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REDUCTION OF THE BERGE-FULKERSON CONJECTURE TO CYCLICALLY 5-EDGE-CONNECTED SNARKS

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ABSTRACT. The Berge-Fulkerson conjecture, originally formulated in the language of mathematical programming, asserts that the edges of every bridgeless cubic (3-valent) graph can be covered with six perfect matchings in such a way that every edge is in exactly two of them. As with several other classical conjectures in graph theory, every counterexample to the Berge-Fulkerson conjecture must be a non 3-edge-colorable cubic graph. In contrast to Tutte's 5-flow conjecture and the cycle double conjecture, no nontrivial reduction is known for the Berge-Fulkerson conjecture. In the present paper, we prove that a possible minimum counterexample to the conjecture must be cyclically 5-edge-connected.

1. Introduction

A well known result of Petersen [16], proven as early as in 1891, states that every bridgeless cubic graph has a perfect matching. Somewhat later, Schönberger [17] proved that any prescribed edge of a bridgeless cubic graph lies in a perfect matching. These are only two of many results about perfect matchings in cubic graphs, but several important problems are still open, and the Berge-Fulkerson conjecture is related to many of them. We can safely say that the Berge-Fulkerson Conjecture belongs to one of the most prominent open problems in entire Graph Theory. The conjecture appeared in a paper of Fulkerson [5] and is also attributed to Berge (see [18]). It suggests the following.

Conjecture 1.1 (Berge-Fulkerson conjecture)**.** Every bridgeless cubic graphs contains a family of six perfect matchings covering each edge exactly twice.

This conjecture belongs to a group of conjectures on cubic graphs with various implication among them. For example, it is a consequence of the Petersen colouring conjecture of Jaeger [9], while it implies the Fan-Raspaud conjecture [3].

It follows from the aforementioned Petersen theorem that every bridgeless cubic graph can be coloured with either with three or four colours. The Berge-Fulkerson conjecture trivially holds for the subclass of 3-edge-colorable cubic graphs – one can obtain six perfect matchings by doubling the three colour classes. Similarly to other conjectures, *snarks*, that is bridgeless cubic graphs that need four colours to be coloured, form the class of cubic graphs for which the problem is really hard to prove.

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Although the fame of the Berge-Fulkerson conjecture may partially be due to the simplicity of the statement, its importance is mainly due to its numerous connections to other problems in graph theory as well as to its relations with other geometric structures. Among others, it admits equivalent formulations in mathematical programming and in terms of configurations of points and lines in the context of algebraic and finite geometry.

More precisely, for any given graph G with edge-set E, we can consider the real vector space $\mathbb{R}^E$. Edmonds' Perfect Matching Polytope Theorem (see [2]) characterizes all vectors in $\mathbb{R}^E$ that can be written as convex combinations of characteristic vectors of perfect matchings of G. In particular, it turns out that, for every bridgeless cubic graph, the constant vector with entries $\frac{1}{3}$ is always a point of the polytope having characteristic vectors of perfect matchings of G as vertices. The Berge-Fulkerson conjecture is equivalent to saying that such a point can always be obtained as a convex combination of very few, indeed at most six, vertices of the polytope.

Another important link can be established with a remarkable geometric configurations of 15 points and 15 lines in the real projective space known as the Cremona-Richmond configuration. This configuration was discovered in 19th century in algebraic geometry, in the study of families of lines on cubic surfaces (see [1]). It also appears in finite geometry, for example in the study of generalized quadrangles (see [15]). In [13], it is proved in detail how the Cremona-Richmond configuration arises in connection with the Berge-Fulkerson conjecture.

Although the Berge-Fulkerson conjecture is almost half century old, only partial results has been achieved – it has been verified for some explicitly defined classes of cubic graphs, see for example [4, 7, 6, 10, 14].

In 1980, Jaeger and Swart conjectured that there is no snark with cyclic connectivity greater than six. Moreover, cyclic connectivity bounds the girth of a graph from above. For many other long-standing conjectures, reductions to cubic graphs with no small cycles and no small cycle separating cuts have been established. For instance, it is known that a potential minimum counterexample to the 5-flow conjecture of Tutte (see [21]) must be a cyclically 6-edge-connected snark [11] with girth at least 11 [12]. For the cycle double conjecture [20, 19] a similar reduction to snarks of girth 12 has been achieved [8]. On the other hand, no non-trivial result reducing the Berge-Fulkerson conjecture to cubic graphs of higher girth or cyclic connectivity has been derived so far. The main result of this paper is the first non-trivial reduction for this conjecture and reads as follows.

Theorem 1.2. *A minimum counterexample to the Berge-Fulkerson conjecture must be a cyclically* 5*-edge-connected snark.*

If Jaeger's and Swart's conjecture is true, Theorem 1.2 implies that a potential counterexample to the Berge-Fulkerson conjecture must have cyclic connectivity 5 or 6.

2. Preliminaries

In this section we introduce notation and auxiliary results that we will use in the following section in order to prove our main result.

A *multipole* (V, E) consists of a set of vertices V and a set of edges E. Each edge has two ends, each of which can, but need not, be incident with a vertex. If an edge has both ends incident with a vertex, it is called a *proper edge*. If one of the ends of an edge is incident with a vertex and the other end is not, the edge is called a *dangling edge*. If none of the ends of an edge is incident with a vertex, the edge in question is called an *isolated edge*. An end of an edge which is not incident with a vertex is called a *semiedge*. A *k-pole* is a multipole with exactly k semiedges. An *ordered k-pole* is a k-pole with a linear order on its semiedges. Note that, throughout the paper, we always consider multipoles with vertices of degree 3 and then, according to our definition, a 0-pole is nothing but a cubic graph.

Along the entire paper, we consider colorings of the edges of a k-pole: for semplicity, it is always implicitly assumed that if we assign a color to a given edge, then the same color is also assigned to all (possible) semiedges of that edge and viceversa. Then, from now on, we indifferently say that a color is assigned either to an edge or to a semiedge.

Definition 2.1. Let $H = (V, E)$ be a multipole. A *Berge-Fulkerson coloring* of H, *BF-coloring* for short, is a function φ which assigns to every element in E a 2-subset of the set of colors $\{1, 2, 3, 4, 5, 6\}$, in such a way that the subsets assigned to any two adjacent edges are disjoint.

It is straightforward that a BF-coloring of a cubic graph G is equivalent to the existence of six perfect matchings of G covering each edge of G twice. Moreover, each of the six color classes induces a perfect matching of G.

Let us recall that a graph G is *cyclically k-edge-connected* if it does not contain an edge-cut $|S|$ such that $|S| < k$ and $G - S$ contains at least two components containing cycles. The *cyclic connectivity* of a graph G is the greatest k such that G is cyclically k-edge-connected.

We now observe that there is no small cycle-separating cut in a smallest potential counterexample to the Berge-Fulkerson conjecture.

Proposition 2.2. Let G be a possible minimum counterexample to the Berge-Fulkerson conjecture. Then G is cyclically 4-edge-connected.

Proof. Suppose that G is a possible minimum counterexample to the Berge-Fulkerson conjecture. As the conjecture is stated for bridgeless graph, G does not contain cycle separating cuts of size 1. Let S be a cycle separating edge-cut in G of size 2. Create two smaller graphs from G in the usual way: remove the edges of S and in each of the resulting components join two 2-valent vertices with an edge, thereby producing two graphs G_1 and G_2, both smaller than G. As G is bridgeless, so is G_i for $i \in \{1, 2\}$. Therefore each G_i admits a BF-coloring. By permuting colors in one of these colorings, we can obtain a BF-coloring of G. Therefore G does not contain cycle-separating 2-edge-cuts. Similarly, if G would contain a cycle-separating 3-edge-cut, one could create two smaller bridgeless cubic graphs by the removal of the edges from S and joining by an edge the three 2-valent vertices with a new vertex in each of the components. Clearly, the two cubic graphs are smaller that G, therefore they admit a BF-coloring. Again, by a permutation of colors in one of the components, we can obtain a BF-coloring of G. The result follows. $\square$

Since the main result of this paper involves 4-edge-cuts, we now analyze the behavior of a BF-coloring on the four semiedges of a 4-pole in more detail. Let H be an ordered 4-pole. First observe that, since H is cubic, each of the six colors occurs an even number of times on the semiedges of H, more precisely, each color occurs 0, 2, or 4 times. Now we show that in every BF-coloring of H, the pairs of colors in the semiedges can be expressed as a composition of two colorings of the semiedges, each of which uses at most two colors and each color appears in at most one of these two colorings. Let us introduce the notation used to describe such colorings. An edge-coloring of the semiedges of an ordered 4-pole where ALL semiedges receive the same color is said to be of *type* A. In all other cases, for $i = 2, 3, 4$, an edge-coloring of the semiedges of an ordered 4-pole such that the first semiedge has the same color as the i-th one is said to be of *type* T_i (see also Figure 1).

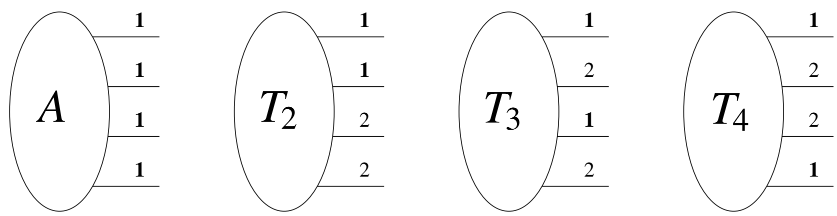

FIGURE 1. Possible types of edge-colorings for the semiedges of an ordered 4-pole.

Let φ be a BF-coloring of an ordered 4-pole H. Recall this means that every edge receives a pair of distinct colors. If all the four semiedges have the same pair of colors in φ, H is said to be of *type* AA. If only one color appears in all semiedges, then φ is of type AT_i for some $i \in \{2, 3, 4\}$. Finally, since every color occurs on an even number of edges, the only remaining cases to consider are the cases when four colors are present in φ on the four semiedges and each of them appears on exactly two semiedges. It can be easily checked that in each of these BF-colorings, we can partition the four colors in two subsets such that the two colors in the first subset give a coloring of type T_i and the other two colors a coloring of type T_j. Such BF-coloring will be denoted as of *type* T_iT_j, for some $i, j \in \{2, 3, 4\}$. Hence, we have proved the following proposition.

Proposition 2.3. Each BF-coloring of the semiedges of an ordered 4-pole is of type XY where $X, Y \in \{A, T_2, T_3, T_4\}$.

For our purposes, we do not distinguish a BF-coloring from another by the specific set of colors used for the semiedges, but only by the type of colorings. Moreover, note that a BF-coloring of type XY and a BF-coloring of type YX for $X, Y \in \{A, T_2, T_3, T_4\}$ are always considered of the same type. Hence, we have exactly ten types of BF-colorings of an ordered 4-pole, namely AA, AT_i for $i \in \{2, 3, 4\}$, and T_iT_j for $i, j \in \{2, 3, 4\}$, $i < j$ (see Figure 2).

We denote by $\mathcal{C}$ the set of these 10 types of BF-colorings, and we denote by $\mathcal{C}(H)$ the set of admissible types of colorings for a given ordered 4-pole H. A priori $\mathcal{C}(H)$ is one of the 2^{10} elements of the power set of $\mathcal{C}$. Nevertheless, we will show that this is not the case, i.e. the number of possible choices for $\mathcal{C}(H)$ is significantly smaller than 2^{10}. This can be seen by employing Kempe switches using whose we can prove that the existence of one type of BF-colorings guarantees the existence of some other BF-colorings. Kempe switches are a common tool when working with the usual edge-colorings of graphs but they can be used in more general form for BF-colorings too. We describe it now in more detail.

2.1. **Kempe chains.** Let φ be a BF-coloring of a 4-pole H. Let s be a semiedge of H and denote by c_1 one of the two colors in $\varphi(s)$ and by c_2 one of the four colors not in $\varphi(s)$. Consider the subgraph of H induced by all edges e such that $\varphi(e) \cap \{c_1, c_2\}$ is not empty. Let K be the connected component of such a subgraph which contains the semiedge s. Clearly, K contains exactly two semiedges, let s' denote the one different from s. Moreover, in φ, every edge of K receives exactly one of the two colors c_1 and c_2 and these colors alternate along the edges of K. The component K will be called a *Kempe chain*. Starting from φ, we can obtain a new BF-coloring of H by performing a *Kempe switch*, that is an interchange of the two colors c_1 and c_2 for all edges of K.

For example, consider a BF-coloring of type AA of an ordered 4-pole H which assigns colors 1 and 2 to all semiedges, like in the top left of Figure 2. Consider the Kempe chain containing the first semiedge and with colors 1 and 3. Such a Kempe chain must contain exactly one of the other three semiedges: assume it contains the i-th one, for $i \in \{2, 3, 4\}$. Then, by a Kempe switch, we obtain a BF-coloring of H of type AT_i. Therefore if AA belongs to $\mathcal{C}(H)$, then at least one of AT_2, AT_3 and AT_4 also belongs to $\mathcal{C}(H)$. All other cases work in the same way and a complete list of all these possible implications is shown in Figure 2.

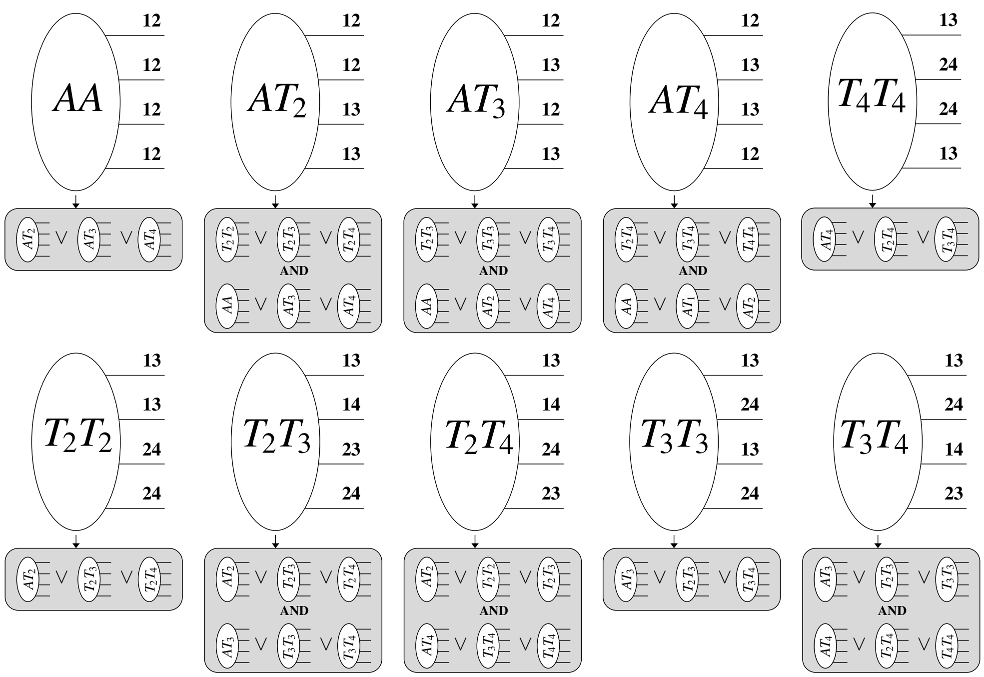

FIGURE 2. The 10 possible combinations of colorings for the semiedges of an ordered 4-pole in a BF-coloring. Below each one of them, we indicate all possible types of BF-colorings that could be obtained by performing a Kempe switch.

Throughout the paper, we will sometimes say that we *glue together* two ordered 4-poles G_1 and G_2. We mean that we consider the cubic graph obtained by the identification of the pair of dangling (or isolated) edges in the same position in the linear orderings of the two 4-poles. By a slight abuse of terminology we will also say that we *glue together* a BF-coloring of G_1 and a BF-coloring of G_2, by meaning that we consider the BF-coloring of the graph G whose restrictions to G_1 and G_2 coincide with the original BF-colorings of G_1 and G_2, respectively.

2.2. **The auxiliary graph M and its admissable subgraphs.** In order to make the presentation more transparent, instead of working directly with $\mathcal{C}(H)$, we prefer to construct an auxiliary graph M and to identify $\mathcal{C}(H)$ with a suitable subgraph of M.

The vertices of M are A, T_2, T_3 and T_4 – they correspond to the four possible types of edge-colorings of the semiedges of a 4-pole such that each color occurs an even number of times, while, in the natural way, each of the ten edges corresponds to a different type of BF-coloring, more precisely, the one obtained by the composition of the two edge-colorings of its semiedges. See Figure 3. (Here the loops are also considered to be edges.)

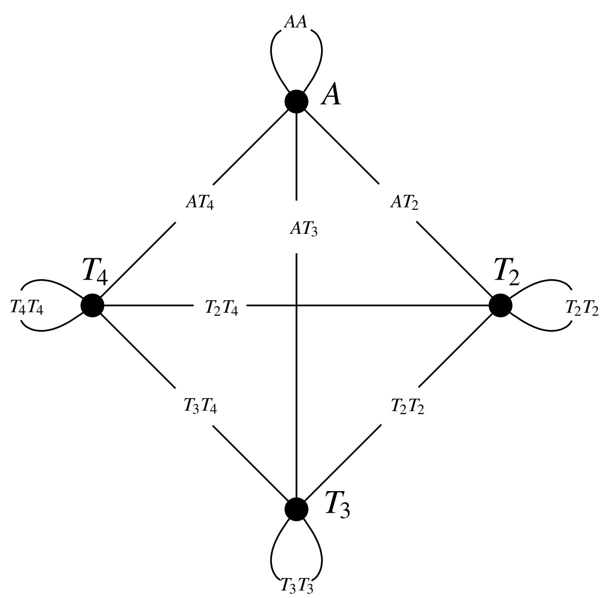

FIGURE 3. The graph M. Every edge stands for a different element of $\mathcal{C}$.

Hence, for any given ordered 4-pole H, the set $\mathcal{C}(H)$ corresponds to a set of edges in M: we will denote by H^* the subgraph of M induced by such set of edges.

The following lemma gives necessary conditions for a subgraph of M in order to be admissible.

Lemma 2.4. *Let H be an ordered 4-pole. Then, the subgraph H^* of M has no vertex of degree 1 and no vertex whose only incident edge is a loop.*

Proof. Let X, Y be two arbitrary elements, possibly the same, of the set $\{A, T_2, T_3, T_4\}$. Consider the vertex of M corresponding to the element X and assume, by contradiction, that XY is the unique edge of M in H^* (note that XY could be a loop of M). Consider a BF-coloring φ of H of type XY. Without loss of generality we can assume that φ assigns to the four semiedges of the ordered 4-pole H the same pair of colours as shown in Figure 2 for type XY. Note that color 5 is unused in all cases. Consider a Kempe chain in H starting from the first semiedge, ending in another semiedge and colored alternately 1 and 5. By a Kempe switch we obtain a BF-coloring of type XZ, where Z is different from Y. Hence, XY and XZ are two distinct edges of H^* incident X, that is X is not a vertex of degree 1 in H^* and it is not incident uniquely to the loop XX. □

2.3. **Acyclic 4-poles.** The main aim of the present paper is proving that a possible minimum counterexample G to the Berge-Fulkerson conjecture would be cyclically 5-edge-connected. Therefore, we need to exclude the presence of cyclic 4-edge-cuts in G.

An (ordered) 4-pole with no cycle is said to be *acyclic*. In this section, we study all possible subgraphs H^*, where H is an acyclic ordered 4-pole.

It is not hard to check that there are only two different acyclic 4-poles: the one consisting of two isolated edges and the one having two adjacent vertices of degree 3 (see Figure 4). Indeed, if a 4-pole has at least 4-vertices then it has a cycle.

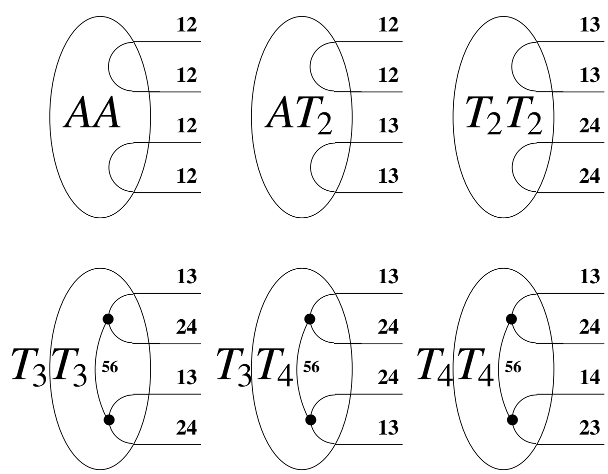

FIGURE 4. Some admissable colorings for two of the possible acyclic ordered 4-poles.

Each of the two acyclic 4-poles gives rise to three different acyclic ordered 4-poles, according to the order of its semiedges. Then, we have exactly six acyclic ordered 4-poles. An easy direct computation shows that for each of them the corresponding subgraph of M is a dumbbell graph (i.e. the graph illustrated in Figure 5).

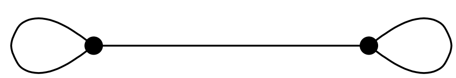

FIGURE 5. The dumbbell graph.

More precisely, every acyclic ordered 4-pole H corresponds to one of the induced dumbbell subgraphs H^* of M:

- if H is an acyclic 4-pole with two isolated edges, then H^* is the dumbbell subgraph of M induced, according to the order of semiedges, by one of the following three subsets $\{A, T_2\}$ (see the first line in Figure 4), $\{A, T_3\}$ or $\{A, T_4\}$ and
- if H is an acyclic 4-pole with two adjacent vertices of degree 3, then H^* is the dumbbell subgraph of M induced, according to the order of semiedges, by one of the following three subsets $\{T_3, T_4\}$ (see the second line in Figure 4), $\{T_2, T_3\}$ or $\{T_2, T_4\}$.

We can summarize all previous considerations in the following lemma by noting that M has exactly six dumbbell graphs.

Lemma 2.5. *Let D be a dumbbell subgraph of M. Then, there exists an acyclic ordered 4-pole H such that $D = H^*$.*

3. Main Result

Now we are in position to prove our main result.

Proof. Assume there exists a counterexample to the Berge-Fulkerson conjecture, and let G be one of minimum order. By Proposition 2.2, G has no cycle-separating 2- and 3-edge-cuts. By contradiction, assume that G is not cyclically 5-edge-connected, that is G has a 4-edge-cut C such that both the two 4-poles separated by C, say G_1 and G_2, contain a cycle. In particular, note that both G_1 and G_2 have order at least four.

Consider the two subgraphs G_1^* and G_2^* of M. First observe that for $i \in \{1, 2\}$, G_i^* has at least one edge. Indeed, if G_i^* contains no edge, this implies that G_i does not admit a BF-coloring. By glueing together G_i and an arbitrary acyclic ordered 4-pole, we obtain a bridgeless cubic graph smaller than G which does not admit a BF-coloring, a contradiction by minimality of G.

Furthermore, since G is a counterexample to the Berge-Fulkerson conjecture, then G_1^* and G_2^* must be edge-disjoint. Otherwise, G_1 and G_2 admit a BF-colorings of the same type and, up to permutation of colors, we can glue such BF-colorings together to obtain a BF-coloring of G.

Now, we prove that G_1^* and G_2^* cannot be vertex-disjoint. Assume this is the case, then at least one of them, say G_1^*, has at most two vertices since M has order four. By Lemma 2.4, G_1^* cannot have just one vertex and then, it has two vertices and it is a dumbbell subgraph of M. By Lemma 2.5 there exists an acyclic ordered 4-pole H such that $H^* = G_1^*$. The graph obtained by glueing together the acyclic ordered 4-pole H and the ordered 4-pole G_2, is longer a counterexample and it is smaller than G, a contradiction.

It follows that there exists a vertex, say X, of M belonging both to G_1^* and G_2^*. The vertex X is incident to four edges in M (one of them is a loop). By Lemma 2.4 exactly two of these edges belong to G_1^* and the remaining two to G_2^*. Without loss of generality, we can assume that XX and XY are edges of G_1^*, that is G_1 admits a BF-coloring of type XX and a BF-coloring of type XY.

Consider a BF-coloring φ of G_1 of type XX. It is obtained by composition of two edge-colorings of the four semiedges, say φ_1 and φ_2, of type X. Without loss of generality, we can assume that the first semiedge, say s, receives colors 1 and 3 in φ, in particular we can assume $\varphi_1(s) = 1$ and $\varphi_2(s) = 3$. Moreover, if $X \neq A$, let 2 and 4 be the other colors assigned to semiedges by φ_1 and φ_2, respectively. Consider a Kempe chain starting from s and with colors 1 and 2, and another Kempe chain starting from s and with colors 3 and 4. Both the corresponding Kempe switches change the BF-coloring φ of type XX to a BF-coloring of type XY: indeed the new BF-coloring cannot be of the same type of φ and we assumed that XY is the only other edge incident to X in G_1^*. In particular, the two Kempe switches change both φ_1 and φ_2 in two edge-colorings of type Y. Since the two Kempe chains are color-independent, by performing the two Kempe switches at the same time, we obtain a BF-coloring of G_1 of type YY. This proves that the dumbbell subgraph induced by XX, XY and YY is a subgraph of G_1^*. It follows, again by Lemma 2.5, that there exists an acyclic ordered 4-pole H such that H^* is a subgraph of G_1^*. The graph obtained by glueing together G_2 and H is a counterexample smaller than G, a contradiction. Hence G cannot admit a 4-edge-cut like C and the assertion follows. □

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