OPTIMAL MANAGEMENT OF PUMPED HYDROELECTRIC PRODUCTION WITH STATE CONSTRAINED OPTIMAL CONTROL

ATHENA PICARELLI AND TIZIANO VARGIOLU

Abstract. We present a novel technique to solve the problem of managing optimally a pumped hydroelectric storage system. This technique relies on representing the system as a stochastic optimal control problem with state constraints, these latter corresponding to the finite volume of the reservoirs. Following the recent level-set approach presented in *O. Bokanowski, A. Picarelli, H. Zidani, State-constrained stochastic optimal control problems via reachability approach, SIAM J. Control and Optim. 54 (5) (2016)*, we transform the original constrained problem in an auxiliary unconstrained one in augmented state and control spaces, obtained by introducing an exact penalization of the original state constraints. The latter problem is fully treatable by classical dynamic programming arguments.

1. Introduction

In the current transition to a low carbon economy, one of the most prominent issues is that the most used renewable energy sources (RES), i.e. photovoltaic and wind, are non-dispatchable: for this reason, there is a strong need to store the energy produced when they are available in order to use it when they are not.

As for today, the most cost-efficient method to store electricity is with pumped hydroelectric storage (PHS) systems, which in 2017 accounted for about 95% of all active tracked storage installations worldwide, with a total installed capacity of over 184 GW [26]. In brief, these systems consist of two or more dam-based hydroelectric plants, linked sequentially so that, while the water used to produce electricity in the lower reservoirs is lost as in traditional hydroelectric plants, upper reservoirs discharge their water in lower reservoirs, where it can be stored and possibly pumped back to upper reservoirs when the electricity price is low (typically in off-peak periods). In this way, a PHS can produce electricity in peak periods, i.e. when its price is high, and recharge the upper reservoir in off-peak periods. These systems can be efficiently coupled with non-dispatchable renewable energy sources (like photovoltaic or wind), thus providing a very effective way to store possible surplus renewable energy when the price is low, and to use it when needed. An alternative is of course to connect the system directly to the power grid: also in this case, PHS is likely to use renewable-generated electricity to pump back water in the upper reservoirs, given the very low marginal generating costs of renewable energy.

In the current literature, the mathematical treatment of PHS is usually done by computing the fair value as the sum of discounted payoffs when operated at optimum[1]. To perform this computation, one can find in literature two alternative approaches. The first one is via operations research techniques by formulating a linear/quadratic programming model, see e.g. [9, 23, 34]. While these techniques allow to model potentially complex networks and constraints, the optimal exercise policy of the system within this approach turns out to be given only by the numerical solution of a linear/quadratic program, from which it is very difficult to extract the policy as a function of the relevant state variables (typically, the spot price of electricity and the reservoir levels).

The second approach is based on stochastic optimal control in continuous time, where with the dynamic programming approach one derives a partial differential equation called the Hamilton-Jacobi-Bellman

Date: June 1, 2020.
Università di Verona, Dipartimento di Scienze Economiche, Via Cantarane 37129, Verona (VR), Italy .
Università degli Studi di Padova, Dipartimento di Matematica, Via Trieste 63, 35121, Padova (PD), Italy.

[1]In this setting, we assume that the investor is risk-neutral. Although here we are not evaluating financial assets, but rather incomes coming from industrial activity, this is in line with all the related literature (see e.g. [25]), and is justified by the following financial argument. Even if the underlying assets are in principle not traded, a risk-neutral evaluation can be applied as long as one can find hedging instruments that can be storable and liquidly traded: see [33] for more economic insight, and [10, Remark 3.6] for the mathematical derivation of such a result for structured products like power plants.

(HJB) equation, i.e. a second-order partial differential equation, see e.g. [18, 29, 33, 35, 16] and references therein. The HJB equation is usually nonlinear, and without an explicit analytical solution: thus, one should recur to numerical methods, and this limits the dimensionality that one can reach. Proof of this is that in all these papers (apart from the notable exception of [18]), the lower reservoir has infinite capacity (e.g. is a sea basin): this entails that their model has one less state variable and much easier state constraints. However, the main advantage of this technique is that one can obtain the optimal pumping/producing strategy as a feedback control, i.e. as a function of the relevant state variables (here being time, electricity price and water levels in the basins).

As said above, our approach is based on optimal stochastic control in continuous time and dynamic programming, leading to a HJB equation. Indeed, dynamic programming techniques are usually applied to prove that the value function associated to optimal control problems is the unique solution of the HJB equation in the viscosity sense and to characterize the optimal policy as feedback of the state variables. However, while this is a well-established research field for generic unconstrained optimal control problems, PHS has the peculiarity that, the reservoirs being finite, the state variables corresponding to their levels have to satisfy given constraints: thus, we have to formulate a state-constrained optimal control problem. There exists a huge literature on state-constrained optimal control problems and their HJB characterization. We refer the reader e.g. to [2, 3, 20, 21] for stochastic control problems and to [13, 19, 30, 31] for deterministic ones. In this case, the characterization of the value function as a viscosity solution of a HJB equation is intricate and usually requires a delicate interplay between the dynamics of the processes involved and the set of constraints, see e.g. [4]. First, some *viability* (or *controllability*) conditions have to be satisfied to guarantee the finiteness of the value function, second, specific properties on the set of admissible controls must hold to ensure the continuity of the value function and its PDE characterization. When the behavior at the boundary of the constrained region is clear, one could think in principle to use the same penalization techniques as in [4]. However, when (as in our case) one does not have natural boundary conditions, this approach would not characterize the solution uniqluely. This often makes the problem not tractable by the classical dynamic programming techniques.

In this paper we follow the alternative approach developed in [8] to provide a fully characterization of the value function and optimal strategy associated to the optimal control problem in a general framework. We pass by a suitable auxiliary reformulation of the problem which allows a simplified treatment of the state constraints. This is achieved by the use of the so called *level-set method*, built to permit a treatment of state constraints by an exact penalization technique. Initially introduced by Osher and Sethian in [27] to model some deterministic front propagation phenomena, the level-set approach has been used in many applications related to controlled systems (see e.g. [1, 17, 22, 24, 32]).

In our case, the level-set method allows to link the original state constrained problem to an auxiliary optimal control problem, referred as the *level-set problem* defined on an augmented state and control space, but without state constraints. This level-set problem has the great advantage of leading to a complete characterization of the original one and of being, at the same time, fully treatable by classical dynamic programming argument under very mild assumptions.

The rest of the paper is organized as follows. We introduce the optimal control problem and the main assumptions in Section 2. In Section 3 we provide a HJB characterization of the associated value function under suitable controllabilty conditions on the system dynamics. Then, under a simplified model, we discuss in Section 4 the main difficulties arising from the presence of state constraints when such assumptions are not satisfied. In Section 5 we present the level set method and provide the main results of the paper. A numerical validation of the proposed approach is provided in Section 6.

2. Formulation of the problem and main assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one-dimensional Brownian motion $B_{\cdot}$ and let $\mathbb{F}$ be the filtration generated by B. We consider a finite time horizon $T > 0$ and the electricity price for the period $[t, T]$ governed by the following stochastic differential equation in $\mathbb{R}$:

$$\mathrm{d}X_s = b(s, X_s)\mathrm{d}s + \sigma(s, X_s)\mathrm{d}B_s \qquad s \in [t, T], \qquad X_t = x. \tag{1}$$

We work under the following assumption:

(H1) $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ and there exists $C_0 \geq 0$ such that for any $x, \bar{x} \in \mathbb{R}$, $t \in [0,T]$ one has

$$|b(t,x) - b(t,\bar{x})| + |\sigma(t,x) - \sigma(t,\bar{x})| \leq C_0|x - \bar{x}|,$$
$$|b(t,x)| + |\sigma(t,x)| \leq C_0(1 + |x|).$$

Under this assumption there exists a unique strong solution, denoted by $X^{t,x}_{\cdot}$, to Equation (1). We also assume the following non negativity condition:

(H2) for any $t \in [0,T], x \geq 0$ one has $X^{t,x}_s \geq 0, \forall s \geq t$ a.s. .

Different price models can be taken into account. In order to guarantee assumptions (H1) and (H2) being satisfied we will focus on the following two dynamics:

a) Price modeled as a Geometric Brownian Motion (GBM): $b(t,x) = bx$ and $\sigma(t,x) = \sigma x$ for some $b, \sigma \geq 0$;
b) Price modeled as an Inhomogeneous Geometric Brownian Motion (IGBM): $b(t,x) = a - bx$ and $\sigma(t,x) = \sigma x$ for some $a, b, \sigma \geq 0$;

Both models provide non-negative prices. In particular, the second model is used in many financial applications where one wants a process exhibiting non-negativity and mean-reversion, for example when modeling interest rates, default intensities, volatilities, etc., see e.g. [12] and references therein.

We assume to have a water storage, composed of two reservoirs. The two reservoirs are filled with rate $\beta_1 : [0,T] \to \mathbb{R}$ and $\beta_2 : [0,T] \to \mathbb{R}$ respectively with β_1, β_2 given positive bounded functions. The volume $Y_{1,\cdot}$ of water remaining at every time in the first (upper) reservoir is controlled by ν_1 and satisfies

$$\mathrm{d}Y_{1,s} = (\beta_1(s) - \nu_{1,s})\mathrm{d}s \qquad s \in [t,T], \qquad Y_{1,t} = y_1, \tag{2}$$

where ν_1 can either be negative (pumping water up) or positive (pulling water down). While for the second (lower) reservoir one has

$$\mathrm{d}Y_{2,s} = (\beta_2(s) + \nu_{1,s} - \nu_{2,s})\mathrm{d}s \qquad s \in [t,T], \qquad Y_{2,t} = y_2, \tag{3}$$

for some positive ν_2. We will assume that the couple (ν_1, ν_2), that is our control, is a progressively measurable process taking values in $U := [-\underline{u}_1, \bar{u}_1] \times [0, \bar{u}_2]$. We denote by $\mathcal{U}$ the set of these controls[2]. For any $t \in [0,T]$, $y \equiv (y_1, y_2)$ and any choice of $\nu \equiv (\nu_1, \nu_2) \in \mathcal{U}$, we denote by $Y^{t,y,\nu}_{\cdot} \equiv (Y^{t,y,\nu}_{1,\cdot}, Y^{t,y,\nu}_{2,\cdot})$ the solution to (2)-(3). For $i \in \{1,2\}$, the value $Y^{t,y,\nu}_{i,\cdot}$ is required to remain non negative and bounded by a maximum value $\bar{y}_i$ (the reservoir capacity) along the interval $[t,T]$. This is expressed by the following constraint on the state:

$$(Y^{t,y,\nu}_{1,s}, Y^{t,y,\nu}_{2,s}) \in [0, \bar{y}_1] \times [0, \bar{y}_2] =: K, \qquad \forall s \in [t,T] \quad \text{a.s.}.$$

The amount of electricity instantaneously produced or consumed by the reservoirs is modeled by the function $\kappa : U \to \mathbb{R}$ defined as follows[3]

$$\kappa(u) := (u_2 + c(u_1)) \qquad \text{with } c(u_1) = \begin{cases} u_1 & \text{if } u_1 \geq 0 \\ \gamma u_1 & \text{if } u_1 < 0 \end{cases} \qquad (\gamma > 1) \tag{4}$$

(more generally, one can consider any concave and Lipschitz continuous function c such that $c(0) = 0$). Here γ is a coefficient which expresses how much does it cost, in terms of expended energy, to store a unit of potential energy back in the upper reservoir. By the physical impossibility of creating perpetual motion, γ must be strictly greater than 1. The aim of the controller is to maximize the cash flow obtained

[2] Physically, a more sophisticated model could be like that in [33], who perform a fine modeling of the physical constraints for the water flows between the higher and the lower basin (this one still having infinite capacity), and consider the water inflow of the upper basin as stochastic. Differently from this, for ease of exposition we treat both the water inflow into the two basins, as well as the constraints on the outflow, as deterministic quantities not depending on water height. Our simplification has no practical consequences when the total height of the higher basin is negligible with respect to its relative height with respect to the lower one; if this is not the case, however, these constraints could be easily included in our model.

[3] A more realistic model would incorporate in the function κ some parameter for the efficiency of the energy production process. We could for instance consider $\kappa(u) = \eta_2 u_2 + \eta_1 c(u_1)$ for some $\eta_1, \eta_2 \leq 1$. To simplify the notation here we assumed $\eta_1 = \eta_2 = 1$.

by selling the produced electricity $\kappa(u)$ at the price x. This results in the following state constrained optimal control problem

$$V(t,x,y) := \sup_{\nu \in \mathcal{U}_{\text{ad}}^{t,y}} \mathbb{E}\left[\int_t^T L(X_s^{t,x}, \nu_s)\mathrm{d}s\right] \tag{5}$$

with $L : [0,+\infty) \times U \to \mathbb{R}$ given by

$$L(x,u) := x\kappa(u)$$

and $\mathcal{U}_{\text{ad}}^{t,y}$ the set of progressively measurable processes in $\mathcal{U}$ such that $Y_s^{t,y,\nu} \in K$ for any $s \in [t,T]$ a.s.. We refer at the function $V : [0,T] \times \mathbb{R} \times K \to \mathbb{R}$ as the value function of the problem. In the sequel, whenever $\mathcal{U}_{\text{ad}}^{t,y} = \emptyset$ we use the convention $V(t,x,y) = -\infty$.

Throughout the paper we will also consider a simplified problem of one single reservoir with level managed by ν_1 taking values in $U = [0,\bar{u}]$. We then restrict to the two-dimensional dynamics $(X_s, Y_{1,s})$ and consider

$$\kappa(u_1) := u_1.$$

To simplify the notation, in this case we will drop the subscript indices. More precisely, one gets the optimal control problem (5) with $L(x,u) := xu$ subject to (1) and

$$\mathrm{d}Y_s = (\beta(s) - \nu_s)\mathrm{d}s \qquad s \in [t,T], \qquad Y_t = y$$

under the state constraint $Y_s^{t,y,\nu} \in K$, $\forall s \in [t,T]$ a.s. with $K := [0,\bar{y}]$.

3. HJB characterization under controllability assumptions

We outline in this section the way dynamic programming techniques apply to state constrained optimal control problems to obtain a HJB characterization of the value function V.

3.1. The single reservoir model.

We start considering the single reservoir model. To have a meaningful problem we impose that

$$\int_0^T \beta(s)\mathrm{d}s \leq \bar{u}T$$

i.e. the entire amount of water entering the basin in the period $[0,T]$ is not greater than the amount that can be withdrawn.[4] In what follows we denote by $Q := (0,+\infty) \times (0,\bar{y})$ the state space and by $\overline{Q}$ its closure.

In order to have a finite value function on $[0,T] \times \overline{Q}$ one needs $\mathcal{U}_{\text{ad}}^{t,y} \neq \emptyset$ for any $t \in [0,T], y \in [0,\bar{y}]$. The following assumption ensures this.

(H3) there exists $\eta > 0$ such that $\eta \leq \beta(t) \leq \bar{u} - \eta, \forall t \in [0,T]$.

Assumption (H3) is known in literature as an "inward pointing" condition (see [30, 31] the earlier introduction of this type of assumptions) and enables to always find, for any level of water y in the reservoir, a suitable control value u steering the level towards the interior of the interval $K = [0,\bar{y}]$. In particular, for $y = \bar{y}$ one has for any $t \in [0,T]$ the existence of $u \in U$ such that $\beta(t) - u < 0$ which pushes down the level of water in the reservoir, while for $y = 0$ the positivity of $\beta(t)$ let the level of water increase for any choice of $u \in U$ such that $\beta(t) - u > 0$.

Remark 3.1. *To guarantee that $\mathcal{U}_{ad}^{t,y} \neq \emptyset$ it is sufficient that $0 \leq \beta(t) \leq \bar{u}, \forall t \in [0,T]$ since, in this case, the control $u_t \equiv \beta(t), \forall t \in [0,T]$ is always admissible. However, the stronger Assumption (H3) is what one needs for the well posedness of the state constrained problem given by Theorem 3.1 below.*

We start with the following result concerning some regularity and growth properties of V.

[4] We are not assuming, as is the case in real dams, that we have the possibility to eliminate the excess water in the reservoir, i.e. to be able to withdraw from the reservoir, if it is close to be full, via a safety discharge, thus not producing electricity. This however typically results in a waste of water, i.e. of potential electricity production, thus of potential profit (recall that we assume the electricity price being nonnegative). Thus, this should always be regarded as suboptimal, and is not modeled here.

Lemma 3.1. *Let assumptions (H1)-(H3) be satisfied. Then, there exists a constant $C \geq 0$ such that for any $t \in [0,T], x, \bar{x} \in [0,+\infty), y \in [0,\bar{y}]$ one has*

$$0 \leq V(t,x,y) \leq C(1+x) \qquad \text{and} \qquad |V(t,x,y) - V(t,\bar{x},y)| \leq C|x-\bar{x}|.$$

Moreover, for any $(x,y) \in \overline{Q}$

$$\lim_{t \to T} V(t,x,y) = 0.$$

Proof. Thanks to Assumption (H3), V is finite on $[0,T] \times \overline{Q}$, and its non negativity follows directly from the definition of L and by Assumption (H2). Under Assumption (H1) it is well known that for any $x, \bar{x} \in \mathbb{R}$

$$\mathbb{E}\left[\sup_{s\in[t,T]} (X_s^{t,x})^2\right] \leq C(1+x^2) \qquad \text{and} \qquad \mathbb{E}\left[\sup_{s\in[t,T]} \left|X_s^{t,x} - X_s^{t,\bar{x}}\right|^2\right] \leq C|x-\bar{x}|^2,$$

where the constant C only depends on T and the constants appearing in Assumption (H1). By the very definition of L, it follows that

$$V(t,x,y) \leq \bar{u}\,\mathbb{E}\left[\int_t^T X_s^{t,x} \mathrm{d}s\right] \leq \bar{u}(T-t)\,\mathbb{E}\left[\sup_{s\in[t,T]} X_s^{t,x}\right]$$

$$|V(t,x,y) - V(t,\bar{x},y)| \leq \sup_{\nu \in \mathcal{U}_{\text{ad}}^{t,y}} \mathbb{E}\left[\int_t^T |L(X_s^{t,x},\nu_s) - L(X_s^{t,\bar{x}},\nu_s)|\mathrm{d}s\right] \leq \bar{u}(T-t)\,\mathbb{E}\left[\sup_{s\in[t,T]} |X_s^{t,x} - X_s^{t,\bar{x}}|\right]$$

from which the first two statements follow. Let $h > 0$, from the previous estimate one also has

$$0 \leq V(T-h,x,y) \leq Ch(1+x)$$

which gives the last result. □

The following theorem allows us to characterize the value function V as the unique solution of a suitable HJB partial differential equation. Due to the strong degeneracy of the diffusion term we do not expect to have classical (i.e. $C^{1,2}([0,T] \times \overline{Q})$) solutions of the equation. For this reason we use the notion of viscosity solutions.

Theorem 3.1. *Let assumptions (H1)-(H3) be satisfied. Then V is the unique continuous viscosity solution with linear growth of the following state constrained HJB equation*

$$-V_t - b(t,x)V_x - \frac{1}{2}\sigma^2(t,x)V_{xx} + \inf_{u \in U^{t,y}} \{-(\beta(t) - u)V_y - L(x,u)\} = 0 \qquad \text{on } [0,T) \times \overline{Q} \tag{6a}$$

$$V(T,x,y) = 0 \qquad \text{on } \overline{Q} \tag{6b}$$

where the set $U^{t,y}$ is defined, for all $t \in [0,T)$, as

$$U^{t,y} := \begin{cases} [0,\beta(t)] & \text{if } y = 0, \\ [0,\bar{u}] & \text{if } y \in (0,\bar{y}), \\ [\beta(t),\bar{u}] & \text{if } y = \bar{y}. \end{cases} \tag{7}$$

The proof follows by the dynamic programming and comparison principle for viscosity solutions of state-constrained HJB equations. We refer the interested reader to [2] and [13] for further details.

Problem (6a) does not admit, in general, an explicit solution. We then need to approximate its solution (i.e. the value function and the optimal feedback) numerically. Here, we use a semi-Lagrangian scheme, see e.g. [15, 16, 11, 14, 28]. In our numerical experiments, we fix the time horizon $T = 1$, the maximal capacity of the reservoir $\bar{y} = 1$ and take the following function

$$\beta(t) = 2\sin(\pi t) + 0.5$$

as the filling rate of the reservoir[5]

[5]This filling rate could represent, for example, a water income distributed from April, 1 ($t = 0$) to September, 30 ($t = 1$) for a glacier-fed water reservoir (of course in the Northern hemisphere). At $t = 0$, snow and ice melting is already significant, and will reach its maximum at $t = 0.5$ (i.e. end of June). From that time on, though temperatures are still high enough to melt snow and ice in significant quantity, their available amount will start to decline, so the resulting water income will decrease, and will do it more and more as temperature starts to fall again.

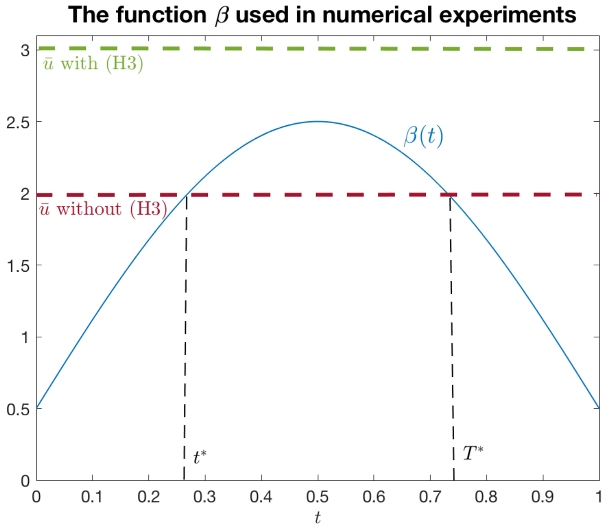

FIGURE 1. The function $\beta(t) = 2\sin(\pi t) + 0.5$ used in our numerical tests.

The function $\beta(\cdot)$ is plotted in Figure 1. In this section we assume that the dam under consideration admits maximal discharge rate $\bar{u} = 3$. It is immediate to observe that this value is sufficiently large to ensure that Assumption (H3) is satisfied, which allows us to use Theorem 3.1 to characterize the solution of our problem.

In Figure 2 we plot a numerical approximation of the value function V at time $t = 0$ (left) and the optimal feedback control (right) obtained solving (6a) for a price process following a GBM. This process has the peculiar property of being an exponential Levy process, thus its log-increments are independent. This makes optimal controls, given the multiplicative structure of the value function, independent of the current price level. To see this, notice that, when the price is modeled as a GBM, the value function can be factorized, as the electricity price depends on the initial value $X_t = x$ just via a multiplication factor: thus we have

$$V(t,x,y) = \sup_{\nu \in \mathcal{U}^{t,y}_{\rm ad}} x\,\mathbb{E}\left[\int_t^T e^{(b-\frac{1}{2}\sigma^2)s+\sigma B_s}\nu_s \mathrm{d}s\right] =: x\,v(t,y) \tag{8}$$

where v satisfies a simplified version of the HJB equation, given by

$$-v_t(t,y) + \inf_{u\in U^{t,y}} \{-bv(t,y) - (\beta(t) - u)v_y(t,y) - u\} = 0 \tag{9}$$

still with terminal condition $v(T,y) = 0$. Given this multiplicative decomposition, here we have the optimal control which ends up being just a feedback control of (t,y), without explicit dependence on x. One can also see from the structure of Equation (9) that the optimal feedback control ν^* here is of bang-bang type and takes the form $\nu^*_t = u^*(t,Y_t)$, with

$$u^*(t,y) := \begin{cases} \max U^{t,y} & \text{if } v_y(t,y) < 1, \\ \min U^{t,y} & \text{if } v_y(t,y) \geq 1, \end{cases} \qquad t\in[0,T],$$

where the specific values of $\max U^{t,y}$ or $\min U^{t,y}$ depend on (t,y) through Equation (7). In terms of the dependence on the y variable, we can see that the graphics reflects quite well the expectations that we have on this model. Indeed, the value function is increasing with respect to the reservoir level y, and the dependence here seems almost linear: this is maybe due to the fact that, regardless on the current level y, we always have the maximum possible flexibility in the control u, and one can let the dam fill or use its flow always at maximum speed in either direction, being the optimal control ν^* of bang-bang type. This is also confirmed at $y = 0$, where the minimum speed is 0, while the maximum speed is $\beta(t)$ instead of $\bar{u}$ (see Eq. (7)), as is evident from the lower left corner of the right panel of Figure 2.

The case of the electricity price modeled as an IGBM is given in Figure 3. With this process, we cannot use the multiplicative decomposition of the value function seen in the GBM case, thus the optimal feedback

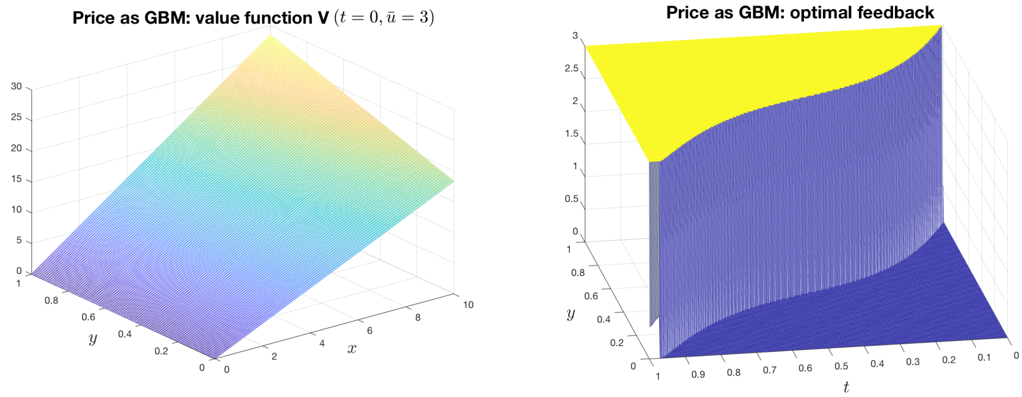

FIGURE 2. Single dam, price modeled as a GBM with $b(t, x) = 0.05x$ and $\sigma(t, x) = 0.1x$, maximal discharge rate $\bar{u} = 3$ (Assumption (H3) is satisfied). Left: numerical approximation of the value function V at $t = 0$. Right: optimal feedback as a function of (t, y). Discretization steps used in numerical tests: $\Delta x = 0.05$, $\Delta y = 0.01$, $\Delta t = 0.002$.

will in general depend on state x. We notice that the value function exhibits a linear growth both in x as in y, as in the GBM case (even if here, differently from the GBM case, we have $V(t, 0, y) > 0$). Here the interpretation is the same as in the GBM case: regardless of the current level y, we always have the maximum possible flexibility in the control u. Also, the optimal control is again of bang-bang type, now of the form $\nu_t^* = u^*(t, X_t, Y_t)$, with

$$u^*(t, x, y) := \begin{cases} \max U^{t,y} & \text{if } V_y(t, x, y) < x, \\ \min U^{t,y} & \text{if } V_y(t, x, y) \geq x, \end{cases} \qquad t \in [0, T],$$

and is found out to increase with respect to x, which is quite natural, as it is optimal to produce more and more as the electricity price increases; conversely, when the price is low, it is optimal to delay production and wait for the price to increase again. Differently from the GBM case, where the future price evolution is totally independent of the current price level itself, here we have a natural concept of "high" and "low" prices, as the IGBM is a mean-reverting process, thus possesses a concept like long-term mean (equal to a/b in our parameterization). Thus, in this case, the intuitive concept above "produce when the price is high and wait when it is low" makes sense when referring to the long-term mean. This can also be observed in the optimal feedback controls in Figure 3: the first two subfigures have $x < a/b$ (equal to 5 there), thus it is optimal to produce only when the dam is filled or the terminal time T is near; instead the latter two subfigures have $x \geq a/b$, thus it is optimal to produce in almost all couples of (t, y). Observe also in Figure 3 the particular behavior of the optimal feedback control for $y = 0$ and $y = \bar{y}$. Indeed, for $y = \bar{y}$ the minimal admissible discharge rate corresponds to the value $\beta(t)$, $t \in [0, T]$, which partially appears on the top-right corner of the first two bottom panels of Figure 3. Instead, for $y = 0$, $\beta(t)$ is the maximal admissible discharge rate, which partially appears in the first three bottom panels and becomes completely visible on the fourth panel of Figure 3.

3.2. **The two reservoirs model.** In the two dams setting, the analogous of Assumption (H3) is the following requirement:

(H3)' there exists $\eta > 0$ such that for all $t \in [0, T]$

$$\eta - \underline{u}_1 \leq \beta_1(t) \leq \bar{u}_1 - \eta, \quad -\bar{u}_1 + \eta \leq \beta_2(t) \leq \bar{u}_2 + \underline{u}_1 - \eta, \quad \eta \leq \beta_1(t) + \beta_2(t) \leq \bar{u}_2 - \eta.$$

Assumption (H3)' can still be seen as an inward pointing condition on the rectangular domain $K = [0, \bar{y}_1] \times [0, \bar{y}_2]$, In particular, the first two conditions ensure for any $t \in [0, T]$ the existence of an inward pointing drift at any point $y \in \partial K \setminus \{(0, 0), (\bar{y}_1, \bar{y}_2)\}$, while the third concerns the case where $y \in \{(0, 0), (\bar{y}_1, \bar{y}_2)\}$, i.e. where both reservoirs are either empty or full. Indeed, in this latter case one has to consider that if the dams are both empty the overall system needs to be filled by some positive inflow $\beta_1(t) + \beta_2(t)$, while if the dams reached their maximal capacity, to let the level decrease in the upper reservoir one has to release water in the lower one at a rate greater than $\beta_1(t)$, so the lower reservoir has to be able to

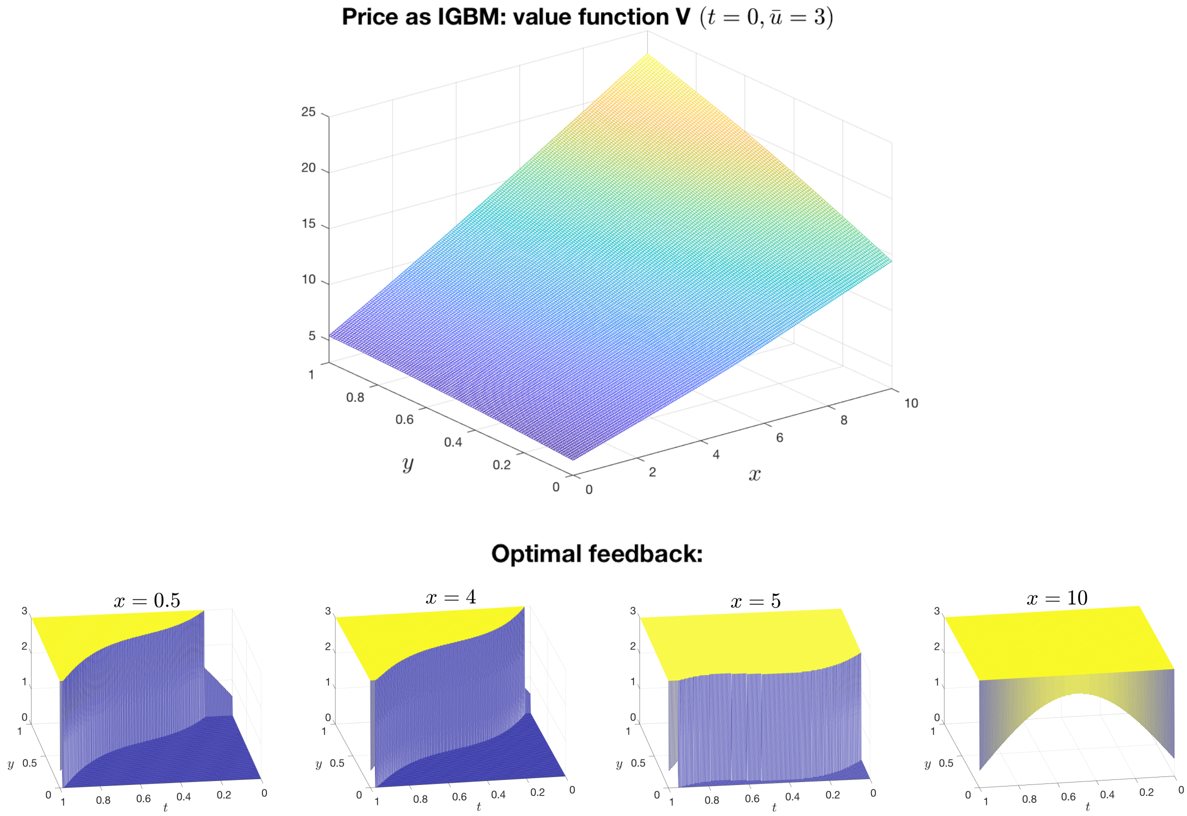

FIGURE 3. Single dam model, price modeled as IGBM with $b(x) = 5 - x$ and $\sigma(x) = 0.1x$, maximal discharge rate $\bar{u} = 3$ (Assumption (H3) is satisfied). Top: numerical approximation of the value function V at $t = 0$. Bottom: optimal feedback as a function of (t, y) for different values of $x = 0.5, 4, 5, 10$ (from left to right). Discretization steps used in numerical tests: $\Delta x = 0.05$, $\Delta y = 0.01$, $\Delta t = 0.002$.

discharge at a rate greater than $\beta_1(t) + \beta_2(t)$. Under assumptions (H1)-(H3)' the value function V is characterized by following HJB equation

$$-V_t - b(t,x)V_x - \frac{1}{2}\sigma^2(t,x)V_{xx} + \inf_{u \in U^{t,y}} \Big\{ -(\beta_1(t) - u_1)V_{y_1} - (\beta_2(t) + u_1 - u_2)V_{y_2} - L(x,u) \Big\} = 0$$

on $(0,T) \times \overline{Q}$ with $Q := (0,+\infty) \times (0,\bar{y}_1) \times (0,\bar{y}_2)$, where this time $U^{t,y}$ is much more involved, being defined as

$$U^{t,y} := U \cap \begin{cases} \{u_1 \le \beta_1(t), u_2 \le \beta_2(t) + u_1\} & \text{if } y_1 = 0, y_2 = 0, \\ \{u_1 \le \beta_1(t)\} & \text{if } y_1 = 0, y_2 \in (0,\bar{y}_2), \\ \{u_1 \le \beta_1(t), u_2 \ge \beta_2(t) + u_1\} & \text{if } y_1 = 0, y_2 = \bar{y}_2, \\ \{u_2 \le \beta_2(t) + u_1\} & \text{if } y_1 \in (0,\bar{y}_1), y_2 = 0, \\ U & \text{if } y_1 \in (0,\bar{y}_1), y_2 \in (0,\bar{y}_2), \\ \{u_2 \ge \beta_2(t) + u_1\} & \text{if } y_1 \in (0,\bar{y}_1), y_2 = \bar{y}_2, \\ \{u_1 \ge \beta_1(t), u_2 \le \beta_2(t) + u_1\} & \text{if } y_1 = \bar{y}_1, y_2 = 0, \\ \{u_1 \le \beta_1(t)\} & \text{if } y_1 = \bar{y}_1, y_2 \in (0,\bar{y}_2), \\ \{u_1 \ge \beta_1(t), u_2 \ge \beta_2(t) + u_1\} & \text{if } y_1 = \bar{y}_1, y_2 = \bar{y}_2. \end{cases} \tag{10}$$

We plot the results obtained in this case under a GBM and IGBM model in Figure 4 and 5, respectively. Here, we take $\beta_1 = \beta_2 = \beta$, $\gamma = 1.5$, $\bar{y}_1 = \bar{y}_2 = 1$, $u_1 \in [-1,3]$ and $u_2 \in [0,5.5]$ which ensures (H3)' being satisfied.

We can observe similar qualitative results as the single dam model. If the price is modeled by a GBM (Figure 4) one gets an optimal feedback independent of x and numerically of bang-bang type, though with an analytical form more difficult than the one-dam case. Similarly to the case of a single reservoir, the optimal control is to wait for the dam to be filled far from the terminal time T and for a low level of water in the system. Observe that, since $\gamma > 1$, water is never pumped from the lower reservoir since this can only be optimal in view of a price-dependent feedback. Moreover, the maximal discharge rate of the upper reservoir reduces from $\bar{u}_1$ to $\beta_1(t)$ when the reservoir is empty (first three panels of Figure 4), while the one of the lower reservoir reduces from $\bar{u}_2$ to $\beta_2(t) + \bar{u}_1$ if the reservoir is empty and the upper one is not (fifth and sixth panels on Figure 4), and further to $\beta_2(t) + \beta_1(t)$ if also the upper reservoir is empty (fourth panel on Figure 4). This is even more visible looking at the last six panels of Figure 5b corresponding to the IGBM case with price above the long-term mean (which is 5 here). We recall that, in this case, the optimal strategy is to discharge with the maximal rate independently of the reservoir level. When the price if modeled by an IGBM, numerically we can observe, as in the one-dam case, a linear-like structure in the value function, both in x as in y (Figure 5a), and a bang-bang-like structure in the optimal controls (Figure 5b), as follows. The optimal control ν_2^* is always in one of the two states "wait for the lower reservoir to be filled" or "discharge it at the maximum possible rate". Instead, the optimal control ν_1^* is always in one of the three states "discharge the upper reservoir at the maximum possible rate", "let it be filled" or "pump water up as much as possible". In particular, if the price is below the long-term mean, it is optimal to let the reservoirs be filled and, for very low levels of the price (see the first three panels of Figure 5b), the water is also pumped back to the upper reservoir at the maximal rate. Observe that if the lower reservoir is empty (first of the three panels) water can be pumped back to the upper one with maximal rate $\min(\beta_2(t), \underline{u}_1)$ instead of $\underline{u}_1$. Moreover, we can see in the three panels on the second line of Figure 5b that, if the lower reservoir is full its minimal discharge rate is $\max(\beta_2(t) - \underline{u}_1, 0)$ unless the upper reservoir is full (third of the three panels) where the minimal rate is $\beta_2(t)$ since water cannot be pumped back.

4. State constrained optimal control: controllability issues

The filling rate β (resp. β_1, β_2) is exogenously given and many structural reasons may prevent Assumption (H3) (resp. (H3)') to be satisfied. Considering again the simpler single dam model, a typical situation could be that the dam was built in the past with a maximum outflow level $\bar{u}$ as suitable to a given inflow intensity $\tilde{\beta}$ which used to be observed in the past, but nowadays, due to climate changes, the inflow intensity has changed from $\tilde{\beta}$ to β, which could potentially be greater than $\bar{u}$ for some given times $t \in [0, T]$. As an example, when the inflow intensity was $\tilde{\beta}(t) = \sin(\pi t) + 0.5$ for $t \in [0, 1]$, the dam could have been built with $\bar{u} = 2$, thus satisfying Assumption (H3). In this case we say that the problem is *controllable*. If we now assume that the current inflow intensity is instead $\beta(t) = 2\sin(\pi t) + 0.5$, we can easily see from Figure 1 that Assumption (H3) is now violated and then controllability is lost.

In order to treat the problem in its full generality we aim in what follows to remove Assumption (H3). Let us then assume that $U = [0, \bar{u}]$ with possibly $\bar{u} < \max_{t\in[0,T]} \beta(t)$ (we keep $\beta(t) \geq 0, \forall t \in [0, T]$, for simplicity). The first difficulty one faces in this scenario is to determine the maximal "controllable region", i.e. the set $\mathcal{D}_t \subseteq K$ such that

$$\mathcal{U}_{\text{ad}}^{t,y} \neq \emptyset \Leftrightarrow y \in \mathcal{D}_t.$$

Roughly speaking $\mathcal{D}_t$ is the largest subset of K where at time t the value function is well defined. Let us assume the function β is qualitatively as the one reported in Figure 1, i.e. such that $\beta(t) \geq \bar{u}$ in the interval $[t^*, T^*]$, for some $0 \leq t^* \leq T^* \leq T$.

Proposition 4.1. *Let*

$$\hat{y}_t := \bar{y} - \max\left(\int_t^{T^*\vee t} (\beta(s) - \bar{u})\mathrm{d}s, 0\right).$$

Then, for any $t \in [0, T]$ one has

$$\mathcal{U}_{ad}^{t,y} \neq \emptyset \Leftrightarrow y \in \mathcal{D}_t := [0, \hat{y}_t].$$

Proof. The result is straightforward if $t \geq T^*$. Let $t^* \leq t < T^*$. If there exists a control $\nu \in \mathcal{U}$ such that $Y_s^{t,y,\nu} \in [0, \bar{y}], \forall s \in [t, T]$ a.s., this certainly implies (taking $s = t$) that $y \in [0, \bar{y}]$. Moreover, taking $s = T^*$ (recalling we are in the case $t < T^*$) it also gives

$$y \leq \bar{y} - \int_t^{T^*} (\beta(r) - \nu_r)\mathrm{d}r \leq \bar{y} - \int_t^{T^*} (\beta(r) - \bar{u})\mathrm{d}r.$$

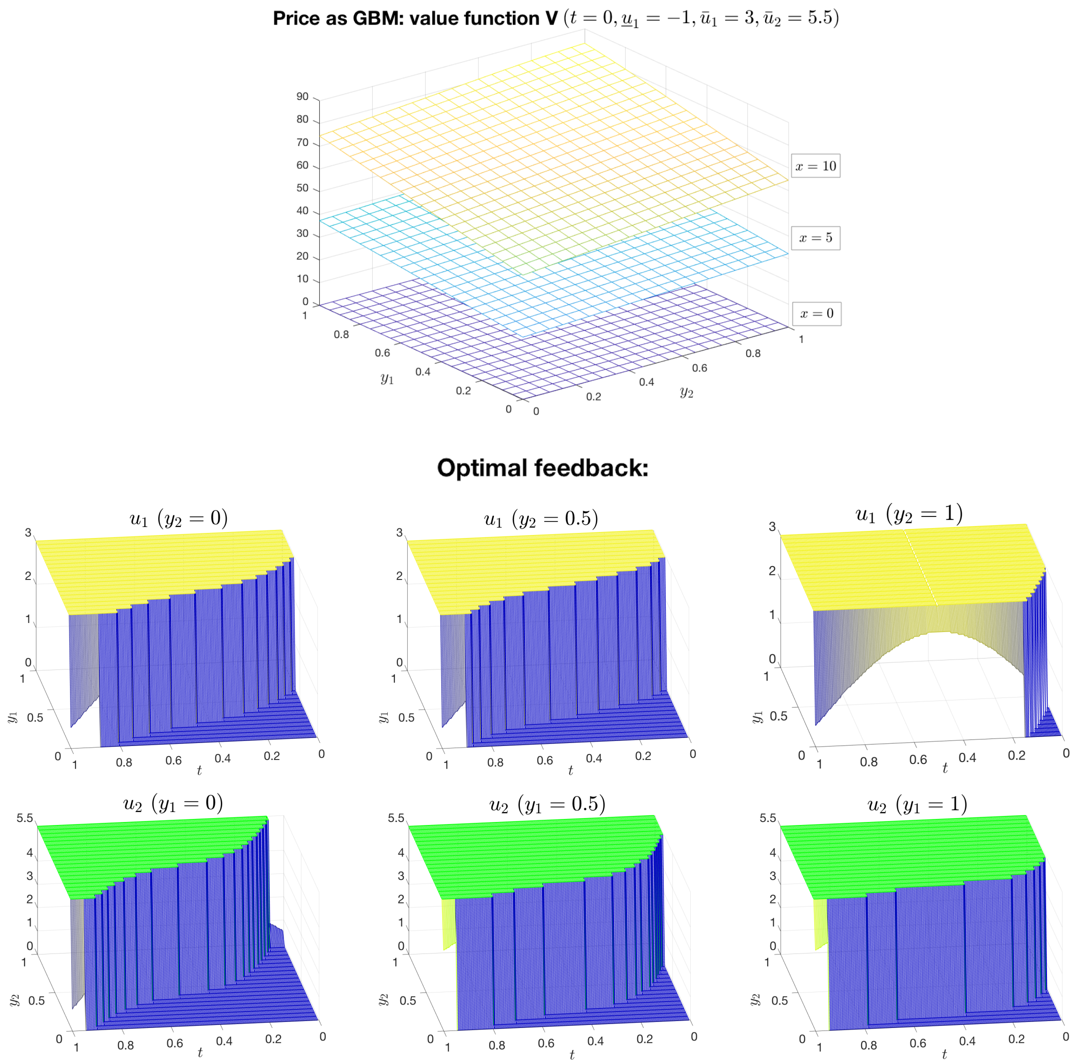

FIGURE 4. Two dams, price modeled as GBM with $b(t,x) = 0.05x$ and $\sigma(t,x) = 0.1x$, maximal/minimal discharge rate $\underline{u}_1 = -1, \bar{u}_1 = 3$ and $\bar{u}_2 = 5.5$ (Assumption (H3)' is satisfied). Top: numerical approximation of the value function V at $t = 0$. Second (resp. third) line: optimal feedback u_1 (resp. u_2) as a function of (t, y_1) (resp. (t, y_2)) for different values of y_2 (resp. y_1). Discretization parameters used in numerical tests: $\Delta x = 0.5$, $\Delta y_1 = \Delta y_2 = 0.05$, $\Delta t = 0.008$.

On the other hand, let $y \in [0, \hat{y}_t]$: defining the control

$$\hat{\nu}_r := \bar{u}\mathbb{1}_{[t,\tau)\cup[t^*,T^*)} + \beta(r)\mathbb{1}_{[\tau,t^*)\cup[T^*,T]} \quad a.s.,$$

where $\tau = \inf\{r \in [t,t^*] : Y_r^{t,y,\bar{u}} < 0\} \wedge t^*$, one has $\hat{\nu} \in \mathcal{U}$ (because $\beta(r) \in U$ for $r \notin [t^*, T^*]$) and $Y_s^{t,y,\hat{\nu}} \in K\, \forall s \in [t,T]$ a.s. thanks to the construction of $\hat{\nu}$.

□

Finding explicitly the sets $\{\mathcal{D}_t, t \in [0,T]\}$ can be in general very complex. An alternative characterization of $\{\mathcal{D}_t, t \in [0,T]\}$ can be obtained using the so-called *level set approach*, see e.g. [6, 7, 17, 22]. In particular, it is possible to define a function $\vartheta : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that for any $t \in [0,T]$

$$y \in \mathcal{D}_t \Leftrightarrow \vartheta(t,y) = 0. \tag{11}$$

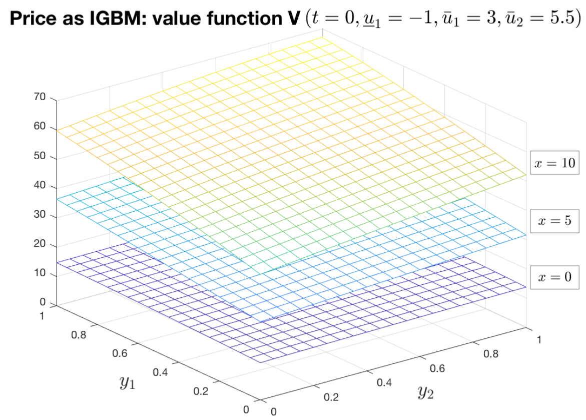

FIGURE 5A. Two dams, price modeled as IGBM with $b(t,x)=5-x$ and $\sigma(t,x)=0.1x$, maximal/minimal discharge rate $\underline{u}_1=-1, \bar{u}_1=3$ and $\bar{u}_2=5.5$ (Assumption (H3)' is satisfied). Numerical approximation of the value function V at $t=0$. Discretization steps used in numerical tests: $\Delta x=0.5$, $\Delta y_1=\Delta y_2=0.05$, $\Delta t=0.008$.

The equivalence above relates the characterization of the set $\mathcal{D}_t$ with the one of the zero level set of the real-valued function ϑ. A possible choice of ϑ is

$$\vartheta(t,y)=\inf_{\nu\in\mathcal{U}}\int_t^T d_K(Y_s^{t,y,\nu})\mathrm{d}s, \tag{12}$$

where $d_K(y)$ denotes the (positive) distance of y to the set K. It is not difficult to show that, for any $(t,y)\in[0,T]\times\mathbb{R}$, problem (12) admits an optimal control $\bar{\nu}\in\mathcal{U}$ such that $\vartheta(t,y)=\int_t^T d_K(Y_s^{t,y,\bar{\nu}})\mathrm{d}s$ (this follows by the same arguments used in the proof of Theorem 5.1) and, as a consequence, (11) holds. The function ϑ is the value function of an (unconstrained) optimal control problem fully characterized by a first order HJB equation. Then, thanks to (11), numerically solving this HJB equation leads to a numerical approximation of the set $\mathcal{D}_t$.

Figure 6 (left) shows the level sets of the numerical approximation of the function $\vartheta(t,\cdot)$ for $t\in[0,T]$. In particular, the white region approximates the set of level zero and therefore represents a numerical approximation of the region $\{\mathcal{D}_t, t\in[0,T]\}$. Of course, the choice of ϑ is not unique and many different functions may provide the characterization given by (11) (see for instance the discussion in [6]).

Once defined the set $\mathcal{D}_t$ for any $t\in[0,T]$, which is in general not trivial, as we have seen, one has to deal with the optimal control problem associated to $V(t,x,y)$ settled in the domain $\{(t,x,y) \mid t\in[0,T], x\in\mathbb{R}^+, y\in\mathcal{D}_t\}$. For a problem of this form, the regularity properties of the value function on its domain of definition, its HJB characterization as well as its numerical approximation are very complex and non standard in optimal control theory. This motivates the study presented in the next section where, to avoid the direct treatment of the state constrained optimal control problem by dynamic programming techniques, we pass to a suitable reformulation of the problem that eliminates this difficulty.

5. Treatment of the state constraints without controllability assumptions

Let us go back to the model with two reservoirs. Motivated by the discussion in the previous section, we follow here the alternative approach presented in [8] to overcome the issues related to the treatment of state constraints when assumptions such as (H3) (and (H3)') are not satisfied.

Let us start introducing the following auxiliary *unconstrained* optimal control problem:

$$W(t,x,y,z):=\inf_{(\nu,\alpha)\in\mathcal{U}\times L^2_{\mathbb{F}}}\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\nu,\alpha},0\Big)+\int_t^T d_K(Y_s^{t,y,\nu})\mathrm{d}s\Big], \tag{13}$$

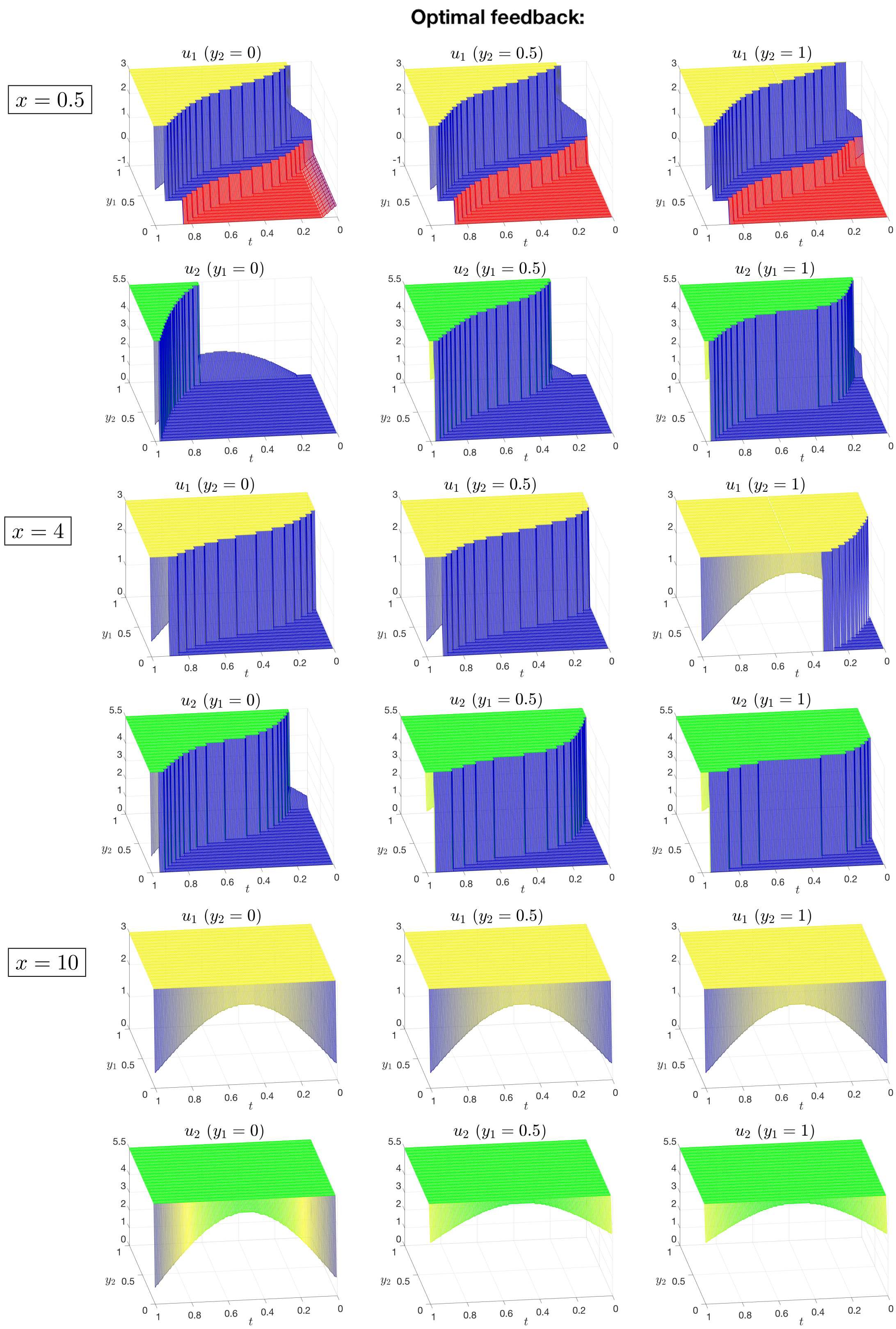

FIGURE 5B. Two dams, price modeled as IGBM with $b(t, x) = 5 - x$ and $\sigma(t, x) = 0.1x$, maximal/minimal discharge rate $\underline{u}_1 = -1, \bar{u}_1 = 3$ and $\bar{u}_2 = 5.5$ (Assumption (H3)' is satisfied). Numerical approximation of the optimal feedbacks u_1 and u_2. Discretization steps used in numerical tests: $\Delta x = 0.5$, $\Delta y_1 = \Delta y_2 = 0.05$, $\Delta t = 0.008$.

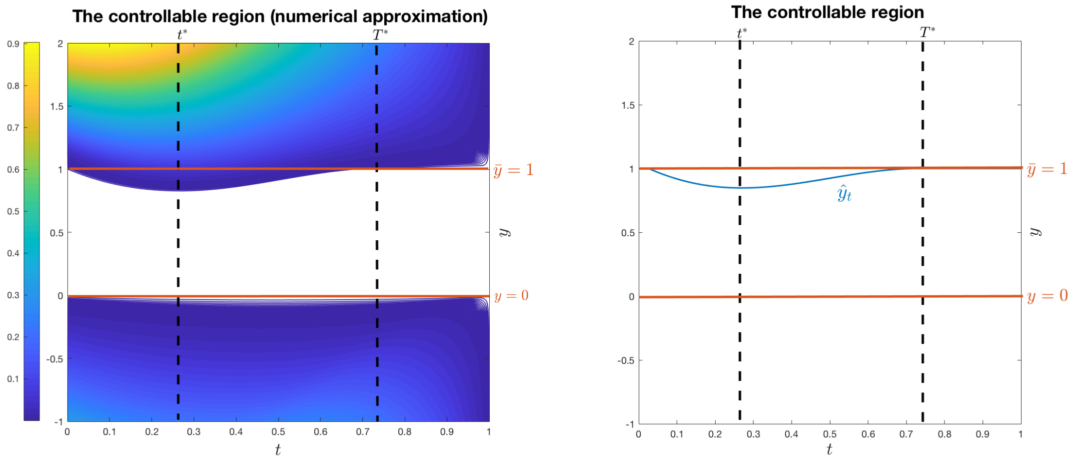

FIGURE 6. The set $\{\mathcal{D}_t, t \in [0,T]\}$ with β as in Figure 1. Left: level sets of the function ϑ (the white region corresponds to the set of level zero). Right: plot of the function $\hat{y}_t$ explicitly defining $\mathcal{D}_t$ in Proposition 4.1.

where $L^2_{\mathbb{F}} \equiv L^2_{\mathbb{F}}([t,T];\mathbb{R})$ is the set of adapted processes α such that $\|\alpha\|_{L^2_{\mathbb{F}}} := \mathbb{E}[\int_t^T |\alpha_s|^2 \mathrm{d}s] < +\infty$ and

$$Z^{t,x,z,\nu,\alpha}_{\cdot} := z - \int_t^{\cdot} \tilde{L}(X^{t,x}_s, \nu_s)\mathrm{d}s + \int_t^{\cdot} \alpha_s \mathrm{d}B_s$$

for $\tilde{L}(x,u) := x(\kappa(u) - \bar{\kappa})$ with $\bar{\kappa} := \max_{u \in U} \kappa(u)$.

We refer to W as the *level set function* for problem (5). The fundamental link of W with the original control problem in Equation (5) can be seen by analyzing the structure of W: in particular, the link is with the level set $\{(t,x,y,z) \mid W(t,x,y,z) = 0\}$. In fact, we can see that in the definition of W we are minimizing two non-negative addends. The latter is linked to the value function θ of the problem in Equation (12): in particular, we have that there exists a $\nu \in \mathcal{U}$ such that $\int_t^T d_K(Y^{t,y,\nu}_s)\mathrm{d}s = 0$ a.s. if and only if $y \in \mathcal{D}_t$, and in that case $\nu \in \mathcal{U}^{t,y}_{\text{ad}}$ too. The former addend instead is zero if and only if $Z^{tax,z,\nu,\alpha}_T \leq 0$ a.s., i.e. $z \leq \int_t^T \tilde{L}(X^{t,x}_s, \nu_s)\mathrm{d}s + \int_t^T \alpha_s \mathrm{d}B_s$ a.s. . Since $\alpha \in L^2_{\mathbb{F}}$ can be controlled too, it makes sense to choose it in such a way that the stochastic integral represents all the randomness of the first integral, i.e. in such a way to obtain that $z \leq \mathbb{E}[\int_t^T \tilde{L}(X^{t,x}_s, \nu_s)\mathrm{d}s] = \mathbb{E}[\int_t^T X^{t,x}_s \kappa(\nu_s)\mathrm{d}s] - \bar{\kappa}\mathbb{E}[\int_t^T X^{t,x}_s \mathrm{d}s]$. Of course, we can pick the largest z having this property when $\nu \in \mathcal{U}^{t,y}_{\text{ad}}$ maximizes the first addend, i.e. when it is also the optimal control of the original problem (5). In conclusion, it turns out that $W(t,x,y,z) = 0$ only if $\mathcal{U}^{t,y}_{\text{ad}}$ is not empty and if

$$z \leq V(t,x,y) - \bar{\kappa}\mathbb{E}\left[\int_t^T X^{t,x}_s \mathrm{d}s\right]$$

The following theorem states rigorously this fundamental link between the original optimal control problem (5) and the auxiliary problem (13).

Theorem 5.1. *The following holds*

$$V(t,x,y) = \sup\left\{z \leq 0 : W(t,x,y,z) = 0\right\} + G(t,x), \tag{14}$$

with $G(t,x) := \bar{\kappa}\,\mathbb{E}\left[\int_t^T X^{t,x}_s \mathrm{d}s\right]$.

Proof. We start proving that

$$V(t,x,y) \leq \sup\left\{z \leq 0 : W(t,x,y,z) = 0\right\} + G(t,x).$$

If $\mathcal{U}_{\rm ad}^{t,y}=\emptyset$ there is nothing to prove. Let us then assume that $\mathcal{U}_{\rm ad}^{t,y}\neq\emptyset$. For any $\nu\in\mathcal{U}_{\rm ad}^{t,y}$, define

$$z^\nu := \mathbb{E}\left[\int_t^T L(X_s^{t,x},\nu_s)\mathrm{d}s\right] - G(t,x) = \mathbb{E}\left[\int_t^T \tilde{L}(X_s^{t,x},\nu_s)\mathrm{d}s\right]\leq 0$$

Since $\int_t^T \tilde{L}(X_s^{t,x},\nu_s)\mathrm{d}s\in L^2_{\mathbb{F}}$, by the martingale representation theorem there exists $\alpha\in L^2_{\mathbb{F}}$ such that a.s.

$$z^u = \int_t^T \tilde{L}(X_s^{t,x},\nu_s)\mathrm{d}s - \int_t^T \alpha_s \mathrm{d}B_s.$$

This implies that

$$0\leq W(t,x,y,z^\nu)\leq \mathbb{E}\Big[\max\Big(Z_T^{t,x,z^\nu,\nu,\alpha},0\Big)+\int_t^T d_K(Y_s^{t,y,\nu})\mathrm{d}s\Big]=0$$

which gives the following inclusion

$$\Big\{z\leq 0: z=\mathbb{E}\Big[\int_t^T L(X_s^{t,x},\nu_s)\mathrm{d}s\Big]-G(t,x),\nu\in\mathcal{U}_{\rm ad}^{t,y}\Big\}\subseteq\Big\{z\leq 0: W(t,x,y,z)=0\Big\}$$

from which the desired inequality follows.

We now prove that

$$V(t,x,y)\geq \sup\Big\{z\leq 0: W(t,x,y,z)=0\Big\}+G(t,x).$$

Let $z\leq 0$ such that $W(t,x,y,z)=0$ (if such z does not exists there is nothing to prove). Let $(\nu^k,\alpha^k)\in\mathcal{U}\times L^2_{\mathbb{F}}$ be a minimizing sequence for $W(t,x,y,z)$. Therefore for any $\varepsilon>0$ there exists k_0 such that $\forall k\geq k_0$ one has

$$\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\nu^k,\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\nu^k})ds\Big]\leq W(t,x,y,z)+\varepsilon=\varepsilon.$$

Being ν^k uniformly bounded in norm $L^2_{\mathbb{F}}$ (because ν takes values in a compact set), there exists a weakly convergent subsequence (for simplicity still indexed by k), i.e.

$$\nu^k\to\hat{\nu}\qquad\text{weakly in } L^2_{\mathbb{F}},$$

for some $\hat{\nu}\in\mathcal{U}$. Applying Mazur's theorem one has that there exists $\tilde{\nu}^k=\sum_{i\geq 0}\lambda_i\nu^{i+k}$ with $\lambda_i\geq 0$ and $\sum_{i\geq 0}\lambda_i=1$ such that

$$\tilde{\nu}^k\to\hat{\nu}\qquad\text{strongly in } L^2_{\mathbb{F}}.$$

We then consider $(\tilde{\nu}^k,\tilde{\alpha}^k)\equiv\sum_{i\geq 0}\lambda_i(\nu^{i+k},\alpha^{i+k})$. Observe that $(\tilde{\nu}^k,\tilde{\alpha}^k)$ still belongs to $\mathcal{U}\times L^2_{\mathbb{F}}$ because U is convex and $L^2_{\mathbb{F}}$ is a convex space. Let us now consider $(Z^{\tilde{\nu}^k,\tilde{\alpha}^k},Y^{\tilde{\nu}^k})$. One has

$$\begin{aligned}\mathbb{E}\left[\left|Z_T^{t,x,z,\hat{\nu},\tilde{\alpha}^k}-Z_T^{t,x,z,\tilde{\nu}^k,\tilde{\alpha}^k}\right|^2\right]&=\mathbb{E}\Big[\Big|\int_t^T X_s^{t,x}\big(\tilde{\nu}_{2,s}^k-\hat{\nu}_{2,s}+c(\tilde{\nu}_{1,s}^k)-c(\hat{\nu}_{1,s})\big)\mathrm{d}s\Big|^2\Big]\\&\leq T\mathbb{E}\Big[\int_t^T\Big|X_s^{t,x}\big(\tilde{\nu}_{2,s}^k-\hat{\nu}_{2,s}+c(\tilde{\nu}_{1,s}^k)-c(\hat{\nu}_{1,s})\big)\Big|^2\mathrm{d}s\Big]\\&\leq T\mathbb{E}\Big[\int_t^T (X_s^{t,x})^2\big(2|\tilde{\nu}_{2,s}^k-\hat{\nu}_{2,s}|^2+2|c(\tilde{\nu}_{1,s}^k)-c(\hat{\nu}_{1,s})|^2\big)\mathrm{d}s\Big]\\&\leq 2T\mathbb{E}\Big[\int_t^T (X_s^{t,x})^2\big(|\tilde{\nu}_{2,s}^k-\hat{\nu}_{2,s}|^2+L_c^2|\tilde{\nu}_{1,s}^k-\hat{\nu}_{1,s}|^2\big)\mathrm{d}s\Big]\end{aligned}$$

where we used the Lipschitz continuity with Lipschitz constant L_c ($=\gamma$ for the particular choice in (4)) of the function c. Moreover, from the strong convergence in $L^2_{\mathbb{F}}$-norm it follows that there exists a subsequence $\tilde{\nu}^{k_n}$ such that $|\tilde{\nu}^{k_n}-\hat{\nu}|^2\to 0$ a.e.. One also has

$$(X_s^{t,x})^2\left(2|\tilde{\nu}_{2,s}^k-\hat{\nu}_{2,s}|^2+2L_c^2|\tilde{\nu}_{1,s}^k-\hat{\nu}_{1,s}|^2\right)\leq CX_s^2\qquad\text{a.s.}$$

for some constant C depending on the uniform bound of elements in $\mathcal{U}$. Being X_s^2 integrable under the classical Assumption (H1), we can apply the dominate convergence theorem on the subsequence ν^{k_n} to get

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_t^T (X_s^{t,x})^2\big(|\tilde\nu_{2,s}^{k_n}-\hat\nu_{2,s}|^2+|\tilde\nu_{1,s}^{k_n}-\hat\nu_{1,s}|^2\big)\mathrm{d}s\Big]=\mathbb{E}\Big[\int_t^T \lim_{n\to\infty}(X_s^{t,x})^2\big(|\tilde\nu_{2,s}^{k_n}-\hat\nu_{2,s}|^2+|\tilde\nu_{1,s}^{k_n}-\hat\nu_{1,s}|^2\big)\mathrm{d}s\Big]=0.$$

In conclusion, there exists a suitable subsequence (indexed with k for simplicity) such that

$$\mathbb{E}\left[\left|Z_T^{t,x,z,\tilde\nu^k,\tilde\alpha^k}-Z_T^{t,x,z,\hat\nu,\tilde\alpha^k}\right|^2\right]\to 0.$$

Similarly one has

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|Y_s^{t,y,\tilde\nu^k}-Y_s^{t,y,\hat\nu}\right|^2\right]\to 0.$$

Then, for any $\varepsilon>0$ there exists k_1 such that $\forall k\geq k_1$ one has

$$\left|\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\tilde\nu^k,\tilde\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\tilde\nu^k})\mathrm{d}s\Big]-\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\hat\nu,\tilde\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\hat\nu})\mathrm{d}s\Big]\right|\leq\varepsilon.$$

Moreover, one has (using the fact that $c(\cdot)$ is concave and $X_s^{t,x}\geq 0$):

$$\begin{aligned}
Z_T^{t,x,z,\tilde\nu^k,\tilde\alpha^k}&=z-\int_t^T X_s^{t,x}\big(\tilde\nu_{2,s}^k+c(\tilde\nu_{1,s}^k)-\bar\kappa\big)\mathrm{d}s+\int_t^T\tilde\alpha_s^k\mathrm{d}B_s\\
&=z-\int_t^T X_s^{t,x}\Big(\sum_{i\geq0}\lambda_i\nu_{2,s}^{i+k}+c\Big(\sum_{i\geq0}\lambda_i\nu_{1,s}^{i+k}\Big)-\bar\kappa\Big)\mathrm{d}s+\int_t^T\sum_{i\geq0}\lambda_i\alpha_s^{i+k}\mathrm{d}B_s\\
&\leq z-\int_t^T X_s^{t,x}\Big(\sum_{i\geq0}\lambda_i\nu_{2,s}^{i+k}+\sum_{i\geq0}\lambda_i c(\nu_{1,s}^{i+k})-\bar\kappa\Big)\mathrm{d}s+\int_t^T\sum_{i\geq0}\lambda_i\alpha_s^{i+k}\mathrm{d}B_s\\
&=z-\sum_{i\geq0}\lambda_i\int_t^T X_s^{t,x}\Big(\nu_{2,s}^{i+k}+c(\nu_{1,s}^{i+k})-\bar\kappa\Big)\mathrm{d}s+\sum_{i\geq0}\lambda_i\int_t^T\alpha_s^{i+k}\mathrm{d}B_s\\
&=\sum_{i\geq0}\lambda_i Z_T^{t,x,z,\nu^{i+k},\alpha^{i+k}}
\end{aligned}$$

and (Y being linear in ν)

$$Y_s^{t,y,\tilde\nu^k}=\sum_{i\geq0}\lambda_i Y_s^{t,y,\nu^{i+k}}.$$

Putting these things together (observing that $\max(z,0)$ is convex and that being K convex also d_K is) one has for $k\geq\max(k_0,k_1)$:

$$\begin{aligned}
\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\hat\nu,\tilde\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\hat\nu})\mathrm{d}s\Big]&\leq\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\tilde\nu^k,\tilde\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\tilde\nu^k})\mathrm{d}s\Big]+\varepsilon\\
&=\mathbb{E}\Big[\max\Big(\sum_{i\geq0}\lambda_i Z_T^{t,x,z,\nu^{i+k},\alpha^{i+k}},0\Big)+\int_t^T d_K(\sum_{i\geq0}\lambda_i Y_s^{t,y,\nu^{i+k}})\mathrm{d}s\Big]+\varepsilon\\
&\leq\sum_{i\geq0}\lambda_i\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\nu^{i+k},\alpha^{i+k}},0\Big)+\int_t^T d_K(Y_s^{t,y,\nu^{i+k}})\mathrm{d}s\Big]+\varepsilon\\
&\leq W(t,x,y,z)+2\varepsilon\\
&=2\varepsilon.
\end{aligned}$$

From the previous inequality we can immediately conclude that $\hat\nu\in\mathcal{U}_{\mathrm{ad}}^{t,y}$, because for any arbitrary $\varepsilon>0$ one has $0\leq\mathbb{E}\Big[\int_t^T d_K(Y_s^{t,y,\hat\nu})\mathrm{d}s\Big]\leq 2\varepsilon$. Moreover, one also has

$$z-\mathbb{E}\left[\int_t^T\tilde L(X_s^{t,x},\hat\nu_s)\mathrm{d}s\right]\leq\mathbb{E}\Big[\max\Big(Z_T^{t,x,z,\hat\nu,\tilde\alpha^k},0\Big)+\int_t^T d_K(Y_s^{t,y,\hat\nu})\mathrm{d}s\Big]\leq 2\varepsilon$$

which gives

$$z \le \mathbb{E}\left[\int_t^T L(X_s^{t,x}, \hat{\nu}_s)\mathrm{d}s\right] + G(t,x)$$

and then $z \le V(t,x,y) - G(t,x)$.

□

Corollary 5.1. *If $W(t,x,y,z)=0$, there exists an optimal $(\hat{\nu},\hat{\alpha}) \in \mathcal{U} \times L^2_{\mathbb{F}}$ that realizes the infimum in* (13).

Proof. Given $\hat{\nu} \in \mathcal{U}^{t,y}_{\text{ad}}$ as in the proof of Theorem 5.1 and $\hat{\alpha} \in L^2_{\mathbb{F}}$ such that

$$\int_t^T \tilde{L}(X_s^{t,x}, \hat{\nu}_s)\mathrm{d}s - \int_t^T \hat{\alpha}_s \mathrm{d}B_s = \mathbb{E}\left[\int_t^T \tilde{L}(X_s^{t,x}, \hat{\nu}_s)\mathrm{d}s\right] \ge z$$

one has

$$\mathbb{E}\left[\max\left(Z_T^{t,x,z,\hat{\nu},\hat{\alpha}}, 0\right) + \int_t^T d_K(Y_s^{t,y,\hat{\nu}})\mathrm{d}s\right] = 0 = W(t,x,y,z).$$

□

Remark 5.1. *The use of the modified cost $\tilde{L}$ instead of L guarantees the non-positivity of the running cost which is a useful property in view of the HJB characterization provided in* [8]*. Notice that the computation of the function $G(t,x)$ typically does not add any difficulty and we can consider it given by the problem data.*

The following regularity result turns out to be particularly useful for the PDE characterization of W.

Proposition 5.1. *The function W is uniformly continuous in (x,y,z) and satisfies $\lim_{t\to T} W(t,x,y,z) = \max(z,0)$ uniformly.*
Moreover, for any $z \ge 0$ one has $W(t,x,y,z) = z + \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[\int_t^T \left(\tilde{L}(X_s^{t,x},\nu_s) + d_k(Y_s^{t,y,\nu})\right)\mathrm{d}s\right]$.

Proof. The uniform continuity of W with respect to (x,y,z) is straightforward. Moreover, by the definition of W, the Lipschitz continuity of d_K and $\tilde{L}$ and classical estimates on the processes $(X_\cdot, Y_\cdot)$, one has for any $h>0$ such that $T-h \in [t,T]$ and $\nu \in \mathcal{U}$,

$$\begin{aligned} W(T-h,x,y,z) - \max(z,0) &\le \mathbb{E}\left[\max\left(Z_T^{T-h,x,z,\nu,0}, 0\right) + \int_{T-h}^T d_K(Y_s^{T-h,y,\nu})\mathrm{d}s\right] - \max(z,0) \\ &\le C\mathbb{E}\left[|Z^{T-h,x,z,\nu,0} - z|\right] + h\left(1 + C\mathbb{E}\left[\sup_{s\in[T-h,T]} |Y_s^{t,y,\nu}|\right]\right) \le Ch(1+|x|+|y|), \end{aligned}$$

where by C we denote any generic nonnegative constant. If $z \le 0$ by the nonegativity of W one immediately has $W(t,x,y,z) - \max(z,0) \ge 0$. Otherwise if $z>0$, by the nonegativity of d_K, the martingale property of stochastic integrals and classical estimates on the process X, one has

$$W(t,x,y,z) - \max(z,0) \ge \inf_{(\nu,\alpha)\in\mathcal{U}\times L^2_{\mathbb{F}}} \mathbb{E}\left[Z_T^{T-h,x,z,\nu,\alpha} - z\right] = \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[Z_T^{T-h,x,z,\nu,0} - z\right] \ge -Ch(1+|x|),$$

which concludes the proof of $\lim_{t\to T} W(t,x,y,z) = \max(z,0)$.
Let us now assume that $z \ge 0$. Minorating the positive part by its argument and using the martingale property of stochastic integrals one always has

$$W(t,x,y,z) \ge z + \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[\int_t^T \left(\tilde{L}(X_s^{t,x},\nu_s) + d_k(Y_s^{t,y,\nu})\right)\mathrm{d}s\right].$$

The reverse inequality is obtained observing that

$$W(t,x,y,z) \le \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[\max\left(z - \int_t^T \tilde{L}(X_s^{t,x},\nu_s)\mathrm{d}s, 0\right) + \int_t^T d_K(Y_s^{t,y,\nu})\mathrm{d}s\right]$$

and using the non negativity of the process $z - \int_t^T \tilde{L}(X_s^{t,x},\nu_s)\mathrm{d}s$.

□

Corollary 5.2. *Let $V(t,x,y)$ be finite. Then, there exists $z^* \le 0$ that realizes the supremum in (14) and if (ν^*, α^*) is an optimal control for the level set problem (13) at point (t,x,y,z^*), then ν^* is admissible and an optimal control for the original optimal control problem (5) at (t,x,y).*

Proof. Being the supremum in (14) defined on a nonempty (because V is finite) set closed and bounded from above, one can always find $z^* \le 0$ realizing this supremum. It follows by the proof of Theorem 5.1 that there exists $\nu^* \in \mathcal{U}_{\mathrm{ad}}^{t,y}$ such that

$$z^* \le \mathbb{E}\left[\int_t^T \tilde{L}(X_s^{t,x}, \nu_s^*)\mathrm{d}s\right] = \mathbb{E}\left[\int_t^T \tilde{L}(X_s^{t,x}, \nu_s^*)\mathrm{d}s\right] - G(t,x).$$

Recalling that $z^* = V(t,x,y) - G(t,x)$ one can immediately conclude that ν^* is an optimal control for the original problem. □

In conclusion, by Theorem 5.1 and Corollary 5.2, on its domain of definition the optimal value function $V(t,x,y)$ and the associated optimal strategy is completely determined by the unconstrained optimal problem (13).

It is possible to show that the value function W is associated to the following HJB equation:

$$\begin{cases} -W_t - b(t,x)W_x - \frac{1}{2}\sigma^2(t,x)W_{xx} - d_K(y) + \sup\limits_{u\in U,\alpha\in\mathbb{R}}\Big\{ -(\beta_1(t)-u_1)W_{y_1} - (\beta_2(t)+u_1-u_2)W_{y_2} \\ \qquad +\tilde{L}(x,u)W_z - \alpha\sigma(t,x)W_{xz} - \frac{1}{2}\alpha^2 W_{zz}\Big\} = 0 \\ W(T,x,y,z) = \max(z,0) \end{cases}$$

We refer the interested readers to [8, Section 4] for the complete characterization of W as the unique continuous viscosity solution of this generalized equation.

6. Numerical simulations

In this section we apply the results of Theorem 5.1 and Corollary 5.2 to numerically approximate the original value function V solution of (5) and the associated optimal feedback strategy. For details and discussions concerning the numerical approximation of this particular type of equations, we refer the interested readers e.g. to [14, Section 9.4] and [5].

We focus on the single dam model as described in Section 2. We recall that in this case the set of admissible controls is defined as

$$\mathcal{U}_{\mathrm{ad}}^{t,y} = \left\{U\text{-valued progressively measurable processes } \nu : Y_s^{t,y,\nu} \in K \text{ for any } s \in [t,T] \text{ a.s.}\right\},$$

where $U = [0,\bar{u}]$, $K = [0,\bar{y}]$ and

$$Y_\cdot^{t,y,\nu} = y + \int_t^\cdot (\beta(s) - \nu_s)\mathrm{d}s.$$

The value function $V : [0,T] \times [0,+\infty) \times K \to \mathbb{R}$ is defined by

$$V(t,x,y) = \sup_{\nu\in\mathcal{U}_{\mathrm{ad}}^{t,y}} \mathbb{E}\left[\int_t^T L(X_s^{t,x}, \nu_s)\mathrm{d}s\right]$$

with $L(x,u) = xu$. Along the entire section we take $\bar{y} = 1$, $T = 1$ and $\beta(t) = 2\sin(\pi t) + 0.5$.

6.1. Electricity price modeled as a GBM.

In order to better understand the technique we use, let us start considering the electricity price evolving as a geometric Brownian motion (GBM). In particular, we consider the same parameters used in Section 3, i.e.

$$b(t,x) = bx \quad \text{and} \quad \sigma(t,x) = \sigma x.$$

with $b := 0.05$ and $\sigma := 0.1$. We point out that in this case we simply have $G(t,x) = \bar{u}\frac{e^{bT}-e^{bt}}{b}x$ for any $(t,x) \in [0,T] \times \mathbb{R}$.

In order to validate the approach presented in the previous section we first suppose that Assumption (H3) is satisfied choosing, for instance, $\bar{u} = 3$. The procedure is outlined in Figure 7. As here V depends linearly on x (recall Equation (8)), to visualize the results we can freeze the variable x and concentrate ourselves just on the dependence on t, y and z. On the top we show the function $W(t,x,y,z)$ (left) and

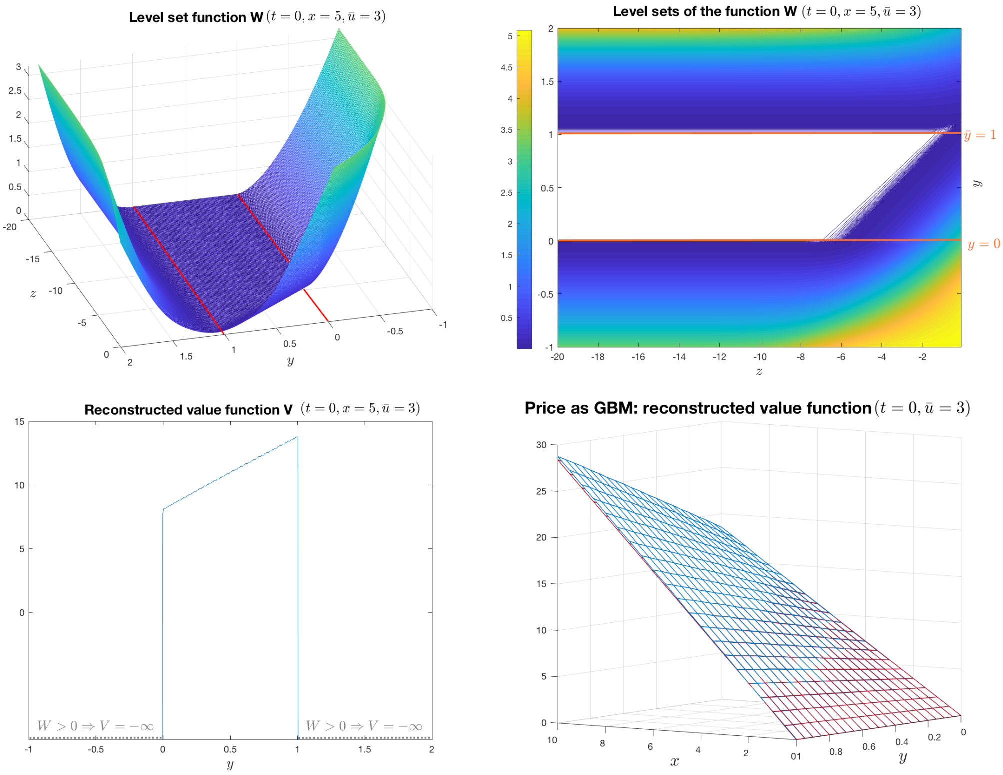

FIGURE 7. Single dam, price modeled as a GBM with $b(t,x) = 0.05x$ and $\sigma(t,x) = 0.1x$, maximal discharge rate $\bar{u} = 3$ (Assumption (H3) is satisfied). Top: the level set function W (left) and its level sets (right) for $x = 5$ and $t = 0$. Bottom: reconstruction of the value function for $t = 0$, $x = 5$ (left) and reconstructed value function at $t = 0$ together with the "original" one (in dark red), obtained as the numerical solution of the HJB Equation (6a) reported on Figure 2 (right). Discretization steps used in numerical tests: $\Delta x = 0.125$, $\Delta y = 0.0125$, $\Delta z = 0.06$, $\Delta t = 0.0125$.

its level sets (right) for $t = 0$, $x = 5$ and $(y, z) \in \mathbb{R} \times (-\infty, 0]$. Then, using (14) we obtain (bottom, left) the value function $V(t, x, y)$ for $t = 0, x = 5$ and $y \in \mathbb{R}$. Observe that in the region outside the interval $K = [0, 1]$ the level-set function W is positive which means, using again (14), that in those points V is equal to $-\infty$, represented by the dashed grey line in the figure. This is aligned with the fact that points outside K are not controllable. The resulting value function $V(t, x, y)$ for $t = 0$, $x \geq 0$ and $y \in [0, 1]$ is finally visible on the bottom-right panel, here plotted together with the original value function to assess the difference. In this case, being Assumption (H3) satisfied, Figure 7 (bottom, right) can be compared with the value function which was directly computed by HJB equation in Figure 2, and we can notice that the reconstructed value is very near to the value obtained with the HJB equation.

We now remove Assumption (H3), taking for instance $\bar{u} = 2$. The decomposition $V(t, x, y) = xv(t, y)$ in Equation (8) holds also in this case, with the consequence that the optimal control is still a feedback control of (t, y) only, but neither V nor v here satisfy an HJB equation (at least with the simple structure of Equation (6a)), and one must reconstruct V via the function W instead. We recall that, in this setting, the plot of the controllable region $\{\mathcal{D}_t, t \in [0, T]\}$ is given in Figure 6. We plot the level sets of W and the reconstructed value function at point $x = 5$ in the first two lines of Figure 8 on the left and right column, respectively. On the first line we choose $t = 0$ and on the second $t = 0.3$. We point out that in the second case the finiteness region for V, i.e. the region where $W = 0$, is reduced to an interval strictly contained in $[0, \bar{y}] \equiv K$. This corresponds to the finding in Figure 6. The reconstructed value function is plotted for $t = 0$ and $t = 0.3$, respectively, at the bottom line of Figure 8.

One can observe that the value function is still increasing with respect to the reservoir level y, but the almost linear slope here seems to have a discontinuity for $y \simeq 0.2$: this is due to the fact that here, for

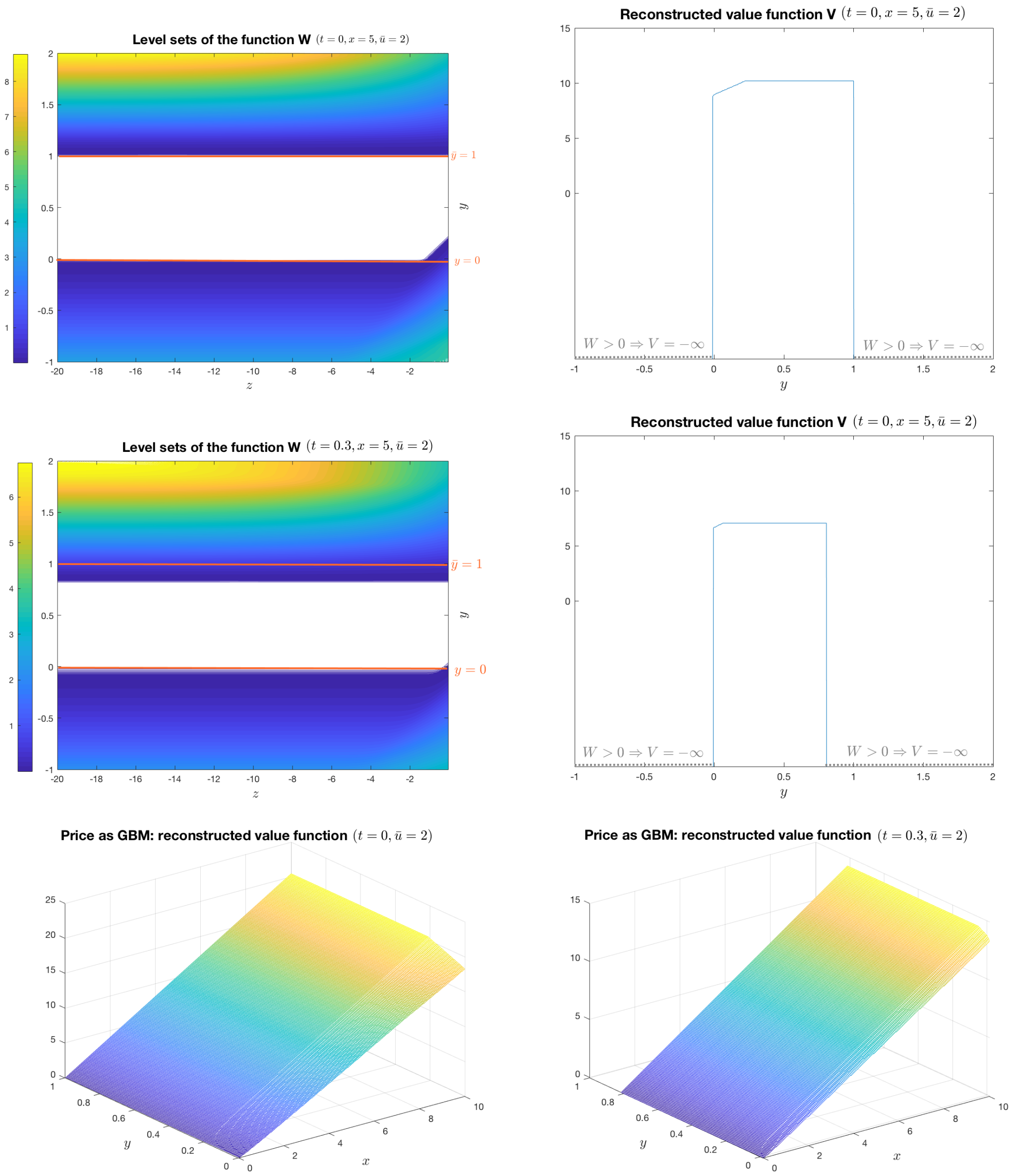

FIGURE 8. Single dam, price modeled as a GBM with $b(t,x) = 0.05x$ and $\sigma(t,x) = 0.1x$, maximal discharge rate $\bar{u} = 2$ (Assumption (H3) is not satisfied). First and second lines: the level sets of the function W (left) and the reconstructed value function (right) for $x = 5$ at two different time $t = 0$ (first line) and $t = 0.3$ (second line). Last line: reconstructed value function V for any $x \geq 0$ at different times $t = 0$ (left) and $t = 0.3$ (right).

levels of y sufficiently high, we lose flexibility in the control u, as one must check whether the strategy which was optimal in the previous case now ends up being not admissible: as a result, one could be forced to empty the dam when it would be not optimal. As a consequence of the lower discharge capacity of the dam and of the necessity to release water for high levels of the reservoir, the value function at $t = 0$ does not surpass the value 25, while under Assumption (H3) the value 30 was almost achieved (compare with Figure 2). Observe that for $t = 0.3$ the value function is defined only for values of y below $\simeq 0.8$. Indeed, as can be observed looking at Figure 6, this is approximately the value of $\hat{y}_t$ (see the definition in Proposition 4.1) at $t = 0.3$ so that for y above this value the function V is not defined being $\mathcal{U}_{\mathrm{ad}}^{t,y} = \emptyset$.

6.2. **Electricity price modeled by inhomogeneous GBM (IGBM) process.** Let us now discuss the case where the electricity price evolves according to an IGBM, obtained by taking

$$b(t,x) = a - bx \quad \text{and} \quad \sigma(t,x) = \sigma x.$$

Considering again the parameters used in Section 2, we take $b := 1$, $\sigma := 0.1$ and $a := 5$. Using the level set method we can approximate the value function and the optimal feedback, in the cases when Assumption (H3) holds (again when $\bar{u} = 3$) and when it does not ($\bar{u} = 2$). First of all, since $\mathbb{E}[X_s^{t,x}] = e^{b(s-t)}x + \frac{a}{b}(e^{bs} - e^{bt})$, we have

$$G(t,x) = \bar{\kappa}\left(\frac{1}{b}\left(x + \frac{a}{b}\right)(e^{b(T-t)} - 1) - \frac{a}{b}(T-t)\right)$$

The reconstructed value function and optimal feedback are plotted in Figures 9 and 10 in the two cases when Assumption (H3) holds ($\bar{u} = 3$) and does not hold ($\bar{u} = 2$), respectively.

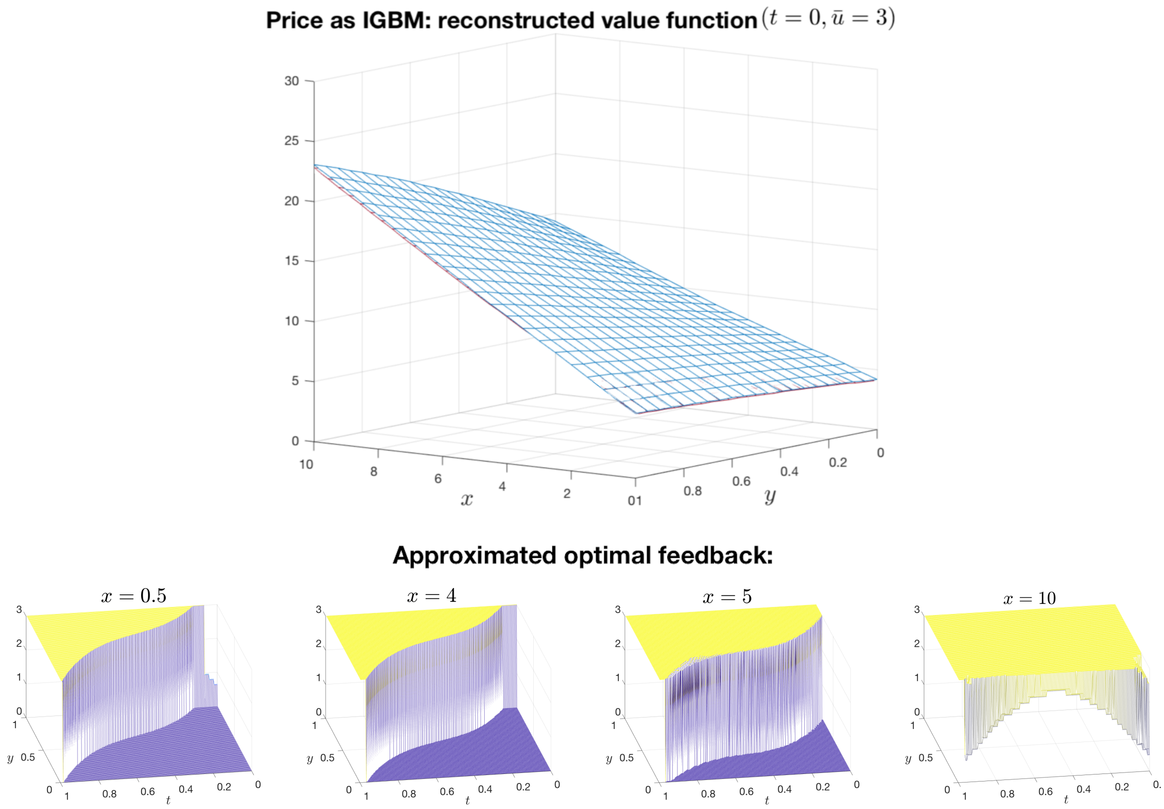

FIGURE 9. Single dam, price modeled as a IGBM with $b(t,x) = 5 - x$ and $\sigma(t,x) = 0.1x$, maximal discharge rate $\bar{u} = 3$ (Assumption (H3) is satisfied). Top: the reconstructed value function together with the "original" one (in dark red), obtained as numerical solution of the HJB Equation (6a) reported in Figure 3. Bottom: the approximated optimal feedback as a function of (t,y) for different values of $x = 0.5, 4, 5, 10$ (from left to right). Discretization steps used in numerical tests: $\Delta x = 0.125$, $\Delta y = 0.0125$, $\Delta z = 0.06$, $\Delta t = 0.0125$.

When $\bar{u} = 3$, we can see from Figure 9 (top) that the reconstructed value function well approximates the one obtained by the direct solution of the HJB equation (Figure 3, top). Also the optimal feedback reconstructed by mean of the level set approach (Figure 9, bottom panels) looks qualitatively close to the one in Figure 3.

When $\bar{u} = 2$, Figure 10 represents the reconstructed value function (top) and the optimal control (four bottom panels). In this case, the value function exhibits a linear growth in x (again here $V(t,0,y) > 0$), while in y we have the same kind of discontinuity in the linear growth that we had with the GBM, i.e. the

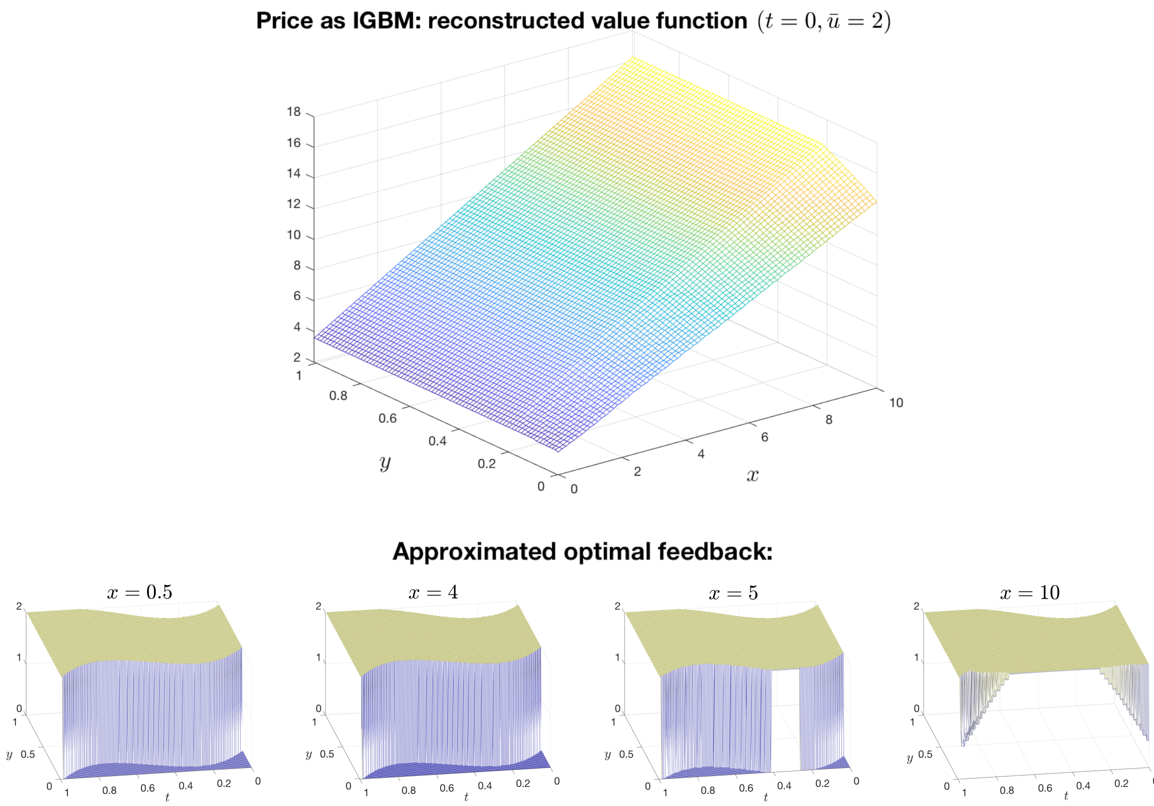

FIGURE 10. Single dam, price modeled as a IGBM with $b(t, x) = 5 - x$ and $\sigma(t, x) = 0.1x$, maximal discharge rate $\bar{u} = 2$ (Assumption (H3) is not satisfied). Top: the reconstructed value function. Bottom: the approximated optimal feedback as a function of (t, y) for different values of $x = 0.5, 4, 5, 10$ (from left to right). Discretization steps used in numerical tests: $\Delta x = 0.125$, $\Delta y = 0.0125$, $\Delta z = 0.06$, $\Delta t = 0.0125$.

slope lowers from $y \simeq 0.2$ on. The interpretation here is similar: for levels of y sufficiently high, we lose flexibility in the control u, as the strategy of letting the dam fill far from the terminal time T for low levels of the price and the reservoir, which was optimal under Assumption (H3) (see Figure 3), does not look admissible any more. Indeed, as the four panels at the bottom of Figure 10 show for large levels of y, one is compelled to discharge the dam even if the price is low. As a consequence, also here the ranges of the value functions are different: when Assumption (H3) is satisfied the value function at $t = 0$ has a maximum value above 20, while in this case it is slightly greater than 16. Even if with a much restricted flexibility, as discussed above, the feedback is still of bang-bang type and increases with respect to x, with the same interpretation given under Assumption (H3). In particular, observe that when the price is high (last bottom panel in Figure 10) it is always optimal to discharge at the maximal admissible rate; this is equal to $\bar{u}$ if the dam is not empty and coincides with $\min(\bar{u}, \beta(t))$ otherwise, explaining the shape of the optimal feedback for $y = 0$. The empty area on the top-right corner of the four bottom panels of Figure 10 coincides with the region where the value function V is not defined (compare with the shape of the admissible region in Figure 6), i.e. at those points no admissible control exists.

7. CONCLUSIONS

In this paper, we have framed the problem of modeling a system of one or two hydroelectric power plants with linked basins, so that, besides discharging water to produce electricity, it is also possible to pump up water in the higher basin in order to store it when the electricity price is lower and use it when price rises up again. This system is framed as a stochastic optimal control problem with state constraints, this latter due to finite reservoirs' volumes, which turn out to be, in general, highly non-trivial. In the case that the system is controllable, i.e. when for each possible state there exists a control which does not make the system violate the state constraints, we show that the value function satisfies a suitable HJB equation. Instead, in the case that the system is not controllable, we show that it is not trivial at all to characterize the states where we are at risk of violating the constraints. We then present a novel level-set approach, which consists in adding one more state variable to the system to account for the constraints' violation. The value function of this augmented control problem is the solution of a HJB equation, and thus it is possible to characterize numerically the states of the original problem where constraints can be violated, as well as the optimal control.

We also present numerical implementations for our findings. Both in the one reservoir as in the two reservoirs case, the resulting optimal controls are of bang-bang type: in fact, the peculiar form of the problem makes optimal to produce electricity when its price is high, and to postpone production when the price is low. Moreover, in the two reservoirs case, there are particular states (notably when the electricity price is particularly low) when it is indeed optimal to pump water in the upper basin and store it for future production. These numerical experiments are performed in the controllable case both solving the original HJB equation and solving the HJB equation for the augmented problem: in this case, we find out that this latter technique gives results which are very near to the original formulation, both in terms of value function as in terms of optimal controls. We also implement the problem in the case when the system is not controllable: in this case of course we have to resort to the HJB equation for the augmented problem. Here we can notice a difference between this case and the controllable one. In fact, in the controllable case the value function exhibits a linear growth in y, while in the uncontrollable case this linear growth exhibits a discontinuity at certain values of y. This is due to the fact that a strategy which is optimal in the controllable case can end up being not admissible in the uncontrollable case: as a result, one could be forced to empty the dam when it would be not optimal in order not to violate the state constraints.

Acknowledgments. The authors wish to thank Olivier Bokanowski, Paolo Falbo, Fulvio Fontini, Jacco Thijssen, Ralf Wunderlich and Hasnaa Zidani, for interesting discussions, and all the participants to the workshop "Investments, Energy, and Green Economy 2019" in Brescia, the XLIII AMASES Conference in Perugia, the workshop "Control Theory and applications" in L'Aquila and the joint Freiburg-Padova-Vienna-Zürich seminar. The second author gratefully acknowledges financial support from the research projects of the University of Padova BIRD172407-2017 "New perspectives in stochastic methods for finance and energy markets" and BIRD190200/19 "Term Structure Dynamics in Interest Rate and Energy Markets: Modelling and Numerics"..

References

[1] A. Altarovici, O. Bokanowski, and H. Zidani. A general Hamilton-Jacobi framework for non-linear state-constrained control problems. ESAIM: Control, Optimisation and Calculus of Variations (COCV), 19:337–357, 2013.

[2] G. Barles and J. Burdeau. The Dirichlet problem for semi-linear second-order degenerate elliptic equations and applications to stochastic exit time control problems. Communications in Partial Differential Equations, 20(1-2):129–178, 1995.

[3] G. Barles and E. Rouy. A strong comparison result for the Bellman equation arising in stochastic exit time control problems and its applications. Comm. Partial Differential Equations, 23(11-12):1995–2033, 1998.

[4] M. Basei, A. Cesaroni, and T. Vargiolu. Optimal exercise of swing contracts in energy markets: an integral constrained optimal control problem. SIAM Journal of Financial Mathematics, 5(1):581–608, 2014.

[5] C. Benezet, J.-F. Chassagneux, and C. Reisinger. A numerical scheme for the quantile hedging problem. preprint https://arxiv.org/abs/1902.11228.

[6] O. Bokanowski, Y. Cheng, and C. Shu. A discontinuous Galerkin scheme for front propagation with obstacles. Numerische Mathematik, 126:1–31, 2014.

[7] O. Bokanowski, N. Forcadel, and H. Zidani. Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. SIAM Journal on Control and Optimization, 48(7):4292–4316, 2010.

[8] O. Bokanowski, A. Picarelli, and H. Zidani. State-constrained stochastic optimal control problems via reachability approach. SIAM Journal on Control and Optimization, 54(5):2568–2593, 2016.

[9] P. Brown, J. P. Lopes, and M. A. Matos. Optimization of pumped storage capacity in an isolated power system with large renewable penetration. IEEE Transactions on Power Systems, 23(2):523–531, 2008.

[10] G. Callegaro, L. Campi, V. Giusto, and T. Vargiolu. Utility indifference pricing and hedging for structured contracts in energy markets. Mathematical Methods of Operations Research, 85(2):265–303, 2017.

[11] F. Camilli, and M. Falcone, An approximation scheme for the optimal control of diffusion processes. ESAIM: Mathematical Modeling and Numerical Analysis 29:97–122, 1995.

[12] L. Capriotti, Y. Jang, and G. Shaimerdenova. Approximation methods for Inhomogeneous Geometric Brownian Motion. International Journal of Theoretical and Applied Finance.

[13] I. Capuzzo-Dolcetta and P.-L. Lions. Hamilton-Jacobi equations with state constraints. Transactions of the American Mathematical Society, 318(2):643–683, 1990.
[14] K. Debrabant, and E.R. Jakobsen. Semi-Lagrangian schemes for linear and fully non-linear diffusion equations Mathematics of Computation, 82:1433–1462, 2013.
[15] Z. Chen and P. A. Forsyth. A semi-lagrangian approach for natural gas storage valuation and optimal operation. SIAM Journal on Scientific Computing, 30(1):339–368, 2007.
[16] Z. Chen and P. A. Forsyth. Pricing hydroelectric power plants with/without operational restrictions: a stochastic control approach. pages 253–281, 2008. in Ehrhardt M, Editor. Nonlinear Models in Mathematical Finance, Nova Science Publisher.
[17] M. Falcone, T. Giorgi, and P. Loreti. Level sets of viscosity solutions: some applications to fronts and rendez-vous problems. SIAM Journal on Applied Mathematics, 54(5):1335–1354, 1994.
[18] B. Felix and C. Weber. Valuation of multiple hyro reservoir storage systems in competitive electricity markets. 2014. EWL Working Paper No. 01/2014. Available at SSRN: https://ssrn.com/abstract=2424674 or http://dx.doi.org/10.2139/ssrn.2424674.
[19] H. Ishii and S. Koike. A new formulation of state constraint problems for first-order PDEs. SIAM Journal on Control and Optimization, 34(2):554–571, 1996.
[20] H. Ishii and P. Loreti. A class of stochastic optimal control problems with state constraint. Indiana University Mathematics Journal, 51(5):1167–1196, 2002.
[21] M. A. Katsoulakis. Viscosity solutions of second-order fully nonlinear elliptic equations with state constraints. Indiana University Mathematics Journal, 43(2):493–520, 1994.
[22] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability under state constraints. SIAM Journal on Control and Optimization, 45(4):1369–1394, 2006.
[23] N. Löhndorf, D. Wozabal, and S. Minner. Optimizing trading decisions for hydro storage systems using approximate dual dynamic programming. Operations Research, 61(4):810–823, 2013.
[24] K. Margellos and J. Lygeros. Hamilton-Jacobi formulation for reach-avoid differential games. IEEE Transactions on Automatic Control, 56:1849–1861, 2011.
[25] R. McDonald and D. Siegel. Investment and the valuation of firms when there is an option to shut down. International Economic Review, 26(2):331–349, 1985.
[26] D. of Energy of the U.S.A. DOE Global Energy Storage Database. `www.energystorageexchange.org`, 2017.
[27] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. Journal of computational physics, 79(1):12–49, 1988.
[28] A. Picarelli, and C. Reisinger. Probabilistic error analysis for some approximation schemes to optimal control problems Syestems & Control Letters 137:1–11, 2020.
[29] A. A. Shardin and R. Wunderlich. Partially observable stochastic optimal control problems for an energy storage. 89(1):280–310, 2017.
[30] H. M. Soner. Optimal control with state-space constraint. I. SIAM Journal on Control and Optimization, 24(3):552–561, 1986.
[31] H. M. Soner. Optimal control with state-space constraint. II. SIAM Journal on Control and Optimization, 24(6):1110–1122, 1986.
[32] H. M. Soner and N. Touzi. A stochastic representation for the level set equations. Communications in Partial Differential Equations, 27(9-10):2031–2053, 2002.
[33] M. Thompson, M. Davison, and H. Rasmussen. Optimal control of hydroelectric facility incorporating pump storage. Operations Research 52(4):546–562, 2004.
[34] M. T. Vespucci, F. Maggioni, M. I. Bertocchi, and M. Innorta. A stochastic model for the daily coordination of pumped storage hydro plants and wind power plants. Ann Oper Res, 193:91–105, 2012.
[35] G. Zhao and M. Davison. Optimal control of hydroelectric facility incorporating pump storage. Renewable Energy, 34(4):1064–1077, 2009.