structure. However, almost–Poisson brackets do not satisfy the Jacobi identity and this has the consequence that, without additional conditions, first integrals in involution do not produce commuting vector fields—an essential ingredient for integrability. The solution we adopt here is to require that *the first integrals in involution have the property that their Hamiltonian vector fields are strongly Hamiltonian* or, equivalently, that the *generators of the invariant tori are strongly Hamiltonian,* see Section 2 for precise statements.

Under such a hypothesis we will recover essentially all of the structure of the symplectic case—from quasi–periodicity of motions (Section 2) to an analogue of the action–angle coordinates (Section 3) and of the isotropic–coisotropic dual pair structure (Section 4). Some comments on the crucial, but strong, hypothesis of strong Hamiltonianity are deferred to the Conclusions.

A preliminary version of this work has appeared in [25].

2 Liouville integrability

A. Preliminaries. In order to formulate a notion of Liouville integrability for the almost–symplectic context it is convenient to first look at this matter from a broader, non–Hamiltonian perspective. As pointed out in particular by Bogoyavlenskij [9], but see also [16], quasi–periodicity of a (not necessarily Hamiltonian) flow can be linked to the presence of a number of first integrals and of a complementary number of commuting dynamical symmetries which preserve these first integrals. We recall here this basic result focusing on the bundle structure of the resulting fibration by invariant tori, which plays an important role in the sequel.

Let M and P be two manifolds[2] and $\pi : M \to P$ a submersion with fiber $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ for some $n \geq 1$. By the Ehresmann fibration theorem, $\pi : M \to P$ is a locally trivial fibration, see e.g. [22]. Therefore, for each $p \in P$ there is a neighbourhood $U \subset P$ of p and a local trivialization of $\pi^{-1}(U)$, that is, a diffeomorphism

$$(\pi, \alpha) \ : \ \pi^{-1}(U) \ \to \ U \times \mathbb{T}^n \, .$$

The fibration $\pi : M \to P$ is said to be a torus bundle, or more precisely a $\mathbb{T}^n$–*bundle*, if there is an atlas of local trivializations with the following property: if $(\pi, \alpha) : \pi^{-1}(U) \to U \times \mathbb{T}^n$ and $(\pi, \alpha') : \pi^{-1}(U') \to U' \times \mathbb{T}^n$ are any two local trivializations of this atlas with $U \cap U' \neq \emptyset$, then in each connected component of $U \cap U'$ the transition functions have the form

$$\alpha' \ = \ Z\alpha + \mathcal{F} \circ \pi \quad (\mathrm{mod} 1) \tag{2.1}$$

for some (constant) matrix $Z \in \mathrm{SL}_{\pm}(\mathbb{Z}, n)$, the set of $n \times n$ unimodular matrices with integral entries, and some function $\mathcal{F} : U \cap U' \to \mathbb{R}^n$. (For background, see e.g. [26]). We shall call *bundle charts* the local trivializations in this atlas, which will in turn be called *bundle atlas*. A bundle atlas can be extended to a maximal bundle atlas, so whenever useful we will assume that the considered bundle atlas is maximal. In each bundle chart, the vector fields $\partial_{\alpha_1}, \dots, \partial_{\alpha_n}$ are tangent to the fibers of π and will be called the *semilocal generators* of the fibration. (By “semilocal” we mean “in a π–saturated open set”).

We may now state the following

Proposition 1 *(See* [9]*) Assume that on a manifold M there are*

H1. *A submersion $F = (F_1, \dots, F_k) : M \to \mathbb{R}^k$ with compact and connected fibers, for some $1 \leq k < \dim M$.*

H2. *$n = \dim M - k$ everywhere linearly independent and pairwise commuting vector fields $Y_1, \dots, Y_n$ which are tangent to the fibers of F:*

$$[Y_i, Y_j] = 0 \quad \text{and} \quad L_{Y_i} F_r = 0 \qquad \forall i, j = 1, \dots, n \, , \quad r = 1, \dots, k \, . \tag{2.2}$$

[2] In the sequel, all functions and geometric objects are tacitly assumed to be smooth. We thus stress smothness only in those few cases where some doubt might arise.

Then:

i. $F : M \to F(M) \subset \mathbb{R}^k$ *is a* $\mathbb{T}^n$*–bundle.*

ii. Any vector field X *on* M *which satisfies*

$$L_X F_r = 0 \quad \text{and} \quad [X, Y_i] = 0 \qquad \forall i = 1, \ldots, n\,, \quad r = 1, \ldots, k$$

is conjugate to a constant vector field on $\mathbb{T}^n$ *by each bundle charts of* $F : M \to F(M)$.

Proof. We only sketch the proof, focusing on the bundle structure of the fibration. This will also fix some notation for the sequel.

(i) A standard argument (see e.g. [1]) shows that the fibers of F, being n–dimensional compact and connected manifolds which have n everywhere independent commuting tangent vector fields, are diffeomorphic to $\mathbb{T}^n$. More precisely, for each $f \in F(M)$ there exist a neighborhood $U \subset F(M)$ of f and a local trivialization of $F^{-1}(U)$, namely a diffeomorphism

$$\mathcal{C} = (F, \alpha) \;:\; F^{-1}(U) \;\to\; U \times \mathbb{T}^n\,.$$

Here, the 'angles' α are constructed as linear combinations of the times along the flows of $Y_1, \ldots, Y_n$, measured from a smooth local section of the fibration, with coefficients which are constant on the fibers of F. Thus, the corresponding tangent vector fields ∂_{α_i} in $F^{-1}(U)$ satisfy[3]

$$\partial_{\alpha_i} \;=\; (L_{ji} \circ F) Y_j\,, \qquad i = 1, \ldots, n \tag{2.3}$$

for some smooth map $L : U \to \mathrm{GL(n)}$ called the 'period–matrix' [14]. Therefore, the transition functions between any two such local trivializations (F, α) and (F, α') with intersecting domains are affine: $\alpha'(F, \alpha) = Z(F)\alpha + \mathcal{F}(F)$ (mod1) for some smooth invertible matrix $Z(F)$ and some function $\mathcal{F}$. The fact that this is a diffeomorphism of $\mathbb{T}^n$ implies that $Z(F) \in \mathrm{SL}_{\pm}(\mathbb{Z}, n)$ and is therefore constant on connected sets. These local trivializations give $F : M \to F(M)$ the structure of a $\mathbb{T}^n$–bundle.

(ii) Since $Y_1, \ldots, Y_n$ span the tangent spaces to the fibers of F, there are functions $c_1, \ldots, c_n$ such that $X = c_j Y_j$. Since $0 = [Y_i, X] = [Y_i, c_j Y_j] = (L_{Y_i} c_j) Y_j$, these functions are constant on the fibers of F. The claim now follows from (2.3). ∎

Remark: In each bundle chart, the vector fields $\partial_{\alpha_1}, \ldots, \partial_{\alpha_n}$ are the infinitesimal generators of an action of $\mathbb{T}^n$. This action is defined in the chart domain and, if $n > 1$, might not be extendable to the whole of M. The obstruction to this is the

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[3]We understand everywhere the summation over repeated indices and tacitly make the convention that the indices i, j, h assume the values $1, \ldots, n$, the indices r, s assume the values $1, \ldots, k = \dim M - n$ and the indices u, v assume the values $1, \ldots, k - n = \dim M - 2n$. Since any almost–symplectic manifold has even dimension, later on we shall write $\dim M = 2d$.

so–called monodromy [14]. This is why $F : M \to F(M)$ need not be a principal bundle.

B. The symplectic context. In the standard symplectic Hamiltonian case, Liouville–integrability links quasi–periodicity to the existence of a number of first integrals with certain involutivity properties. The complementary number of commuting vector fields $Y_1, \dots, Y_n$ as in Proposition 1 are derived from these first integrals through the symplectic structure. Consider, for example, the following version of 'Liouville integrability', which is due to Nekhoroshev [24] and is general enough for our purposes:

Liouville integrability: *Let M be a symplectic manifold of dimension $2d$ and let $n \le d$ be a positive integer. Assume that there are $2d-n$ functions $F_1, \dots, F_{2d-n}$ such that $F = (F_1, \dots, F_{2d-n}) : M \to \mathbb{R}^{2d-n}$ is a submersion with compact and connected fibers, and the first n functions are in involution with all others:*

$$\{F_i, F_r\} \;=\; 0 \qquad \text{for all} \;\; i = 1, \dots, n \quad \text{and} \quad r = 1, \dots, 2d-n\,. \tag{2.4}$$

Then:

L1. *The level sets of F are diffeomorphic to $\mathbb{T}^n$ and, if H is any function such that $\{H, F_r\} = 0$ for all $r = 1, \dots, 2d-n$, then the flow of X_H is conjugate to a linear flow on each of them.*

L2. *Action–angle coordinates can be introduced.*

L3. *The fibration F has a polar foliation, together which it constitutes an isotropic–coisotropic dual pair.*

For precise formulations of statements L2 and L3 see e.g. [12, 19, 15, 13] and references therein. Note that this integrability criterion reduces to the standard complete integrability if $n = d$ and that it is essentially equivalent to other slightly more general versions of Liouville integrability, such as that due to Mischenko and Fomenko [23].

The relation of the above Liouville integrability criterion to Proposition 1 is that the Hamiltonian vector fields of the first n functions $F_1, \dots, F_n$ play the role of the symmetries $Y_1, \dots, Y_n$. Conditions (2.4) are satisfied because in a symplectic manifold the Poisson brackets satisfy

$$L_{X_F} G \;=\; \{G, F\} \qquad \text{and} \qquad [X_F, X_G] \;=\; X_{\{G,F\}} \tag{2.5}$$

for any two functions F and G.

As is well known, the second property (2.5) expresses the fact that, in a symplectic manifold, the Lie algebra of functions is homomorphic to that of Hamiltonian vector fields. The basic difficulty in extending Liouville integrability to the almost–symplectic context is that, if the two–form is not closed, then this Lie algebra homomorphism is lost.

C. The almost–symplectic context. An almost–symplectic structure σ on a manifold M defines an *almost–Poisson bracket* on M, namely a bilinear and antisymmetric map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies Leibnitz rule, via

$$\{F, G\} := \sigma(X_F, X_G). \tag{2.6}$$

As in the symplectic case, $L_{X_F} G = \{G, F\}$ and so F is a first integral of X_G if and only if $\{F, G\} = 0$. However, the nonclosedness of σ reflects on the fact that the almost–Poisson bracket does not satisfy the Jacobi identity, see e.g. [10]. This implies that the algebra of functions is not a Lie algebra when equipped with the almost–Poisson bracket (2.6) and that the second property (2.5) is not satisfied by all functions F and G.

However, this property turns out to be satisfied by those functions whose Hamiltonian vector fields are strongly Hamiltonian, as it follows from the following

Proposition 2 *Let (M, σ) be an almost–symplectic manifold. If two vector fields Y and Z are symmetries of σ (namely $L_Y \sigma = L_Z \sigma = 0$) then $[Y, Z] = X_{\sigma(Y,Z)}$.*

Proof. $d(\sigma(Y, Z)) = L_Z(i_Y \sigma) - i_Z d(i_Y \sigma) = i_Y(L_Z \sigma) + i_{[Z,Y]}\sigma = -i_{[Y,Z]}\sigma$. ■

This makes clear that if, in the above Liouville integrability criterion, σ is assumed to be almost–symplectic and *the Hamiltonian vector fields of the functions $F_1, \ldots, F_n$ which are in involution with all others are assumed to be strongly Hamiltonian*, then Conclusion L1 remains true.

In the next sections we will show that, under these same hypotheses, also Conclusions L2 and L3 of the Liouville integrability criterion remain, *mutatis mutandis*, true. For clarity, we shall however carry on this analysis in the setting of Proposition 1.

Remark: It is possible to prove that the set $\mathcal{I}$ of all functions on M whose Hamiltonian vector fields are strongly Hamiltonian form a Lie algebra with respect to the almost–Poisson bracket (2.6) of (M, σ), namely, if F and G are in $\mathcal{I}$ then so is $\{F, G\}$ and the restriction to $\mathcal{I}$ of the almost–Poisson bracket satisfies the Jacobi identity, see [25]. By Proposition 2, then, this Lie algebra is homomorphic to the Lie algebra of strongly Hamiltonian vector fields on M.

3 Action–angle coordinates.

A. The actions. We now show that, in the almost–symplectic context, the strong Hamiltonianity of the vector fields $Y_1, \ldots, Y_n$ of Proposition 1 implies that the fibration by invariant tori acquires a rich almost–symplectic geometry, which is completely analogous to that of the symplectic case. To begin with, in this section we prove that there is an analogue of the action–angle coordinates.

From now on, we consider an almost–symplectic manifold (M,σ) equipped with the almost–Poisson bracket (2.6); by saying that two functions are in involution we mean that their almost–Poisson bracket vanishes.

Proposition 3 *Under hypotheses H1 and H2 of Proposition 1, assume also that:*

H3. *M is an almost–symplectic manifold and the vector fields $Y_1,\dots,Y_n$ are strongly Hamiltonian.*

Then:

i. The Hamiltonians of $Y_1,\dots,Y_n$ are constant on each fiber of F and are pairwise in involution.

ii. The torus bundle $F : M \to F(M)$ has a bundle atlas in which all semilocal generators are strongly–Hamiltonian. Their Hamiltonians, called 'actions', are pairwise in involution.

Proof. (i) Consider a bundle chart (F,α) of $F : M \to F(M)$ and let G_i be the Hamiltonian of Y_i, $i=1,\dots,n$.[4] Since $[Y_i,Y_j]=0$, from Proposition 2 we have $L_{Y_i}G_j = \{G_j,G_i\} = \sigma(Y_j,Y_i) =: c_{ij}$, a constant on connected set. Using (2.3) and the fact that the period matrix L is constant on the fibers of F, this gives $\frac{\partial G_i}{\partial \alpha_j} = L_{jh}c_{hi}$ for all i,j. But a function on the torus may have constant derivative if and only if this constant is zero. Thus, $\{G_i,G_j\}=0$.

(ii) The proof of the fact that the semilocal generators can be chosen to be strongly Hamiltonian is essentially an analogue of the proof of the existence of the action–angle coordinates in the symplectic case, and therefore requires some knowledge of the techniques used in that proof. Specifically, we need to recall some details of the construction of the angles α and of the period matrix L (see any proof of the Liouville–Arnold theorem for details). The commuting flows of $Y_1,\dots,Y_n$ define an action $\Phi : (\tau,m) \mapsto \Phi_\tau(m)$ of $\mathbb{R}^n$ on M via

$$\Phi_{(\tau_1,\dots,\tau_n)}(m) \;:=\; \Phi^{Y_1}_{\tau_1}\circ\cdots\circ\Phi^{Y_n}_{\tau_d}(m)\,.$$

Choose a local section $s : U \to M$ of the fibration F, where U is an open set in $F(M)$. The covering map

$$\mathcal{C} \;:\; U\times\mathbb{R}^n \to F^{-1}(U)\,, \qquad (f,\tau)\mapsto \Phi_\tau(s(f))$$

provides local (even though not semilocal) coordinates near any point of $F^{-1}(U)$, that we denote (F,T) so as to distinguish them from the coordinates (f,τ) in the covering $U\times\mathbb{R}^n$. Note that, since τ_i is the time along the flow of Y_i, $\mathcal{C}^*(Y_i|_{F^{-1}(U)}) = \partial_{\tau_i}$.

The restriction of σ to the coordinate domain $F^{-1}(U)$ can be written as

$$U_{rj}dF_r\wedge dT_j + \frac{1}{2}V_{rs}dF_r\wedge dF_s + \frac{1}{2}W_{ij}dT_i\wedge dT_j$$

[4]For shortness, we do not distinguish between functions on M and their local representatives in the bundle charts.

with functions $V_{rs} = -V_{sr}$, U_{rj} and $W_{ij} = -W_{ji}$. But $W_{ij} = \sigma(\partial_{T_i}, \partial_{T_j}) = \sigma(Y_i, Y_j) = \{G_i, G_j\} = 0$. Moreover, $U_{rj} = \sigma(\partial_{F_r}, \partial_{T_j}) = \sigma(\partial_{F_r}, Y_j) = \frac{\partial G_j}{\partial F_r}$. Since $G_1, \dots, G_n$ are constant on the tori, we see that the restriction of σ is

$$dG_i \wedge dT_i + \frac{1}{2} V_{rs} dF_r \wedge dF_s \,.$$

Using the fact that $\Phi_\tau^* \sigma = \sigma$ for all $\tau \in \mathbb{R}^n$, which is implied by $L_{Y_i}\sigma = 0$ for all i, it is easy to show that V_{rs} is invariant under Φ_τ, namely $V_{rs} = V_{rs} \circ \Phi_\tau$ for all τ. Thus, $V_{rs} = v_{rs} \circ F$ for some functions $v_{rs} : U \to \mathbb{R}^d$ and $\sigma|_{F^{-1}(U)}$ pulls back to the two–form

$$\tilde{\sigma} \;=\; dg_i \wedge d\tau_i + \frac{1}{2} v_{rs} df_r \wedge df_s \tag{3.1}$$

on $U \times \mathbb{R}^n$, where of course $g_i = G_i \circ \mathcal{C}$.

The angles $\alpha = (\alpha_1, \dots, \alpha_n)$ are now constructed, exactly as in the symplectic case, with a linear change of coordinates $(f, \tau) \mapsto (f, L(f)\tau)$ on the covering which, by composition with $\mathcal{C}$, gives another covering map $(F, \alpha) : U \times \mathbb{R}^n \to F^{-1}(U)$. The construction of the angles is then completed by a quotient over $\mathbb{Z}^n$, but we need not recall the details here. Instead we show that, restricting U if necessary, the vector fields $\partial_{\alpha_1}, \dots, \partial_{\alpha_n}$ are strongly Hamiltonian.

Fix $i \in \{1, ..., n\}$. That $i_{\partial_{\alpha_i}} d\sigma = 0$ follows from the fact that ∂_{α_i} is a linear combination of the strongly Hamiltonian vector fields Y_i. Since σ is constant along the fibers of F, the Hamiltonianity of ∂_{α_i} in some set $F^{-1}(\hat{U})$ with $\hat{U} \subset U$ open is equivalent to the closedness of $i_{\partial_{\alpha_i}}\sigma$, that is, to the closedness of $i_{\mathcal{C}^*\partial_{\alpha_i}}\tilde{\sigma}$. Since $\mathcal{C}^*\partial_{\alpha_i} = L_{ji}\partial_{\tau_j}$, from (3.1) we see that $i_{\mathcal{C}^*\partial_{\alpha_i}}\tilde{\sigma}$ is closed if and only if

$$d(L_{ji} dg_i) \;=\; 0 \,,$$

namely, written in coordinates

$$\frac{\partial L_{ji}}{\partial f_r} \frac{\partial g_j}{\partial f_s} \;=\; \frac{\partial L_{ji}}{\partial f_s} \frac{\partial g_j}{\partial f_r} \qquad \forall\, r, s = 1, \dots, 2d - n \,. \tag{3.2}$$

Observe now that, if (f, τ) and (f, τ') are two preimages of a point of $F^{-1}(U)$, namely $\mathcal{C}(f, \tau') = \mathcal{C}(f, \tau)$, and if $Z_1, Z_2 \in T_{(f,\tau)}(\mathbb{R}^d \times \mathbb{T}^d)$ and $Z'_1, Z'_2 \in T_{(f,\tau')}(\mathbb{R}^d \times \mathbb{T}^d)$ are two pairs of tangent vectors which are mapped onto the same tangent vectors in $T_{\mathcal{C}(f,\tau)}M$, that is

$$T_{(f,\tau)}\mathcal{C} \cdot Z_p \;=\; T_{(f,\tau')}\mathcal{C} \cdot Z'_p \qquad \forall\, p = 1, 2 \,,$$

then $\tilde{\sigma}_{(f,\tau)}(Z_1, Z_2) \;=\; \tilde{\sigma}_{(f,\tau')}(Z'_1, Z'_2)$. On account of the definition of the period matrix, the preimages under $\mathcal{C}$ of the section $s : U \to M$ are the sections of $U \times \mathbb{R}^n$ parametrized by $\nu \in \mathbb{Z}^n$ and given by

$$U \ni f \;\mapsto\; (f, L(f)\nu) \,.$$

For each $\nu \in \mathbb{Z}^d$, a basis of linearly independent vector fields tangent to the section through $(f, L(f)\nu)$ is given by

$$Z_r^\nu \;=\; \partial_{f_r} \;+\; \nu_j \frac{\partial L_{hj}}{\partial f_r}(f)\partial_{\tau_h}\,, \qquad r = 1, \dots, 2d-n\,.$$

Since $T_{(f,\tau)}\mathcal{C} \cdot Z_r^\nu = \partial_{F_r}$ for all ν and all r, we conclude that

$$\tilde\sigma(Z_r^\nu, Z_s^\nu) \;=\; \sigma(\partial_{F_r}, \partial_{F_s}) \;=\; v_{rs} \qquad \forall\, r, s = 1, \dots, 2d-n\,.$$

But directly computing $\tilde\sigma(Z_j^\nu, Z_h^\nu)$ using (3.1) one easily verifies that $\tilde\sigma(Z_r^\nu, Z_s^\nu) = v_{rs}$ for all $\nu \in \mathbb{Z}^n$ if and only if (3.2) is satisfied.

Finally, the fact that the Hamiltonians of $\partial_{\alpha_1}, \dots, \partial_{\alpha_n}$ are pairwise in involution is proven with the same argument used in the proof of statement i, taking into account that $[\partial_{\alpha_i}, \partial_{\alpha_j}] = 0$ for all i, j. ■

The reason for the terms 'actions' used for the Hamiltonians of $\partial_{\alpha_1}, \dots, \partial_{\alpha_n}$ is that, in (symplectic) Hamiltonian mechanics, an action is a function whose Hamiltonian vector field is tangent to the invariant tori and has periodic flow with unit period.

Remarks: (i) Statement i. of this Proposition is the analogue of the fact that, in the standard Liouville–Arnold theorem, if the invariant sets are compact then the assumption of involution of the n integrals of motion may be replaced by the (apparently weaker) requirement of the commutativity of their Hamiltonian vector fields. In that case, this follows from the fact that any Hamiltonian action of a compact Lie group is Poisson [21].

(ii) The main difference between the proof of Proposition 3 and the analogous proof for the symplectic case is that here we cannot use a Lagrangian section of the fibration by the invariant tori. We note also that, as in the symplectic case, it would be possible to prove more, that is, that the period matrix L is a function of the actions alone.

(iii) Statement ii. implies that the fibers of F are, at least semilocally, the orbits of a (strongly Hamiltonian) action of $\mathbb{T}^n$. See also the Conclusions.

B. The action–angle coordinates. Let (M, σ) be an almost–symplectic manifold of dimension $2d$ and $F : M \to F(M)$ a $\mathbb{T}^n$–bundle. A system of (semilocal) *action–angle coordinates* for a bundle chart $(F, \alpha) : F^{-1}(U) \to U \times \mathbb{T}^n$ is a diffeomorphism

$$(a, b, \alpha) \;:\; F^{-1}(U) \;\to\; \mathcal{A} \times \mathcal{B} \times \mathbb{T}^n$$

where $\mathcal{A} = a(U) \subset \mathbb{R}^n$ and $\mathcal{B} = b(U) \subset \mathbb{R}^{2d-2n}$ are open sets and $a = (a_1, \dots, a_n)$ are actions for this bundle chart.

Proposition 4 *Under hypotheses H1, H2 and H3:*

i. In a neighbourhood of each point of M there exists a system of action–angle coordinates (a, b, α). The representative of σ in these coordinates has the form

$$da_i \wedge d\alpha_i \;+\; \frac{1}{2}A_{ij}da_i \wedge da_j \;+\; B_{iu}da_i \wedge db_u \;+\; \frac{1}{2}C_{uv}db_u \wedge db_v \tag{3.3}$$

where the matrices $A = -A^T$, B and $C = -C^T$ smoothly depend on (a, b) and C is everywhere nonsingular.

ii. The transition functions between any two systems of action–angle coordinates (a, b, α) and (a', b', α') with intersecting domains have the form

$$a' \;=\; Z^{-T}a + z\,, \qquad b' \;=\; b'(a, b)\,, \qquad \alpha' \;=\; Z\alpha + \mathcal{F}(a, b) \tag{3.4}$$

where $z \in \mathbb{R}^n$ and $Z \in \mathrm{SL}_\pm(n, \mathbb{Z})$ are constant in each connected component of the domains intersection, Z^{-T} is the inverse of the transpose of Z, and the function $\mathcal{F}$ is independent of the angles.

Proof. (i) From Proposition 3 we know that, given any $m \in M$, there is neighbourhood U of m and a bundle chart $(F, \alpha) : F^{-1}(U) \to U \times \mathbb{T}^n$ with strongly Hamiltonian semilocal generators $\partial_{\alpha_1}, \ldots, \partial_{\alpha_n}$. Their Hamiltonians $a_1, \ldots, a_n$ are constant along the fibers and, since σ is nondegenerate, are functionally independent. Therefore, by restricting U if necessary, we can construct a diffeomorphism $(\hat{a}, \hat{b}) : U \to \hat{U} \subset \mathbb{R}^{2d-n}$ such that $a = \hat{a} \circ F$. In this way we obtain semilocal coordinates $(a, b, \alpha) : F^{-1}(U) \to \hat{U} \times \mathbb{T}^n$, where $b = \hat{b} \circ \pi$. The local representative $\tilde{\sigma}$ of σ in these coordinates has no terms in $d\alpha_i \wedge d\alpha_j$ because $\sigma(\partial_{a_i}, \partial_{a_j}) = 0$. Taking also into account that each action a_i is the Hamiltonian of the corresponding generator ∂_{α_i} we see that $\tilde{\sigma}$ can be written in the form (3.3) for some matrices A, B and C which smoothly depend on the coordinates. Impose now $i_{\partial_{\alpha_i}} d\tilde{\sigma} = 0$ for all i. The vanishing of the terms of $i_{\partial_{\alpha_i}} d\tilde{\sigma}$ containing some $da_i \wedge da_j$, some $da_i \wedge db_u$ or some $db_u \wedge db_v$ gives that A, B and C are α–independent. Since

$$\det \begin{pmatrix} A & B & \mathbf{1} \\ -B^T & C & 0 \\ -\mathbf{1} & 0 & 0 \end{pmatrix} \;=\; \det C$$

the nondegeneracy of σ implies $\det C \neq 0$.

(ii) By part i., the local representative $\tilde{\sigma}'$ of σ in the coordinates (a', b', α') has the form

$$\tilde{\sigma}' \;=\; da'_i \wedge d\alpha'_i \;+\; \frac{1}{2}A'_{ij}da'_i \wedge da'_j \;+\; B'_{iu}da'_i \wedge db'_u \;+\; \frac{1}{2}C'_{uv}db'_u \wedge db'_v \tag{3.5}$$

with certain matrices A', B' and C' independent of the angles. By (2.1), the transition functions have the form $a' = a'(a, b)$, $b' = b'(a, b)$ and $\alpha' = Z\alpha + \mathcal{F}(a, b)$.

Expressing all the differentials of a', b', α' in terms of those of a, b, α and equalling the result to $\tilde\sigma$ as in (3.3) shows that the terms in $\tilde\sigma'$ which contain some $db_u \wedge d\alpha_j$ come from $da_i \wedge d\alpha'_i$ and their sum equals

$$\frac{\partial a'_i}{\partial b_u}\frac{\partial \alpha'_i}{\partial \alpha_j} db_u \wedge d\alpha_j \;=\; \left(Z^T \frac{\partial a'}{\partial b}\right)_{ju} db_u \wedge d\alpha_j \,.$$

Since Z is invertible, vanishing of these terms implies $\frac{\partial a'}{\partial b} = 0$. The terms which contain some $da_i \wedge d\alpha_j$ also come from $da'_i \wedge d\alpha'_i$ and their sum equals

$$\frac{\partial a'_i}{\partial a_j}\frac{\partial \alpha'_i}{\partial \alpha_h} da_j \wedge d\alpha_h \;=\; \left(Z^T \frac{\partial a'}{\partial a}\right)_{jh} da_j \wedge d\alpha_h \,.$$

Since this must equal $da_i \wedge d\alpha_i$, we see that $\frac{\partial a'}{\partial a} = Z^{-T}$. Thus $a' = Z^{-T} a + \text{const}$. ■

At variance with the symplectic case, it is not possible to further put the expression (3.3) of σ into a normal form with $A = B = 0$ and C equal to the symplectic identity. In particular, in the completely integrable case with $n = d$, (3.3) becomes $da_i \wedge d\alpha_i + \frac{1}{2} A_{ij} da_i \wedge da_j$.

The form of the transition functions (3.4), where the actions in one chart are functions only of the actions in the other chart, implies that there is a foliation of M with leaves of dimension $2d - n$ which, locally, are the level sets of the (local) actions. We now study the almost–symplectic geometry of this foliation.

4 The dual pair structure.

A. The setting. We now study the almost–symplectic geometry of the fibration by the invariant tori of Proposition 3. For greater clarity we consider a sligthly more general case, that is, we consider an almost–symplectic manifold (M, σ) of dimension $2d$ and a fibration $\pi : M \to P$ with fiber $\mathbb{T}^n$ for some $1 \le n \le d$, where P is a manifold of dimension $2d - n$, which is such that

H4. *$\pi : M \to P$ is a $\mathbb{T}^n$–bundle.*

H5. *M is almost–symplectic and $\pi : M \to P$ has a bundle atlas in which all semilocal generators are strongly Hamiltonian.*

By Proposition 3, under hypotheses H1, H2 and H3 the fibration $F : M \to F(M)$ of the previous section satisfies hypotheses H4 and H5. Conversely, every fibration $\pi : M \to P$ which satisfies hypotheses H4 and H5 can be described, at least semilocally, by a map $F : M \to \mathbb{R}^n$ which satisfies H1, H2 and H3—just take local coordinates on the base P. In particular, we can equip M with an atlas of action–angle charts as in Proposition 4.

B. The isotropic–coisotropic dual pair. An almost–symplectic structure σ on a manifold M defines an orthogonality relation on each tangent space T_mM: if E is a subspace of T_mM, then its σ–orthogonal is $E^\sigma = \{v \in T_mM \,:\, \sigma_m(u,v) = 0 \; \forall u \in E\}$. One can then define isotropic, coisoptropic and Lagrangian submanifolds of M exactly as in the symplectic case, for which we refer e.g. to [21].

Furthermore, given a distribution $\mathcal{D}$ on M, the *polar* distribution $\mathcal{D}^\sigma$ is the distribution on M whose fibers are the σ–orthogonals to those of $\mathcal{D}$. A foliation of M will be called *σ–complete* if the distribution of its tangent spaces has a Frobenius–integrable polar distribution; the integral foliation is then called *polar foliation.*

Proposition 5 *Under hypotheses H4 and H5:*

i. The fibers of $\pi : M \to P$ are isotropic.

ii. $\pi : M \to P$ has a polar foliation, whose leaves are locally the level sets of the actions and are coisotropic.

iii. If the space of the leaves of the foliation polar to $\pi : M \to P$ is a manifold, then it has an affine structure.

Proof. We can use local action–angle coordinates. (i) This follows from (3.3). (ii) The tangent spaces to the level sets of constant actions are spanned by the vector fields $\partial_{\alpha_i}, \ldots, \partial_{\alpha_n}, \partial_{b_1}, \ldots, \partial_{b_{2d-2n}}$ and are σ–orthogonal to the fibers of π because, as follows from (3.3), $\sigma(\partial_{\alpha_i}, \partial_{\alpha_j}) = \sigma(\partial_{\alpha_i}, \partial_{b_u}) = 0$ for all i, j and u. As we have already noted, the transition functions (3.4) imply that the level sets of the local actions are independent of the particular choice of the actions and globalize to a foliation of M. Coisotropy is obvious. (iii) This follows from (3.4), since an affine structure on a manifold is given by an atlas with affine transition functions. ■

C. The almost–Poisson structure of the base manifold. In order to conclude our analysis, it remains to discuss the structure of the base manifold P.

The (local) Casimirs, the characteristic distribution and the rank of an almost–Poisson bracket $\{\ ,\ \}$ on a manifold P can be defined exactly as in the Poisson case. The almost–Poisson bracket associates a Hamiltonian vector field to any function F, as the unique vector field Y_F such that $\{F, G\} = L_{Y_F}G$ for all functions G. A (local) *Casimir* of $\{\ ,\ \}$ is then any function F (defined in some open set) whose Hamiltonian vector field is zero, that is, which is in involuton with any other function. The *characteristic space* of $\{\ ,\ \}$ at a point $p \in P$ is the subspace of T_pP spanned by the germs of Hamiltonian vector fields. The dimension of this subspace is the *rank* of $\{\ ,\ \}$ at p. Clearly, the rank at a point equals $\dim P$ minus the number of independent germs of Casimirs at that point.

At variance with the Poisson case, the characteristic distribution of an almost–Poisson manifold need not be Frobenius–integrable, in which case there is no almost–Poisson analogue of the symplectic foliation of a Poisson manifold. In the present case, however, we have the following

Proposition 6 *Under hypotheses H4 and H5:*

i. *The base P of $\pi : M \to P$ has an almost–Poisson structure with bracket $\{\ ,\ \}_P$ such that*

$$\{F, G\}_P \circ \pi \ := \ \{F \circ \pi, G \circ \pi\}_M \tag{4.1}$$

for all functions $F, G : P \to \mathbb{R}$.

ii. *$\{\ ,\ \}_P$ has constant rank $2d - 2n$.*

iii. *If $a_1, \ldots, a_n$ are (local) actions of $\pi : M \to P$, then the functions $\hat{a}_1, \ldots, \hat{a}_n$ such that $a_i = \hat{a}_i \circ \pi$ are (local) Casimirs of $\{\ ,\ \}_P$.*

iv. *The characteristic distribution of $\{\ ,\ \}_P$ is Frobenius integrable and its leaves carry an almost–symplectic structure, the Poisson bracket of which coincides with the restriction of $\{\ ,\ \}_P$.*

Proof. (i) In order to show that equation (4.1) defines an almost–Poisson bracket on P it suffices to show that, in M, the almost–Poisson bracket of any two first integrals of π is a first integral of π. (A first integral of a fibration is any function constant on its fibers, that is, the lift of a function defined on the base manifold.) Using action–angle coordinates one sees that a function on M is a first integrals of π if and only if its local representative is independent of the angles α. The claim then follows from the fact that, since the local representative of σ is independent of α, the local representative of the almost–Poisson bracket of two first integrals is also independent of α.

(ii) and (iii) Consider a system of action–angle coordinates (a, b, α) and let $a_i = \hat{a}_i \circ \pi$. In view of (4.1), in order to prove that $\hat{a}_i$ is a Casimir of P we may show that $\{a_i, a_j\}_M = \{a_i, b_u\}_M = 0$ for all i and u. We already know that $\{a_i, a_j\}_M = 0$. Since the Hamiltonian vector field of a_i is ∂_{α_i}, $\{a_i, b_u\}_M = -i_{\partial_{\alpha_i}} db_u = 0$. Thus, P has at any points at least n independent germs of Casimirs. The nondegeneracy of σ (and hence of the matrix C, see the proof of Proposition 4) implies that there are no more than n of them.

(iv) The foliation of M polar to π, being locally given by the constancy of the actions, projects onto a foliation of P which is tangent to the characteristic distribution, which is therefore integrable. An almost–symplectic 2–form on the leaves of this foliation is obtained by projecting the local 2–forms on M which, in each system of action–angle coordinates (a, b, a), have the form $\frac{1}{2} C_{uv}(a, b) db_u \wedge db_v$, where C is the matrix entering the expression (3.3) of σ. That this operation actually produces a 2–form is proven by observing that, if (a, b, a) and (a', b', a') are two systems of action–angle coordinates, in which the representatives of σ are (3.3) and (3.5), respectively, then

$$C'\Big(Z^{-T} a + z, b'(a, b)\Big) \ = \ \left(\frac{\partial b'}{\partial b}(a, b)\right)^T C(a, b)\, \frac{\partial b'}{\partial b}(a, b) \tag{4.2}$$

as is easily seen by using the fact that the transition functions have the form (3.4). The nondegeneracy of this 2–form follows form the invertibility of the matrix C. ∎

D. Dynamics. Finally, we characterize the Hamiltonian vector fields which are tangent to the fibers of $\pi : M \to P$.

Proposition 7 *Under hypotheses H4 and H5, a Hamiltonian vector field X_H is tangent to the fibers of π if and only if its Hamiltonian H is a first integral of the foliation polar to π. In that case X_H is strongly Hamiltonian and its local representative in any system of action–angle coordinates is*

$$-\frac{\partial h}{\partial a_i}(a)\,\partial_{\alpha_i}$$

where $h = h(a)$ is the local representative of H, which depends only on the actions.

Proof. Let us work with action–angle coordinates $\mathcal{C} = (a, b, \alpha)$. If $X = X^{a_i}\partial_{a_i} + X^{b_u}\partial_{b_u} + X^{\alpha_i}\partial_{\alpha_i}$ is the local representative of X_H, then using $\mathcal{C}^*(i_{X_H}\sigma) = -dh$ with $\mathcal{C}^*\sigma$ as in (3.3) gives

$$\frac{\partial h}{\partial \alpha} = -X^a\,, \qquad \frac{\partial h}{\partial a} = X^\alpha + AX^a + BX^b\,, \qquad \frac{\partial h}{\partial b} = CX^b - B^T X^a\,.$$

Since C is invertible, these equations show that $X^a = X^b = 0$ if and only if h depends only on a. In that case, $X^\alpha = \frac{\partial h}{\partial a}$ and $i_X d\sigma = 0$ because X is a linear combination of $\partial_{\alpha_1}, \ldots, \partial_{\alpha_d}$, which are strongly Hamiltonian. ■

Thus, any Hamiltonian vector field which is tangent to the fibers of π has quasi–periodic flow with frequencies which depend only on the actions.

5 Conclusions

We have shown that, under the hypothesis of strong Hamiltonianity of the generators of the invariant tori, the structure of an integrable almost–symplectic Hamiltonian system is completely analogous to that of the symplectic case. Specifically, there are two σ–orthogonal foliations of the phase space, one formed by the isotropic invariant tori and the other by the coisotropic level sets of the (local) actions. Moreover, the leaves of the latter foliation are union of invariant tori, and all the tori in a given coisotropic leaf carry motions with equal frequencies.

The hypothesis of strong Hamiltonianity is clearly rather strong. For instance, nonholonomic mechanical systems are Hamiltonian with respect to an almost–symplectic structure but, usually, not strongly Hamiltonian [6]. In fact, we do not know of any integrable nonholonomic system in which the generators of the invariant tori are strongly Hamiltonian. However, this hypothesis is not new: as we have already mentioned, it is *de facto* encountered in the context of reduction of almost–Dirac manifolds [11, 8, 7], of which almost–symplectic manifolds are a special case.

Simple examples show that some stronger property than just the Hamiltonianity of the generators is actually necessary in the present context. For instance, consider the almost–symplectic form

$$\sigma \;=\; z_1 da_1\wedge d\alpha_1 \;+\; da_2\wedge d\alpha_2 \;+\; dz_1\wedge dz_2$$

on $M=\mathbb{T}^2\times\mathbb{R}^2\times\mathbb{R}^2_+\ni(\alpha,a,z)$. The submersion $F=(a_1,a_2,z_1,z_2)$ and the vector fields $Y_1=z_1^{-1}\partial_{\alpha_1}$ and $Y_2=\partial_{\alpha_2}$ satisfy the hypotheses of Proposition 1. Both Y_1 and Y_2 are Hamiltonian, with Hamiltonian $-a_1$ and respectively $-a_2$, but Y_1 is not strongly Hamiltonian. One of the two generators of the tori $F=\mathrm{const}$, namely ∂_{α_1}, is not Hamiltonian. Therefore, there is no notion of 'actions' and of action–angle coordinates and most of the statements of Propositions 3–7 are meaningless. Note that the foliation $F=\mathrm{const}$ is isotropic and does have a polar foliation, which is given by $(a_1,a_2)=\mathrm{const}$. However, the Hamiltonian vector field Y_1+Y_2 has quasi–periodic flow tangent to the tori $F=\mathrm{const}$ with frequencies which are not constant on the leaves of the polar foliation.

Nevertheless, it is conceivable that our study could be based on some property weaker than the strong Hamiltonianity of the generators. For instance, in the case of periodic flow, it is $n=1$ and the existence of action–angle coordinates essentially reduces to the so–called 'period–energy relation', see e.g. [27, 20, 17, 4]. As shown in [17] such a relation is valid if, and in fact is equivalent to, $L_X\bar\sigma=0$, where $\bar\sigma$ is the average of σ over the flow of X. It is conceivable that, in the case of quasi–periodic flow, one may reproduce the construction of this article under the hypothesis $L_X\bar\sigma=0$, where now $\bar\sigma$ is the average of σ over the invariant tori.

As a final remark we note that, even though we have not stressed this point of view within our approach, the strong Hamiltonianity of the torus generators implies that the invariant tori are the orbits of semilocal[5] strongly Hamiltonian actions of $\mathbb{T}^n$. Here, by a (strongly) Hamiltonian action of a Lie group on an almost–symplectic manifold we mean an action whose infinitesimal generators are (strongly) Hamiltonian vector fields. Now, any Hamiltonian action on an almost–symplectic manifold defines a momentum map $J:M\to\mathfrak{g}^*$ in the usual way: for any $\xi\in\mathfrak{g}$ and any $m\in M$, $<J(m),\xi>$ is the value at m of the Hamiltonian J_ξ of the infinitesimal generator ξ^M (here, of course, $\mathfrak{g}$ denotes the Lie algebra of the group). But moreover, if the action is strongly Hamiltonian, then its momentum map is a constant of motion for the Hamiltonian vector field of any G–invariant function H:

$$\frac{d}{dt}(J_\xi\circ\Phi_t^{X_H})=(L_{X_H}J_\xi)\circ\Phi_t^{X_H}=-(L_{\xi^M}H)\circ\Phi_t^{X_H}=0\,.$$

Within our setting, the (local) actions could then be regarded as the components of the momentum map of a strongly Hamiltonian $\mathbb{T}^n$–action. (For the use of this point of view in the proof of the standard Liouville–Arnold theorem on complete integrability, see e.g. [5]).

[5]See the Remarks at the end of Sections 2.A and 3.A.

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