On the maximum fraction of edges covered by t perfect matchings in a cubic bridgeless graph

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Abstract

A conjecture of Berge and Fulkerson (1971) states that every cubic bridgeless graph contains 6 perfect matchings covering each edge precisely twice, which easily implies that every cubic bridgeless graph has three perfect matchings with empty intersection (this weaker statement was conjectured by Fan and Raspaud in 1994). Let m_t be the supremum of all reals $\alpha \leq 1$ such that for every cubic bridgeless graph G, there exist t perfect matchings of G covering a fraction of at least α of the edges of G. It is known that the Berge-Fulkerson conjecture is equivalent to the statement that $m_5 = 1$, and implies that $m_4 = \frac{14}{15}$ and $m_3 = \frac{4}{5}$. In the first part of this paper, we show that $m_4 = \frac{14}{15}$ implies $m_3 = \frac{4}{5}$, and $m_3 = \frac{4}{5}$ implies the Fan-Raspaud conjecture, strengthening a recent result of Tang, Zhang, and Zhu. In the second part of the paper, we prove that for any $2 \leq t \leq 4$ and for any real τ lying in some appropriate interval, deciding whether a fraction of more than (resp. at least) τ of the edges of a given cubic bridgeless graph can be covered by t perfect matching is an NP-complete problem. This resolves a conjecture of Tang, Zhang, and Zhu.

Keywords: Berge-Fulkerson conjecture, perfect matchings, cubic graphs.
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1 Introduction

Most of the notation used in this paper is standard (see [1] or [2] for any undefined terminology). A *perfect matching* of a graph G is a spanning subgraph of G in which each vertex has degree precisely one. In this paper we will only deal with *cubic bridgeless* graphs, that is graphs in which each vertex has degree 3 and such that each component is 2-edge-connected. We are interested in the following conjecture of Berge and Fulkerson [5].

Conjecture 1 (Berge-Fulkerson, 1971)**.** *If G is a cubic bridgeless graph, then there exist six perfect matchings of G such that each edge of G belongs to exactly two of them.*

This conjecture is equivalent to the existence of five perfect matchings of G such that any three of them have empty intersection. Therefore, a weaker statement is the following conjecture of Fan and Raspaud [4].

Conjecture 2 (Fan-Raspaud, 1994)**.** *If G is a cubic bridgeless graph, then there exist three perfect matchings M_1, M_2 and M_3 of G such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

In this paper we will also consider the following middle step between these two conjectures.

Conjecture 3. *If G is a cubic bridgeless graph, then there exist four perfect matchings such that any three of them have empty intersection.*

Following the notation introduced in [8] we define $m_t(G)$ as the maximum fraction of edges in G that can be covered by t perfect matchings and we define m_t as the infimum of $m_t(G)$ over all cubic bridgeless graphs G.

The second author [9] proved that the Berge-Fulkerson conjecture is equivalent to the conjecture that the edge-set of every cubic bridgeless graph can be covered by 5 perfect matchings, i.e. $m_5 = 1$. Kaiser, Král', and Norine [8] proved that the infimum m_2 is a minimum, attained by the Petersen graph (i.e. $m_2 = \frac{3}{5}$), and Patel [10] conjectured that the values of m_3 and m_4 are also attained by the Petersen graph. In other words:

Conjecture 4 (Patel, 2006)**.** $m_3 = \frac{4}{5}$.

Conjecture 5 (Patel, 2006)**.** $m_4 = \frac{14}{15}$.

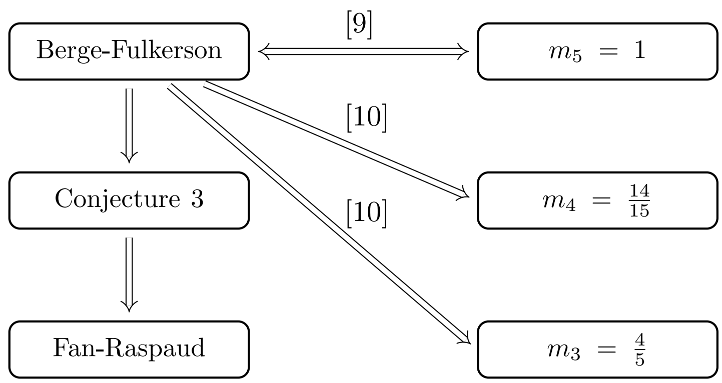

Figure 1: A diagram of the implications presented in Section 1

Patel proved [10] that the Berge-Fulkerson conjecture implies Conjectures 4 and 5. These implications are summed up in Figure 1.

Tang, Zhang, and Zhu [11] recently showed that weighted versions of Conjectures 4 and 5 imply the Fan-Raspaud conjecture: for instance, they prove that if for any cubic bridgeless graph G and any edge-weighting of G, a weighted version of $m_3(G)$ is at least $\frac{4}{5}$, then the Fan-Raspaud conjecture holds. In Section 2, we show that it is enough to consider only the edge-weighting where every edge has weight 1. More precisely, we show that $m_4 = \frac{14}{15}$ implies Conjecture 3 and $m_3 = \frac{4}{5}$ implies the Fan-Raspaud conjecture. We also show that $m_4 = \frac{14}{15}$ implies $m_3 = \frac{4}{5}$.

Tang, Zhang, and Zhu [11] conjectured that for any real $\frac{4}{5} < \tau \leq 1$, determining whether a cubic bridgeless graph G satisfies $m_3(G) \geq \tau$ is an NP-complete problem. In Section 3 we prove this conjecture together with similar statements for $m_2(G)$ and $m_4(G)$.

2 Main results

Given two cubic graphs G and H and two edges xy in G and uv in H, the *glueing*, or *2-cut-connection*, of (G, x, y) and (H, u, v) is the graph obtained from G and H by removing edges xy and uv, and connecting x and u by an edge, and y and v by an edge. We call these two new edges the *clone edges of xy or uv* in the resulting graph. Note that if G and H are cubic and bridgeless, then the resulting graph is also cubic and bridgeless. In the present paper H will always be K_4 or the Petersen graph, which are both arc-transitive (for any two pairs of adjacent vertices u_1, u_2 and v_1, v_2, there is an automorphism that maps u_1 to v_1 and u_2 to v_2). In this case the choice of uv and the order of each pair (x, y) and (u, v) are not relevant, so we simply say that *we glue H on the edge xy of G*.

In what follows, we will need to glue several graphs on each edge of a given graph G. This has to be understood as follows: given copies $H_1, \ldots, H_k$ ($k \geq 2$) of K_4 or the Petersen graph, *glueing $H_1, \ldots, H_k$ on the edge e of G* means glueing H_k on some clone edge of e in the glueing of $H_1, \ldots, H_{k-1}$ on the edge e of G (see Figure 2).

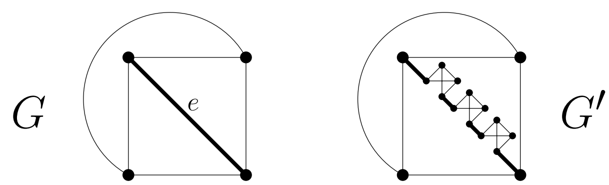

Figure 2: The graph G' is obtained by glueing 3 copies of K_4 on the edge e of G. The clone edges of e in G' are drawn as thick edges.

Note that each perfect matching in the graph G' resulting from the glueing of $H_1, \ldots, H_k$ ($k \geq 1$) on some edge e of G either contains all clone edges of e, or none of them (since each pair of such edges forms a 2-edge-cut in G', and the intersection of every perfect matching with an edge-cut has the same parity as the edge-cut). It follows that each perfect matching M' of G' corresponds to a perfect matching M of G and perfect matchings M_i of H_i, for each $1 \leq i \leq k$. We call each of these perfect matchings the *restriction* of M to $G, H_1, \ldots, H_k$, respectively.

The following is a well-known property of the Petersen graph.

Lemma 6. *The Petersen graph P has exactly 6 distinct perfect matchings. Moreover the following properties hold:*

- *Every edge of P is covered by exactly two distinct perfect matchings;*
- *Every two distinct perfect matchings of P intersect in exactly one edge (and therefore cover exactly 9 edges);*
- *Every three distinct perfect matchings of P cover exactly 12 edges;*
- *Every four distinct perfect matchings of P cover exactly 14 edges.*

It was proved in [11] that a fractional version of Conjecture 4 ($m_3 = \frac{4}{5}$) implies the Fan-Raspaud conjecture. We strengthen this result by showing that not only the integral conjecture itself implies the Fan-Raspaud conjecture, but it implies an even stronger statement (which will be needed in what follows).

Theorem 7. *Conjecture 4 implies that any cubic bridgeless graph G has three perfect matchings M_1, M_2, M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$ and $|M_1 \cup M_2 \cup M_3| \geq \frac{4}{5}|E(G)|$. In particular, Conjecture 4 implies the Fan-Raspaud Conjecture.*

Proof. Let G be a cubic bridgeless graph, and let G' be the graph obtained by glueing $|E(G)|$ copies of the Petersen graph on each edge of G (see Figure 3). The number of edges of G' can be easily computed as $15|E(G)|^2 + |E(G)|$. If Conjecture 4 holds, then G' has a set $\mathcal{M}' = \{M'_1, M'_2, M'_3\}$ of perfect matchings such that $M'_1 \cup M'_2 \cup M'_3$ contains at least $\frac{4}{5}|E(G')| = 12|E(G)|^2 + \frac{4}{5}|E(G)|$ edges of G'. Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be the restriction of $\mathcal{M}'$ to G. We now prove that $|M_1 \cup M_2 \cup M_3| \geq \frac{4}{5}|E(G)|$ and $M_1 \cap M_2 \cap M_3 = \emptyset$.

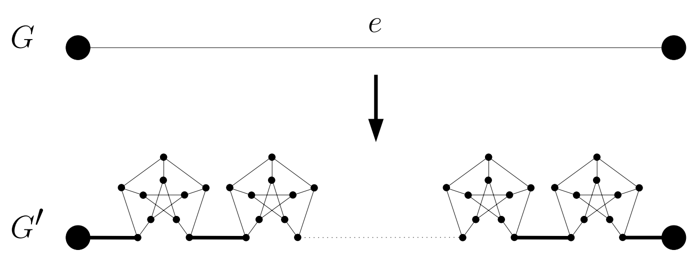

Figure 3: G' is obtained by glueing multiple copies of the Petersen graph on each edge of G. The thick edges are the clone edges of e in G'.

Note that by Lemma 6, at most 12 edges are covered by the restriction of $\mathcal{M}'$ to each of the $|E(G)|^2$ copies of the Petersen graph glued on the edges of G. Since $M'_1 \cup M'_2 \cup M'_3$ contains at least $12|E(G)|^2 + \frac{4}{5}|E(G)|$ edges of G', it follows that $|M_1 \cup M_2 \cup M_3| \geq \frac{4}{5}|E(G)|$.

By Lemma 6, for each edge e of G lying in $M_1 \cap M_2 \cap M_3$, at most 9 edges are covered by the restriction of $\mathcal{M}'$ to each of the $|E(G)|$ copies of the Petersen graph glued on e. If $M_1 \cap M_2 \cap M_3 \neq \emptyset$, this implies that $M'_1 \cup M'_2 \cup M'_3$ contains at most $12|E(G)|(|E(G)| - 1) + 9|E(G)| + |E(G)| = 12|E(G)|^2 - 2|E(G)|$ edges, a contradiction. $\square$

Let $\mathcal{M}$ be a set of perfect matchings of a graph G. For $e \in E(G)$, we define the *depth* of e in $\mathcal{M}$, denoted by $dp_e(\mathcal{M})$, as the number of perfect matchings of $\mathcal{M}$ containing e. The *edge-depth* of $\mathcal{M}$, denoted by $dp(\mathcal{M})$, is the maximum depth of an edge of G in $\mathcal{M}$. The same proof as that of Theorem 7, considering four perfect matchings instead of three, shows the following connection between Conjecture 5 ($m_4 = \frac{14}{15}$) and Conjecture 3.

Theorem 8. *Conjecture 5 implies that any cubic bridgeless graph G has a set $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ of four perfect matchings such that $dp(\mathcal{M}) \leq 2$ and $|M_1 \cup M_2 \cup M_3 \cup M_4| \geq \frac{14}{15}|E(G)|$. In particular, Conjecture 5 implies Conjecture 3.*

We now use Theorem 8 to show that $m_4 = \frac{14}{15}$ implies $m_3 = \frac{4}{5}$.

Theorem 9. *Conjecture 5 implies Conjecture 4.*

Proof. Assume that Conjecture 5 holds and let G be any cubic bridgeless graph. By Theorem 8, G has a set $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ of four perfect matchings such that $dp(\mathcal{M}) \leq 2$ and $|M_1 \cup M_2 \cup M_3 \cup M_4| \geq \frac{14}{15}|E(G)|$. For $i \geq 0$, let ε_i be the fraction of edges of G covered precisely i times. Consider the set E_1 of edges covered exactly once by $\mathcal{M}$, and remove from $\mathcal{M}$ the perfect matching, say M_1, containing the smallest number of edges of E_1. Then the fraction of edges of G covered by $M_2 \cup M_3 \cup M_4$ is at least $\frac{3}{4}\varepsilon_1 + \varepsilon_2$.

Since $dp(\mathcal{M}) \leq 2$ and every perfect matching contains a third of the edges, we have $\varepsilon_1 + 2\varepsilon_2 = \frac{4}{3}$. Combining this with the assumption that $\varepsilon_1 + \varepsilon_2 \geq \frac{14}{15}$, we obtain $\frac{3}{4}\varepsilon_1 + \varepsilon_2 \geq \frac{4}{5}$, which concludes the proof. $\square$

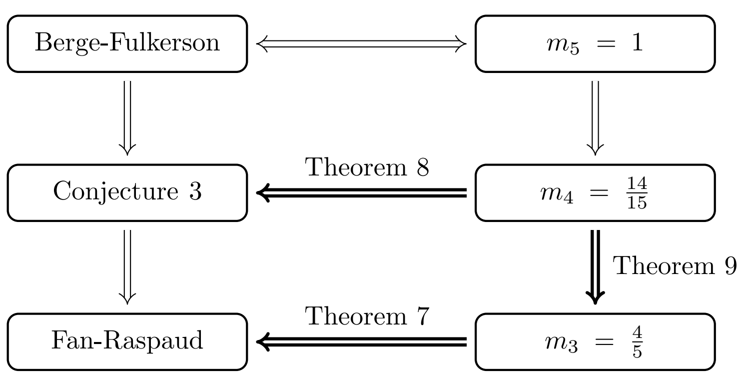

Figure 4: New diagram of implications after Section 2

We summarize in Figure 4 the implications which follow from the results of this section. An important remark is that in order to prove that a given cubic bridgeless graph H satisfies $m_3(H) \geq \frac{4}{5}$, in Theorem 9 we really use the assumption that $m_4(G) \geq \frac{14}{15}$ *for every cubic bridgeless graph* G. We do not know how to prove the stronger statement that if $m_4(G) = \frac{14}{15}$, then $m_3(G) \geq \frac{4}{5}$. In the remainder of this section, we provide weaker bounds relating $m_k(G)$ and $m_{k+1}(G)$, for any $k \geq 2$.

Theorem 10. *If G is a cubic bridgeless graph, then $m_2(G) \geq \frac{1}{2}\, m_3(G) + \frac{1}{6}$.*

Proof. Let M_1, M_2, M_3 be perfect matchings of G covering a fraction of $m_3(G)$ of the edges of G. For $i \geq 0$, let ε_i be the fraction of edges of G covered precisely i times by M_1, M_2, M_3. We have $(\mathbf{1})$ $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = m_3(G)$ and $(\mathbf{2})$ $\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 = 1$. The combination $\frac{1}{2}\,(\mathbf{1}) + \frac{1}{6}\,(\mathbf{2})$ gives $\frac{2}{3}\,\varepsilon_1 + \frac{5}{6}\,\varepsilon_2 + \varepsilon_3 = \frac{1}{2}\, m_3(G) + \frac{1}{6}$. We remove from M_1, M_2, M_3 the perfect matching, say M_1, containing the least number of edges covered precisely once by M_1, M_2, M_3. It follows that the fraction of edges of G covered by M_2, M_3 is at least $\frac{2}{3}\,\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \geq \frac{1}{2}\, m_3(G) + \frac{1}{6}$. □

Theorem 11. *If G is a cubic bridgeless graph and $k \geq 4$ is an integer, then $m_{k-1}(G) \geq \frac{k-1}{k}\, m_k(G) + \frac{1}{3k}$.*

Proof. Let $M_1, \ldots, M_k$ be perfect matchings of G covering a fraction of $m_k(G)$ of the edges of G. We say that a vertex of G has type (x, y, z) if the three edges incident to v are covered x, y, and z times (respectively) by $M_1, \ldots, M_k$. Let a, b, c, d, e, f be the fractions of vertices of type $(0, 0, k)$, $(0, 1, k-1)$, $(0, \geq 2, \geq 2)$, $(1, 1, k-2)$, $(1, \geq 2, \geq 2)$, and $(\geq 2, \geq 2, \geq 2)$, respectively. Any vertex of G is of one of these six types, so we have $(\mathbf{1})$ $a+b+c+d+e+f = 1$. Let n and m denote the number of vertices and edges of G, respectively. Then by definition of $m_k(G)$, the number of edges covered by $M_1, \ldots, M_k$ is $m \cdot m_k(G) = \frac{1}{2}(an+2bn+2cn+3dn+3en+3fn)$. Since G is cubic, $m = \frac{3}{2}n$ and so, $(\mathbf{2})$ $\frac{a}{3}+\frac{2b}{3}+\frac{2c}{3}+d+e+f = m_k(G)$. Similarly, the fraction of edges covered precisely once by $M_1, \ldots, M_k$ is equal to $\frac{b}{3}+\frac{2d}{3}+\frac{e}{3}$, so we can remove one of the perfect matchings, say M_1, in such way that $M_2, \ldots, M_k$ cover a fraction of at least $m_k(G) - \frac{1}{k}\,(\frac{b}{3}+\frac{2d}{3}+\frac{e}{3})$ of the edges of G. Note that the combination $(\mathbf{2}) - \frac{1}{3}\,(\mathbf{1})$ gives $m_k(G) - \frac{1}{3} = \frac{b}{3}+\frac{c}{3}+\frac{2d}{3}+\frac{2e}{3}+\frac{2f}{3} \geq \frac{b}{3}+\frac{2d}{3}+\frac{e}{3}$, so it follows that $m_{k-1}(G) \geq \frac{k-1}{k}\, m_k(G) + \frac{1}{3k}$. □

In particular, this shows that any cubic bridgeless graph G whose edge-set can be covered by 4 perfect matchings satisfies $m_3(G) \geq \frac{5}{6}$. It follows that the conjecture stating that every cubic bridgeless graph G satisfies $m_3(G) \geq \frac{4}{5}$ only needs to be verified for graphs whose edge-set cannot be covered by 4 perfect matchings (some results on this class of graphs can be found in [3] and [6]).

Recall that Kaiser, Kráľ, and Norine [8] proved that every cubic bridgeless graph G satisfies $m_2(G) \geq \frac{3}{5}$. An interesting problem is to characterize graphs for which equality holds (the Petersen graph is an example). The next theorem implies that, again, these graphs are such that their edge-set cannot be covered by 4 perfect matchings (indeed, if the edge-set of a cubic bridgeless graph G can be covered by 4 perfect matchings, then $m_4(G) = 1$, and Theorem 12 below implies that $m_2(G) \geq \frac{11}{18} > \frac{3}{5}$).

Theorem 12. *If G is a cubic bridgeless graph, then $m_2(G) \geq \frac{5}{12}\, m_4(G) + \frac{7}{36}$.*

Proof. Let $M_1, \ldots, M_4$ be perfect matchings of G covering a fraction of $m_4(G)$ of the edges of G. Let a, b, c, d, e, f be the fractions of vertices defined in the proof of Theorem 11, with $k = 4$. Observe that $e = f = 0$, thus we have (**1**) $a + b + c + d = 1$ and (**2**) $\frac{a}{3} + \frac{2b}{3} + \frac{2c}{3} + d = m_4(G)$. Let ε_i be the fraction of edges of G covered precisely i times by M_1, M_2, M_3, M_4.

Note that after the removal of two perfect matchings M_i and M_j ($i \neq j$) from $M_1, \ldots, M_4$, the edges that were covered only by M_i, or only by M_j, or only by M_i and M_j are not covered anymore. If we sum the fractions of edges of these three types, for any of the six pair $\{i, j\}$ we obtain $3\varepsilon_1 + \varepsilon_2$ (every edge covered exactly once is counted three times). Note that $\varepsilon_1 = \frac{b}{3} + \frac{2d}{3}$ and $\varepsilon_2 = \frac{2c}{3} + \frac{d}{3}$, so $3\varepsilon_1 + \varepsilon_2 = b + \frac{2c}{3} + \frac{7d}{3}$. Thus we can choose a pair $\{i, j\}$, such that after the removal of M_i and M_j from $M_1, \ldots, M_4$, a fraction of at least $m_4(G) - \frac{1}{6}(b + \frac{2c}{3} + \frac{7d}{3})$ of the edges of G is still covered. The combination $\frac{7}{2}\,(\mathbf{2}) - \frac{7}{6}\,(\mathbf{1})$ gives $\frac{7}{2}\, m_4(G) - \frac{7}{6} = \frac{7b}{6} + \frac{7c}{6} + \frac{7d}{3} \geq b + \frac{2c}{3} + \frac{7d}{3}$, so it follows that $m_2(G) \geq \frac{5}{12}\, m_4(G) + \frac{7}{36}$. □

3 Complexity

A cubic bridgeless graph G satisfies $m_3(G) = 1$ if and only if it is 3-edge-colorable, and deciding this is a well-known NP-complete problem (see [7]). It was proved in [3] that deciding whether a cubic bridgeless graph G satisfies $m_4(G) = 1$ is also an NP-complete problem.

In this section, we prove that determining whether $m_2(G)$, $m_3(G)$, and $m_4(G)$ are more than any given constant (lying in some appropriate interval) is also an NP-complete problem. In the case of $m_3(G)$ (see Theorem 14), this solves a conjecture of Tang, Zhang, and Zhu [11].

Theorem 13. *For any constant $\frac{3}{5} < \tau < \frac{2}{3}$, deciding whether a cubic bridgeless graph G satisfies $m_2(G) > \tau$ (resp. $m_2(G) \geq \tau$) is an NP-complete problem.*

Proof. The proof proceeds by reduction from the 3-edge-colorability of cubic bridgeless graphs, which is a well-known NP-complete problem [7]. Note that our problem is clearly in NP, since any set of two perfect matchings whose union covers a fraction of more than (resp. at least) τ of the edges is a certificate that can be checked in polynomial time.

For any cubic bridgeless graph G with m edges, we construct (in polynomial time) a cubic bridgeless graph G' (of size polynomial in m), such that G is 3-edge-colorable if and only if $m_2(G') > \tau$ (resp. $m_2(G') \geq \tau$).

Let $a = \lfloor \frac{m}{2} \rfloor + 1$ and $b = \lfloor a \cdot \frac{4-6\tau}{15\tau-9} \rfloor$. It can be checked that

$$\tau < \tfrac{4a+9b+2/3}{6a+15b+1} < \tau + \tfrac{2a}{m(6a+15b+1)}.$$

The graph G' is obtained from G by glueing a copies of K_4 and b copies of the Petersen graph on every edge of G. Note that G' has precisely $m(6a+15b+1)$ edges.

Assume first that G is 3-edge-colorable. Then G has two perfect matchings M_1 and M_2 such that $M_1 \cap M_2 = \emptyset$. We construct two perfect matchings M_1' and M_2' of G' by combining M_1 and M_2, respectively, with perfect matchings of each copy of K_4 and the Petersen graph glued on the edges of G. Since no edge of G is contained both in M_1 and M_2, these two perfect matchings can be combined with perfect matchings covering precisely 4 edges of each copy of K_4 (see Figure 5, left and center) and 9 edges of each copy of the Petersen graph (by Lemma 6). It follows that G' contains two perfect matchings M_1' and M_2' covering $m(4a+9b) + \frac{2}{3}\,m$ edges of G'. Therefore, we have $m_2(G') \geq \frac{4a+9b+2/3}{6a+15b+1} > \tau$, as desired.

Assume now that G is not 3-edge-colorable, and take any two perfect matchings M_1' and M_2' of G'. Let M_1 and M_2 be the restriction of M_1' and M_2' to G, respectively. Since G is not 3-edge-colorable, M_1 and M_2 have non-empty intersection, so G has an edge e lying both in M_1 and M_2. We consider the union of the restriction of M_1' and the restriction of M_2' to each copy of K_4 or the Petersen graph glued on the edges of G. As before, this union covers at most 4 edges in each copy of K_4, and at most 9 edges in each copy

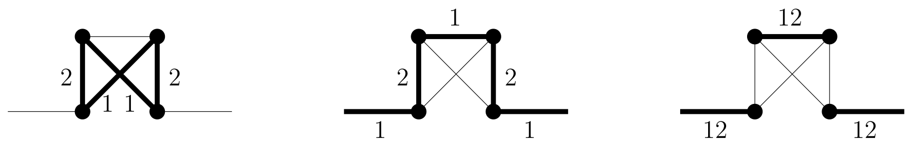

Figure 5: Perfect matchings covering a copy of K_4 glued on some edge of G. Edges are labelled i if they are covered by M_i' in G'.

of the Petersen graph. However, it can be checked that since e is contained in $M_1 \cap M_2$, at most 2 edges can be covered in each copy of K_4 glued on e (see Figure 5, right). It follows that $M_1' \cup M_2'$ covers at most $m(4a + 9b + \frac{2}{3}) - 2a$ edges of G'. Therefore, $m_2(G') \leq \frac{4a+9b+2/3}{6a+15b+1} - \frac{2a}{m(6a+15b+1)} < \tau$, as desired. $\square$

Theorem 14. *For any constant $\frac{4}{5} < \tau < 1$, deciding whether a cubic bridgeless graph G satisfies $m_3(G) > \tau$ (resp. $m_3(G) \geq \tau$) is an NP-complete problem.*

Proof. The proof follows the lines of the proof of Theorem 13. Here we define $a = \lfloor \frac{3m}{2} \rfloor + 1$, where m is the number of edges of G, and $b = \lfloor a \cdot \frac{6-6\tau}{15\tau-12} \rfloor$. It can be checked that

$$\tau < \frac{6a+12b+1}{6a+15b+1} < \tau + \frac{2a}{m(6a+15b+1)}.$$

The graph G is 3-edge-colorable if and only if it contains 3 perfect matchings with pairwise empty intersection. It follows that as above, if G is 3-edge-colorable we can find 3 perfect matchings of G' such that their restrictions cover 6 edges in each copy of K_4 (see Figure 6, left) and 12 edges in each copy of the Petersen graph (by Lemma 6) glued on the edges of G.

Figure 6: Perfect matchings covering a copy of K_4 glued on some edge of G. Edges are labelled i if they are covered by M_i' in G'.

If G is not 3-edge-colorable, then for any 3 perfect matchings of G' there is an edge of G covered at least twice by their restrictions to G. This implies that at most 4 edges are covered in each copy of K_4 glued on this edge (see Figure 6, right) and the result follows. □

Theorem 15. *For any constant* $\frac{14}{15} < \tau < 1$*, deciding whether a cubic bridgeless graph* G *satisfies* $m_4(G) > \tau$ *(resp.* $m_4(G) \geq \tau$*) is an NP-complete problem.*

Proof. Instead of reducing from 3-edge-colorability, we have to make a small modification here. The proof proceeds by reduction from the problem of deciding whether a cubic bridgeless graph G satisfies $m_4(G) = 1$, which is an NP-complete problem [3]. We define $a = \lfloor \frac{m}{2} \rfloor + 1$, where m is the number of edges of G, and $b = \lfloor a \cdot \frac{6-6\tau}{15\tau-14} \rfloor$. It can be checked that

$$\tau < \tfrac{6a+14b+1}{6a+15b+1} < \tau + \tfrac{2a}{m(6a+15b+1)}.$$

As above, given a cubic bridgeless graph G, we construct G' by glueing a copies of K_4 and b copies of the Petersen graph on every edge of G.

If $m_4(G) = 1$, then G contains 4 perfect matchings such that any edge of G is covered by at least 1 of the 4 perfect matchings, and avoided by at least 2 of the 4 perfect matchings. It follows that G' contains 4 perfect matchings such that their restrictions cover 6 edges in each copy of K_4 (see Figure 7, left and center) and 14 edges in each copy of the Petersen graph (by Lemma 6) glued on the edges of G.

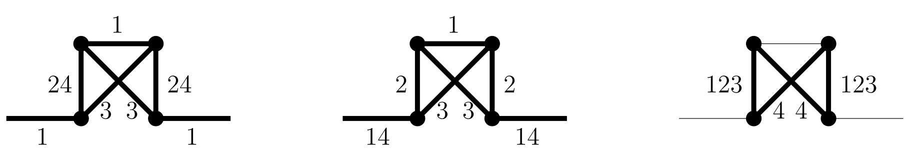

Figure 7: Perfect matchings covering a copy of K_4 glued on some edge of G. Edges are labelled i if they are covered by M'_i in G'.

If $m_4(G) \neq 1$, then for any 4 perfect matchings of G' there is an edge of G avoided by each of their restrictions to G. This implies that at most 4 edges are covered in of each copy of K_4 glued on this edge (see Figure 7, right) and the result follows. □

Recall that the Berge-Fulkerson Conjecture is equivalent to $m_5 = 1$ (see [9]), so if this conjecture is true we cannot prove hardness results similar to Theorems 13, 14, and 15, for m_k when $k \geq 5$.

4 Conclusion

Let $\mathcal{F}_{3/5}$ be the family of cubic bridgeless graphs G for which $m_2(G) = \frac{3}{5}$. A nice problem left open by the results in the previous section is the following.

Problem 16. *What is the complexity of deciding whether a cubic bridgeless graph belongs to* $\mathcal{F}_{3/5}$*?*

By Theorem 12, the edge-set of a graph of $\mathcal{F}_{3/5}$ cannot be covered by 4 perfect matchings. Using arguments similar to that of [8], it can be proved that any graph $G \in \mathcal{F}_{3/5}$ has a set $\mathcal{M}$ of at least three perfect matchings, such that for any $M \in \mathcal{M}$ there is a set $\mathcal{M}_M$ of at least three perfect matchings of G satisfying the following: for any $M' \in \mathcal{M}_M$, $|M \cup M'| = \frac{3}{5}|E(G)|$. However, this necessary condition is not sufficient: it is not difficult to show that it is satisfied by the dodecahedron and by certain families of snarks.

An interesting question is whether there exists any *3-edge-connected* cubic bridgeless graph $G \in \mathcal{F}_{3/5}$ that is different from the Petersen graph.

Similarly, we do not know if it can be decided in polynomial time whether a cubic bridgeless graph G satisfies $m_3(G) = \frac{4}{5}$ (resp. $m_4(G) = \frac{14}{15}$), but the questions seem to be significantly harder than for m_2 (since it is not known whether $m_3 = \frac{4}{5}$ and $m_4 = \frac{14}{15}$).

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