Methodol Comput Appl Probab (2019) 21:331–366
https://doi.org/10.1007/s11009-018-9650-3

Option Pricing with Fractional Stochastic Volatility and Discontinuous Payoff Function of Polynomial Growth

Viktor Bezborodov[1] · **Luca Di Persio**[1] · **Yuliya Mishura**[2]

Received: 14 August 2016 / Revised: 30 May 2018 /
Accepted: 29 June 2018 / Published online: 6 August 2018

Abstract We consider the pricing problem related to payoffs of polynomial growth that can have discontinuities of the 1st kind. The asset price dynamic is modeled within the Black-Scholes framework characterized by a stochastic volatility term driven by a fractional Ornstein-Uhlenbeck process. In order to solve the aforementioned problem, we consider three approaches. The first one consists in a suitable transformation of the initial value of the asset price, in order to eliminate possible discontinuities. Then we discretize both the Wiener process and the fractional Brownian motion and estimate the rate of convergence of the related discretized price to its real value whose closed-form analytical expression is usually difficult to obtain. The second approach consists in considering the conditional expectation with respect to the entire trajectory of the fractional Brownian motion (fBm). Here we derive a presentation for the option price which involves only an integral functional depending on the fBm trajectory, and then discretize the fBm and estimate the rate of convergence of the associated numerical scheme. In both cases the rate of convergence is the same and equals

The research of Viktor Bezborodov was supported by the Department of Computer Science of the University of Verona, Luca Di Persio was supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) within the project "Stochastic Partial Differential Equations and Stochastic Optimal Transport with Applications to Mathematical Finance", and Yuliya Mishura is thankful to the University of Verona for the hospitality during her visit.

✉ Yuliya Mishura
myus@univ.kiev.ua

Viktor Bezborodov
viktor.bezborodov@univr.it

Luca Di Persio
luca.dipersio@univr.it

[1] The Department of Computer Science Universita degli Studi di Verona - Cà Vignal 2, The University of Verona, Verona, Italy

[2] The Faculty of Mechanics and Mathematics Volodymyrska 64, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

n^{-rH}, where n is the partition size, H is the Hurst index of the fBm, and r is the Hölder exponent of the volatility function. The third method consists in calculating the density of the integral functional depending on the trajectory of the fBm via Malliavin calculus and providing the option price in terms of the associated probability density.

Keywords Option pricing · Stochastic volatility · Black–Scholes model · Wiener process · Discontinuous payoff function · Polynomial growth · Rate of convergence · Discretization · Conditioning · Malliavin calculus · Stochastic derivative · Skorokhod integral

Mathematics Subject Classification (2010) 91G20 · 91B24 · 91B25 · 91G60 · 60G22 · 60H07

1 Introduction

Starting with the pioneering works by Hull and White (1987) and Heston (1993), stochastic volatility models for assets prices have been a subject of intensive research activity, which is still vibrant from analytical, computational and statistical points of view. Of course, option pricing is one of most relevant problems. In the latter context, stochastic volatility models are widely used because of their flexibility. Concerning the stochastic volatility modeling, note that there are approaches involving Gaussian (Nicolato and Venardos 2003; Schobel and Zhu 1999), non-Gaussian (Barndorff-Nielsen and Shephard 2001, 2002), jump-diffusion and Lèvy processes (Cont and Tankov 2004; Kyprianou and Schoutens 2005), as well as time series (Carrasco and Chen 2002; Palm 1996; Shephard 1996). Our references are not in any way intended to be exhaustive or complete, we only illustrate the availability of various approaches. We would also like to mention the books (Fouque et al. 2000; Kahl 2008; Knight and Satchell 2011) and references therein, as well as the paper (Altmayer and Neuenkirch 2015) which in some sense was a starting point for our considerations. A useful decomposition formula for option prices that is valid even when the Malliavin regularity conditions are not satisfied, was obtained in Alòs (2012). Furthermore, the models of financial market where the asset price includes stochastic volatility with long memory in the volatility process is a subject of extensive research activity, see, e.g., Bollerslev and Mikkelsen (1996), where a wide class of fractionally integrated GARCH and EGARCH models for characterizing financial market volatility was studied, Comte et al. (2012) for affine fractional stochastic volatility models, and Chronopoulou and Viens (2012), where the Heston model with fractional Ornstein–Uhlenbeck stochastic volatility was studied. As was mentioned in Comte et al. (2012), long memory included into the volatility model allows to explain some option pricing puzzles such as steep volatility smiles in long term options and co-movements between implied and realized volatility. Although the long memory effect corresponds to the case $H > 1/2$, empirical evidence suggests values $H < 1/2$, ("rough volatility"), see for example (Bayer et al. 2016) or discussion in Funahashi and Kijima (2017), where an approximation formula is proposed. When dealing with "rough volatility", one may find useful the decomposition approach developed in Bergomi and Guyon (2012). Concerning the approach that permits to get the semi-closed form solution for the call option price in Heston model with jumps in the asset price and semimartingale stochastic volatility with long memory, see Pospíšil and Sobotka (2016). Note again that as a rule, the option pricing in stochastic volatility models needs some approximation procedures including Monte–Carlo methods.

The present paper contains a comprehensive and diverse approach to the exact and approximate option pricing of the asset price model that is described by the linear model with stochastic volatility driven by the fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0, 1)$. We assume that the Wiener processes, one of which is driving the asset price and another one is the underlying Wiener process for the fractional Brownian motion driving stochastic volatility, are correlated with a constant correlation coefficient. The significant novelty of our approach is that we consider three possible levels of the representation and approximation of the option price, with the corresponding rate of convergence of discretized option price to the original one. Another novelty is that we can rigorously treat the class of discontinuous payoff functions of polynomial growth. As an example, our model allows to analyze linear combinations of a digital and call option. Moreover, we provide, for the first time in literature to the best of our knowledge, rigorous estimates for the rates of convergence of option prices for polynomial discontinuous payoffs f and Hölder volatility coefficients, a crucial feature considering settings for which exact pricing is not possible.

The first level of approximation corresponds to the case when the price is presented as the functional of both driving stochastic processes, the Wiener process and the fBm. We dicretize and simulate both the trajectories of the Wiener process and of the fBm (the double discretization) and estimate the rate of convergence for the discretized model. In these settings we apply the Malliavin calculus technique, following Altmayer and Neuenkirch (2015), to transform the option price to the form that does not contain discontinuous functions. The second level corresponds to the case when we discretize and simulate only the trajectories of the fBm involved in Ornstein–Uhlenbeck stochastic volatility process (the single discretization), basically conditioning on the stochastic volatility process, then calculating the corresponding option price as a functional of the fBm trajectory, and finally estimating the rate of convergence of the discretized price. This approach allows to simulate only the trajectories of the fBm. Corresponding simulations are presented and compared to those obtained by the first level. We conjecture that the single discretization gives better simulation results. In general conditioning is widely used in option pricing in various situations, see, e.g., Bertholon et al. (2007) and references therein, so, it is not surprising that it helps here as well. The third level potentially permits to avoid simulations, because it is possible to provide an analytical expression for the option price, as an integral including the density of the functional which depends on stochastic volatility. However, the density we obtain within the Malliavin calculus framework is rather complicated from the computational point of view.

Taking into account previously mentioned approaches and techniques, the subject of the present paper is a financial market characterized by a finite maturity time T and composed by a risk free bond, or bank account, $\beta = \{\beta_t, t \in [0, T]\}$, whose dynamic reads as $\beta_t = e^{\beta t}$, where $\beta \in \mathbb{R}^+$ represents the risk free interest rate, and a risky asset $S = \{S_t, t \in [0, T]\}$ whose stochastic price dynamic is defined over the probability space $\{\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P}\}$ by the following system of stochastic differential equations

$$dS_t = bS_t dt + \sigma(Y_t) S_t dW_t, \tag{1}$$

$$dY_t = -\alpha Y_t dt + dB_t^H, \ t \in [0, T]. \tag{2}$$

Here $W = \{W_t, t \in [0, T]\}$ is a standard Wiener process, $b \in \mathbb{R}$, $\alpha > 0$ are constants, and $Y = \{Y_t, t \in [0, T]\}$ characterizes the stochastic volatility term of our model, being the argument of the function σ. The process Y is Ornstein-Uhlenbeck, driven by a fractional Brownian motion $B^H = \{B_t^H, t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$, and is assumed to be correlated with W. Recall that an fBm is a centered Gaussian process with covariance function $\mathrm{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. Moreover, due to the Kolmogorov theorem, any fBm has a modification such that almost surely its trajectory is a Hölder function up to order H. In what follows we shall consider such a modification. Recall that for $H > \frac{1}{2}$, fBm has a long memory. This is suitable for stochastic volatility which represents the memory of the model. On the other hand, the case $H < \frac{1}{2}$ corresponds to rough volatility. We would also like to emphasize that a market model described by the system of Eqs. 1 and 2 is incomplete because of two sources of uncertainty, whether or not it is arbitrage-free. Therefore, in what follows we focus our attention on the so called physical, or *real world*, measure, instead of using an equivalent martingale one. Note, however, that in the case when the market is indeed arbitrage-free and there exists a minimal martingale measure, the stock prices evaluated w.r.t. the minimal martingale measure, resp. w.r.t. the objective measure, differ only by the non-random coefficient $e^{(b-\beta)t}$, as it happens in the standard Black–Scholes framework. A discussion of arbitrage–free property of the market under consideration (with additional restrictions of the coefficients), a presentation of the class of martingale measures, and a formula for the minimal martingale measure is contained in Section 2. For more details on the arbitrage-free property of the markets with stochastic volatility see, e.g., Kuchuk-Iatsenko and Mishura (2015). Concerning the payoff function, we consider a measurable one defined by $f : \mathbb{R}^+ \to \mathbb{R}^+$, and depending on the value S_T of the stock at maturity time T. Our main goal is to calculate and approximate $\mathrm{E}f(S_T)$ using the aforementioned levels, also providing rigorous estimates for the corresponding rate of convergence for the first and second levels.

The paper is organized as follows: in Section 2 we give additional assumptions on the components of the model and formulate auxiliary results; Section 3 contains the necessary elements of the Malliavin calculus that will be used later; Section 4 contains the main results on the rate of convergence of the discretized option pricing approach; Section 5 contains the main results concerning the rate of convergence of the discretized option pricing problem when conditioning on the trajectory of the fBm; Section 6 is devoted to the analytical derivation of the option price in terms of the density of the volatility functional; the proofs are collected in Section 7; finally, Section 8 provides the numerical simulations associated to the approaches described in Sections 4 and 5.

2 Payoff Function: Additional Assumptions, Auxiliary Properties. Discussion of Asset Price Model, Absence of Arbitrage, Incompleteness

2.1 Assumptions on the Payoff Function and Volatility Coefficient

Throughout the paper we assume that the payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(A)

(i) f is a measurable function of polynomial growth,

$$f(x) \le C_f(1 + x^p), \quad x \ge 0,$$

for some constants $C_f > 0$ and $p > 0$.

(ii) The Function f is locally Riemann integrable, possibly, having discontinuities of the first kind.

Moreover we assume that the function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(**B**) there exists $C_\sigma > 0$ such that

(i) σ is bounded away from 0, $\sigma(x) \geq \sigma_{\min} > 0$;

(ii) σ has moderate polynomial growth, i.e., there exists $q \in (0, 1)$ such that

$$\sigma(x) \leq C_\sigma(1 + |x|^q),\ \ x \in \mathbb{R};$$

(iii) σ is uniformly Hölder continuous, so that there exists $r \in (0, 1]$ such that

$$|\sigma(x) - \sigma(y)| \leq C_\sigma |x - y|^r,\ \ x, y \in \mathbb{R};$$

(iv) $\sigma \in C(\mathbb{R})$ is differentiable a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$, and its derivative is of polynomial growth: there exists $q' > 0$ such that

$$|\sigma'(x)| \leq C_\sigma(1 + |x|^{q'}),$$

a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$.

Remark 1 1) Concerning the relations between properties (ii) and (iii), note that we allow $r = 1$ in (iii) whereas (ii) follows from (iii) only in the case $r < 1$.

2) Concerning the relations between properties (iii) and (iv), neither of these properties implies the other one unless $r = 1$. Indeed, on the one hand, a typical trajectory of a Wiener process is Hölder up to order $\frac{1}{2}$ but nowhere differentiable. On the other hand, even continuous differentiability does not imply the uniform Hölder property.

3) Concerning the assumption (i), we need it for theoretical calculations because in the process of smoothing the payoff function we divide on σ.

2.2 Properties of Fractional Ornstein–Uhlenbeck Process

According to Norros et al. (1999), fBm admits a compact interval representation via some Wiener process B, specifically,

$$B_t^H = \int_0^t k(t,s)dB_s,$$

$$k(t,s) = c_H\left(\left(\frac{t}{s}\right)^{H-1/2}(t-s)^{H-1/2} - (H-1/2)s^{-H+1/2}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right)\mathbb{1}_{s<t},$$

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{1/2} \tag{3}$$

For $H > 1/2$ the kernel $k(t,s)$ is simplified to

$$k(t,s) = \left(H - \frac{1}{2}\right)c_H s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du\mathbb{1}_{s<t}. \tag{4}$$

The processes B, B^H and Y generate the same flow of σ-fields. Denote this flow as

$$\mathbb{F}^H = \{\mathcal{F}_t^H = \sigma\{B_s^H, 0 \leq s \leq t\}, t \geq 0\}.$$

Our next assumption is that the processes B and W are correlated as follows
(**C**) $\mathrm{E}(B_t W_t) = \rho t$, with some constant $\rho \in [-1, 1]$.

It means that $W_t = \rho B_t + \sqrt{1-\rho^2} V_t$, $t \geq 0$, where V is a Wiener process, independent of B. For technical reasons we introduce the notation $\mu = \sqrt{1-\rho^2}$.

The next result is almost evident, however, we formulate it and give a short proof for the reader's convenience.

Lemma 2 (i) Eq. 2 *has a unique solution of the form*

$$Y_t = Y_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} dB_s^H := Y_0 e^{-\alpha t} + B_t^H - \alpha \int_0^t e^{-\alpha(t-s)} B_s^H ds. \tag{5}$$

Moreover, for any $a > 0$ *and any* $\varrho < 2$

$$\mathrm{E} \exp\{a \sup_{t \in [0,T]} |Y_t|^{\varrho}\} < \infty. \tag{6}$$

(ii) Eq. 1 *has a unique solution of the form*

$$S_t = S_0 \exp\left\{bt + \int_0^t \sigma(Y_s) dW_s - \frac{1}{2} \int_0^t \sigma^2(Y_s) ds\right\}. \tag{7}$$

(iii) *For any* $m \in \mathbb{Z}$ *we have* $\mathrm{E}(S_T)^m < \infty$*, and consequently for any* $m > 0$ *the following relation holds:* $\mathrm{E}(f(S_T))^m < \infty$*. Moreover, for any* $m \in \mathbb{Z}$ *we have* $\sup_{t \in [0,T]} \mathrm{E} \exp\left\{m \int_0^t \sigma(Y_s) dB_s\right\} < \infty$*, and* $\mathrm{E} \exp\left\{m \int_0^T \sigma^2(Y_s) ds\right\} < \infty$*.*

Remark 3 We can generalize assertion (ii) of Lemma 2 to the following one: for any function $\psi = \psi(x) : \mathbb{R} \to \mathbb{R}$ of polynomial growth $\sup_{t \in [0,T]} \mathrm{E}(|\psi(S_t)|) < \infty$. Also, it follows from (i) that for any function $\psi = \psi(x) : \mathbb{R} \to \mathbb{R}$ of polynomial growth $\sup_{t \in [0,T]} \mathrm{E}(|\psi(Y_t)|) < \infty$, and in particular, $\sup_{t \in [0,T]} \mathrm{E}(|\sigma(Y_t)|) < \infty$.

2.3 Arbitrage-free Property, Class of Martingale Measures, Incompleteness, Minimal Martingale Measure

We formulate the arbitrage properties of the market (1)–(2) in the following statement.

Theorem 4 *Let the volatility coefficient* σ *satisfy assumption* (**B**). *Then the market described by* (1)–(2) *has the following properties:*

(i) *It is arbitrage-free and incomplete.*

(ii) *Any probability measure* Q *with Radon-Nikodym derivative*

$$\frac{d\mathrm{Q}}{d\mathrm{P}} = \exp\left\{\int_0^T \nu_1(s) dB_s + \int_0^T \nu_2(s) dV_s - 1/2 \Sigma_{i=1}^2 \int_0^T \nu_i^2(s) ds\right\} \tag{8}$$

with non-anticipative bounded coefficients ν_i *satisfying the equation*

$$\rho \nu_1(s) + \mu \nu_2(s) = \frac{\beta - b}{\sigma(Y_s)}, \tag{9}$$

is a martingale measure.

(iii) *Taking* $\nu_1(s) = \rho \frac{\beta - b}{\sigma(Y_s)}$ *and* $\nu_2(s) = \mu \frac{\beta - b}{\sigma(Y_s)}$ *in* (8)*, we get the minimal martingale measure.*

3 Elements of Malliavin Calculus and Application to Option Pricing

In what follows, we recall some basic definitions and results about Malliavin calculus. Our main reference here is Nualart (2006). Let $\widetilde{W} = \{\widetilde{W}(t), t \in [0, T]\}$ be a Wiener process on the standard probability space $\{\Omega, \mathcal{F}, \mathrm{F} = \{\mathcal{F}_t^{\widetilde{W}}\}, t \in [0, T], \mathrm{P}\}$, where $\Omega = C([0, T], \mathbb{R})$. Denote by $\widehat{C}^\infty(R)$ the set of all infinitely differentiable functions with the derivatives of polynomial growth at infinity.

Definition 5 Random variables ξ of the form $\xi = h(\widetilde{W}(t_1), \dots, \widetilde{W}(t_n))$,

$$h = h(x^1, \dots, x^n) \in \widehat{C}^\infty(\mathbb{R}^n),\ t_1, \dots t_n \in [0, T],\ n \geq 1$$

are called smooth. Denote by $\mathcal{S}$ the class of smooth random variables.

Definition 6 Let $\xi \in \mathcal{S}$. The stochastic derivative of ξ at t is the random variable

$$D_t\xi = \sum_{i=1}^n \frac{\partial h}{\partial x^i}(\widetilde{W}(t_1), \dots, \widetilde{W}(t_n))\mathbb{1}_{t\in[0,t_i]}, \quad t \in [0, T].$$

Considered as an operator from $L^2(\Omega)$ to $L^2(\Omega; L^2[0, T])$, D is closable. We use the same notation D for its closure. D is known as the Malliavin derivative, or the stochastic derivative. The domain of the operator of the stochastic derivative is a Hilbert space $\mathrm{D}^{1,2}$ of random variables, on which the inner product (which coincides with the operator norm) is given by

$$\langle \xi, \eta\rangle_{1,2} = \mathrm{E}(\xi\eta) + \mathrm{E}(\langle D\xi, D\eta\rangle_H), \quad H = L^2([0, T], \mathbb{R}).$$

Thus, the stochastic derivative operator D is closed, unbounded and defined on a dense subset of the space $L^2(\Omega)$ (see Nualart 2006). The following statement is known as the chain rule.

Proposition 7 *(*Nualart 2006*, Proposition 1.2.3). Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $\xi = (\xi_1, ..., \xi_m)$ is a random vector whose components belong to* $\mathrm{D}^{1,2}$. *Then $\varphi(\xi) \in \mathrm{D}^{1,2}$ and*

$$D\varphi(\xi) = \sum_{i=1}^m \partial_i\varphi(\xi)D\xi_i.$$

Denote by δ the operator adjoint to D and by Dom δ its domain. The operator δ is unbounded in H with values in $L^2(\Omega)$ and such that

(i) Dom δ consists of square-integrable random processes $u \in H$, satisfying

$$\left|\mathrm{E}\,(\langle D\xi, u\rangle_H)\right| \leq C(\mathrm{E}(\xi^2))^{1/2},$$

for any $\xi \in \mathrm{D}^{1,2}$, where C is a constant depending on u;

(ii) If u belongs to Dom δ, then $\delta(u)$ is an element of $L^2(\Omega)$ and

$$\mathrm{E}\,(\xi\delta(u)) = \mathrm{E}\,(\langle D\xi, u\rangle_H)$$

for any $\xi \in \mathrm{D}^{1,2}$.

The operator δ is closed. Consider the space $L^{1,2} = L^2([0,T], \mathrm{D}^{1,2})$ with the norm $||\cdot||_{L^{1,2}}$, where

$$||u||^2_{L^{1,2}} = \mathrm{E}\left(\int_0^T u_t^2 dt + \int_0^T \int_0^T (D_s u_t)^2 \, dt ds\right).$$

If $u \in L^{1,2}$, then the integral $\delta(u)$ is correctly defined, the notation $\delta(u) = \int_0^T u_t dW_t$ is used and

$$\mathrm{E}\left(\int_0^T u_t dW_t\right)^2 \le ||u||^2_{L^{1,2}}$$

(see Nualart 2006). In this case operator $\delta(u)$ is called the Skorokhod integral of the process u and is denoted by

$$\delta(u) = \int_0^T u_t d\widetilde{W}_t.$$

To apply Malliavin calculus to the asset price S, note that we have a two-dimensional case with two independent Wiener processes (V, B). With evident modifications, denote by (D^V, D^B) the stochastic derivative with respect to the two-dimensional Wiener process (V, B). Denote also

$$X(t) = \log S(t) = \log S_0 + bt - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds + \int_0^t \sigma(Y_s)dW_s = \log S_0 + bt - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds + \mu\int_0^t \sigma(Y_s)dV_s + \rho\int_0^t \sigma(Y_s)dB_s.$$

Lemma 8 (i) *The stochastic derivatives of the fBm B^H equal to*

$$D_u^V B_t^H = 0, \quad D_u^B B_t^H = k(t,u)\mathbb{1}_{u<t}.$$

(ii) *The stochastic derivatives of Y equal to*

$$D_u^V Y_t = 0, \quad D_u^B Y_t = \left(k(t,u) - \alpha\int_u^t e^{-\alpha(t-s)}k(s,u)ds\right)\mathbb{1}_{u<t}. \tag{10}$$

In particular, for $H > 1/2$ the stochastic derivative of Y can be simplified to

$$D_u^B Y_t = c_H e^{-\alpha t} u^{1/2-H}\int_u^t e^{\alpha s} s^{H-1/2}(s-u)^{H-3/2} ds \mathbb{1}_{u<t}. \tag{11}$$

(iii) *The stochastic derivatives of X equal to*

$$D_u^V X_t = \mu\sigma(Y_u)\mathbb{1}_{u<t}, \; D_u^B X_t = \left(-\int_u^t \sigma(Y_s)\sigma'(Y_s)D_u^B Y_s ds + \int_u^t \sigma'(Y_s)D_u^B Y_s dW_s + \rho\sigma(Y_u)\right)\mathbb{1}_{u<t}. \tag{12}$$

Lemma 9 *The laws of S_T and X_T are absolutely continuous with respect to the Lebesgue measure.*

From now on, we denote C any constant whose value is not important and can change from line to line and even inside the same line. Throughout the paper, C cannot depend on n, t, s, but can depend on σ, H, T, Y_0, S_0, α, b, p, r, q, q', f and other parameters specified in the problem. In what follows we need the statement contained in the next remark.

Remark 10 The chain rule of stochastic differentiation can be extended to a wider class of functions in the following way. Applying Proposition 1.2.4 from Nualart (2006) and the related remark, we get that if the function φ is Lipschitz continuous and has a derivative a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$, and the law of the r.v. ξ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then $\varphi(\xi)$ has a stochastic derivative and $D\varphi(\xi) = \varphi'(\xi)D\xi$ a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$.

In particular, consider the stochastic differentiation of the functions of the Ornstein–Uhlenbeck process Y. Let φ be locally Lipschitz, with both and φ and φ' being of polynomial growth. Then $\varphi_n(x) = \varphi(x)\mathbb{1}_{|x|\le n} + (\varphi(-n))\mathbb{1}_{x<-n} + \varphi(n)\mathbb{1}_{x>n}$ is Lipschitz, has a derivative φ_n' a.e. w.r.t. the Lebesgue measure. Moreover, there exists a polynomial $\bar{\varphi}$ with non-negative coefficients such that

$$|\varphi_n(x)| + |\varphi_n'(x)| \le \bar{\varphi}(|x|),\ \ x \in \mathbb{R},$$

which implies that

$$\mathrm{E}|\varphi(Y_s) - \varphi_n(Y_s)|^2 \le 4\mathrm{E}\left(\bar{\varphi}^2(|Y_s|)\mathbb{1}_{Y_s\notin[-n,n]}\right) \to 0.$$

Also, it follows from (10) that in fact $D^B Y_s$ is in $L^2([0, T])$. Furthermore,

$$\mathrm{E}\left(\max_{s\in[0,T]} \bar{\varphi}^2(|Y_s|)\right) < \infty,$$

due to the fact that $\max\limits_{s\in[0,T]} |Y_s|$ has exponential moments. Therefore,

$$\begin{aligned}
&\mathrm{E}\left(\int_0^T (\varphi'(Y_s)D_u^B Y_s - D_u^B \varphi_n(Y_s))^2 du\right)\\
&\le 4\mathrm{E}\left(\max_{s\in[0,T]} \bar{\varphi}^2(|Y_s|)\mathbb{1}_{\max\limits_{s\in[0,T]} Y_s\notin[-n,n]} \int_0^s (D_u^B Y_s)^2 du\right) \to 0.
\end{aligned}$$

Previous results, together with the fact that D is closed, imply that $D_u^B\varphi(Y_s) = \varphi'(Y_s)D_u^B Y_s$.

Let us introduce the following notations: $g(y) = f(e^y)$, $F(x) = \int_0^x f(z)dz$ and let $G(y) = \int_0^y g(z)dz$, $x \ge 0$, $y \in \mathbb{R}$. Also, let

$$Z_T = \int_0^T \sigma^{-1}(Y_u)dV_u. \tag{13}$$

Note that Z_T is well defined because of condition (**B**), (i). Now, analogously to Altmayer and Neuenkirch (2015), we are in position to transform the option price in such a way that it does not contain discontinuous functions.

Lemma 11 *Under conditions* (**A**) *and* (**B**) *the option price* $\mathrm{E} f(S_T) = \mathrm{E} g(X_T)$ *can be represented as*

$$\mathrm{E} f(S_T) = \mathrm{E}\left(\frac{F(S_T)}{S_T}\left(1+\frac{Z_T}{T}\right)\right). \tag{14}$$

Alternatively,

$$\mathrm{E} g(X_T) = \frac{1}{T}\mathrm{E}\left(G(X_T)Z_T\right). \tag{15}$$

4 The Rate of Convergence of Approximate Option Prices in the Case When Both Wiener Process and Fractional Brownian Motions are Discretized

In the present section we provide our first approach (first level) to the numerical approximation of the solution for the option pricing problem. In particular, we are going to provide a double discretization procedure with respect to both the Wiener process and the fBm, also estimating the rate of convergence for the corresponding approximated option prices to the real value given by $\mathrm{E} f(S_T)$.

To pursue latter aim, let us introduce the following notation. For any $n \in \mathbb{N}$ consider equidistant partition of the interval $[0, T]$: $t_i = t_i(n) = \frac{iT}{n}$, $i = 0, 1, 2, ..., n$. Then we define the discretizations of Wiener processes W, V, B and the fractional Brownian motion B^H:

$$\Delta P_i = P\left(t_{i+1}\right) - P\left(t_i\right), i = 0, 1, 2, ..., n, \ P = W, V, B, B^H.$$

Discretized processes Y and X, corresponding to a given partition have the form

$$\begin{aligned} Y^n_{t_j} &= Y_0 e^{-\alpha t_j} + B^H_{t_j} - \frac{\alpha T}{n}\sum_{i=0}^{j-1} e^{-\alpha(t_j - t_i)} B^H_{t_i}, \\ X^n_{t_j} &= X_0 + b t_j - \frac{T}{2n}\sum_{i=0}^{j-1}\sigma^2\left(Y^n_{t_i}\right) + \sum_{i=0}^{j-1}\sigma\left(Y^n_{t_i}\right)\Delta W_i \\ &= X_0 + b t_j - \frac{1}{2}\int_0^{t_j}\sigma^2(Y^n_s)ds + \int_0^{t_j}\sigma(Y^n_s)dW_s, \quad j = 0, ..., n, \end{aligned}$$

where we take $Y^n_s = Y^n_{t_i}$ for $s \in \left[t_i, t_{i+1}\right)$. The discretization of Z_T from (13) is $Z^n_T = \int_0^T \frac{1}{\sigma(Y^n_s)} dV_s$. Also, we define $S^n_{t_j} = \exp\left\{X^n_{t_j}\right\}$. The three lemmas below contain the auxiliary bounds that are necessary in order to establish the main result.

Lemma 12 (i) *For any* $\theta > 0$ *there exists a constant* C *depending on* θ *such that for any* $s, t \in [0, T]$

$$\mathrm{E}|Y_t - Y_s|^\theta \le C\,|t-s|^{\theta H}.$$

(ii) *For any* $\theta > 0$ *there exists a constant* C *depending on* θ *such that for any* $0 \le j \le n$

$$\mathrm{E}\left|Y_{t_j} - Y^n_{t_j}\right|^\theta \le C n^{-\theta H}.$$

(iii) *For any* $\theta > 0$ *there exists a constant* C *depending on* θ *such that for any* $s \in [0, T]$

$$\mathrm{E}|Y_s - Y^n_s|^\theta = \mathrm{E}|Y_s - Y^n_{t_i}|^\theta \le C n^{-\theta H}.$$

(iv) *Approximating process has uniformly bounded moments: for any* $\theta > 0$

$$\sup_{s\in[0,T]} \mathrm{E}|Y_s^n|^\theta < \infty, \tag{16}$$

and bounded exponential moments: for any $a > 0$ *and any* $\varrho < 2$

$$\mathrm{E}\exp\{a \sup_{t\in[0,T]} |Y_t^n|^\varrho\} < \infty. \tag{17}$$

Remark 13 Using (16) and (17), we can prove similarly to Lemma 2 and 3 that for any $m \in \mathbb{Z}$

$$\sup_{n\geq 1}\sup_{0\leq j\leq n} \mathrm{E}\left(S_{t_j}^n\right)^m < \infty, \sup_{n\geq 1}\sup_{t\in[0,T]} \mathrm{E}\exp\left\{m\int_0^t \sigma(Y_s^n)dB_s\right\}$$
$$< \infty, \ \sup_{n\geq 1}\mathrm{E}\exp\left\{m\int_0^T \sigma^2(Y_s^n)ds\right\} < \infty, \tag{18}$$

and

$$\sup_{n\geq 1}\sup_{t\in[0,T]} \mathrm{E}\exp\left\{m\left(\int_0^t \sigma(Y_s^n)dB_s - \frac{1}{2}\int_0^t \sigma^2(Y_s^n)ds\right)\right\} < \infty.$$

Lemma 14 *There exists a constant* $C > 0$ *such that for any* $n \geq 1$

$$\mathrm{E}(X_T - X_T^n)^2 \leq Cn^{-2rH}, \tag{19}$$

and

$$\mathrm{E}(Z_T - Z_T^n)^2 \leq Cn^{-2rH}. \tag{20}$$

Lemma 15 *Under conditions* (**A**) *and* (**B**) *we have the following upper bound: there exists a constant* $C_F > 0$ *such that*

$$\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right|^2 \leq C_F \cdot n^{-2rH}.$$

Using previous lemmas, we are now in a position to state the main result of this section, namely to provide the rate of convergence of the discretized option price to the exact one represented by $\mathrm{E}f(S_T)$, under the double discretization.

Theorem 16 *Let conditions* (**A**) *and* (**B**) *hold. There exists a constant* $C > 0$ *not depending on* n *such that*

$$\left|\mathrm{E}f(S_T) - \mathrm{E}\left(\frac{F(S_T^n)}{S_T^n}\left(1 + \frac{Z_T^n}{T}\right)\right)\right| \leq Cn^{-rH}.$$

5 The Rate of Convergence of Approximate Option Prices in the Case When only Fractional Brownian Motion is Discretized

The present section is devoted to the implementation of the second approach (second level) to approximate the option price. It is based on the fact that the logarithm of the asset price is conditionally Gaussian given the trajectory of the fractional Brownian motion. It allows to exclude Wiener process W from the consideration and to calculate the option price explicitly in terms of the trajectory of fBm B^H. Respectively, we can discretize and simulate only the trajectories of B^H (the single discretization). Theorem 18 gives the explicit option pricing formula as the functional of the trajectory of fBm B^H, and Theorem 19 gives the rate of convergence. Comparing to Theorem 16, we see that the rate of convergence admits a bound of the same order, influenced by the behavior of volatility.

Let us introduce the following notations: define a covariance matrix

$$C_{X,Z} = \begin{pmatrix} \mu^2\sigma_Y^2 & \mu T \\ \mu T & \sigma_Z^2 \end{pmatrix},$$

and let

$$\sigma_Y^2 = \int_0^T \sigma^2(Y_s)ds, \quad m_Y = X_0 + bT - \frac{1}{2}\sigma_Y^2 + \rho\int_0^T \sigma(Y_s)dB_s,$$

$$\sigma_Z^2 = \int_0^T \sigma^{-2}(Y_s)ds, \quad \Delta = |C_{X,Z}| = \mu^2(\sigma_Y^2\sigma_Z^2 - T^2).$$

Evidently, $\Delta \geq 0$. We assume additionally that the following assumption is satisfied.

(**D**) $\Delta = \sigma_Y^2\sigma_Z^2 - T^2 > 0$ with probability 1, in particular, $\mu > 0$.

Note that the random vector

$$(X_T, Z_T) = \left(X_0 + bT - \frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \rho\int_0^T \sigma(Y_s)dB_s + \mu\int_0^T \sigma(Y_s)dV_s,\ \int_0^T \sigma^{-1}(Y_s)dV_s \right) \tag{21}$$

is Gaussian conditionally to the σ-field $\mathcal{F}_T^H$. The conditional mean vector equals to $(m_Y, 0)$, and the conditional covariance matrix is $C_{X,Z}$. Next lemma presents common conditional density of (X_T, Z_T). Note that under assumption (**D**) the distribution of (X_T, Z_T) is non-degenerate in $\mathbb{R}^2$.

Lemma 17 *Let assumption* (**D**) *hold. Then the common conditional density $p_{X,Z}(x, z)$ of (X_T, Z_T), conditionally to the given trajectory $\{Y_t, t \in [0, T]\}$, equals*

$$p_{X,Z}(x, z) = \frac{1}{2\pi\Delta^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2\Delta}\left(\sigma_Z^2(x - m_Y)^2 + \mu^2\sigma_Y^2 z^2 - 2T\mu(x - m_Y)z\right)\right\}. \tag{22}$$

The next result states that the option price can be presented as a functional of σ_Y^2 and $\int_0^T \sigma(Y_s)dB_s$ only.

Theorem 18 *Under conditions* (**A**)–(**D**) *the following equality holds:*

$$\begin{aligned} \mathrm{E}g(X_T) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} G(x)\mathrm{E}\left(\frac{(x - m_Y)}{\mu^2\sigma_Y^3}\exp\left\{-\frac{(x - m_Y)^2}{2\mu^2\sigma_Y^2}\right\}\right)dx \\ &= (2\pi)^{-\frac{1}{2}}\mathrm{E}\left((\sigma_Y)^{-1}\int_{\mathbb{R}} G((x + m_Y)\sigma_Y)\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}}dx\right). \end{aligned} \tag{23}$$

In order to state the main result of the present section, let us define the following quantities

$$\sigma^2_{Y,n} = \int_0^T \sigma^2(Y^n_s)ds, \quad m_{Y,n} = X_0 + bT - \frac{1}{2}\sigma^2_{Y,n} + \rho\sum_{k=0}^n \sigma(Y_{t^n_k})\Delta B_k$$
$$= X_0 + bT - \frac{1}{2}\sigma^2_{Y,n} + \rho\int_0^T \sigma(Y^n_s)dB_s.$$

Theorem 19 *Under conditions* (**A**) – (**D**) *we have*

$$\left|\mathrm{E}g(X_T) - (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} G(x)\mathrm{E}\left(\frac{(x-m_{Y,n})}{\mu^2\sigma^3_{Y,n}}\exp\left\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma^2_{Y,n}}\right\}\right)dx\right| \le Cn^{-rH}. \tag{24}$$

Without any doubt, the form of density is much simplified in the case $\rho = 0$, i.e., processes W and B are independent, because in this case the option price can be presented as the functional of σ^2_Y only. The great advantage of this situation is that we can discretize just the trajectories of Y. And although this case is perhaps more particular and not so common, we still prefer to consider it, since similar methods can be applied also in the case of a weak dependence between W and B. In particular, if $\rho = 0$, we have

$$C_{X,Z} = \begin{pmatrix}\sigma^2_Y & T\\ T & \sigma^2_Z\end{pmatrix}, \quad m_Y = X_0 + bT - \frac{1}{2}\sigma^2_Y, \quad m_{Y,n} = X_0 + bT - \frac{1}{2}\sigma^2_{Y,n},$$

and (24) transforms into

$$\left|\mathrm{E}g(X_T) - (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} G(x)\mathrm{E}\left(\frac{(x-m_{Y,n})}{\sigma^3_{Y,n}}\exp\left\{-\frac{(x-m_{Y,n})^2}{2\sigma^2_{Y,n}}\right\}\right)dx\right| \le Cn^{-rH}.$$

6 Option Price in Terms of the Density of the Integrated Stochastic Volatility

Consider for simplicity the case when $\rho = 0$. Applying Theorem 18 and equality (23), we clearly see that the option price depends on the random variable $\sigma^2_Y = \int_0^T \sigma^2(Y_s)ds$. Therefore it is natural to derive the density of this random variable. Since σ^2_Y depends on the whole trajectory of the fBm B^H on $[0, T]$, we apply Malliavin calculus in an attempt to find the density. First, establish some auxiliary results. For any $\varepsilon > 0$ and $\delta > 0$ introduce the stopping times $\tau_\varepsilon = \inf\{t > 0 : |B^H_t| \ge \varepsilon\}$ and $\nu_\delta = \inf\{t > 0 : |Y_t - Y_0| \ge \delta\}$.

Lemma 20 *For any $l > 0$ the negative moment is well defined:* $\mathrm{E}(\nu_\delta)^{-l} < \infty$.

Now we introduce additional assumptions on σ.

(**E**) The function $\sigma \in C^{(2)}(\mathbb{R})$, its derivative σ' is strictly nonnegative, $\sigma'(x) > 0,\ x \in \mathbb{R}$, and σ', σ'' are of polynomial growth.

Lemma 21 *Under assumptions* (**B**) *and* (**E**) *the stochastic process*

$$\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}=\left\{\frac{D_t^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2},\ t\in[0,T]\right\}$$

belongs to the domain Dom δ of the Skorokhod integral δ.

Denote $\eta=(||D^B\sigma_Y^2||_H^2)^{-1},\ l(u,s)=c_He^{-\alpha s}\int\limits_u^s e^{\alpha v}v^{H-1/2}(v-u)^{H-3/2}dv,\ \kappa(y)=\sigma(y)\sigma'(y)$.

Theorem 22 (i) *The density $p_{\sigma_Y^2}$ of the random variable σ_Y^2 is bounded, continuous and given by the following formulas*

$$p_{\sigma_Y^2}(u)=\mathrm{E}\left[\mathbb{1}_{\sigma_Y^2>u}\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}\right)\right],\tag{25}$$

where the Skorokhod integral is in fact reduced to a Wiener integral,

$$\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}\right)=2\eta\int_0^T\kappa(Y_s)\left(\int_0^s u^{1/2-H}l(u,s)dB_u\right)ds-\int_0^T D_u^B\eta D_u^B(\sigma_Y^2)du.$$

(ii) *The option price $\mathrm{E}g(X_T)$ can be represented as the integral with respect to the density $p_{\sigma_Y^2}(u)$ defined by (25) as follows:*

$$\mathrm{E}g(X_T)$$
$$=(2\pi)^{-\frac{1}{2}}T\int\limits_{\mathbb{R}}G(x)\int_{\mathbb{R}}\frac{(x+u/2-X_0-bT)}{u^3}\exp\left\{-\frac{(x+u/2-X_0-bT)^2}{2u^2}\right\}p_{\sigma_Y^2}(u)du.$$

7 Proofs

Proof of Lemma 2. (i) The representation (5) for the fractional Ornstein-Uhlenbeck process Y is well known, see, e.g., Cheridito et al. (2003). It is a continuous Gaussian process with $\sup_{t\in[0,T]}\mathrm{E}(Y_t)^2<\infty$. The finiteness of any exponential moments of the form (6) follows from Fernique (1975), or (Ledoux 1996, Theorem 4.1).

(ii), (iii) To establish the representation (7) for S, we need only to prove that the integrals $\int_0^t\sigma(Y_s)dW_s$ and $\int_0^t\sigma(Y_s)S_sdW_s$ are well defined, while the form of the representation is straightforward, because W is a square-integrable martingale with respect to the flow $\mathbb{F}^{B,V}$ generated by the couple of independent processes B and V. Concerning $\int_0^t\sigma(Y_s)dW_s$, it follows from (6) and condition (**B**), (i) that $\int_0^t\mathrm{E}\sigma^2(Y_s)ds\le C\int_0^t\mathrm{E}(1+|Y_s|^{2q})ds<\infty$, consequently $\int_0^t\sigma(Y_s)dW_s$ is well defined. Moreover, the following moments of any order are finite: $\sup_{t\in[0,T]}\mathrm{E}\sigma^{2n}(Y_t)<\infty$. Additionally, for any $\varrho<2$ exponential inequality (6) follows from Fernique (1975). Therefore, taking into account that $q<1$, we get that for any $k>0$

$$\mathrm{E}\exp\left\{k\int_0^t\sigma^2(Y_s)ds\right\}\le\mathrm{E}\exp\left\{C_\sigma k\int_0^t(1+|Y_s|^{2q})ds\right\}\le C\mathrm{E}\exp\left\{C_\sigma Tk\sup_{s\in[0,T]}|Y_s|^{2q}\right\}<\infty.\tag{26}$$

It follows immediately from (26) and from Novikov's condition that for any $n \in \mathbb{Z}$

$$\mathrm{E}\exp\left\{2n\int_0^t \sigma(Y_s)dW_s - 2n^2\int_0^t \sigma^2(Y_s)ds\right\} = 1,$$

consequently

$$\begin{aligned}\sup_{t\in[0,T]} \mathrm{E}S_t^n &\le C \sup_{t\in[0,T]} \mathrm{E}\exp\left\{n\int_0^t \sigma(Y_s)dW_s - \frac{n}{2}\int_0^t \sigma^2(Y_s)ds\right\} \\ &\le C \sup_{t\in[0,T]} \left(\left(\mathrm{E}\exp\left\{2n\int_0^t \sigma(Y_s)dW_s - 2n^2\int_0^t \sigma^2(Y_s)ds\right\}\right)^{1/2}\right. \\ &\qquad \left.\times\left(\mathrm{E}\exp\left\{(2n^2-n)\int_0^t \sigma^2(Y_s)ds\right\}\right)^{1/2}\right) \\ &= C \sup_{t\in[0,T]} \left(\mathrm{E}\exp\left\{(2n^2-n)\int_0^t \sigma^2(Y_s)ds\right\}\right)^{1/2} < \infty. \end{aligned} \tag{27}$$

Further, applying both final and intermediate bounds from (27), we get that

$$\int_0^T \mathrm{E}(\sigma^2(Y_s)S_s^2)ds \le T\left(\sup_{t\in[0,T]} \mathrm{E}S_t^4 \sup_{t\in[0,T]} \mathrm{E}\sigma^4(Y_t)\right)^{\frac{1}{2}} < \infty, \tag{28}$$

and finally the proof of (ii) follows from (27) and (28). To establish (iii), it sufficient to prove that for any $m \in \mathbb{Z}$, $\sup_{t\in[0,T]} \mathrm{E}\exp\{m\int_0^t \sigma(Y_s)dB_s\} < \infty$. But we can proceed as before: it follows from (26) that for any $n \in \mathbb{Z}$ and any $t \in [0, T]$

$$\mathrm{E}\exp\left\{m\int_0^t \sigma(Y_s)dB_s - \frac{1}{2}m^2\int_0^t \sigma^2(Y_s)ds\right\} = 1,$$

and consequently

$$\begin{aligned}\sup_{t\in[0,T]} \mathrm{E}\exp\left\{m\int_0^t \sigma(Y_s)dB_s\right\} &\le \sup_{t\in[0,T]} \left(\mathrm{E}\exp\left\{2m\int_0^t \sigma(Y_s)dB_s - 2m^2\int_0^t \sigma^2(Y_s)ds\right\}\right)^{1/2} \\ &\qquad \times\left(\mathrm{E}\exp\left\{2m^2\int_0^t \sigma^2(Y_s)ds\right\}\right)^{1/2} < \infty. \end{aligned}$$

□

Remark 23 Analyzing the bounds obtained in (27), we can immediately conclude that for any $m \in \mathbb{Z}$

$$\sup_{t\in[0,T]} \mathrm{E}\exp\left\{m\left(\int_0^t \sigma(Y_s)dW_s - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds\right)\right\} < \infty,$$

and

$$\sup_{t\in[0,T]} \mathrm{E}\exp\left\{m\left(\int_0^t \sigma(Y_s)dB_s - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds\right)\right\} < \infty.$$

Proof of Theorem 4*.* Let a probability measure Q be defined via (8) with functions ν_i satisfying (9). Then the discounted process

$$D_t := e^{-\beta t} S_t \cdot \frac{d\mathrm{Q}}{d\mathrm{P}}\bigg|_{\mathcal{F}_t}$$

gets a form

$$D_t = 1 + \int_0^t (\rho\sigma(Y_s) + \nu_1(s))\, dB_s + \int_0^t (\mu\sigma(Y_s) + \nu_2(s))\, dV_s,$$

therefore it is a martingale w.r.t. the measure P. Therefore, $e^{-\beta t} S_t$ is a martingale w.r.t. the measure Q, and we established both (i) and (ii). To check (iii), note that the minimal martingale measure, according to Schweizer (1995), is defined via the relation

$$\frac{d\mathrm{Q}}{d\mathrm{P}}\Big|_{\mathcal{F}_t} = 1 - \int_0^t Z_s \alpha(s) dM_s,$$

where the price process has a form $S_t = S_0 + M_t + \int_0^t \alpha(s) d\langle M\rangle_s$. In our case $M_t = \int_0^t S_s\sigma(Y_s)dW_s$, $\int_0^t \alpha(s)d\langle M\rangle_s = (b-\beta)\int_0^t S_s ds$, therefore $\alpha(s) = \frac{b-\beta}{S_s\sigma^2(Y_s)}$, whence $1 - \int_0^t Z_s\alpha(s)dM_s = 1 + \int_0^t Z_s \frac{\beta-b}{\sigma(Y_s)} dW_s$. It means that the minimal martingale measure Q (that is unique) has a Radon–Nikodym derivative

$$\frac{d\mathrm{Q}}{d\mathrm{P}}\Big|_{\mathcal{F}_t} = \exp\left\{ \int_0^t \frac{\beta - b}{\sigma(Y_s)} dW_s - 1/2 \int_0^t \frac{(\beta-b)^2}{\sigma^2(Y_s)} ds \right\},$$

and this equality is satisfied if we choose $\nu_1(s) = \rho\frac{\beta-b}{\sigma(Y_s)}$ and $\nu_2(s) = \mu\frac{\beta-b}{\sigma(Y_s)}$. The theorem is proved. □

Proof of Lemma 8. Statement (i) follows directly from the definition of stochastic derivative and from the fact that B and V are independent. Similarly, the first equality in (10) is obvious since Y is independent of V. Furthermore, integrating by parts (5) and taking into account representation (3), we get the following equalities

$$Y_t = Y_0 e^{-\alpha t} + B_t^H - \alpha e^{-\alpha t}\int_0^t e^{\alpha s} B_s^H ds, \tag{29}$$

whence

$$D_u^B Y_t = \left(k(t,u) - \alpha e^{-\alpha t} \int_u^t e^{\alpha s} k(s,u) ds \right) \mathbb{1}_{u<t},$$

where the kernel k was introduced in (3). Now, let $H > 1/2$. Then we can use representation (4). Note that the derivative k_s' of the kernel k equals

$$k_s'(s,u) = c_H u^{\frac{1}{2}-H} s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \mathbb{1}_{u<s}.$$

It is an integrable function, therefore we can integrate by parts once again and get

$$D_u^B Y_t = e^{-\alpha t} \int_u^t e^{\alpha s} k_s'(s,u) ds = c_H e^{-\alpha t} u^{\frac{1}{2}-H} \int_u^t e^{\alpha s} s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \mathbb{1}_{u<t}.$$

So, we get (10) and (11). The first equation from (12) follows from the definition of stochastic derivative. Further,

$$D_u^B X_t = \left(-\frac{1}{2} \int_0^t D_u^B(\sigma^2(Y_s)) ds + \int_0^t D_u^B(\sigma(Y_s)) dW_s \right) \mathbb{1}_{u<t}.$$

By Remark 10 and since Y_s has a Gaussian distribution and the function σ is continuous and a.e. differentiable, we can apply the chain rule to $\sigma(Y_s)$. Moreover, the result can be written in the standard form, so that $D_u^B(\sigma(Y_s)) = \sigma'(Y_s)D_u^B(Y_s)$, and

$$D_u^B(\sigma^2(Y_s)) = 2\sigma(Y_s)\sigma'(Y_s)D_u^B(Y_s) \tag{30}$$

a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$. Besides, similarly to proof of Lemma 2, we can apply properties (**B**), (ii) and (iv), which together with (10) ensure that the integrals in (12) exist, whence the proof follows. □

Proof of Lemma 9. Conditionally on the σ-field $\mathcal{F}_T^H$, X_T is a Gaussian random variable. Therefore, for any Borel set $A \subset \mathbb{R}$ of zero Lebesgue measure, we have

$$\mathrm{P}\{X_T \in A\} = \mathrm{E}(\mathbb{1}_{X_T\in A}) = \mathrm{E}\left(\mathrm{E}\left(\mathbb{1}_{X_T\in A}|\mathcal{F}_T^H\right)\right) = 0.$$

The absolute continuity of the law of S_T follows from that of X_T since $S_T = \exp\{X_T\}$. □

Proof of Lemma 11. Let the function H be locally Lipschitz and $H'(x) = h(x)$ a.e. with respect to the Lebesgue measure. Assume additionally that h is of exponential growth. Hence it follows from Remark 10 that

$$D_u^V H(X_T) = h(X_T)D_u^V X_T.$$

We will now establish that $H(X_T) \in \mathrm{D}^{1,2}$, where we consider stochastic differentiation w.r.t. V. Indeed, h is of exponential growth,

$$h(x) \le C_h(1 + e^{p_h|x|}),$$

and

$$H(x) = \int_0^x h(y)dy \le C_h|x|(1 + e^{p_h|x|}) \le C_h(1 + e^{(p_h+1)|x|}).$$

Furthermore, since

$$e^{(p_h+1)|X_T|} = (S_T)^{p_h+1} \vee (S_T)^{-p_h-1},$$

we get from (27) that $\mathrm{E}H^2(X_T) < \infty$. Additionally,

$$\mathrm{E}\int_0^T \left(h(X_T)D_u^V X_T\right)^2 du = \mu^2\mathrm{E}\left(h^2(X_T)\int_0^T \sigma^2(Y_u)du\right) =$$

$$\le C\left(\mathrm{E}h^4(X_T)\int_0^T \mathrm{E}\sigma^4(Y_u)du\right)^{1/2} < \infty.$$

Therefore

$$H(X_T) \in \mathrm{D}^{1,2}. \tag{31}$$

Having established both the existence and the form of the stochastic derivative, together with (31), we can proceed as in the proof of Proposition 4.1 (Altmayer and Neuenkirch 2015). Namely, the Skorokhod integral is the adjoint operator to the Malliavin derivative, therefore

$$\begin{aligned}\mathrm{E}h(X_T) &= \frac{1}{T}\mathrm{E}\left(\int_0^T h(X_T)D_u^V X_T \frac{1}{D_u^V X_T}du\right) = \frac{1}{T\mu^2}\mathrm{E}\left(\int_0^T D_u^V H(X_T)\frac{1}{\sigma(Y_u)}du\right)\\ &= \frac{1}{T\mu^2}\mathrm{E}\left(H(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dV_u\right) = \frac{1}{T}\mathrm{E}\left(H(X_T)Z(T)\right). \qquad (32)\end{aligned}$$

The function G is locally Lipschitz and $G'(x) = g(x)$ a.e. with respect to the Lebesgue measure. Moreover, g is of exponential growth, namely,

$$g(x) \leq C_f(1 + e^{p|x|}),$$

therefore (15) follows directly from (32).

To establish (14), we start with the identity

$$G(x) = \frac{F(e^x)}{e^x} + \int_0^x \frac{F(e^y)}{e^y}dy - F(1),$$

then we rewrite it, applying (15), as follows:

$$\begin{aligned}\mathrm{E}f(S_T) &= \mathrm{E}g(X_T) = \frac{1}{T}\mathrm{E}\left(G(X_T)Z_T\right)\\ &= \frac{1}{T}\mathrm{E}\left(\left(\frac{F(S_T)}{S_T} + \int_0^{X_T}\frac{F(e^y)}{e^y}dy - F(1)\right)Z_T\right) = \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T}Z_T\right)\\ &\quad + \frac{1}{T}\mathrm{E}\left(\int_0^{X_T}\frac{F(e^y)}{e^y}dyZ_T\right) - \frac{1}{T}\mathrm{E}(F(1)Z_T)\\ &= \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T}Z_T\right) + \frac{1}{T}\mathrm{E}\left(\int_0^{X_T}\frac{F(e^y)}{e^y}dyZ_T\right).\end{aligned}$$

Applying Eq. 32 to $h(x) = \frac{F(e^x)}{e^x}$, we get

$$\mathrm{E}\left(\frac{F(S_T)}{S_T}\right) = \frac{1}{T}\mathrm{E}\left(\int_0^{X_T}\frac{F(e^y)}{e^y}dyZ_T\right).$$

Hence

$$\mathrm{E}f(S_T) = \frac{1}{T}\mathrm{E}\left(\frac{F(S_T)}{S_T}Z_T\right) + \mathrm{E}\left(\frac{F(S_T)}{S_T}\right) = \mathrm{E}\left(\frac{F(S_T)}{S_T}\left(1 + \frac{Z_T}{T}\right)\right).$$

□

Proof of Lemma 12. Since the process Y as well Y^n are Gaussian, it is sufficient to consider throughout the proof only the case $\theta = 2$.

(i) The increment of Y can be presented as

$$Y_t - Y_s = B_t^H - B_s^H - \alpha(e^{-\alpha t} - e^{-\alpha s})\int_0^s e^{\alpha u} B_u^H du + e^{-\alpha t}\int_s^t e^{\alpha u} B_u^H du.$$

Since $|e^{-\alpha t} - e^{-\alpha s}| \le \alpha|t-s|$ for $t, s \ge 0$, we have

$$\begin{aligned}\mathrm{E}\,(Y_t - Y_s)^2 &\le C\left(\mathrm{E}\left(B_t^H - B_s^H\right)^2 + |t-s|^2\mathrm{E}\left(\int_0^s e^{\alpha u} B_u^H du\right)^2 + \mathrm{E}\left(\int_s^t e^{\alpha u} B_u^H du\right)^2\right)\\ &\le C\left(|t-s|^2 + |t-s|^{2H}\right) \le C|t-s|^{2H}.\end{aligned}$$

(ii) Define the approximation $e_n(s) = e^{-\alpha t_i} B_{t_i}^H, \quad s \in [t_i, t_{i+1}),\ 0 \le i \le n-1$. Then, similarly to above calculations,

$$\mathrm{E}\left(Y_{t_j} - Y_{t_j}^n\right)^2 = e^{-2\alpha t_j}\mathrm{E}\left(\int_0^{t_j}\left(e^{\alpha s} B_s^H - e_n(s)\right)ds\right)^2 \le \frac{C}{n^2} + \frac{C}{n^{2H}} \le \frac{C}{n^{2H}}.$$

(iii) Now, let $s \in [t_i, t_{i+1})$ and $\theta \ge 1$. Then it follows from (i) and (ii) that

$$\mathrm{E}|Y_s - Y_s^n|^\theta = \mathrm{E}|Y_s - Y_{t_i}^n|^\theta \le C\mathrm{E}|Y_s - Y_{t_i}|^\theta + C\mathrm{E}|Y_{t_i} - Y_{t_i}^n|^\theta \le Cn^{-\theta H}.$$

Statement (iv) follows immediately from Lemma 2, statement (i), and from statement (iii) above. □

Proof of Lemma 14. Let us start with (19). Taking into account condition (**B**), (ii) and (iii), we can write

$$\begin{aligned}\mathrm{E}(X_T - X_T^n)^2 &= \mathrm{E}\left[-\frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \int_0^T \sigma(Y_s)dW_s + \frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s\right]^2\\ &\le 2\mathrm{E}\left[\frac{1}{2}\int_0^T(\sigma^2(Y_s) - \sigma^2(Y_s^n))ds\right]^2 + 2\mathrm{E}\left[\int_0^T(\sigma(Y_s) - \sigma(Y_s^n))dW_s\right]^2\\ &\le \frac{T}{2}\int_0^T \mathrm{E}(\sigma^2(Y_s) - \sigma^2(Y_s^n))^2 ds + 2\int_0^T \mathrm{E}(\sigma(Y_s) - \sigma(Y_s^n))^2 ds\\ &= \frac{T}{2}\int_0^T \mathrm{E}\left[|\sigma(Y_s) - \sigma(Y_s^n)|^2|\sigma(Y_s) + \sigma(Y_s^n)|^2\right]ds + 2\int_0^T \mathrm{E}(\sigma(Y_s) - \sigma(Y_s^n))^2 ds\\ &\le C\int_0^T\left(\mathrm{E}\left(|Y_s - Y_s^n|^{2r}\left(C + |Y_s|^{2q} + |Y_s^n|^{2q}\right)\right) + \mathrm{E}\left|Y_s - Y_s^n\right|^{2r}\right)ds\\ &\le C\int_0^T\left(\mathrm{E}\left|Y_s - Y_s^n\right|^{4r}\mathrm{E}\left(C + |Y_s|^{4q} + |Y_s^n|^{4q}\right)\right)^{1/2} ds. \qquad (33)\end{aligned}$$

Lemma 2 (i), and Lemma 12 (ii) imply that for any $\theta \ge 1$

$$\sup_{n\in\mathbb{N}, s\in[0,T]} \mathrm{E}\left(|Y_s|^\theta + |Y_s^n|^\theta\right) < \infty. \tag{34}$$

Moreover, it follows from Lemma 12 that for any $s \in [0, T]$ and $\theta \ge 1$

$$\mathrm{E}|Y_s - Y_s^n|^{\theta r} \le C n^{-\theta r H}. \tag{35}$$

Set $\theta = 4q$ in (34) and $\theta = 4$ in (35) and substitute the result into the right-hand side of (33):

$$\mathrm{E}(X_T - X_T^n)^2 \le C \int_0^T \left(\mathrm{E}\left(Y_s - Y_s^n\right)^{4r}\right)^{\frac{1}{2}} ds \le C n^{-2rH},$$

so that (19) is proved. Now we move on to (20). Taking into account condition (**B**), (i), we get that

$$\left|\frac{1}{\sigma(x)} - \frac{1}{\sigma(y)}\right| \le \frac{|\sigma(x) - \sigma(y)|}{\sigma(x)\sigma(y)} \le \frac{|\sigma(x) - \sigma(y)|}{\sigma_{\min}^2},$$

whence

$$\begin{aligned}\mathrm{E}(Z_T - Z_T^n)^2 &= \int_0^T \left(\frac{1}{\sigma(Y)} - \frac{1}{\sigma(Y_s^n)}\right)^2 ds \\ &\le \frac{1}{\sigma_{\min}^2} C_\sigma \int_0^T \mathrm{E}\left(Y_s - Y_s^n\right)^{2r} ds.\end{aligned}$$

We can apply (35) with $\theta = 2$ to the last inequality and conclude this part of the proof exactly as it was done for (19). □

Proof of Lemma 15. We can write

$$\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right|^2 \le 2\mathrm{E}\left|\frac{F(S_T)}{S_T} - \frac{F(S_T)}{S_T^n}\right|^2 + 2\mathrm{E}\left|\frac{F(S_T)}{S_T^n} - \frac{F(S_T^n)}{S_T^n}\right|^2 := 2I_1 + 2I_2. \tag{36}$$

Now we estimate the right-hand side of (36) term by term. For I_1 we have

$$I_1 = \mathrm{E}\Big(F(S_T)\big((S_T)^{-1} - (S_T^n)^{-1}\big)\Big)^2 \le \Big(\mathrm{E}(F(S_T))^4 \mathrm{E}\big((S_T)^{-1} - (S_T^n)^{-1}\big)^4\Big)^{1/2}. \tag{37}$$

On the one hand, since f and F both have a polynomial growth, $\mathrm{E}(F(S_T))^4 < \infty$ according to Remark 3. On the other hand,

$$\mathrm{E}\big((S_T)^{-1} - (S_T^n)^{-1}\big)^4 = S_0^{-4} e^{-4bT} \tag{38}$$

$$\times \mathrm{E}\left(\exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s\right\} - \exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s)ds - \int_0^T \sigma(Y_s)dW_s\right\}\right)^4.$$

Using the inequalities

$$|e^x - e^y| \le (e^x + e^y)|x - y|, \quad x, y \in \mathbb{R},$$

$$(x + y)^{2n} \le C(n)(x^{2n} + y^{2n}), \quad x, y \in \mathbb{R}, n \in \mathbb{N},$$

along with results outlined in Remarks 3 and 13, the Burkholder–Davis–Gundy and Hölder inequalities, condition (**B**), (ii) and (iii), and relation (34) with $\nu = 16q$, we get from (38) that

$$
\begin{aligned}
&\mathrm{E}\big((S_T)^{-1} - (S_T^n)^{-1}\big)^4 \\
&\le C\mathrm{E}\left(\exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s\right\} - \exp\left\{\frac{1}{2}\int_0^T \sigma^2(Y_s)ds - \int_0^T \sigma(Y_s)dW_s\right\}\right)^4 \\
&\le C\mathrm{E}\left(\left(\exp\left\{2\int_0^T \sigma^2(Y_s^n)ds - 4\int_0^T \sigma(Y_s^n)dW_s\right\} + \exp\left\{2\int_0^T \sigma^2(Y_s)ds - 4\int_0^T \sigma(Y_s)dW_s\right\}\right)\right. \\
&\quad \left.\times\left(\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s - \frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \int_0^T \sigma(Y_s)dW_s\right)^4\right) \\
&\le C\left(\mathrm{E}\left(\exp\left\{4\int_0^T \sigma^2(Y_s^n)ds - 8\int_0^T \sigma(Y_s^n)dW_s\right\}\right.\right. \\
&\quad \left.\left. + \exp\left\{4\int_0^T \sigma^2(Y_s)ds - 8\int_0^T \sigma(Y_s)dW_s\right\}\right)\right)^{1/2} \\
&\quad \times\left[\mathrm{E}\left(\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s - \frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \int_0^T \sigma(Y_s)dW_s\right)^8\right]^{1/2} \\
&\le C\left[\mathrm{E}\left(\frac{1}{2}\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma(Y_s^n)dW_s - \frac{1}{2}\int_0^T \sigma^2(Y_s)ds + \int_0^T \sigma(Y_s)dW_s\right)^8\right]^{1/2} \\
&\le C\left[\mathrm{E}\left(\int_0^T \sigma^2(Y_s^n)ds - \int_0^T \sigma^2(Y_s)ds\right)^8 + \mathrm{E}\left(\int_0^T \sigma(Y_s^n)dW_s - \int_0^T \sigma(Y_s)dW_s\right)^8\right]^{1/2} \\
&\le C\left[T^7\mathrm{E}\left(\int_0^T \left(\sigma^2(Y_s^n) - \sigma^2(Y_s)\right)^8 ds\right) + C\mathrm{E}\left(\int_0^T \left(\sigma(Y_s^n) - \sigma(Y_s)\right)^2 ds\right)^4\right]^{1/2} \\
&= C\left[T^7\left(\int_0^T \mathrm{E}\left\{(\sigma(Y_s^n) - \sigma(Y_s))(\sigma(Y_s^n) + \sigma(Y_s))\right\}^8 ds\right)\right. \\
&\quad \left. + CT^3\mathrm{E}\left(\int_0^T \left(\sigma(Y_s^n) - \sigma(Y_s)\right)^8 ds\right)\right]^{1/2}
\end{aligned}
$$

$$\leq C\left[\mathrm{E}\left(\int_0^T\left(\mathrm{E}|Y_s-Y_s^n|^{16r}\mathrm{E}\left(C+|Y_s|^{16q}+|Y_s^n|^{16q}\right)\right)^{1/2}ds\right)\right.$$

$$\left.+\,\mathrm{E}\left(\int_0^T\left|Y_s-Y_s^n\right|^{8r}ds\right)\right]^{1/2}$$

$$\leq C\left[\int_0^T\left(\left(\mathrm{E}\left|Y_s-Y_s^n\right|^{16r}\right)^{\frac{1}{2}}+\mathrm{E}\left|Y_s-Y_s^n\right|^{8r}\right)ds\right]^{1/2}. \tag{39}$$

Applying (35) consequently with $\theta = 8$ and $\theta = 16$ we get that the last expression in (39) does not exceed $C\left(\frac{1}{n}\right)^{4rH}$, thus from (37) we obtain

$$I_1 \leq Cn^{-2rH}. \tag{40}$$

Now we continue with I_2 from (36):

$$I_2 \leq \left[\mathrm{E}(F(S_T)-F(S_T^n))^4\right]^{1/2}\left[\mathrm{E}(S_T^n)^{-4}\right]^{1/2}.$$

The second multiplier is bounded according to Remark 13, therefore it follows from condition (A), (i), that

$$I_2 \leq C\left[\mathrm{E}(F(S_T)-F(S_T^n))^4\right]^{1/2} = C\left[\mathrm{E}\left(\int_{S_T\wedge S_T^n}^{S_T\vee S_T^n} f(x)dx\right)^4\right]^{1/2}$$

$$\leq C(C_f)^2\left[\mathrm{E}\left(|S_T-S_T^n|^4(1+S_T^p+(S_T^n)^p)^4\right)\right]^{1/2} \leq C\left[\mathrm{E}|S_T-S_T^n|^8\mathrm{E}(1+S_T^p+(S_T^n)^p)^8\right]^{1/4}.$$

According to Lemma 2 and Remark 13,

$$\sup_{n\in\mathbb{N}}\mathrm{E}(1+S_T^p+(S_T^n)^p)^8 < \infty,$$

whence we get that

$$I_2 \leq C\left(\mathrm{E}|S_T-S_T^n|^8\right)^{1/4}.$$

To evaluate the right-hand side of this inequality, we can proceed as in the proof of (39) and the subsequent inequalities, because neither the opposite sign of the exponents nor the 8th power instead of the 4th lead to serious discrepancies in the estimations. Therefore we get

$$I_2 \leq C\left[\int_0^T\left(\left(\mathrm{E}\left|Y_s-Y_s^n\right|^{32r}\right)^{\frac{1}{2}}+\mathrm{E}\left|Y_s-Y_s^n\right|^{16r}\right)ds\right]^{\frac{1}{8}} \leq C\left(n^{-16rH}\right)^{1/8} = Cn^{-2rH}. \tag{41}$$

Bounds (40) and (41) complete the proof. □

Proof of Theorem 16. By Lemma 11 we can write

$$\left|\mathrm{E}f(X_T) - \mathrm{E}\left(\frac{F(S_T^n)}{S_T^n}\left(1+\frac{Z_T^n}{T}\right)\right)\right| = \mathrm{E}\left|\left(\frac{F(S_T)}{S_T}\left(1+\frac{Z_T}{T}\right)\right) - \left(\frac{F(S_T^n)}{S_T^n}\left(1+\frac{Z_T^n}{T}\right)\right)\right|$$
$$\leq \frac{1}{T}\mathrm{E}\left|\frac{F(S_T)}{S_T}\left(Z_T - Z_T^n\right)\right| + \mathrm{E}\left|\left(1+\frac{Z_T^n}{T}\right)\left(\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right)\right|$$
$$\leq \frac{1}{T}\left(\mathrm{E}\left(\frac{F(S_T)}{S_T}\right)^2 \mathrm{E}\left(Z_T - Z_T^n\right)^2\right)^{1/2} + \left(\mathrm{E}\left(\frac{F(S_T)}{S_T} - \frac{F(S_T^n)}{S_T^n}\right)^2 \mathrm{E}\left(1+\frac{Z_T^n}{T}\right)^2\right)^{1/2}.$$

According to Lemma 2, 3 and the Cauchy–Schwartz inequality, $\mathrm{E}\left(\frac{F(S_T)}{S_T}\right)^2 < \infty$. Obviously,

$$\sup_{n\geq 1} \mathrm{E}\left(Z_T^n\right)^2 < \frac{T}{\sigma_{min}^2}.$$

Now the proof follows from Lemma 14 and Lemma 15. □

Proof of Lemma 17. The proof immediately follows from the general formula for the density of a two dimensional Gaussian vector:

$$f(x_1, x_2) = \frac{(1-\varrho^2)^{-1/2}}{2\pi\sigma_1\sigma_2} \times \exp\left(-\frac{1}{2(1-\varrho^2)}\left[\frac{(x_1-m_1)^2}{\sigma_1^2} + \frac{(x_2-m_2)^2}{\sigma_2^2} - \frac{2\varrho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2}\right]\right) \quad (42)$$

with the mean vector and covariance matrix respectively

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \begin{pmatrix} \sigma_1^2 & \varrho\sigma_1\sigma_2 \\ \varrho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

In our case the covariance matrix equals

$$C_{X,Z} = \begin{pmatrix} \mu^2\sigma_Y^2 & \mu T \\ \mu T & \sigma_Z^2 \end{pmatrix},$$

the mean vector $(m_Y, 0) = (\log S_0 + bT - \frac{1}{2}\sigma_Y^2 + \rho\int_0^T \sigma(Y_s)dB_s, 0)$, and $\varrho = \frac{T}{\sigma_Y\sigma_Z}$. Now (22) follows immediately from (42). □

Proof of Theorem 18. Applying Lemma 11, equality (15), and Lemma 17, we get that

$$T\mathrm{E}g(X_T) = \mathrm{E}\left(G(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dV_u\right) = \mathrm{E}\left(\mathrm{E}\left(G(X_T)\int_0^T \frac{1}{\sigma(Y_u)}dV_u \middle| \mathcal{F}_T^H\}\right)\right)$$
$$= \mathrm{E}\left(\mathrm{E}\left(\int_{\mathbb{R}^2} G(x)zp_{X,Z}(x,z)dxdz \middle| \mathcal{F}_T^H\}\right)\right) = \mathrm{E}\int_{\mathbb{R}^2} G(x)zp_{X,Z}(x,z)dxdz$$
$$= \mathrm{E}\int_{\mathbb{R}} G(x)\left(\int_{\mathbb{R}} zp_{X,Z}(x,z)dz\right)dx. \quad (43)$$

The inner integral can be significantly simplified. Indeed, denote $\tilde{x} = x - m_Y$. Then

$$\begin{aligned}\int_{\mathbb{R}} z p_{X,Z}(x,z)dz &= \frac{1}{2\pi\Delta^{\frac{1}{2}}}\int_{\mathbb{R}} z\exp\Big\{-\frac{1}{2\Delta}\Big(\sigma_Z^2\tilde{x}^2+\mu^2\sigma_Y^2z^2-2\mu T\tilde{x}z\Big)\Big\}dz\\ &= \frac{1}{2\pi\Delta^{\frac{1}{2}}}\int_{\mathbb{R}} z\exp\Big\{-\frac{1}{2\Delta}\Big(\Big(\mu\sigma_Y z-\frac{T\tilde{x}}{\sigma_Y}\Big)^2-\frac{T^2\tilde{x}^2}{\sigma_Y^2}+\sigma_Z^2\tilde{x}^2\Big)\Big\}dz\\ &= \frac{1}{2\pi\Delta^{\frac{1}{2}}}\exp\Big\{-\frac{\tilde{x}^2}{2\Delta}\frac{\sigma_Y^2\sigma_Z^2-T^2}{\sigma_Y^2}\Big\}\int_{\mathbb{R}} z\exp\Big\{-\frac{1}{2\Delta}\Big(\mu\sigma_Y z-\frac{T\tilde{x}}{\sigma_Y}\Big)^2\Big\}dz\\ &= \frac{1}{2\pi\Delta^{\frac{1}{2}}}\exp\Big\{-\frac{\tilde{x}^2}{2\mu^2\sigma_Y^2}\Big\}\int_{\mathbb{R}} z\exp\Big\{-\Big(\frac{\mu\sigma_Y}{\sqrt{2}\Delta^{\frac{1}{2}}}z-\frac{T\tilde{x}}{\sqrt{2}\Delta^{\frac{1}{2}}\sigma_Y}\Big)^2\Big\}dz.\end{aligned}$$

Since

$$\int_{\mathbb{R}} xe^{-(ax-b)^2}dx = \frac{b}{a^2}\sqrt{\pi},$$

we obtain

$$\int_{\mathbb{R}} zp_{X,Z}(x,z)dz = \frac{T\tilde{x}}{\mu^2\sigma_Y^3\sqrt{2\pi}}\exp\Big\{-\frac{\tilde{x}^2}{2\mu^2\sigma_Y^2}\Big\}. \tag{44}$$

Combining (43) and (44), we get the proof. □

Proof of Theorem 19. To simplify notation, without loss of generality, let us assume that $X_0 + bT = 0$. Then, using (23), we get that

$$\begin{aligned}&\Big|\mathrm{E}g(X_T)-(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} G(x)\mathrm{E}\Big(\frac{(x-m_{Y,n})}{\mu^2\sigma_{Y,n}^3}\exp\Big\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma_{Y,n}^2}\Big\}\Big)dx\Big|\\ &= (2\pi)^{-\frac{1}{2}}\Big|\int_{\mathbb{R}} G(x)\mathrm{E}\Big(\frac{(x-m_Y)}{\mu^2\sigma_Y^3}\exp\Big\{-\frac{(x-m_Y)^2}{2\mu^2\sigma_Y^2}\Big\}\\ &\quad -\frac{(x-m_{Y,n})}{\mu^2\sigma_{Y,n}^3}\exp\Big\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma_{Y,n}^2}\Big\}\Big)dx\Big|\\ &\le (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} G(x)\Big[\mathrm{E}\Big(\frac{1}{\mu^2}\Big|\frac{x-m_Y}{\sigma_Y^3}-\frac{x-m_{Y,n}}{\sigma_{Y,n}^3}\Big|\exp\Big\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma_{Y,n}^2}\Big\}\Big)\\ &\quad +\mathrm{E}\Big|\frac{(x-m_Y)}{\mu^2\sigma_Y^3}\Big(\exp\Big\{-\frac{(x-m_Y)^2}{2\mu^2\sigma_Y^2}\Big\}-\exp\Big\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma_{Y,n}^2}\Big\}\Big)\Big|\Big]dx\\ &:= (2\pi)^{-\frac{1}{2}}\Big(\int_{\mathbb{R}} G(x)(J_1^n(x)+J_2^n(x))dx\Big).\end{aligned}$$

To bound $J_1^n(x)$ from above, denote

$$E_{exp}^n(x) = \Big(\mathrm{E}\exp\Big\{-\frac{(x-m_{Y,n})^2}{\mu^2\sigma_{Y,n}^2}\Big\}\Big)^{1/2},\ E_{exp}(x) = \Big(\mathrm{E}\exp\Big\{-\frac{(x-m_Y)^2}{\mu^2\sigma_Y^2}\Big\}\Big)^{1/2}.$$

By the Cauchy–Schwarz inequality,

$$J_1^n(x) \le \frac{1}{\mu^2}\mathrm{E}\left(\left|\frac{x-m_Y}{\sigma_Y^3} - \frac{x-m_{Y,n}}{\sigma_{Y,n}^3}\right|^2\right)^{1/2} E_{exp}^n(x). \tag{45}$$

Furthermore,

$$\begin{aligned}\frac{x-m_Y}{\sigma_Y^3} - \frac{x-m_{Y,n}}{\sigma_{Y,n}^3} &= \frac{1}{2}\sigma_Y^{-3}(\sigma_Y^2-\sigma_{Y,n}^2) + (x+\frac{1}{2}\sigma_{Y,n}^2)(\sigma_Y^{-3}-\sigma_{Y,n}^{-3})\\ &+ \rho\left(\int_0^T(\sigma(Y_s)-\sigma(Y_s^n))dB_s\right)(\sigma_Y^{-3}-\sigma_{Y,n}^{-3}).\end{aligned} \tag{46}$$

Since $|a_1^3-a_2^3| \le |a_1^2-a_2^2|(a_1+a_2)$, $a_1, a_2 > 0$, and also the lower bounds $\sigma_Y^2 \ge T\sigma_{\min}^2$, $\sigma_{Y,n}^2 \ge T\sigma_{\min}^2$ hold, we can conclude that

$$\begin{aligned}\left|\sigma_Y^{-3}-\sigma_{Y,n}^{-3}\right| &\le \left|\sigma_Y^{-2}-\sigma_{Y,n}^{-2}\right|\left(\sigma_{Y,n}^{-1}+\sigma_Y^{-1}\right) = \frac{\left|\sigma_Y^2-\sigma_{Y,n}^2\right|}{\sigma_Y^2\sigma_{Y,n}^2}\left(\sigma_{Y,n}^{-1}+\sigma_Y^{-1}\right)\\ &\le \frac{2\left|\sigma_Y^2-\sigma_{Y,n}^2\right|}{\sigma_{Y,n}^2 T^{\frac{3}{2}}\sigma_{\min}^3} \le C|\sigma_Y^2-\sigma_{Y,n}^2|.\end{aligned} \tag{47}$$

By the penultimate bound from (47) and since $\sigma_{Y,n}^2$ is bounded from below uniformly in n, we get

$$\begin{aligned}|(x+\frac{1}{2}\sigma_{Y,n}^2)(\sigma_Y^{-3}-\sigma_{Y,n}^{-3})| &\le \frac{(|x|+\frac{1}{2}\sigma_{Y,n}^2)|\sigma_Y^2-\sigma_{Y,n}^2|}{\sigma_{Y,n}^2 T^{\frac{3}{2}}\sigma_{\min}^3}\\ \le C\left(\frac{|x|}{\sigma_{Y,n}^2}+\frac{1}{2}\right)|\sigma_Y^2-\sigma_{Y,n}^2| &\le C(1+|x|)\left|\sigma_Y^2-\sigma_{Y,n}^2\right|.\end{aligned} \tag{48}$$

Summarizing (46)–(48) we get

$$\left|\frac{x-m_Y}{\sigma_Y^3} - \frac{x-m_{Y,n}}{\sigma_{Y,n}^3}\right| \le C\left(1+|x|+\int_0^T(\sigma(Y_s)-\sigma(Y_s^n))dB_s\right)|\sigma_Y^2-\sigma_{Y,n}^2|. \tag{49}$$

Furthermore, applying Burkholder–Davis–Gundy inequality and Remark 3 together with Lemma 12, we obtain

$$\begin{aligned}&\mathrm{E}\left(\left(\int_0^T(\sigma(Y_s)-\sigma(Y_s^n))dB_s\right)(\sigma_Y^2-\sigma_{Y,n}^2)\right)^2\\ &\le \left(\mathrm{E}\left(\int_0^T(\sigma(Y_s)-\sigma(Y_s^n))dB_s\right)^4 \mathrm{E}\left(\sigma_Y^2-\sigma_{Y,n}^2\right)^4\right)^{1/2}\\ &\le C\left(\mathrm{E}\left(\int_0^T(\sigma(Y_s)-\sigma(Y_s^n))^2ds\right)^2 \mathrm{E}\left(\sigma_Y^2-\sigma_{Y,n}^2\right)^4\right)^{1/2}\\ &\le C\left(\mathrm{E}\left(\sigma_Y^2-\sigma_{Y,n}^2\right)^4\right)^{1/2}.\end{aligned} \tag{50}$$

Finally, we get from (45) – (50) that

$$J_1^n(x) \le C(1+|x|)(\mathrm{E}(\sigma_{Y,n}^2-\sigma_Y^2)^4)^{1/4}E_{exp}^n(x). \tag{51}$$

Similarly to (33) and (39), by applying condition (**B**), Lemma 12, (iii) and (iv), together with the standard Hölder's inequality, we get

$$\begin{aligned}\mathrm{E}(\sigma_{Y,n}^2-\sigma_Y^2)^4 &= \mathrm{E}\left(\int_0^T(\sigma^2(Y_s^n)-\sigma^2(Y_s))ds\right)^4 \le C\mathrm{E}\int_0^T(\sigma^2(Y_s^n)-\sigma^2(Y_s))^4ds\\ &\le C_\sigma C\int_0^T\left[\mathrm{E}(Y_s^n-Y_s)^{8r}\mathrm{E}(\sigma(Y_s^n)+\sigma(Y_s))^8\right]^{1/2}ds\\ &\le C\int_0^T\left[\mathrm{E}(Y_s^n-Y_s)^{8r}\right]^{1/2}ds \le Cn^{-4rH}.\end{aligned} \tag{52}$$

Combining inequality (52) with (51) we get

$$J_1^n(x) \le Cn^{-rH}(1+|x|)E_{exp}^n(x),$$

and consequently

$$\int_{\mathbb{R}} G(x)J_1^n(x)dx \le Cn^{-rH}\int_{\mathbb{R}} G(x)(1+|x|)E_{exp}^n(x)dx. \tag{53}$$

Let us show that the integral in the right–hand side of (53) is bounded in $n \ge 1$. In this connection, denote $\varkappa = T\mu^2\sigma_{\min}^2$. Applying the standard Hölder inequality together with polynomial growth of $G(x)$ and relations (18) from Remark 13, we get that

$$\begin{aligned}&\int_{\mathbb{R}} G(x)(1+|x|)E_{exp}^n(x)dx\\ &\le \left(\int_{\mathbb{R}} G^2(x)(1+|x|)^2e^{-(2p+1)|x|}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} e^{(2p+1)|x|}\mathrm{E}\exp\left\{-\frac{(x-m_{Y,n})^2}{\mu^2\sigma_{Y,n}^2}\right\}dx\right)^{1/2}\\ &\le C\left(\mathrm{E}\int_{\mathbb{R}} e^{(2p+1)|x+m_{Y,n}|}\exp\left\{-\frac{x^2}{\varkappa}\right\}dx\right)^{1/2} \le C\left(\mathrm{E}\int_{\mathbb{R}} e^{(2p+1)(x+m_{Y,n})}\exp\left\{-\frac{x^2}{\varkappa}\right\}dx\right)^{1/2}\\ &+C\left(\mathrm{E}\int_{\mathbb{R}} e^{-(2p+1)(x+m_{Y,n})}\exp\left\{-\frac{x^2}{\varkappa}\right\}dx\right)^{1/2} = C\left(\mathrm{E}e^{(2p+1)m_{Y,n}}\int_{\mathbb{R}} e^{(2p+1)x}\exp\left\{-\frac{x^2}{\varkappa}\right\}dx\right)^{1/2}\\ &+C\left(\mathrm{E}e^{-(2p+1)m_{Y,n}}\int_{\mathbb{R}} e^{-(2p+1)x}\exp\left\{-\frac{x^2}{\varkappa}\right\}dx\right)^{1/2} \le C.\end{aligned} \tag{54}$$

Construction of an upper bound for $J_2^n(x)$ is similar. Indeed,

$$|\exp\{-u^2\}-\exp\{-v^2\}| \le (\exp\{-u^2\}+\exp\{-v^2\})|u-v|(|u|+|v|). \tag{55}$$

In our case

$$u = \frac{x-m_Y}{\sqrt{2}\mu\sigma_Y},\ \ v = \frac{x-m_{Y_n}}{\sqrt{2}\mu\sigma_{Y_n}}.$$

Note that

$$\mathrm{E}|u|^2 \le C\mathrm{E}\left(|x| + \sigma_Y + \left|\int_0^T \sigma(Y_s)dB_s\right|\right)^2 \le C(1+|x|)^2,$$

and the moments of higher order of u and v can be bounded similarly. For $|u^2 - v^2|$, in the same way as in (46), (47) and (48) we have

$$|u - v| \le C\left(1 + |x| + \left|\int_0^T \sigma(Y_s^n)dB_s\right|\right)|\sigma_Y - \sigma_{Y,n}| + C\left|\int_0^T (\sigma(Y_s) - \sigma(Y_s^n))dB_s\right|. \tag{56}$$

Additionally,

$$|\sigma_Y - \sigma_{Y,n}| \le \frac{|\sigma_Y^2 - \sigma_{Y,n}^2|}{\sigma_Y + \sigma_{Y,n}} \le C|\sigma_Y^2 - \sigma_{Y,n}^2|.$$

Hence, applying Hölder, Minkowski, and Burkholder–Davis–Gundy inequalities together with condition (**B**), (iii), we get from (56)

$$\begin{aligned}(\mathrm{E}|u - v|^8)^{1/8} &\le C(1+|x|)(\mathrm{E}(\sigma_Y^2 - \sigma_{Y,n}^2)^8)^{1/8} \\ &\quad + \left(\mathrm{E}\left(\int_0^T \sigma(Y_s^n)dB_s\right)^{16}\right)^{1/16} (\mathrm{E}(\sigma_Y^2 - \sigma_{Y,n}^2)^{16})^{1/16} \\ &\quad + C\left(\mathrm{E}\left(\int_0^T (\sigma(Y_s) - \sigma(Y_s^n))dB_s\right)^8\right)^{1/8} \\ &\le C(1+|x|)(\mathrm{E}(\sigma_Y^2 - \sigma_{Y,n}^2)^{16})^{1/16} \\ &\quad + C\left(\mathrm{E}\left(\int_0^T (\sigma(Y_s) - \sigma(Y_s^n))^2 ds\right)^4\right)^{1/8}. \end{aligned} \tag{57}$$

Proceeding as in (52), we get

$$(\mathrm{E}(\sigma_Y^2 - \sigma_{Y,n}^2)^{16})^{1/16} \le \left(\mathrm{E}\left(\int_0^T (\sigma^2(Y_s) - \sigma^2(Y_s^n))ds\right)^8\right)^{1/16} \le Cn^{-rH},$$

and

$$\left(\mathrm{E}\left(\int_0^T (\sigma(Y_s) - \sigma(Y_s^n))^2 ds\right)^4\right)^{1/8} \le Cn^{-rH}.$$

Now we continue, preserving for the moment the notations u and v and taking into account (55)–(57):

$$\begin{aligned} J_2^n(x) &= \mathrm{E}\left(\left|\frac{u}{\sigma_Y}\right| |\exp\{-u^2\} - \exp\{-v^2\}|\right) \le \varkappa^{-1}\left(\mathrm{E}|u|^2 \mathrm{E}\left(|\exp\{-u^2\} - \exp\{-v^2\}|\right)^2\right)^{1/2} \\ &\le C(1+|x|)\left(\mathrm{E}\left(|\exp\{-u^2\} - \exp\{-v^2\}|\right)^2\right)^{1/2} \\ &\le C(1+|x|)\left(\mathrm{E}\left(|\exp\{-u^2\} + \exp\{-v^2\}|\right)^4\right)^{1/4} \left(\mathrm{E}\left(|u - v|\right)^8\right)^{1/8} \left(\mathrm{E}\left(|u| + |v|\right)^8\right)^{1/8} \\ &\le C(1+|x|)^3 n^{-rH}\left(\mathrm{E}\left(|\exp\{-u^2\} + \exp\{-v^2\}|\right)^4\right)^{1/4}. \end{aligned} \tag{58}$$

Together with (58) this implies that

$$\int_{\mathbb{R}} G(x)J_2(x)dx \leq Cn^{-rH}\int_{\mathbb{R}}(1+|x|)^3 G(x) \times\left(\mathrm{E}\left(\exp\left\{-\frac{(x-m_Y)^2}{2\mu^2\sigma_Y^2}\right\}+\exp\left\{-\frac{(x-m_{Y,n})^2}{2\mu^2\sigma_{Y,n}^2}\right\}\right)\right)^4\Bigg)^{\frac{1}{4}}dx. \quad (59)$$

The fact that the integral in the right-hand side of (59) is bounded in n, can be established via the same approach as applied to the integral $\int_{\mathbb{R}} G(x)(1+|x|)E^n_{exp}(x)dx$ in (54). □

Proof of Lemma 20. For any $\delta > 0$ choose $\varepsilon = \varepsilon(\delta)$ in such a way that $\varepsilon(2+\alpha|Y_0|) < \delta$. Then we get from the representation (29) that for $0 \leq t \leq \tau_\varepsilon \wedge \varepsilon$

$$\begin{aligned}|Y_t - Y_0| &\leq |Y_0|(1-e^{-\alpha t}) + |B_t^H| + \alpha e^{-\alpha t}\int_0^t e^{\alpha s}|B_s^H|ds\\ &\leq |Y_0|\alpha\varepsilon + \varepsilon + \varepsilon e^{-\alpha t}(e^{\alpha t}-1) \leq \varepsilon(2+\alpha|Y_0|) < \delta.\end{aligned}$$

Therefore for $\varepsilon < \frac{\delta}{2+\alpha|Y_0|}$ we have that $\nu_\delta > \tau_\varepsilon \wedge \varepsilon$. So, it is sufficient to prove that for any $\varepsilon > 0$ and any $l > 0$

$$\mathrm{E}(\tau_\varepsilon \wedge \varepsilon)^{-l} < \infty. \quad (60)$$

Now, for $v < \varepsilon$

$$\mathrm{P}\{\tau_\varepsilon \wedge \varepsilon < v\} = \mathrm{P}\{\tau_\varepsilon < v\} = \mathrm{P}\{\sup_{0\leq t\leq v}|B_t^H| \geq \varepsilon\}.$$

Furthermore, it follows from the self-similarity and symmetry of the fBm that $\mathrm{P}\{\sup_{0\leq t\leq v}|B_t^H| \geq \varepsilon\} \leq 2\mathrm{P}\{\sup_{0\leq t\leq 1} B_t^H \geq \frac{\varepsilon}{v^H}\}$. Let us denote $\vartheta = \mathrm{E}\sup_{0\leq t\leq 1} B_t^H$. Then, according to inequality (2.2) from Talagrand (1994), for $v < \varepsilon$ such that additionally $\frac{\varepsilon}{v^H} > \vartheta$ we have

$$\mathrm{P}\{\sup_{0\leq t\leq v} B_t^H \geq \frac{\varepsilon}{v^H}\} \leq \exp\left\{-\frac{\left(\frac{\varepsilon}{v^H}-\vartheta\right)^2}{2}\right\} = \exp\left\{-\frac{(\varepsilon-\vartheta v^H)^2}{2v^{2H}}\right\},$$

and (60) follows since

$$\mathrm{E}(\tau_\varepsilon\wedge\varepsilon)^{-l} = \int_0^\infty \mathrm{P}\{(\tau_\varepsilon\wedge\varepsilon)^{-l} > u\}du = \int_0^\infty \mathrm{P}\{\tau_\varepsilon\wedge\varepsilon < \frac{1}{u}\}du = \int_0^\infty \frac{1}{v^2}\mathrm{P}\{\tau_\varepsilon\wedge\varepsilon < v\}du < \infty.$$

□

Remark 24 Exponential bounds for the distribution of τ_ε allow to prove that $\mathrm{E}(\tau_\varepsilon \wedge \varepsilon \wedge a)^{-l} < \infty$ for any $a, l > 0$.

Proof of Lemma 21. As it follows from Proposition 2.1.1 and Exercise 2.1.1 in Nualart (2006), it is sufficient to show that

$$\sigma_Y^2 \in \mathrm{D}^{2,4} \quad (61)$$

and that

$$\mathrm{E}\left(||D^B\sigma_Y^2||_H\right)^{-8} < \infty. \quad (62)$$

Recall that $\kappa(x) = \sigma(x)\sigma'(x)$. It follows from conditions (**B**) and (**E**) that κ and κ' are functions of polynomial growth, $\kappa(x) > 0$. Recall the notation $l(u,s) =$

$c_H e^{-\alpha s} \int\limits_u^s e^{\alpha v} v^{H-1/2}(v-u)^{H-3/2} dv$. Taking into account (30) and (10), we write the stochastic derivative as

$$\begin{aligned} D_u^B(\sigma_Y^2) &= D_u^B(\int\limits_0^T \sigma^2(Y_s)ds) = 2\int\limits_0^T \kappa(Y_s) D_u^B Y_s ds \\ &= 2c_H u^{1/2-H} \int\limits_u^T \kappa(Y_s) e^{-\alpha s} \int\limits_u^s e^{\alpha v} v^{H-1/2}(v-u)^{H-3/2} dv ds \\ &= 2u^{1/2-H} \int\limits_u^T \kappa(Y_s) l(u,s) ds. \end{aligned}$$

Therefore, the iterated derivative equals

$$D_z^B(D_u^B(\sigma_Y^2)) = 2u^{1/2-H} z^{1/2-H} \int\limits_{u\vee z}^T \kappa'(Y_s) l(z,s) l(u,s) ds. \tag{63}$$

Since the right-hand side of (63) is in $H \otimes H$ and the corresponding integral has moments of any order due to polynomial growth of κ', (61) follows.

To prove (62), note that

$$D_u^B(\sigma_Y^2) \geq C \int\limits_u^T \kappa(Y_s)(s-u)^{H-1/2} ds,$$

whence

$$||D^B \sigma_Y^2||_H^2 = \int\limits_0^T \left(D_u^B \sigma_Y^2\right)^2 du \geq C \int\limits_0^T du \left(\int\limits_u^T \kappa(Y_s)(s-u)^{H-1/2} ds\right)^2.$$

Now, let $\sigma'(Y_0) = \sigma_0 > 0$. Choose $\delta > 0$ so that for $y \in [Y_0 - \delta, Y_0 + \delta]$ we have $\sigma'(y) > \frac{\sigma_0}{2}$. Then choose $\varepsilon = \varepsilon(\delta)$ as in the proof of Lemma 20, and take $\zeta = \tau_\varepsilon \wedge \varepsilon \wedge \frac{T}{2}$.

Then

$$\begin{aligned} \int\limits_0^T du \left(\int\limits_u^T \kappa(Y_s)(s-u)^{H-1/2} ds\right)^2 &\geq C \int\limits_0^{\frac{1}{3}\zeta} du \left(\int\limits_{\frac{2}{3}\zeta}^{\zeta} \kappa(Y_s)(s-u)^{H-1/2} ds\right)^2 \\ &\geq C \int\limits_0^{\frac{1}{3}\zeta} du \left(\int\limits_{\frac{2}{3}\zeta}^{\zeta} \sigma_{\min}\sigma_0 \left(\frac{1}{3}\zeta\right)^{H-1/2} ds\right)^2 = C\zeta^{2+2H}. \end{aligned}$$

It follows immediately from Lemma 20 and 24 that

$$\mathrm{E}\left(||D^B \sigma_Y^2||_H\right)^{-8} \leq C\mathrm{E}\zeta^{-8-8H} \leq C\mathrm{E}\left(\tau_\varepsilon \wedge \varepsilon \wedge \frac{T}{2}\right)^{-8-8H} < \infty.$$

□

Proof of Theorem 22. From Lemma 21 and Proposition 2.1.1, Nualart (2006) we get the first part of equality (25):

$$p_{\sigma_Y^2}(u) = \mathrm{E}\left[\mathbb{1}_{\sigma_Y^2>u}\delta\left(\frac{D^B\sigma_Y^2}{||D\sigma_Y^2||_H^2}\right)\right].$$

To get the second part, note that $\eta := \left(||D\sigma_Y^2||_H\right)^{-2}$ admits stochastic derivative and, according to Proposition 1.3.3 from Nualart (2006), the following holds

$$\delta\left(\frac{D^B\sigma_Y^2}{||D^B\sigma_Y^2||_H^2}\right) = \int_0^T \eta D_u^B(\sigma_Y^2)dB_u = \eta\int_0^T D_u^B(\sigma_Y^2)dB_u$$
$$-\int_0^T D_u^B\eta D_u^B(\sigma_Y^2)du = 2\eta\int_0^T u^{1/2-H}\int_u^T \kappa(Y_s)l(u,s)dsdB_u$$
$$-\int_0^T D_u^B\eta D_u^B(\sigma_Y^2)du.$$

According to Lemma 2.10 from León and Nualart (1998), we can apply the Fubini theorem for the Skorokhod integral. We have

$$\int_0^T u^{1/2-H}\int_u^T \kappa(Y_s)l(u,s)dsdB_u = \int_0^T \kappa(Y_s)(\int_0^s u^{1/2-H}l(u,s)dB_u)ds,$$

where the interior integral is a Wiener one.

Finally, taking into account that $m_Y = X_0 + bT - \frac{1}{2}\sigma_Y^2$, we get

$$\mathrm{E}\frac{(x-m_Y)}{\sigma_Y^3\sqrt{2\pi}}\exp\left\{-\frac{(x-m_Y)^2}{\sigma_Y^2}\right\}$$
$$= \int_{\mathbb{R}} \frac{(x+u/2-X_0-bT)}{u^3\sqrt{2\pi}}\exp\left\{-\frac{(x+u/2-X_0-bT)^2}{u^2}\right\}p_{\sigma_Y^2}(u)du.$$

Combining this with (23), we get the proof. □

8 Simulations

In this section we use the discretization schemes proposed in 4 and 5 to simulate the option price. We treat double and the single discretization, respectively, and we also compare it to the direct Monte Carlo average. Note that in Theorem 16 and 19 we prove convergence of the expectation, in other words, we prove that the bias is going to zero as the partition size n increases. The tables below suggest that the variance of the estimate expectedly remain on about the same level.

The values of b, α and T are the same in all simulations, and equal $b = 0.2, \alpha = 0.6, T = 1$.

We present here the results of simulations for different functions σ and f. Tables from 1 to 6 demonstrate results of simulations for different n. All numbers have 6 digits after the decimal point. The tables include the outcomes of computations based on (14) (the double discretization, 'dd' in the tables), (23) (the single discretization based on conditioning, 'sd' in the tables), and the direct averaging ('da' in the tables). The mean squared errors are also included. For the single discretization the values in the mean squared errors column (column

Table 1 $f(s) = (s-1)_+ + \mathbb{1}_{s>1}, \sigma(y) = \sqrt{|y| + 0.55}$, $H = 0.6$

n	dd	dd_error	sd	sd_error	da	da_error
100	0.951019	14.363830	0.956867	0.008173	0.956300	6.128996
300	0.973156	15.927797	0.956734	0.007935	0.965695	6.652094
900	0.982417	21.924333	0.956843	0.007999	0.970280	7.991714
2700	0.935197	19.155798	0.957068	0.007875	0.938319	7.233949
8100	1.006177	19.731601	0.956925	0.007998	0.987641	7.904191
24300	0.953743	16.835284	0.957012	0.007977	0.949628	6.776577

5) are obtained by taking same expression as in (23) and replace the mean of the sample by the variance of the sample.

In Tables 1 and 2 f has a linear growth and σ takes moderate values (note that the larger values $\sigma(Y_t)$ takes, the higher the variance of S_T is going to be). With $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, G is given by $G(x) = \mathbb{1}_{x>0}(\exp(x) - 1)$. We see that the single discretization performs better than the other methods considered here, while the double discretization results in a higher mean squared error than the direct averaging.

In Table 3 the results of the single discretization computations for various functions σ are given. Note that the choices of σ result in $\sigma(Y_t)$ taking higher values (on average) and therefore in a high variance for S_t. Because of that, both the double discretization and the direct averaging do not work well under these circumstances. It is therefore noteworthy to point out that the single discretization shows a very robust performance here.

In Tables 4 and 5 σ is as in Tables 1 and 2, respectively, while f is an indicator of an interval. With $f(s) = \mathbb{1}_{s\in[1,2]}$ we have $G(x) = \max(0, \min(x, \log(2)))$. The single discretization again performs better here. Unlike previously, using double discretization results in a smaller mean squared error than the direct averaging.

Table 6 confirms a good overall performance of the single discretization, and that the single discretization works well in settings where the direct averaging and the double discretization would require a huge number of trials.

Table 7 shows the results of simulations for when f grows faster than linearly: $f(s) = ((s-1)_+)^{3/2} + \mathbb{1}_{s>1}$. In this case

$$G(x) = \begin{cases} x + \int_0^x (e^z - 1)^{3/2} dz, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{64}$$

Single discretization method again shows a robust performance here, while double discretization and direct averaging have high variance.

Table 2 $f(s) = (s-1)_+ + \mathbb{1}_{s>1}, \sigma(y) = \cos(8y) + 1.2$, $H = 0.6$

n	dd	dd_error	sd	sd_error	da	da_error
100	1.005087	39.331162	0.973939	0.010322	0.987789	11.421569
300	0.967322	24.129696	0.973903	0.010282	0.972769	8.368354
900	1.014731	38.291297	0.973755	0.010448	0.984578	10.110878
2700	0.973777	26.510487	0.973658	0.010284	0.969705	9.600445
8100	0.967524	26.988813	0.973172	0.010471	0.963961	9.561456
24300	0.954076	24.880125	0.973564	0.010226	0.969705	9.872354

Table 3 The single discretization only, $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $H = 0.4$

n	$\sigma(y) = 2\|y\| + 2.2$	error	$\sigma(y) = 5\sin(y) + 5.05$	error	$\sigma(y) = 10\sqrt{y^2+1}$	error
100	1.196685	0.000532	1.221111	0.000004	1.219343	0.000000
300	1.197129	0.000511	1.221111	0.000006	1.219582	0.000000
900	1.197123	0.000508	1.221124	0.000005	1.219601	0.000000
2700	1.197186	0.000508	1.221130	0.000004	1.219290	0.000000
8100	1.197294	0.000510	1.221140	0.000004	1.219335	0.000000
24300	1.197131	0.000512	1.221117	0.000005	1.219385	0.000000

Table 4 $f(s) = \mathbb{1}_{s\in[1,2]}, \sigma(y) = \sqrt{|y| + 0.55}$, $H = 0.8$

n	dd	dd_error	sd	sd_error	da	da_error
100	0.173459	0.085342	0.172270	0.003021	0.171200	0.141891
300	0.170377	0.083396	0.171784	0.002989	0.171875	0.142334
900	0.172513	0.084694	0.172209	0.002994	0.171275	0.141940
2700	0.170567	0.083248	0.171828	0.002964	0.174650	0.144147
8100	0.172327	0.084907	0.171824	0.002967	0.168925	0.140389
24300	0.172714	0.084648	0.171822	0.002933	0.170300	0.141298

Table 5 $f(s) = \mathbb{1}_{s\in[1,2]}, \sigma(y) = \cos(8y) + 1.2$, $H = 0.8$

n	dd	dd_error	sd	sd_error	da	da_error
100	0.162822	0.191493	0.163004	0.018580	0.164850	0.137674
300	0.167634	0.199874	0.163181	0.018253	0.162775	0.136279
900	0.165612	0.196169	0.163669	0.018401	0.163775	0.136953
2700	0.163298	0.193913	0.164235	0.019251	0.166050	0.138477
8100	0.164814	0.193648	0.164236	0.018682	0.165650	0.138210
24300	0.162855	0.191648	0.164496	0.018850	0.165525	0.138126

Table 6 $f(s) = \mathbb{1}_{s\in[1,2]}, \sigma(y) = 2|y| + 2.2$, $H = 0.8$

n	dd	dd_error	sd	sd_error	da	da_error
100	0.009435	0.004724	0.009434	0.000074	0.009125	0.009042
300	0.008986	0.004406	0.009259	0.000072	0.009025	0.008944
900	0.009631	0.004674	0.009296	0.000072	0.009525	0.009434
2700	0.009373	0.004600	0.009241	0.000070	0.009300	0.009214
8100	0.009703	0.004775	0.009247	0.000071	0.009250	0.009164
24300	0.009217	0.004554	0.009206	0.000071	0.009650	0.009557

Table 7 The single discretization only, $f(s) = ((s-1)_+)^{3/2} + \mathbb{1}_{s>1}$, $H = 0.7$

n	$\sigma(y) = \sqrt{(\lvert y\rvert + 0.55)}$	error	$\sigma(y) = \cos(8y) + 1.2$	Error
100	1.914334	0.009626	2.326036	0.021973
300	1.923092	0.009487	2.327452	0.022298
900	1.919237	0.009389	2.321790	0.022594
2700	1.917376	0.009398	2.318913	0.022438
8100	1.920846	0.009454	2.313767	0.022488
24300	1.921878	0.009446	2.317305	0.022312

The simulations are done in R. For each estimate 40000 trials were used. To simulate the fBm, we used a modified version of the a function from the package dvfBm R kindly provided by J.-F. Coeurjolly in private conversation. For simulations related to the double discretization we take the average of the value under the expectation in the right hand side of (14) over $4 * 10^4$ trials. For simulations related to the single discretization, we replace infinite interval of integration in the right hand side of (23) with a finite one, making sure that the integral over the complement is small. To be more precise, assume, for technical simplicity, that $f(y) \leq y^k$ for some $k \in \mathbb{N}$. Then, for example, the right-hand tail of the integral in (23) can be bounded from above in the following way:

$$
\begin{aligned}
&(2\pi)^{-\frac{1}{2}}\mathrm{E}\left((\sigma_Y)^{-1}\int_A^{\infty} G((x+m_Y)\sigma_Y)\frac{x}{\mu^2}e^{-\frac{x^2}{2\mu^2}}dx\right)\\
&\leq (2\pi)^{-\frac{1}{2}}\mathrm{E}\left((\sigma_Y)^{-1}\int_A^{\infty} e^{k(x+m_Y)\sigma_Y}e^{\frac{x}{\mu^2}-\frac{-Ax}{4\mu^2}}dx\right)\\
&\leq (2\pi)^{-\frac{1}{2}}\mathrm{E}\left((\sigma_Y)^{-1}e^{km_Y\sigma_Y}\left(\frac{A}{4\mu^2}-\mu^{-2}-k\sigma_y\right)^{-1}\right).
\end{aligned}
$$

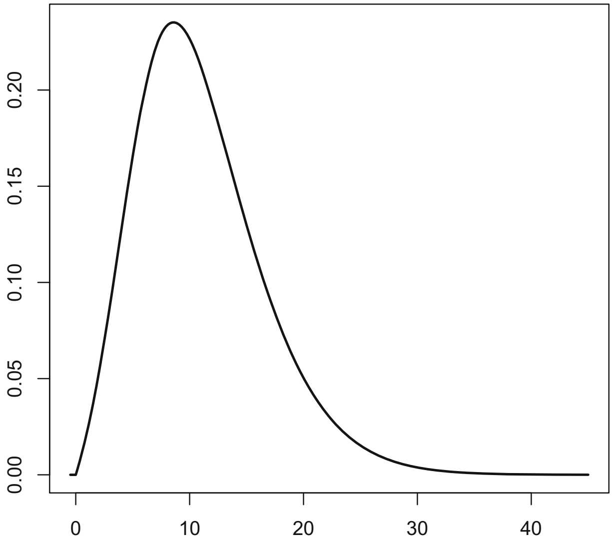

Fig. 1 The picture shows the plot of the integrand in (24), $n = 900$, $H = 0.6$, $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $\sigma(y) = 2|y| + 2.2$, $G(x) = \mathbb{1}_{x>0}(\exp(x) - 1)$

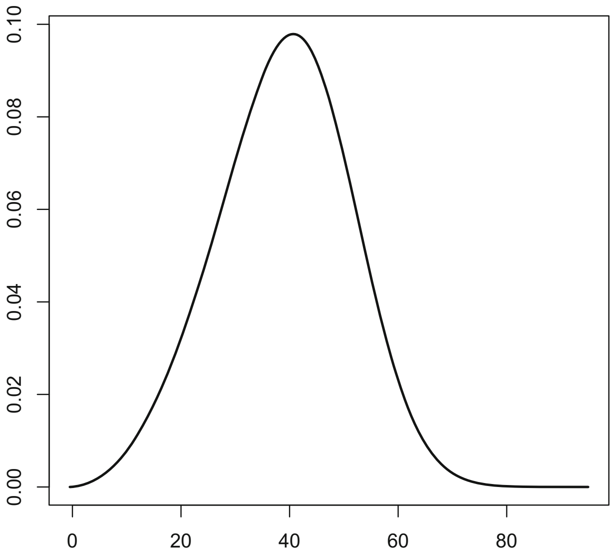

Fig. 2 The picture shows the plot of the integrand in (24), $n = 24100$, $H = 0.4$, $f(s) = (s-1)_+ + \mathbb{1}_{s>1}$, $\sigma(y) = 5\sin(y) + 5.05$, $G(x) = \mathbb{1}_{x>0}(\exp(x) - 1)$

We discretize the finite interval; the partition size varies, but is at least 2000. For each x from the partition, we take the average over the same $4 * 10^4$ trials of the value under the expectation in (23).

Figures 1, 2, and 3 display the integrand in (24) from various simulations.

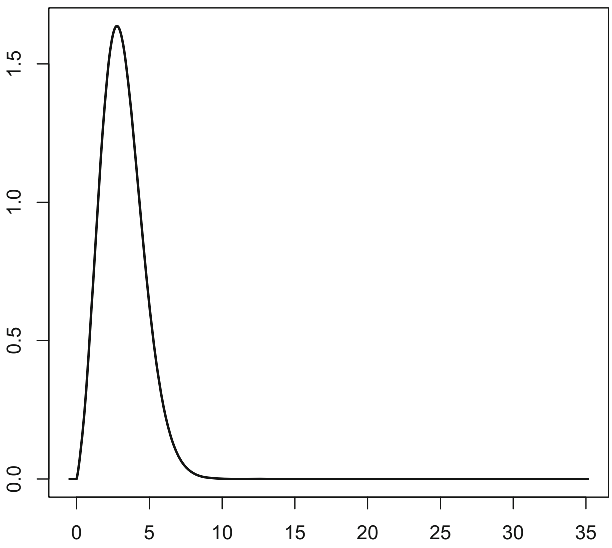

Fig. 3 The picture shows the plot of the integrand in (24), $n = 24100$, $H = 0.7$, $f(s) = \mathbb{1}_{s\in[1,2]}$, $\sigma(y) = \cos(8y) + 1.2$, $G(x) = \max(0, \min(x, \log(2)))$

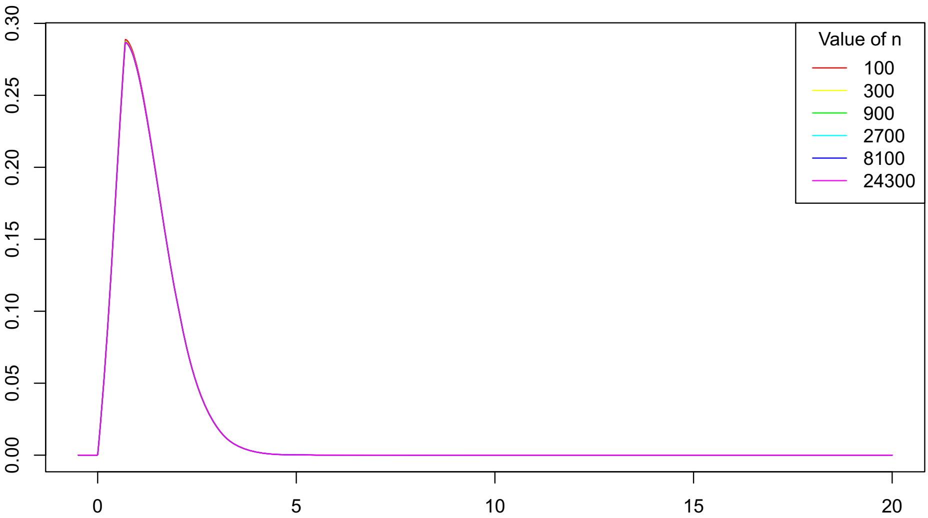

Fig. 4 The picture shows the plot of the integrands in (24) for different n, $H = 0.8$, $f(s) = ((s-1)_+)^{3/2} + \mathbb{1}_{s>1}$, $\sigma(y) = \sqrt{(|y| + 0.55)}$, G is given by (64). Since the functions are very close to each other, their plots overlap

The above pictures give plot for a single choice of n. It's worth pointing out that the resulting function changes very little with n as can be seen on Fig. 4. Each curve almost coincides with previous ones, so the curve for $n = 24300$ almost completely conceals the plots for the other values of n. In fact the absolute value of the difference between each two functions does not exceed 0.0023 in our simulations. This pattern is also observed for other choices of f and σ considered in this section.

As a conclusion, the single method discretization allows to achieve a higher precision. It keeps showing consistent results and a relatively small error in situations where the other considered methods suffer from high variance.

References

Alòs E (2012) A decomposition formula for option prices in the Heston model and applications to option pricing approximation. Finance Stoch 16(3):403–422

Altmayer M, Neuenkirch A (2015) Multilevel MOnte Carlo quadrature of discontinuous payoffs in the generalized Heston model using Malliavin integration by parts. SIAM J Financial Math 6(1):22–52

Barndorff-Nielsen OE, Shephard N (2001) Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. J R Stat Soc Ser B Stat Methodol 63(2):167–241

Barndorff-Nielsen OE, Shephard N (2002) Econometric analysis of realized volatility and its use in estimating stochastic volatility models. J R Stat Soc Ser B Stat Methodol 64(2):253–280

Bayer C, Friz P, Gatheral J (2016) Pricing under rough volatility. Quant Finan 16(6):887–904

Bergomi L, Guyon J (2012) Stochastic volatility's orderly smiles. Risk 25(5):60–66

Bertholon H, Monfort A, Pegoraro F (2007) Pricing and inference with mixtures of conditionally normal processes. Direction Générale des études et des relations internationales, Banque de France, pp 1–59

Bollerslev T, Mikkelsen HO (1996) Modeling and pricing long memory in stock market volatility. J Econ 73(1):151–184

Carrasco M, Chen X (2002) Mixing and moment properties of various GARCH and stochastic volatility models. Economet Theor 18(1):17–39

Cheridito P, Kawaguchi H, Maejima M (2003) ORnstein-Uhlenbeck processes. Electron J Probab 8(3):14 (electronic)
Chronopoulou A, Viens FG (2012) Estimation and pricing under long-memory stochastic volatility. Ann Finance 8(2-3):379–403
Comte F, Coutin L, Renault E (2012) Affine fractional stochastic volatility models. Ann Finance 8(2-3):337–378
Cont R, Tankov P (2004) Financial modelling with jump processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton
Fernique X (1975) Regularité des trajectoires des fonctions aléatoires gaussiennes. In: École d'Été de Probabilités de Saint-Flour, IV-1974, vol 480. Springer, Berlin, pp 1–96. Lecture Notes in Math
Fouque J-P, Papanicolaou G, Sircar KR (2000) Derivatives in financial markets with stochastic volatility. Cambridge University Press, Cambridge
Funahashi H, Kijima M (2017) Does the Hurst index matter for option prices under fractional volatility. Ann Finance 13(1):55–74
Heston SL (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev Financ Stud 6(2):327–343
Hull JC, White A (1987) The pricing of options on assets with stochastic volatilities. J Financ 42(2):281–300
Kahl C (2008) Modelling and simulation of stochastic volatility in finance. Universal-Publishers
Knight J, Satchell S (eds) (2011) Forecasting volatility in the financial markets, 3rd edn. Butterworth-Heinemann, Oxford
Kuchuk-Iatsenko S, Mishura Y (2015) Pricing the E,uropean call option in the model with stochastic volatility driven by Ornstein-Uhlenbeck process. Exact formulas Mod Stoch Theory Appl 2(3):233–249
Kyprianou AE, Schoutens W (eds) (2005) Exotic option pricing and advanced Lévy models. Wiley, Chichester
Ledoux M (1996) Isoperimetry and gaussian analysis. In: Lectures on probability theory and statistics, pp 165–294. Springer
León JA, Nualart D (1998) Stochastic evolution equations with random generators. Ann Probab 26(1):149–186
Nicolato E, Venardos E (2003) Option pricing in stochastic volatility models of the O,rnstein-Uhlenbeck type. Math Financ 13(4):445–466
Norros I, Valkeila E, Virtamo J (1999) An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5(4):571–587
Nualart D (2006) The Malliavin calculus and related topics. Probability and its Applications (New York), 2nd edn. Springer, Berlin
Palm FC (1996) 7 GARCH models of volatility. Handbook of statistics 14:209–240
Pospíšil J, Sobotka T (2016) Market calibration under a long memory stochastic volatility model. Appl Math Finance 23(5):323–343
Schobel R, Zhu J (1999) Stochastic volatility with an Ornstein–Uhlenbeck process: an extension. Ann Finance 3(1):23–46
Schweizer M (1995) On the minimal martingale measure and the Föllmer-Schweizer decomposition. Stochastic Anal Appl 13(5):573–599
Shephard N (1996) Statistical aspects of ARCH and stochastic volatility. Monographs on Statistics and Applied Probability 65:1–68
Talagrand M (1994) Sharper bounds for Gaussian and empirical processes. Ann Probab 22(1):28–76