Nowhere-zero 5-flows on cubic graphs with oddness 4

Giuseppe Mazzuoccolo[*], Eckhard Steffen[†]

Abstract

Tutte's 5-Flow Conjecture from 1954 states that every bridgeless graph has a nowhere-zero 5-flow. It sufficies to prove the conjecture for cyclically 6-edge-connected cubic graphs. We prove that every cyclically 6-edge-connected cubic graph with oddness at most 4 has a nowhere-zero 5-flow. This implies that every minimum counterexample to the 5-flow conjecture has oddness at least 6.

1 Introduction

An integer nowhere-zero k-flow on a graph G is an assignment of a direction and a value of $\{1, \dots, (k-1)\}$ to each edge of G such that the Kirchhoff's law is satisfied at every vertex of G. This is the most restrictive definition of a nowhere-zero k-flow. But it is equivalent to more flexible definitions, see e.g. [11]. One of the most famous conjectures in graph theory is Tutte's 5-flow conjecture which is open for more than 60 years now.

Conjecture 1.1 ([13]) *Every bridgeless graph has a nowhere-zero 5-flow.*

Seymour [10] proved that every bridgeless graph has a nowhere-zero 6-flow. So far this is the best approximation to the 5-flow conjecture, which is equivalent to its restriction to cubic graphs.

*Università degli Studi di Verona, Dipartimento di Informatica, Strada le Grazie 15, 37134 Verona, Italy; giuseppe.mazzuoccolo@univr.it

†Paderborn Institute for Advanced Studies in Computer Science and Engineering, Universität Paderborn, Warburger Straße 100, 33098 Paderborn, Germany; es@upb.de

Petersen [9] proved in 1891 that every bridgeless cubic graph has a 1-factor, i.e. a spanning 1-regular subgraph. Therefore, such graphs have a 2-factor as well. The *oddness* of a bridgeless cubic graph G is the minimum number of odd components of a 2-factor G, and it is denoted by $\omega(G)$. The following three statements are equivalent: (i) $\omega(G) = 0$; (ii) G is 3-edge-colorable; (iii) G has a nowhere-zero 4-flow. Bridgeless cubic graphs which are not 3-edge colorable are also called *snarks*. Hence, a possible counterexample to the 5-flow-conjecture is a snark.

The oddness is a classical parameter to measure how far a cubic bridgeless graph is from being 3-edge-colorable. Some of the main conjectures in graph theory are verified for oddness 2 and 4: for instance, the Cycle Double Cover Conjecture holds true for snarks of oddness 4 (see [5]) and the Fan-Raspaud Conjecture has been recently verified for snarks of oddness 2 (see [8]). Here, we produce an analogous result for the 5-flow conjecture.

It is easy to see that snarks with oddness 2 have a nowhere-zero 5-flow. In [12] it is shown that if the cyclic connectivity of a cubic graph G is at least $\frac{5}{2}\omega(G) - 3$, then G has a nowhere-zero 5-flow. This result implies that cyclically 7-edge-connected cubic graphs with oddness at most 4 have a nowhere-zero 5-flow. However, currently no cyclically 7-edge-connected snark is known. It is even conjectured by Jaeger and Swart [4] that such snarks do not exist. However, there are infinitely many cyclically 6-edge-connected snarks, and by a result of Kochol [6], it suffices to prove the 5-flow conjecture for cyclically 6-edge-connected snarks.

The following is the main theorem of the paper.

Theorem 1.2 *Let G be a cyclically 6-edge-connected cubic graph. If $\omega(G) \leq 4$, then G has a nowhere-zero 5-flow.*

We summarize some structural properties of a possible minimum counterexample to the 5-flow conjecture.

Corollary 1.3 *If G is a possible minimum counterexample to the 5-flow conjecture, then*

- *G is a cubic graph [10].*
- *G is cyclically 6-edge connected [6].*
- *the cyclic connectivity of G is at most $\frac{5}{2}\omega(G) - 4$ [12].*

- *G has girth at least 11 [7].*
- *G has oddness at least 6.*

So far, no cubic graph is known that satisfies all items of Corollary 1.3.

2 Balanced valuations and flow partitions

In this section, we recall the concept of flow partitions, which was introduced by the second author in [12].

Let G be a graph and $S \subseteq V(G)$. The set of edges with precisely one end in S is denoted by $\partial_G(S)$.

An *orientation* D of G is an assignment of a direction to each edge. For $S \subseteq V(G)$, $D^-(S)$ ($D^+(S)$) is the set of edges of $\partial_G(S)$ whose head (tail) is incident to a vertex of S. The oriented graph is denoted by $D(G)$, $d^-_{D(G)}(v) = |D^-(\{v\})|$ and $d^+_{D(G)}(v) = |D^+(\{v\})|$ denote the *indegree* and *outdegree* of vertex v in $D(G)$, respectively. The degree of a vertex v in the undirected graph G is $d^+_{D(G)}(v) + d^-_{D(G)}(v)$, and it is denoted by $d_G(v)$.

Let k be a positive integer, and φ a function from the edge set of the directed graph $D(G)$ into the set $\{0, 1, \dots, k-1\}$. For $S \subseteq V(G)$ let $\delta\varphi(S) = \sum_{e \in D^+(S)} \varphi(e) - \sum_{e \in D^-(S)} \varphi(e)$. The function φ is a k-flow on G if $\delta\varphi(S) = 0$ for every $S \subseteq V(G)$. The *support* of φ is the set $\{e \in E(G) : \varphi(e) \neq 0\}$, and it is denoted by *supp*(φ). A k-flow φ is a nowhere-zero k-flow if *supp*$(\varphi) = E(G)$.

We will use balanced valuations of graphs, which were introduced by Bondy [1] and Jaeger [2]. A *balanced valuation* of a graph G is a function f from the vertex set $V(G)$ into the real numbers, such that $|\sum_{v \in X} f(v)| \leq |\partial_G(X)|$ for all $X \subseteq V(G)$. Jaeger proved the following fundamental theorem.

Theorem 2.1 ([2]) *Let G be a graph with orientation D and $k \geq 3$. Then G has a nowhere-zero k-flow if and only if $f(v) = \frac{k}{k-2}(2d^+_{D(G)}(v) - d_G(v))$, for all $v \in V(G)$, is a balanced valuation of G.*

In particular, Theorem 2.1 says that a cubic graph G has a nowhere-zero 5-flow if and only if there is a balanced valuation of G with values in $\{\pm\frac{5}{3}\}$.

Let G be a bridgeless cubic graph, and $\mathcal{F}_2$ be a 2-factor of G with odd circuits $C_1, \dots, C_{2t}$, and even circuits $C_{2t+1}, \dots, C_{2t+l}$ ($t \geq 0$, $l \geq 0$), and let $\mathcal{F}_1$ be the complementary 1-factor.

A *canonical* 4-edge-coloring, denoted by c, of G with respect to $\mathcal{F}_2$ colors the edges of $\mathcal{F}_1$ with color 1, the edges of the even circuits of $\mathcal{F}_2$ with 2 and 3, alternately, and the edges of the odd circuits of $\mathcal{F}_2$ with colors 2 and 3 alternately, but one edge which is colored 0. Then, there are precisely $2t$ vertices $z_1, \dots, z_{2t}$ where color 2 is missing (that is, no edge which is incident to z_i has color 2).

The subgraph which is induced by the edges of colors 1 and 2 is union of even circuits and t paths P_i of odd length and with $z_1, \dots, z_{2t}$ as ends. Without loss of generality we can assume that P_i has ends z_{2i-1} and z_{2i}, for $i \in \{1, \dots, t\}$.

Let M_G be the graph obtained from G by adding two edges f_i and f'_i between z_{2i-1} and z_{2i} for $i \in \{1, \dots, t\}$. Extend the previous edge-coloring to a proper edge-coloring of M_G by coloring f'_i with color 2 and f_i with color 4. Let $C'_1, \dots, C'_s$ be the cycles of the 2-factor of M_G induced by the edges of colors 1 and 2 ($s \geq t$). In particular, C'_i is the even circuit obtained by adding the edge f'_i to the path P_i, for $i \in \{1, \dots, t\}$. Finally, for $i \in \{1, \dots, t\}$ let C''_i be the 2-circuit induced by the edges f_i and f'_i. We construct a nowhere-zero 4-flow on M_G as follows:

- for $i \in \{1, \dots, 2t+l\}$ let (D_i, φ_i) be a nowhere-zero flow on the directed circuit C_i with $\varphi_i(e) = 2$ for all $e \in E(C_i)$;

- for $i \in \{1 \dots, s\}$ let (D'_i, φ'_i) be a nowhere-zero flow on the directed circuit C'_i with $\varphi'_i(e) = 1$ for all $e \in E(C'_i)$;

- for $i \in \{1, \dots, t\}$ let (D''_i, φ''_i) be a nowhere-zero flow on the directed circuit C''_i (choose D''_i such that f'_i receives the same direction as in D'_i) with $\varphi''_i(e) = 1$ for all $e \in \{f_i, f'_i\}$.

Then,

$$(D, \varphi) = \sum_{i=1}^{2t+l} (D_i, \varphi_i) + \sum_{i=1}^{s} (D'_i, \varphi'_i) + \sum_{i=1}^{t} (D''_i, \varphi''_i)$$

is the desired nowhere-zero 4-flow on M_G.

By Theorem 2.1, $w(v) = 2(2d^+_{D(M_G)}(v) - d_{M_G}(v))$ is a balanced valuation of M_G. It holds that $|2d^+_{D(M_G)}(v) - d_{M_G}(v)| = 1$, and hence, $w(v) \in \{\pm 2\}$ for all vertices v. The vertices of M_G, and therefore, of G as well, are partitioned into two classes $A = \{v | w(v) = -2\}$ and $B = \{v | w(v) = 2\}$. We call the elements of A (B) the white (black) vertices of G, respectively.

Definition 2.2 *Let G be a bridgeless cubic graph and $\mathcal{F}_2$ a 2-factor of G. A partition of $V(G)$ into two classes A and B constructed as above with a canonical 4-edge-coloring c, the 4-flow (D, φ) on M_G and the induced balanced valuation w of M_G is called a* ***flow partition*** *of G w.r.t. $\mathcal{F}_2$. The partition is denoted by $P_G(A, B)(= P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w))$.*

Lemma 2.3 *Let G be a bridgeless cubic graph and $P_G(A, B)$ be a flow partition of $V(G)$ which is induced by a canonical nowhere-zero 4-flow with respect to a canonical edge-coloring c. Let x, y be the two vertices of an edge e. If $e \in c^{-1}(1) \cup c^{-1}(2)$, then x and y belong to different classes, i.e. $x \in A$ if and only if $y \in B$.*

From a flow partition $P_G(A, B)(= P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w))$ we easily obtain a flow partition $P_G(A', B')(= P_G(A', B', \mathcal{F}_2, c, (D', \varphi'), w'))$ such that the colors on the vertices of P_i are switched. Let (D', φ') be the nowhere-zero 4-flow on M_G obtained by using the same 2-factor $\mathcal{F}_2$, the same 4-edge-coloring c of G and the same orientations for all circuits, but for one $i \in \{i, \dots, t\}$ use opposite orientation of C_i' and C_i'' with respect to the one selected in (D, φ).

Lemma 2.4 *Let G be a bridgeless cubic graph and $P_G(A, B)$ be the flow partition which is induced by the nowhere-zero 4-flow (D, φ). If $P_G(A', B')$ is the flow partition induced by the nowhere-zero 4-flow (D', φ'), then $A \setminus V(P_i) = A' \setminus V(P_i)$, $B \setminus V(P_i) = B' \setminus V(P_i)$, $A \cap V(P_i) = B' \cap V(P_i)$ and $B \cap V(P_i) = A' \cap V(P_i)$.*

3 Proof of Theorem 1.2

Suppose to the contrary that the statement is not true. Then there is a cyclically 6-edge-connected cubic graph G with oddness 4, which has no nowhere-zero 5-flow. Let $\mathcal{F}_2$ be a 2-factor of G with precisely four odd circuits $C_1, \dots, C_4$. Let c be a canonical 4-edge coloring of G and z_1, z_2, z_3, z_4 be the four vertices where color 2 is missing. Let $Z = \{z_1, z_2, z_3, z_4\}$. Note, that in any flow partition which depends on $\mathcal{F}_2$ and c, the vertices z_1 and z_2 (and z_3 and z_4 as well) belong to different color classes. By Lemma 2.4 there are flow partitions $P_G(A, B)$ and $P_G(A', B')$ of G such that $\{z_1, z_3\} \subseteq A$, and $\{z_1, z_4\} \subseteq A'$. Hence, $\{z_2, z_4\} \subseteq B$ and $\{z_2, z_3\} \subseteq B'$.

Let w be the function with $w(v) = -\frac{5}{3}$ if $v \in A$ and $w(v) = \frac{5}{3}$ if $v \in B$, and w' be a function with $w'(v) = -\frac{5}{3}$ if $v \in A'$ and $w'(v) = \frac{5}{3}$ if $v \in B'$.

We will prove that w or w' is a balanced valuation of G, and therefore, G has a nowhere-zero 5-flow by Theorem 2.1. Hence, there is no counterexample and Theorem 1.2 is proved.

3.1 Z-separating edge-cuts

Since G has no nowhere-zero 5-flow, w and w' are not balanced valuations of G. Then there are $S \subseteq V(G)$, $S' \subseteq V(G)$ with $|\sum_{v\in S} w(v)| > |\partial_G(S)|$, and $|\sum_{v\in S'} w'(v)| > |\partial_G(S')|$.

We will prove some properties of the edge-cuts $\partial_G(S)$ and $\partial_G(S')$. We deduce the results for S only. The results for S' follow analogously. If $S = V(G)$, then $|\sum_{v\in S} w(v)| = 0 = |\partial_G(S)|$. Therefore, S, S' are a proper subset of V. If $|S| = 1$, then $|\sum_{v\in S} w(v)| = \frac{5}{3} \leq 3 = |\partial_G(S)|$. Since G is cyclically 6-edge-connected, it has no non-trivial 3-edge-cut and no 2-edge-cut. Hence, we assume that $|\partial_G(S)| \geq 4$ in the following.

Let k (k') be the absolute value of the difference between the number of black and white vertices in S (S'). Hence, $\frac{5}{3}k > |\partial_G(S)|$, and $\frac{5}{3}k' > |\partial_G(S')|$.

For $i \in \{0, 1, 2, 3\}$, let $c_i = |\partial_G(S) \cap c^{-1}(i)|$ and $c'_i = |\partial_G(S') \cap c^{-1}(i)|$.

Claim 3.1 *$|\partial_G(S)| \equiv k \pmod 2$, $|\partial_G(S')| \equiv k' \pmod 2$*

Proof. If k is even, then $|S \cap A|$ and $|S \cap B|$ have the same parity, and if k is odd, then they have different parities. Since S is the disjoint union of $S \cap A$ and $S \cap B$ it follows that k and $|S|$ have the same parity. Since G is cubic it follows that $|\partial_G(S)| \equiv k \pmod 2$. □

Let q_A (q_B) be the number of white (black) vertices of S where color 2 is missing. Let $q = |q_A - q_B|$. Since Z has two black and two white vertices, it follows that $q \leq 2$.

Claim 3.2 *$|S \cap Z| = 2 = q$, and $|S' \cap Z| = 2 = q'$.*

Proof. Since $c^{-1}(1)$ is a 1-factor of G, Lemma 2.3 implies that $k \leq c_1$. Hence,

$$c_1 > \frac{3}{5}|\partial_G(S)|. \quad (1)$$

Furthermore, Lemma 2.3 implies that $k \leq c_2 + q$. Hence,

$$c_2 + q > \frac{3}{5}|\partial_G(S)|. \quad (2)$$

Suppose to the contrary, that $|S \cap Z| \neq 2$. Thus, $q \leq 1$ and $c_2 + 1 \geq k$. Hence, $|\partial_G(S)| \geq c_1 + c_2 \geq 2k - 1$. The relation $\frac{5}{3}k > |\partial_G(S)| \geq 2k - 1$ gives $k < 3$ and then $|\partial_G(S)| < 5$. If $|\partial_G(S)| = 4$, then $k \leq 2$, and it follows that $\frac{5}{3}k \leq |\partial_G(S)|$, a contradiction. Thus, $|S \cap Z| = 2$, and therefore, $q \in \{0, 2\}$. If $q = 0$, then $|\partial_G(S)| \geq c_1 + c_2 \geq 2k$, a contradiction. Hence, $q = 2$. □

Claim 3.3 *$|\partial_G(S)| = 6$, $c_1 = 4$ and $c_2 = 2$, and $|\partial_G(S')| = 6$, $c'_1 = 4$ and $c'_2 = 2$.*

Proof. If $|\partial_G(S)| = 4$, then $|\partial_G(S)| < \frac{5}{3}k$ implies $k \geq 3$. Hence, recalling that $c_1 \geq k$ and $c_2 + q \geq k$, we have $c_1 = 3$ and $c_2 = 1$. The edge of $\partial_G(S) \cap c^{-1}(2)$ is contained in a circuit of $\mathcal{F}_2$ whose edges are not in $c^{-1}(1)$. Hence, $2 \geq c_1 \geq k$, a contradiction. If $|\partial_G(S)| = 5$, then $c_1 + c_2 \leq 5$ but (1) and (2) give $c_1 \geq 4$ and $c_2 \geq 2$, respectively, a contradiction.

Now suppose to the contrary that $|\partial_G(S)| > 6$. Since $c_1 > \frac{3}{5}|\partial_G(S)|$, $c_2 > \frac{3}{5}|\partial_G(S)| - 2$, and $c_1 + c_2 \leq |\partial_G(S)|$, it follows that $|\partial_G(S)| > \frac{6}{5}|\partial_G(S)| - 2$. Therefore, $|\partial_G(S)| < 10$. If $|\partial_G(S)| = 7$, then $c_1 \geq 5$ and $c_2 \geq 3$, a contradiction. If $|\partial_G(S)| = 8$, then $c_1 = 5$ and $c_2 = 3$, a contradiction to Claim 3.1 since $c_1 \equiv k \pmod 2$. If $|\partial_G(S)| = 9$, then $c_1 \geq 6$ and $c_2 \geq 4$, a contradiction. Hence, $|\partial_G(S)| = 6$ and $c_1 \geq 4$ and $c_2 \geq 2$. That leaves the unique possibility $c_1 = 4$ and $c_2 = 2$. □

Claim 3.4 *$G[S]$ and $G[S']$ are connected.*

Proof. If $G[S]$ is not connected, then there exists a partition of S in two subsets S_1 and S_2 such that there is no edge connecting S_1 and S_2. It follows that $\partial_G(S) = \partial_G(S_1) \cup \partial_G(S_2)$. Since G does not have a 2-edge-cut or a non-trivial 3-edge-cut, it follows that $|\partial_G(S_1)| = |\partial_G(S_2)| = 3$ and $|S_1| = |S_2| = 1$, that is $|S| = 2$. Hence, $\frac{5}{3}k \leq \frac{10}{3} < |\partial_G(S)| = 6$, a contradiction. □

Definition 3.1 *A 6-edge-cut E of G is* ***bad*** *with respect to a flow partition $P_G(A^*, B^*)$ if it satisfies the following two conditions:*

i) $|E \cap c^{-1}(1)| = 4$ and $|E \cap c^{-1}(2)| = 2$,

ii) E partitions the vertices z_1, z_2, z_3 and z_4 into two sets $\{z_{i_1}, z_{i_2}\}$, $\{z_{i_3}, z_{i_4}\}$, which are in different components of $G - E$ and $\{z_{i_1}, z_{i_2}\} \subseteq A^*$ or $\{z_{i_1}, z_{i_2}\} \subseteq B^*$.

Note that $\{z_{i_1}, z_{i_2}\} \subseteq A^*$ if and only if $\{z_{i_3}, z_{i_4}\} \subseteq B^*$. Further, only condition *ii*) depends on the flow partition. Condition i) depends on the canonical 4-edge-coloring of G which is unchanged along the proof. From the previous results we deduce:

Claim 3.5 *$\partial_G(S)$ is bad w.r.t. $P_G(A, B)$ and $\partial_G(S')$ is bad w.r.t. $P_G(A', B')$.*

Bad 6-edges-cuts are the only obstacles in G for having a nowhere-zero 5-flow. In order to deduce the desired contradiction we will show that all 6-edge-cuts are not bad with respect to either $P_G(A, B)$ or $P_G(A', B')$.

Recall that, z_1 and z_3 receive the same color in $P_G(A, B)$, and that z_1 and z_4 receive the same color in $P_G(A', B')$. For $i \in \{2, 3, 4\}$, let $\mathcal{S}_i = \{V : V \subseteq V(G)$ and $\{z_1, z_i\} \subseteq V\}$ and $\mathcal{E}_i = \{E : E \subseteq E(G), V \in \mathcal{S}_i$ and $E = \partial_G(V)\}$ be the corresponding set of edge-cuts. Since z_1 and z_2 have different colors in both $P_G(A, B)$ and $P_G(A', B')$, all edge-cuts in $\mathcal{E}_2$ are not bad with respect to $P_G(A, B)$ and with respect to $P_G(A', B')$.

For $i \in \{3, 4\}$, by Claim 3.5 there is a 6-edge-cut $E_i \in \mathcal{E}_i$ which is bad. By Claim 3.4, $G - E_3$ consists of two components with vertex sets X and Y, i.e. $X \cup Y = V(G)$. Analogously, $G - E_4$ consists of two components with vertex sets X' and Y'. Let $U_1 = X \cap X'$, $U_2 = Y \cap Y'$, $U_3 = X \cap Y'$ and $U_4 = Y \cap X'$. Thus, $z_i \in U_i$ for $i \in \{1, \ldots, 4\}$, see Figure 1.

Claim 3.6 *$|\partial_G(U_i)| \geq 5$. In particular, $|\partial_G(U_i)| = 5$ if and only if $G[U_i]$ is a path with two edges, one of color 0 and one of color 3.*

Proof. If $G[U_i]$ has a circuit, then $|\partial_G(U_i)| \geq 6$ since G is cyclically 6-edge-connected. If this is not the case, then $G[U_i]$ is a forest, say with n vertices. Hence, $|\partial_G(U_i)| \geq n + 2$. Since $\partial_G(U_i) \subseteq E_3 \cup E_4$, it follows that $\partial_G(U_i) \subseteq c^{-1}(1) \cup c^{-1}(2)$. Two edges $z_i x_i$ and $z_i y_i$ which are incident to z_i are colored with color 0 and 3, respectively. Hence, $\{x_i, y_i\} \subseteq U_i$, $n \geq 3$, and $|\partial_G(U_i)| \geq 5$. If $|\partial_G(U_i)| = 5$, then $|U_i| = 3$, and $G[U_i]$ is a path with two edges, one of color 0 and one of color 3. □

Claim 3.7 *$|\partial_G(U_i)| = 5$ for at most two of the four subsets U_i. Furthermore, if there are i, j such that $i \neq j$ and $|\partial_G(U_i)| = |\partial_G(U_j)| = 5$, then $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$.*

Proof. Since E_3 and E_4 are bad, each of them has exactly two edges of color 2 and four edges of color 1. Hence, each of them intersects with at most

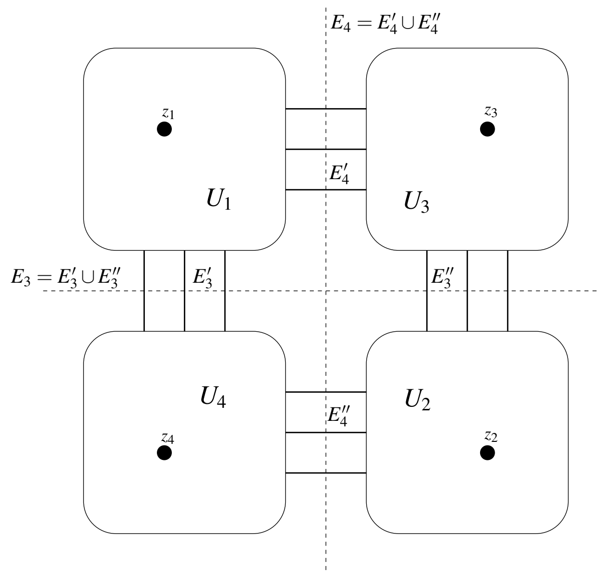

Figure 1: Z-separating 6-edge cuts

one circuit of $\mathcal{F}_2$. For each $i \in \{1, \ldots, 4\}$, $|E(G[U_i]) \cap c^{-1}(0)| = 1$, and hence, there are j_1, j_2 such that $j_1 \neq j_2$ and U_{j_1}, U_{j_2} contain an odd circuit of $\mathcal{F}_2$. Since G is cyclically 6-edge-connected it follows that $|\partial_G(U_{j_1})| \geq 6$ and $|\partial_G(U_{j_2})| \geq 6$.

Let $i, j \in \{1, \ldots, 4\}$ such that $i \neq j$ and $|\partial_G(U_i)| = |\partial_G(U_j)| = 5$. For symmetry, it suffices to prove that $\{i, j\} \neq \{1, 3\}$. Suppose to the contrary that $\{i, j\} = \{1, 3\}$. By Claim 3.6, $G[U_1]$ and $G[U_3]$ are paths of length two with edges colored 0 and 3. Further, $\partial_G(U_1)$ consists of three edges of color 1 and two edges of color 2, which belong to the odd circuit C_1 of $\mathcal{F}_2$. Analogously, the two edges of color 2 of $\partial_G(U_3)$ belong to the odd circuit C_3 of $\mathcal{F}_2$. Hence, both pairs of edges of color 2 in $\partial_G(U_1)$ and $\partial_G(U_3)$ belong to E_3 and they are distinct, a contradiction since E_3 has only two edges of color 2. $\square$

For $i \neq j$ let $\partial_G(U_i, U_j)$ be the set of edges with one vertex in U_i and the other one in U_j.

Claim 3.8 *The following relations hold:*

- $|\partial_G(U_i, U_j)| = 0$, *for* $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$.
- $|\partial_G(U_i, U_j)| = 3$, *for* $\{i, j\} \in \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$.

Proof. Recall that $|E_3| = |E_4| = 6$. Hence, $|E_3 \cup E_4| \leq 12$. Due to Claim 3.7, we can assume that $|\partial_G(U_1)| \geq 5$, $|\partial_G(U_2)| \geq 5$, $|\partial_G(U_3)| \geq 6$ and $|\partial_G(U_4)| \geq 6$. By adding up, we obtain $\sum_{i=1}^{4} |\partial_G(U_i)| \geq 22$, where each edge of E_3 and E_4 is counted exactly twice. Hence, $|E_3 \cup E_4| \geq 11$. If $|E_3 \cup E_4| = 11$, then exactly one edge, say e, belongs to $E_3 \cap E_4$. If $e \in \partial_G(U_1, U_2)$, then $\partial_G(U_3)$ and $\partial_G(U_4)$ are distinct sets of cardinality at least 6. Hence, $|E_3 \cup E_4| > 12$, a contradiction. If $e \in \partial_G(U_3, U_4)$, then $\partial_G(U_1, U_4)$ or $\partial_G(U_2, U_3)$ has cardinality at most 2, say, without loss of generality, $\partial_G(U_1, U_4)$. For the same reason, $\partial_G(U_1, U_3)$ or $\partial_G(U_2, U_4)$ has cardinality at most 2. If $|\partial_G(U_1, U_3)| \leq 2$, then $|\partial_G(U_1)| \leq 4$, and if $|\partial_G(U_2, U_4)| \leq 2$, then $|\partial_G(U_4)| \leq 5$, a contradiction (in both cases). Hence, $|E_3 \cup E_4| = 12$, and therefore, $|\partial_G(U_i, U_j)| = 0$ for $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$.

Now, $|\partial_G(U_i, U_j)| = 3$, for $\{i, j\} \in \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ can be deduced easily. □

Let $E_3' = E_3 \cap \partial(U_1)$, $E_3'' = E_3 \cap \partial(U_2)$, and $E_4' = E_4 \cap \partial(U_1)$, $E_4'' = E_4 \cap \partial(U_2)$, see Figure 1.

Let $H = G[c^{-1}(1) \cup c^{-1}(2)]$. The components of H are even circuits and the two paths P_1 and P_2, where P_1 has the end vertices z_1, z_2, and P_2 has the end vertices z_3, z_4. The paths P_1 and P_2 intersect both $E_3 = E_3' \cup E_3''$ and $E_4 = E_4' \cup E_4''$ an odd number of times, since both, E_3 and E_4, separate their ends. For symmetry, we can assume that $P_1 \cap E_3'$ and $P_1 \cap E_4''$ are even, and hence, $P_1 \cap E_3''$ and $P_1 \cap E_4'$ are odd. Furthermore, we assume that $P_2 \cap E_3''$ and $P_2 \cap E_4''$ are even, and hence, $P_2 \cap E_3'$ and $P_2 \cap E_4'$ are odd. Note, that every other possible choice produces an analogous configuration. The 6-edge-cut $E_3' \cup E_4'$ contains an odd number of edges of $E(P_1) \cup E(P_2)$. Since $E_3' \cup E_4' \subseteq E(H)$, it follows that an odd number of edges of $E_3' \cup E_4'$ are not in $E(P_1) \cup E(P_2)$, a contradiction, since all other components of H are circuits, and they intersect every edge-cut an even number of times.

Hence, at least one of E_3 and E_4 is not bad, contradicting our assumption that both of them are bad.

References

[1] J.A. Bondy, Balanced Colourings and the four colour conjecture, Proc. Amer. Math. Soc. **33** (1972) 241-244

[2] F. Jaeger, Balanced valuations and flows in multigraphs, Proc. Amer. Math. Soc. **55** (1975) 237-242

[3] F. Jaeger, Nowhere-zero flow problems, in: L. W. Beineke, R. J. Wilson eds., Topics in Graph Theory 3, Academic Press, London (1988) 70-95

[4] F. Jaeger, T. Swart, Conjecture 2, in: M. Deza, I.G. Rosenberg (Eds.), Combinatorics 79, Annals of Discrete Mathematics, Vol. 9, North-Holland, Amsterdam, 1980, p. 305.

[5] R. Haggkvist, S. McGuinness, Double covers of cubic graphs with oddness 4, J. Combin. Theory, Ser. B **93** (2005) 251-277

[6] M. Kochol, Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks, J. Combin. Theory, Ser. B **90** (2004) 139-145

[7] M. Kochol Smallest counterexample to the 5-flow conjecture has girth at least eleven, J. Combin. Theory Ser. B **100** (2010) 381-389

[8] E. Macajova, M. Skoviera Sparsely intersecting perfect matchings in cubic graphs, Combinatorica **34** (2014) 61-94

[9] J. Petersen, Die Theorie der regulären graphs, Acta Mathematica **15** (1891) 193 - 220

[10] P. D. Seymour, Nowhere-zero 6-flows, J. Combin. Theory, Ser. B **30** (1981) 130-135

[11] P. D. Seymour, Nowhere-zero flows, in Handbook of Combinatorics (Vol. 1), R. L. Graham, M. Grötschel, L. Lovász (eds.) Elsevier Science B.V. Amsterdam (1995) 289-299

[12] E. Steffen, Tutte's 5-flow conjecture for highly cyclically connected cubic graphs, Discrete Math. **310** (2010) 385-389

[13] W. T. Tutte, A contribution to the theory of chromatic polynomials, Canadian J. Math. **6** (1954) 80-91