An algorithm and new bounds for the circular flow number of snarks

Jan Goedgebeur[1,2]*, Davide Mattiolo[3], Giuseppe Mazzuoccolo[4]

[1] Deparment of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium.

[2] Computer Science Department, University of Mons, Place du Parc 20, 7000 Mons, Belgium.

[3] Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/b, 41126 Modena, Italy.

[4] Dipartimento di Informatica, Università degli Studi di Verona, Strada le Grazie 15, 37134 Verona, Italy.

May 25, 2020

Abstract

It is well known that the circular flow number of a bridgeless cubic graph can be computed in terms of certain partitions of its vertex set with prescribed properties. In the present paper, we first study some of these properties that turn out to be useful in order to design a more efficient algorithm for the computation of the circular flow number of a bridgeless cubic graph. Using this algorithm, we determine the circular flow number of all snarks on up to 36 vertices as well as the circular flow number of various famous snarks. After that, as combination of the use of our algorithm with new theoretical results, we present an infinite family of snarks of order $8k+2$ whose circular flow numbers meet a general lower bound presented by Lukot'ka and Škoviera in 2008. In particular this answers a question proposed in their paper. Moreover, we improve the best known upper bound for the circular flow number of Goldberg snarks and we conjecture that this new upper bound is optimal.

Keywords: circular flow, cubic graph, bisection, snark, algorithm, dot product

1 Introduction

Given a real number $r \geq 2$, a circular nowhere-zero r-flow in a graph $G = (V, E)$ is an orientation of G together with a flow function $f\colon E \to [1, r-1]$ such that, at every

*Supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO)

vertex, the sum of all incoming flow values equals the sum of all outgoing ones. The circular flow number of a graph G, denoted by $\Phi_c(G)$, is the least real number r such that G admits a circular nowhere-zero r-flow. This parameter was explicitely introduced in [8] and shown to be a minimum and a rational number for every bridgeless graph. Tutte's 5-flow Conjecture [22] is one of the most important and outstanding conjectures in the theory of flows in graphs. The conjecture claims that every bridgeless graph admits a (circular) nowhere-zero 5-flow and it is well known that it is equivalent to its restriction to cubic graphs. When dealing with such graphs, the circular flow number is strictly related to the chromatic index, namely a cubic graph G is 3-edge-colorable if and only if $\Phi_c(G) \leq 4$. (The chromatic index of a graph is the minimum number of colors required for a proper edge-coloring of the graph.) Therefore a counterexample to Tutte's conjecture, if any, must be found inside the class of non-3-edge-colorable cubic graphs. In particular, it is known that such a counterexample must be found inside the class of snarks, i.e. cyclically 4-edge-connected cubic graphs with girth at least 5 which do not admit a 3-edge-coloring. In contrast to 3-edge-colorable cubic graphs, whose circular flow number can be either 3 or 4, it was proved in [15] that for every rational number $r \in (4, 5]$ there is a snark G such that $\Phi_c(G) = r$. Due to Tutte's 5-flow Conjecture, snarks with circular flow number exactly 5 have a certain interest and they were studied in [6, 9, 18]. One of the aims of this paper is to design an algorithm that computes the circular flow number of a cubic graph. For reasons explained later, our algorithm works for all bridgeless cubic graphs having circular flow number strictly less than 5; if a bridgeless cubic graph has circular flow number at least 5, the algorithm only says that it has $\Phi_c(G) \geq 5$. Clearly, if Tutte's 5-flow Conjecture holds, then it can be applied to all bridgeless cubic graphs. Our algorithm works using a well-known relation between nowhere-zero flows and bisections (see the definition in Section 2) in cubic graphs, see for instance [7]. Note also that an equivalent formulation of circular flows in terms of balanced valuations is given in [12]. For cubic graphs, a balanced valuation can be viewed as a bisection with, in addition, a weight function which assigns to every vertex a weight in the set $\{-p, p\}$ (where p is a positive real number) such that all vertices in the same bisection class receive the same weight value (see Proposition 4 in [12]). Jaeger proved that determining the maximum possible value of p among all possible balanced valuations of a cubic graph G, denoted by $p_{max}(G)$, is equivalent to determining the circular flow number of the graph. Indeed, the following easy relation holds:

$$\frac{\Phi_c(G)}{\Phi_c(G) - 2} = p_{max}(G).$$

Section 2 is devoted to the description of properties of bisections that turn out to be useful for the design of our algorithm, which is presented in Section 3. Using our implementation of this algorithm, we determined the circular flow number of all snarks on up to 36 vertices as well as the circular flow number of various famous snarks. The results of these computations can also be found in Section 3.

Two of the main results are given in Section 4. In this section we first improve the previous known upper bound from [16] for the circular flow number of Goldberg snarks. Moreover, in [17], Lukot'ka and Škoviera proved a general lower bound for the circular flow number of a snark in terms of its order and, at the end of their paper, they suggest that there might exist an infinite family of snarks of order $8k+2$ whose circular flow numbers meet the general lower bound presented. Here, we confirm this by constructing such an infinite family. The paper ends with Section 5 where we present two new conjectures about the circular flow number of snarks.

2 Useful Properties of Good Bisections

A 2-bisection of a graph G is a partition of its vertex set $V(G) = \mathcal{B} \cup \mathcal{W}$ such that $|\mathcal{B}| = |\mathcal{W}|$ and each connected component of both induced subgraphs $G[\mathcal{B}]$ and $G[\mathcal{W}]$ has at most 2 vertices. We will refer to the vertices of $\mathcal{B}$, resp. $\mathcal{W}$, as **black**, resp. **white**, vertices. We can define the set $\partial(X) = \{uv \in E(G)\colon u \in X \text{ and } v \notin X\}$ and $\Delta(X) = |b_X - w_X|$, where $b_X = |\mathcal{B} \cap X|$ and $w_X = |\mathcal{W} \cap X|$.

A 2-bisection is said to be **orientable** if and only if

$$\frac{|\partial(X)|}{\Delta(X)} \geq 1$$

for every $X \subseteq V(G)$.

It is easy to check that, for any $r < 5$, any circular nowhere-zero r-flow of a cubic graph G induces a 2-bisection of G: color a vertex white or black according to the number of inner edges (1 or 2, respectively) in the orientation of G corresponding to the flow with a positive value for every edge. In order to compute the circular flow number of G, when $\Phi_c(G) < 5$, we can compute, for all 2-bisections of G, the minimum ratio

$$\frac{|\partial(X)|}{\Delta(X)} \tag{1}$$

and then search for the maximum among these values, see for instance [7] and [12].

As already remarked, the well-known relation between the ratio (1) and the circular flow number of G, if $\Phi_c(G) < 5$, is the following.

$$\max_{\text{2-bisection of } G} \left(\min_{X \subset V(G)} \frac{|\partial(X)|}{\Delta(X)} \right) = \frac{\Phi_c(G)}{\Phi_c(G) - 2} \tag{2}$$

The left term is exactly the parameter $p_{max}(G)$ defined in the introduction. We would like to stress that if G has circular flow number at least 5, it is not true in general that its flow naturally induces a 2-bisection. For instance, the Petersen graph does not admit a 2-bisection at all, and there exist other bridgeless cubic graphs, admitting a 2-bisection, with the property that no 5-flow induces a 2-bisection (see for instance [1, 7, 21] for a more general discussion about bisections in cubic graphs). Theoretically, in order to manage these sporadic cases, we should admit bisections with (at most) three vertices in each connected component induced by a monochromatic class, but this would turn out to be unnecessary if Tutte's 5-flow Conjecture is true. Moreover, for all graphs G with circular flow number at least 5 which were determined in this paper (except for the Petersen graph), we could easily establish that $\Phi_c(G) = 5$ since they admit a 2-bisection for which the minimum ratio $\frac{|\partial(X)|}{\Delta(X)}$ is equal to $\frac{5}{3}$. Therefore we do not present a more general version of Algorithm 1 considering 3-bisections here.

For any fixed 2-bisection of a graph, if it does exist, we call the subsets that minimize the ratio (1) **good**. Moreover, we call the 2-bisections that maximize the left term in (2) **optimal**.

If $\Delta(X) = 0$, we define its ratio to be ∞, hence we will look for subsets of $V(G)$ such that $\Delta(X) > 0$. In particular, if X is a proper subset of $V(G)$ it follows that $\frac{|\partial(X)|}{\Delta(X)} = \frac{|\partial(\bar{X})|}{\Delta(\bar{X})}$, where $\bar{X}$ denotes $V(G) - X$, and so, for a given 2-bisection, we can always find at least a good subset of order at most $\frac{|V(G)|}{2}$. From now on, we will also assume without loss of generality to have more black vertices than white ones in X, i.e. $b_X > w_X$.

Lemma 2.1. *Consider a graph G having a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and a subset $X \subseteq V(G)$. Suppose that there is $X \subseteq V$ such that $G[X]$ is disconnected with components $A_1, \ldots, A_n$. Then there is one of those components A such that*

$$\frac{|\partial(A)|}{\Delta(A)} \leq \frac{|\partial(X)|}{\Delta(X)}.$$

Proof There is $A \in \{A_1, \ldots, A_n\}$ such that $\frac{|\partial(A)|}{\Delta(A)} \leq \frac{|\partial(A_i)|}{\Delta(A_i)}$ for each i. Therefore from $|\partial(A)|\Delta(A_i) \leq |\partial(A_i)|\Delta(A)$ and summing up all such inequalities we get

$$\frac{|\partial(A)|}{\Delta(A)} \leq \frac{\sum_{i=1}^{n} |\partial(A_i)|}{\sum_{i=1}^{n} \Delta(A_i)} \leq \frac{|\partial(X)|}{\Delta(X)}.$$

■

Applying the previous lemma we can conclude that, if $X \subseteq V$ is a good subset such that $G[X]$ is not connected, then all its connected components are good as well.

Consider a graph G with a 2-bisection. For a subset $X \subseteq V(G)$ let us denote by $\partial_V(X) = \{v \in X\colon \deg_{G[X]}(v) < 3\} = \{v \in X\colon \exists\, w \in V(G) - X \text{ such that } vw \in \partial(X)\}$.

Lemma 2.2. *Consider a bridgeless cubic graph G having a 2-bisection. Consider a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and one of its good subsets $X \subseteq V(G)$, with $b_X > w_X$ and $\frac{|\partial(X)|}{\Delta(X)} > 1$. Then, $\partial_V(X)$ is a subset of black vertices.*

Proof We want to show that there are no white vertices in $\partial_V(X)$. Suppose by contradiction that there is a white vertex $v \in \partial_V(X)$.

If v is incident to a unique edge of $\partial(X)$, then setting $Y := X - v$,

$$\frac{|\partial(Y)|}{\Delta(Y)} = \frac{|\partial(X)| + 1}{\Delta(X) + 1} < \frac{|\partial(X)|}{\Delta(X)}$$

which is a contradiction since X is good.

If, on the other hand, v is incident to two edges of $\partial(X)$, then setting $Y := X - v$,

$$\frac{|\partial(Y)|}{\Delta(Y)} = \frac{|\partial(X)| - 1}{\Delta(X) + 1} < \frac{|\partial(X)| + 1}{\Delta(X) + 1} < \frac{|\partial(X)|}{\Delta(X)}$$

and again we have a contradiction.

■

Remark 2.3. *If $X \subseteq V(G)$ is a good subset of vertices in a 2-bisection, then also $\bar{X}$ is good. In particular, if the 2-bisection is optimal then both $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic (in particular if one is white the other is black).*

Corollary 2.4. *Consider a bridgeless cubic graph G with $\Phi_c(G) < 5$. Consider an optimal 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$, and let $X \subseteq V(G)$ be a good subset. Then there is no couple of adjacent vertices v, w with the same color such that*

$$v \in X \text{ and } w \in \bar{X}.$$

Remark 2.5. *We have proved that, for a given optimal 2-bisection of a bridgeless cubic graph G with circular flow number less than 5 and among all* ***good*** *subsets of vertices*

- *there is at least one of them, say X, that induces a connected subgraph;*
- *we can search it among all subsets with cardinality at most $\frac{|V(G)|}{2}$, since the ratio of a subset equals the ratio of its complement;*
- *the boundaries $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic of different colors.*

The main idea of the algorithm presented in the following section is to only process sets X which satisfy the three properties in Remark 2.5. In order to assure that this produces consistent results, we need to stress that in every 2-bisection (not necessarily optimal) there exists a set X (not necessarily good) which satisfies all three properties and such that the ratio $\frac{|\partial(X)|}{\Delta(X)}$ is less than or equal to the ratio for a good set in an optimal 2-bisection. The critical property is the one on monochromatic boundaries, since it follows by Lemma 2.2 where we need to assume $\frac{|\partial(X)|}{\Delta(X)} > 1$. Indeed, in principle, it could be that in a non-orientable 2-bisection no good subsets satisfy the third property in Remark 2.5 and, at the same time, all subsets satisfying such properties have a ratio larger than the minimum one in an optimal 2-bisection. The following lemma excludes this possibility.

Lemma 2.6. *Consider a bridgeless cubic graph G having a 2-bisection. Consider a 2-bisection $V(G) = \mathcal{B} \cup \mathcal{W}$ and a good subset $X \subseteq V(G)$, with $b_X > w_X$ and $\frac{|\partial(X)|}{\Delta(X)} \leq 1$. Then, there exists a subset X' of X such that $\partial_V(X')$ is a subset of black vertices and $\frac{|\partial(X')|}{\Delta(X')} \leq 1$.*

Proof If $\partial_V(X)$ is a subset of black vertices, then trivially we can take $X' = X$. Assume there is a white vertex v in $\partial_V(X)$.

If v is incident to a unique edge of $\partial(X)$, then, since $\frac{|\partial(X)|}{\Delta(X)} \leq 1$:

$$\frac{|\partial(X - v)|}{\Delta(X - v)} = \frac{|\partial(X)| + 1}{\Delta(X) + 1} \leq 1.$$

If, on the other hand, v is incident to two edges of $\partial(X)$ then,

$$\frac{|\partial(X - v)|}{\Delta(X - v)} = \frac{|\partial(X)| - 1}{\Delta(X) + 1} < \frac{|\partial(X)| + 1}{\Delta(X) + 1} \leq 1.$$

By repeatedly removing vertices in this way, we obtain a subset X' of X which satisfies the required properties. ■

3 Algorithm and Computational Results

The pseudocode of our algorithm to compute the circular flow number of a bridgeless cubic graph is shown in Algorithm 1. The notation and definitions of $\partial(X)$ and $\Delta(X)$ are as in Section 2. Furthermore, we also use several properties of *good* subsets from the previous section to speed up the algorithm (cf. Remark 2.5). It is also possible to give an optional input parameter r to the algorithm in case you only want to know if $\Phi_c(G) \geq r$ or not. This is usually faster than computing the exact value of $\Phi_c(G)$.

Our algorithm is exponential as it takes exponential time to generate all 2-bisections and exponential time to generate all subsets of a given bisection. However, in practice the bounding criteria allow to prune many of the subset searches and if you only want to know if $\Phi_c(G) \geq r$, the algorithm stops as soon as a 2-bisection with a `min_fraction` larger than $\frac{r}{r-2}$ is found.

We implemented Algorithm 1 in the programming language C. The source code of the program can be obtained from [10]. In [4] Brinkmann et al. determined all snarks on up to 36 vertices. Using our algorithm, we determined all snarks of circular flow number 5 on up to 36 vertices in [9]. We now also determined the circular flow number of all other snarks on up to 36 vertices and the results can be found in Table 1.

Using our algorithm, we also determined the circular flow number of various famous named snarks. The results are summarized in Table 2 together with the circular flow number of the Flower snarks, which was already determined by Lukot'ka and Škoviera in [17], of the Generalized Blanuša snarks, which was already determined by Lukot'ka in [16], and of some Goldberg snarks. We remark that the circular flow number of the Goldberg snarks of order $16k+8$ is not completely determined for $k \geq 4$. It is only known to belong to the interval $[4 + \frac{1}{2k+1}, 4 + \frac{1}{k+1}]$, see Section 4.2 for details.

Algorithm 1 Compute the circular flow number of a (bridgeless) cubic graph G

Optional input: value for r
if r is defined **then**
  `test_lower_bound` := 1 // i.e. only test if $\Phi_c(G) \geq r$
else
  `test_lower_bound` := 0 // i.e. compute $\Phi_c(G)$
end if
`max_min_fraction` := 0
for every 2-bisection $(\mathcal{B}, \mathcal{W})$ of G **do**
  `min_fraction` := ∞
  for every subset $X \subseteq V(G)$ for which: $2 \leq |X| \leq \frac{|V(G)|}{2}$ **and** $G[X]$ is connected **and** $\partial_V(X)$ and $\partial_V(\bar{X})$ are monochromatic of different colors **do**
    Compute $|\partial(X)|$ and $\Delta(X)$
    if $\frac{|\partial(X)|}{\Delta(X)} <$ `min_fraction` **then**
      `min_fraction` := $\frac{|\partial(X)|}{\Delta(X)}$
      if `min_fraction` $\leq$ `max_min_fraction` **then**
        abort subset search
      end if
      if `test_lower_bound` **and** `min_fraction` $\leq \frac{r}{r-2}$ **then**
        abort subset search // since we are searching for a `min_fraction` $> \frac{r}{r-2}$
      end if
    end if
  end for
  if `test_lower_bound` **and** `min_fraction` $> \frac{r}{r-2}$ **then**
    return $\Phi_c(G) < r$
  end if
  if `min_fraction` > `max_min_fraction` **then**
    `max_min_fraction` := `min_fraction`
  end if
end for
if `test_lower_bound` **then**
  return $\Phi_c(G) \geq r$ // i.e. `max_min_fraction` $\leq \frac{r}{r-2}$
else
  return $\Phi_c(G) = \frac{2 \cdot \texttt{max\_min\_fraction}}{\texttt{max\_min\_fraction} - 1}$
end if

4 Improving Bounds for the Circular Flow Number of Some Snarks

4.1 Snarks having minimum possible circular flow number

In [17] a lower bound on the circular flow number that depends only on the order of a graph is given, that is:

Order	Circular flow number					Total
	$4+1/4$	$4+1/3$	$4+1/2$	$4+2/3$	5	
10					1	1
18			2			2
20			6			6
22			20			20
24			38			38
26		57	223			280
28		1 258	1 641		1	2 900
30		10 500	17 897		2	28 399
32		60 008	233 042		9	293 059
34	3 627	372 708	3 457 227		25	3 833 587
36	199 338	3 339 506	56 628 773	17	98	60 167 732

Table 1: The values of the circular flow number of all snarks on up to 36 vertices.

Name	Order	Φ_c
(Generalized) Blanuša snarks [2], [23]	$8k+2$	4+1/2 [16]
Flower snark J_{2k+1} [14]	$8k+4$	$4+1/k$ [17]
Goldberg snark G_3 [11]	24	$4+1/2$
Goldberg snark G_5 [11]	40	$4+1/3$
Goldberg snark G_7 [11]	56	$4+1/4$
Goldberg snark G_{2k+1} [11]	$16k+8$	$[4+1/(2k+1), 4+1/(k+1)]$
Loupekine snark 1 and 2 [13]	22	$4+1/2$
Celmins-Swart snarks 1 and 2 [5]	26	$4+1/2$
Double star snark [14]	30	$4+1/3$
Szekeres snark [20]	50	$4+1/2$
Watkins snark [23]	50	$4+1/3$

Table 2: The values of the circular flow number of various famous snarks.

Theorem 4.1 (Lukot'ka and Škoviera [17])**.** *Let G be a connected bridgeless cubic graph of order at most $8k+4$ that does not admit any 3-edge-coloring. Then*

$$\Phi_c(G) \geq 4 + \frac{1}{k}.$$

In the same paper it is shown that Flower snarks form a family of snarks of order $8k+4$ that attain this bound with equality, more precisely the Flower snark J_{2k+1} has $8k+4$ vertices and circular flow number $4+\frac{1}{k}$, which shows that the upper bound given in [19] was indeed the optimal one.

The paper also reports that Edita Máčajová (using a computer search from [18]) determined that the two Blanuša snarks on $18 = 8 \cdot 2 + 2$ vertices have circular flow number $4+\frac{1}{2}$ and that there are exactly 57 snarks on $26 = 8 \cdot 3 + 2$ vertices with circular flow number $4+\frac{1}{3}$. In [17] Lukot'ka and Škoviera mention that this strongly suggests that there exists an infinite family of snarks of order $8k+2$ with circular flow number $4+\frac{1}{k}$. They also report that they are not aware of any graphs of order $8k$ or $8k-2$ with circular flow number $4+\frac{1}{k}$.

In Table 1 from the previous section, we determined the circular flow number of all snarks on up to 36 vertices. The graphs from Table 1 with the minimum circular flow number for each order can be downloaded from the *House of Graphs* [3] at `http://hog.grinvin.org/Snarks`. As can be seen from that table, none of the snarks on up to 36 vertices of order $8k$ or $8k-2$ has circular flow number $4+\frac{1}{k}$.

We now present, for every positive integer k, a family $\mathcal{S}=\{S_k\}$ consisting of snarks of order $8k+2$ and having circular flow number $4+\frac{1}{k}$. Every snark S_k is obtained by performing a dot product of S_{k-1} and a copy of the Petersen graph P_k with two adjacent vertices removed in a suitable way. We recall the definition of dot product of two connected cubic graphs, say G and H, on at least 6 vertices (see also Figure 1). Consider $G'=G-\{ab,cd\}$, where ab and cd are independent edges of G. Let $H'=H-\{x,y\}$, where x and y are adjacent vertices in H, and let u,v and w,z be the other two neighbours of x and y, respectively. Then the dot product $G\cdot H$ is defined as the graph

$$(V(G)\cup V(H'),E(G')\cup E(H')\cup\{au,bv,cw,dz\}).$$

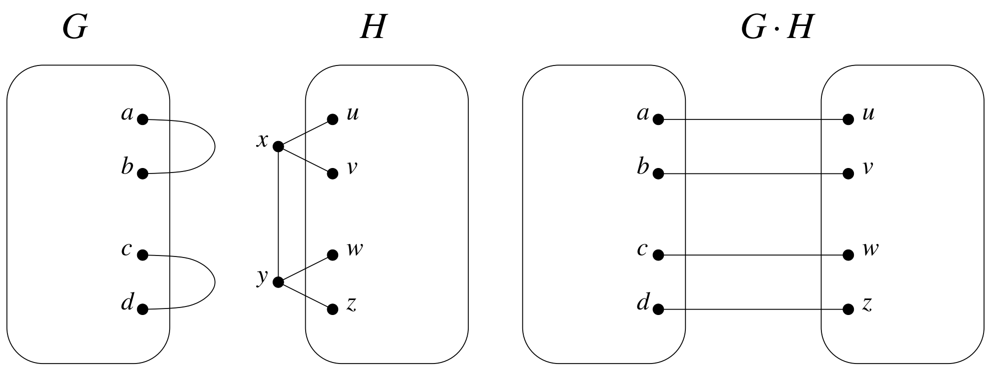

Figure 1: The dot product operation.

Indeed, if the way in which vertices a,b,c,d and u,v,w,z are linked is not specified, there are several ways to form the dot product for selected edges ab, cd and vertices x and y. This order will be relevant in our construction as only one specific way seems to work for our aims.

In what follows, we always consider the edge set of $G\cdot H$ as partitioned in three subsets $E(G')$, $E(H')$ and $\{au,bv,cw,dz\}$, and, with a slight abuse of terminology, we refer to the edges of $G\cdot H$ in $E(G')$, $E(H')$ and $\{au,bv,cw,dz\}$ as edges of G in $G\cdot H$, edges of H in $G\cdot H$, and new edges of $G\cdot H$, respectively.

We inductively define the snark S_k as follows:

- Let S_1 be the Petersen graph.
- Let S_2 be the Blanuša snark obtained by performing the dot product between two copies P_1 and P_2 of the Petersen graph where, we select a pair of edges of P_1 at

distance 1 (where by distance we mean the number of edges of the shortest path connecting two ends of those edges) and a pair of adjacent vertices of P_2.

- For $k \geq 3$, S_k is a dot product of S_{k-1} and a copy P_k of the Petersen graph, where we select a pair of adjacent vertices of the Petersen graph (by symmetry every pair) and two independent edges of S_{k-1}: one in the copy of P_{k-1} and the other one in the set of new edges of S_{k-1} as illustrated in Figure 2 (bold edges).

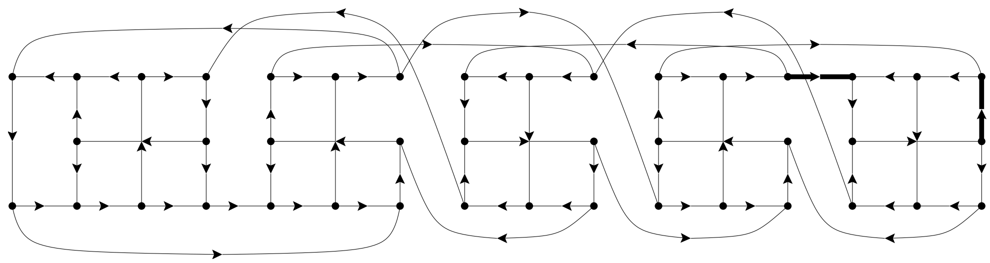

Figure 2: The Snark S_5.

It is easy to check that S_2 has order $18 = 8 \cdot 2 + 2$ and that it has circular flow number $4 + \frac{1}{2}$.

Theorem 4.2. *For any positive integer k, S_k is a snark of order $8k+2$ with circular flow number $4 + \frac{1}{k}$.*

Proof It is well known that the dot product of two snarks is a snark (see [14]), and an easy computation shows that the order of S_k is equal to $8k + 2$. Theorem 4.1 gives the lower bound $4 + \frac{1}{k}$ on the circular flow number, for all snarks of order $8k + 2$. Then, we only need to show that a nowhere-zero flow with maximum flow value $3 + \frac{1}{k}$ can be defined in S_k. We construct such a flow in the following way. First, we exhibit a 4-flow f_k of S_k which has flow value zero only for a specific edge e (the dashed edge in Figure 4).

Let ϕ be the 4-flow in P defined as in Figure 3 (right) and let ϕ^{-1} be the 4-flow in P obtained from ϕ by reversing the orientation of every edge. Moreover let ϕ_k be the 4-flow in P_k defined as follows:

$$\phi_k = \begin{cases} \phi & \text{if } k \text{ is even,} \\ \phi^{-1} & \text{otherwise.} \end{cases}$$

We construct such a 4-flow f_k in S_k as follows:

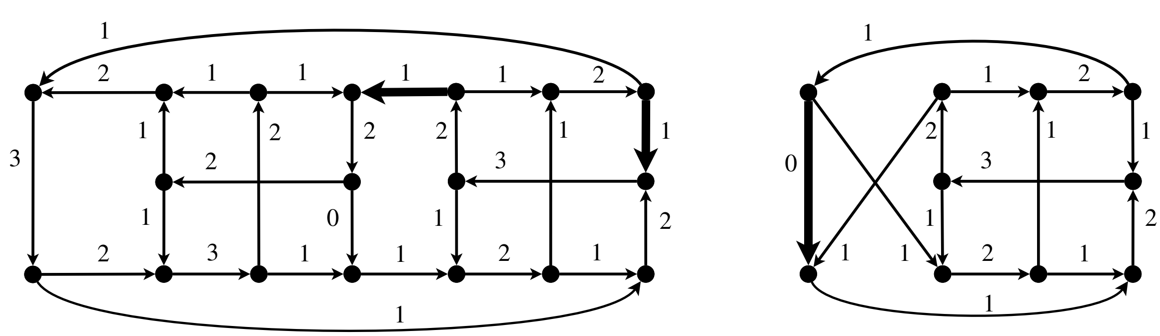

Figure 3: The 4-flow f_2 in S_2 (left) and the 4-flow ϕ in P (right).

- Fix on S_2 the 4-flow f_2 in Figure 3 (left);
- For $k \geq 3$, the dot product $S_{k-1} \cdot P_k$ can be performed in such a way that the vertices x, y such that $xy \in E(P_k)$ and $\phi_k(xy) = 0$ are removed. Then, we define f_k to be the unique 4-flow in S_k such that $f_k = f_{k-1}$ when restricted to the edges of S_{k-1} in S_k and $f_k = \phi_k$ when restricted to the edges of P_k in S_k. The iterative construction works as the edges that will be selected when performing the dot product $S_k \cdot P_{k+1}$ still have flow value 1 and the right orientation.

The 4-flow f_k has the desired properties (Figure 4 shows the flow f_5 in S_5.)

Then, we construct a set of k oriented cycles in the orientation induced by f_k, say $C_1, \dots, C_k$, in S_k such that:

- the edge e belongs to every C_i;
- every edge of S_k having flow value 3 in f_k belongs to exactly one of the cycles C_i's.

Such properties assure that if we construct a flow f'_k starting from f_k and by adding a flow equal to $\frac{1}{k}$ along every oriented cycle C_i then we obtain a nowhere-zero $(4 + \frac{1}{k})$-flow of S_k. Indeed, the former property implies that the edge e has flow value $k \cdot \frac{1}{k} = 1$ in f'_k, and the latter one implies that every other edge has flow value in the interval $[1, 3 + \frac{1}{k}]$.

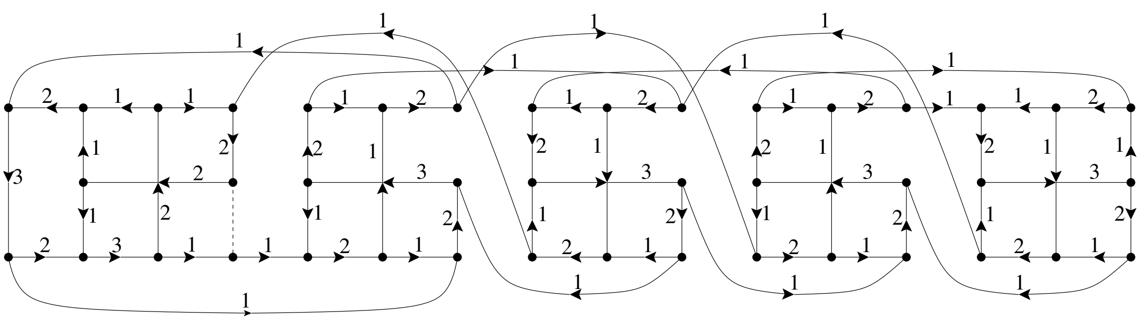

Figure 4: A 4-flow in S_5: the dashed edge is the unique one with flow value zero.

Figure 5 shows the set of five cycles for the case $k = 5$. We refer to this example to briefly explain the general construction of the cycles $C_1, \dots, C_k$.

For any $k > 1$, the cycle C_1 of S_k is the highlighted cycle in the first graph of Figure 5. Indeed, note that C_1 does not depend on how many times a dot product is performed to obtain S_k. Moreover, it contains the two leftmost edges with flow value 3 in f_k.

Every other C_i, for $1 < i < k$, is constructed analogously as it is shown for the second, third and fourth cycle in Figure 5. In particular, note that also in this case C_i does not depend on the value of k, if $k > i$, and the unique edge of S_k with flow value 3 in the cycle C_i does not belong to any cycle C_j with $i \neq j$.

Finally, we construct the last cycle C_k in analogy to the construction of the fifth cycle in Figure 5. Again the unique edge with flow value 3 in the cycle C_k does not belong to any cycle C_j with $j \neq k$.

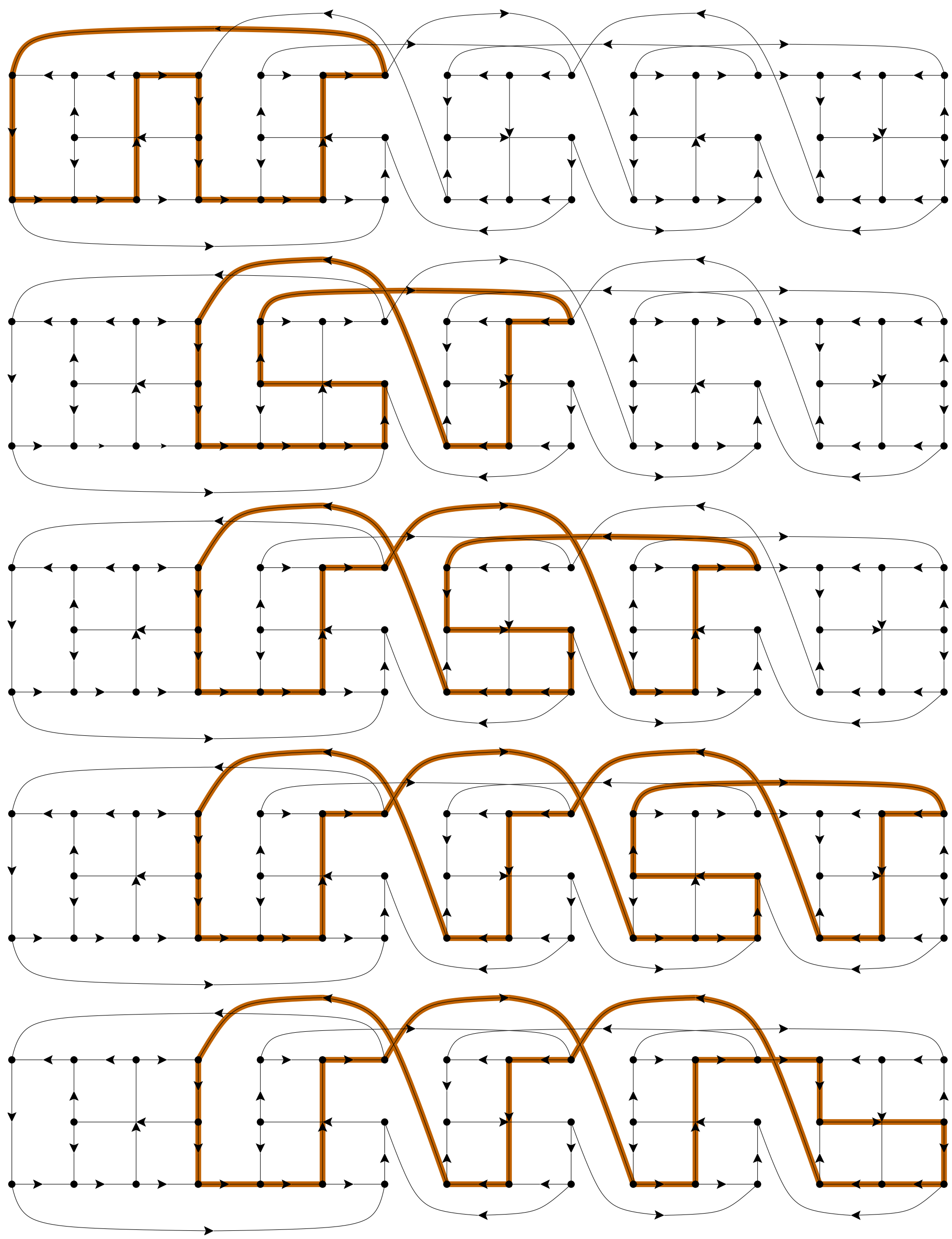

Figure 5: Cycles C_1, C_2, C_3, C_4 and C_5 in S_5.

All of these k oriented cycles pass through the dashed edge e in Figure 4 and, as remarked, every edge of S_k with flow value 3 belongs to exactly one of them. Then, we can construct a nowhere-zero $(4 + \frac{1}{k})$-flow of S_k and the assertion follows. ■

For the sake of completeness, we remark that all snarks S_k constructed here are permutation snarks of order $8k+2$, i.e. S_k admits a 2-factor consisting of two chordless cycles of length $4k+1$.

4.2 A new upper bound for the Circular Flow Number of Goldberg Snarks

The Goldberg snarks $\{G_{2k+1}\}_{k\in\mathbb{N}}$ are another classical family of snarks. The snark G_{2k+1} is constructed in the following way. Let P^- be the Petersen graph minus two vertices at distance 2, take $2k+1$ copies $P_1^-, \ldots, P_{2k+1}^-$ of P^- and glue them together as shown in Figure 6.

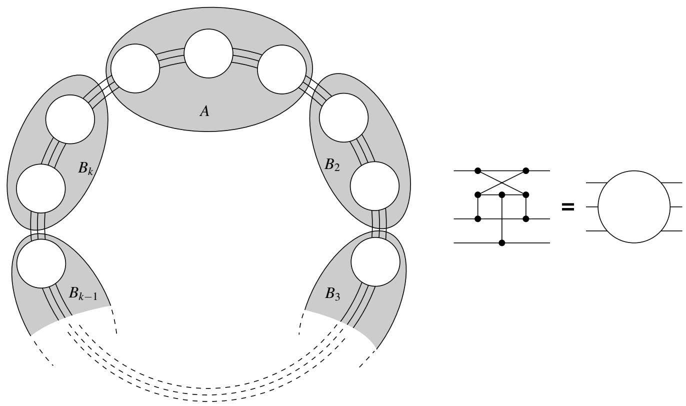

Figure 6: The Goldberg snark G_{2k+1} on $8(2k+1)$ vertices.

In [16] the circular flow number of the Goldberg snark G_{2k+1} is shown to be inside the interval $[4+\frac{1}{2k+1}, 4+\frac{1}{k}]$. By using the algorithm described in Section 3, we have shown (see Table 2) that $\Phi_c(G_{2k+1}) = 4+\frac{1}{k+1}$ for $k = 1, 2, 3$. Here, we show that the value $4+\frac{1}{k+1}$ is an upper bound for the circular flow number of G_{2k+1} for all possible k, thus improving the best known upper bound.

Proposition 4.3. *Let G_{2k+1} be the Goldberg snark of order $8(2k+1)$. Then,*

$$\Phi_c(G_{2k+1}) \leq 4 + \frac{1}{k+1}.$$

Proof The Goldberg snark G_{2k+1} consists of $2k+1$ copies of the Petersen graph minus two vertices at distance 2 glued together as shown in Figure 6. Define the multipole A to be three consecutive blocks of G_{2k+1} and each multipole B_t to be two consecutive blocks of G_{2k+1}, for $t = 2, \ldots, k$. We define on these multipoles the nowhere-zero circular flow represented in Figures 7 and 8. Note that the flow value on each edge of these multipoles is between 1 and $3+\frac{1}{k+1}$.

Moreover, we can glue them together as shown in Figure 6, in such a way that the Goldberg snark G_{2k+1} is constructed. It follows that a nowhere-zero circular $(4+\frac{1}{k+1})$-flow is defined in G_{2k+1}, for every positive integer k. ■

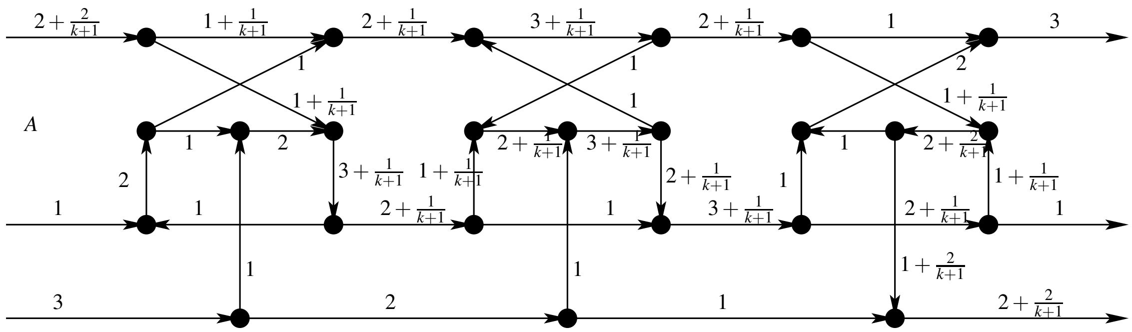

Figure 7: Flow in the multipole A.

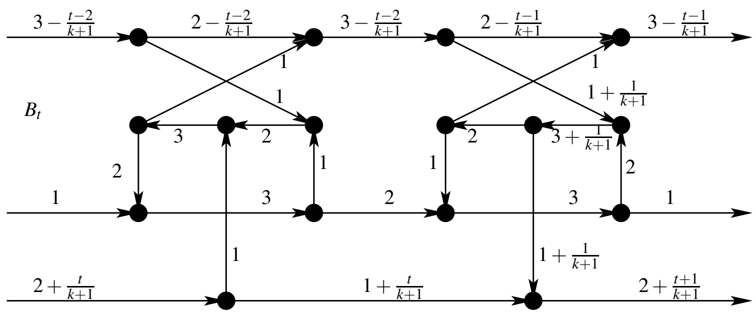

Figure 8: Flow in the multipole B_t.

5 Open Problems and New Conjectures

We conclude our paper with some open problems and conjectures.

We verified by computer that none of the cyclically 4-edge-connected cubic graphs G without a 3-edge-coloring, having girth at least 4 and on at most 32 vertices such that $|V(G)| \equiv 0$ or $6 \mod 8$ has a circular flow number that attains the lower bound of Lukot'ka and Škoviera [17] from Theorem 4.1 (cf. Table 1). This fact implies that none of the non-3-edge-colorable cubic graphs of order at most 32 such that $|V(G)| \equiv 0$ or $6 \mod 8$ has a circular flow number attaining the bound from Theorem 4.1. Indeed, assume that there exists such a graph G having a 3-edge-cut (and the property that $\Phi_c(G)$ attains the bound of Theorem 4.1). Then, a non-3-edge-colorable smaller graph can be constructed by contracting one of the two sides of the 3-edge-cut, say H, and $\Phi_c(H) \leq \Phi_c(G)$ holds, because G could be seen as an expansion of H. Hence, either we get a contradiction with Theorem 4.1 if H has much fewer vertices than G, or, iterating this process, we get a cyclically 4-edge-connected cubic graph with no 3-edge-coloring having circular flow number that attains the bound of Theorem 4.1, in contradiction with our computational results. Note that a similar argument applies to 2-edge-cuts.

Computational evidence suggests the following strengthened version of Theorem 4.1:

Conjecture 5.1. *Let G be a connected bridgeless cubic graph of order at most $8k+8$ that does not admit any 3-edge-coloring. Then*

$$\Phi_c(G) \geq 4 + \frac{1}{k}.$$

By using the algorithm presented in this paper, we verified that the circular flow number of the Goldberg snarks G_3, G_5 and G_7 meet the upper bound given by Proposition 4.3. This seems to suggest that the following conjecture could be true:

Conjecture 5.2. *Let G_{2k+1} be the Goldberg snark on $8(2k+1)$ vertices. Then*

$$\Phi_c(G_{2k+1}) = 4 + \frac{1}{k+1}$$

for every positive integer k.

Acknowledgements

Most computations for this work were carried out using the Stevin Supercomputer Infrastructure at Ghent University.

References

[1] M. Abreu, J. Goedgebeur, D.Labbate, G. Mazzuoccolo, Colourings of cubic graphs inducing isomorphic monochromatic subgraphs, J. Graph Theory, 92 (2019), 415-444.

[2] D. Blanuša, Problem cetiriju boja, Glasnik Mat. Fiz. Astr. Ser. II. 1 (1946), 31-42.

[3] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot, House of Graphs: a database of interesting graphs, *Disc. Appl. Math.*, 161 (2013), 311–314. Available at `http://hog.grinvin.org/`.

[4] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, Generation and properties of snarks, J. Combin. Theory Ser. B., 103 (2013), 468–488.

[5] U. A. Celmins and E. R. Swart, The Constructions of Snarks, Research Report CORR 79-18, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1979.

[6] L. Esperet, G. Mazzuoccolo, M. Tarsi, The structure of graphs with circular flow number 5 or more, and the complexity of their recognition problem, J. of Combinatorics, 7 (2016), 453–479.

[7] L. Esperet, G. Mazzuoccolo, M. Tarsi, Flows and bisections in cubic graphs, J. Graph Theory 86 (2017), 149–158.

[8] L. A. Goddyn, M. Tarsi, C. Q. Zhang, On (k,d)-colorings and fractional nowhere-zero flows, J. Graph Theory, 28 (1998), 155–161.

[9] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, A unified approach to construct snarks with circular flow number 5, arXiv:1804.00957 [math.CO].

[10] J. Goedgebeur, D. Mattiolo, G. Mazzuoccolo, Program to compute the circular flow number of a cubic graph, `https://caagt.ugent.be/cfn/`, 2019.

[11] M.K. Goldberg, Construction of class 2 graphs with maximum vertex degree 3, J. Combin. Theory Ser. B, 31 (1981), 282–291.

[12] F. Jaeger, Balanced valuations and flows in multigraphs, Proc. Amer. Math Soc. 55 (1976), 237–242.

[13] R. Isaacs, Loupekhine's snarks: a bifamily of non-Tait-colorable graphs, Technical Report No. 263, Department of Mathematical Sciences, Johns Hopkins University, 1976.

[14] R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colorable, Am. Math. Monthly 82 (1975), 221–239.

[15] R. Lukot'ka and M. Škoviera, Snarks with given real flow numbers, J. Graph Theory, 68 (2011), 189–201.

[16] R. Lukot'ka, Circular flow number of generalized Blanuša snarks, Discrete Math., 313 (2013), 975-981.

[17] R. Lukot'ka and M. Škoviera, Real flow number and the cycle rank of a graph, J. Graph Theory, 59 (2008), 11-16.

[18] E. Máčajová and A. Raspaud, On the strong circular 5-flow conjecture, J. Graph Theory, 52 (2006), 307–316.

[19] E. Steffen, Circular flow numbers of regular multigraphs, J. Graph Theory, 36 (2001), 24–34.

[20] G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Austral. Math. Soc. 8 (1973), 367–387.

[21] M. Tarsi, Bounded-excess flows in cubic graphs, arXiv:1807.04196 [math.CO].

[22] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canadian Journal of Mathematics, 6 (1954), 80–91.

[23] J. J. Watkins, Snarks, Ann. New York Acad. Sci. 576 (1989), 606-622.