Calc. Var. (2020) 59:58
https://doi.org/10.1007/s00526-020-1718-6

Calculus of Variations

Optimal control of multiagent systems in the Wasserstein space

Chloé Jimenez[1] · Antonio Marigonda[2] · Marc Quincampoix[1]

Received: 27 September 2018 / Accepted: 31 January 2020

Abstract

This paper concerns a class of optimal control problems, where a central planner aims to control a multi-agent system in $\mathbb{R}^d$ in order to minimize a certain cost of Bolza type. At every time and for each agent, the set of admissible velocities, describing his/her underlying microscopic dynamics, depends both on his/her position, and on the configuration of all the other agents at the same time. So the problem is naturally stated in the space of probability measures on $\mathbb{R}^d$ equipped with the Wasserstein distance. The main result of the paper gives a new characterization of the value function as the unique viscosity solution of a first order partial differential equation. We introduce and discuss several equivalent formulations of the concept of viscosity solutions in the Wasserstein spaces suitable for obtaining a comparison principle of the Hamilton Jacobi Bellman equation associated with the above control problem.

Mathematics Subject Classification 49J15 · 49J52 · 49L25 · 49K27 · 49K21 · 49Q20 · 34A60 · 93C15

Communicated by L. Ambrosio.

This research benefited from the support of the FMJH 'Program Gaspard Monge for optimization and operations research and their interactions with data science', and from the support from EDF and/or Thales, PGMO project VarPDEMFG. This research was partially funded by research contract AFOSR-FA9550-18-1-0254.

✉ Antonio Marigonda
antonio.marigonda@univr.it

Chloé Jimenez
chloe.jimenez@univ-brest.fr

Marc Quincampoix
marc.quincampoix@univ-brest.fr

1 Univ Brest, UMR CNRS 6205, Laboratoire de Mathématiques de Bretagne Atlantique, 6 Avenue Le Gorgeu, 2920 Brest, France

2 Department of Computer Science, University of Verona, Strada Le Grazie 15, 37134 Verona, Italy

Published online: 02 March 2020

1 Introduction

There has been an increasing interest of the mathematical control theory community for the so-called *multi-agent* systems, i.e., systems on a reference space X that are composed by a number of agents so huge, that at each time only a *statistical description* of the state is available. A common way to model such kind of system is to consider a *macroscopic* point of view, where the state of the system is described by a (time-evolving) *Borel measure* on X, i.e. the underlying space where the agents move.

If μ_t is a measure on X, and A is a Borel subset of X, the quantity $\mu_t(X)$ measures the total number of agent of the systems at time t, and the quotient $\dfrac{\mu_t(A)}{\mu_t(X)}$ represents the fractions of the total amount of agents that are present in A at the time t. The case in which the system is *isolated*, i.e., the total amount of agents is fixed in time, is of relevant interest. Indeed, in this case, since $\mu_t(X)$ is constant, we can always normalize the measure μ_t assuming $\mu_t(X) = 1$, i.e., $\mu_t \in \mathscr{P}(X)$ the set of Borel probability measures on X. Thus the macroscopic evolution is described by a curve $t \mapsto \mu_t$ in the space of probability measures. In the case $X = \mathbb{R}^d$, a stronger mass-preservation property (i.e., that *locally* there are neither creation nor destruction of agents,), can be obtained assuming that the trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ of the system, seen as a family of measures on X indexed by the time parameter, is expressed by the *continuity equation*

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0, \tag{1}$$

where $v_t(\cdot)$ is a time dependent Borel vector field on $\mathbb{R}^d$, and the PDE must be understood in the sense of distributions.

Under mild integrability properties on $v_t(\cdot)$, it is possible to prove that every solution $t \mapsto \mu_t$ of the above PDE possesses a continuous representative, where continuity is taken w.r.t. the weak* topology induced by the duality with continuous and bounded functions on $\mathbb{R}^d$, thus it make sense to couple the PDE with an initial condition in order to study the macroscopic evolution of the system.

It is natural to introduce now a *cost function* on the system, and study various kinds of optimization problems. More precisely, we are interested in studying the *optimal control problem* where a central planner try to minimize a given cost function on the system by acting on the agents. Another interesting problem—out of the scope of the present paper—concerns the *Nash equilibrium configurations* when each agent try to minimize its individual cost, possibly depending by the configuration of all the other agents. This case is a *mean field game* problem, in the sense of [7,10,24].

The individual motion of each agent can be suject to nonholonomic constraints coming from both *local* conditions, i.e., depending only on its instantaneous position, and from *nonlocal* conditions, i.e., depending on the overall configuration of the agents present in the system. The simplest possible case of nonoholonomic constraint coming from local conditions is the presence of a *maximum speed* for the agents depending on its instantaneous position. In this case, the admissible velocities for the agents passing through the point $x \in \mathbb{R}^d$ are contained in a closed ball $F(x) = \overline{B(0, g(x))} \subseteq \mathbb{R}^d$, where $g : \mathbb{R}^d \to [0, +\infty[$ is a function pointwise giving the speed limit. *Anisotropic* speed limit, i.e., limits depending not only on the position but also on the direction, can be modeled similarly replacing the profile of the ball with a suitable compact convex set.

In general, in presence of nonholonomic constraints on the dynamics of the agent, for instance when the dynamics of each agent is expressed by a set-valued map F with values in

$\mathbb{R}^d$, a natural requirement on the macroscopical vector field $v_t(\cdot)$ is to be a selection of the same set-valued map F.

One of the most interesting features of the generalized control problem in the space of probability measures in this formulation, which does not appear in the classical formulation, is the possibility to take into account internal interactions between the agents, usually leading to *nonlocal nonholonomic* constraints. Indeed, in the analysis of multi-agent systems, like e.g., cell populations, fish swarms, insect colonies, human crowds, bird flocks, the collective behavior is deeply influenced by complex interactions that usually arise among the subjects. These interactions can be added both in the cost function, and in the dynamics. In the latter case, this amounts to allow the set-valued map to depend not only on the position in $\mathbb{R}^d$, but also by the current state of the system, i.e., considering set-valued maps F defined on $\mathscr{P}(\mathbb{R}^d) \times \mathbb{R}^d$ with values in $\mathbb{R}^d$.

An example can be given by penalizing the speed of the agents if the overall current configuration is far from a *fixed* ideal travelling configuration which, for instance, guarantees the safety of the swarm/collective. Denoted by $\boldsymbol{\mu}^{\boldsymbol{R}} = \{\mu_t^R\}_{t\in[0,T]}$ the ideal travelling configuration, we can consider for instance

$$F(\mu, x) = \overline{B\left(0, \frac{1}{1 + W_2^2(\mu, \mu_t^R)}\right)},$$

where $W_2(\cdot)$ denotes the Wasserstein distance between probability measures, and study the problem to achieve a desired configuration in minimum time. The optimal strategy in this simple case will be a compromise between reaching first the ideal travelling profile to travel at the fastest speed possibile (in this case 1), and letting the single agents free to move toward the goal as fast as they can (in general, with speed less than 1).

Summarizing, in the general case the dynamics of the system consists of the continuity equations (1) coupled with

$$v_t(x) \in F(\mu_t, x) \text{ for } \mu_t\text{-almost } x \in \mathbb{R}^d \text{ and for a.e. } t \geq 0. \tag{2}$$

This feature leads to the conclusion that, in presence of interactions, the description of the collective behavior cannot be reduced by the simple superposition of individual behaviors.

Indeed, in [16] it was addressed the problem to identify with geometrical tools and study the macroscopical dynamics of a system where the microscopical agents were subjected by a nonholonomic constraint modeled by a differential inclusion. However, it was made the strong simplifying assumptions of *no interactions* between the agents, and therefore the map F was assumed to depend only on the variable $x \in \mathbb{R}^d$. In this paper, among the other results, we provide an exension of the superposition principle to microscopical dynamics governed by differential inclusions also in the case with interactions. Comparing to [16], this extensions requires the use of appropriate fixed point argument, due to the fact that the evolution of each agent is affected by the evolution of the others. This difficulty did not appear in the case treated in [16].

The problem of rigorously approximate the control problem for the real-world multi-agent discrete system with its *mean field limit*, i.e., the corresponding problem stated in the space of measures, is of fundamental importance both from the theoretical and from the applicative point of views. This problem can be traced back to [27], and a systematical survey of related results can be found in [26]. This problem was addressed also in [13] for some models coming from flocking models, in order to reduce the dimensionality of the problem of the *kinetic* formulation. In [20] and [19] it is rigorously justified the use of mean-field approximations in optimal control of multiagent systems of first order. The reader can find a comprehensive overview of the literature about kinetic formulation and applications, together with some insights on research perspective, in the recent survey [1].

Closer to the problem studied in the present paper, in [8] and in [9] necessary conditions are studied for control problems in the Wasserstein space. The first paper still in connection with mean-field limit and the second one directly in the Wasserstein space. Both papers provide such conditions in form of an extended Pontryagin Maximum Principle in the Wasserstein space, however in order to obtain well-posedness of the adjoint equation heavy regularity assumptions on the problem are needed.

Instead of a mean-field approximation approach, we study directly control systems stated in the space of probability measure. This is motivated for instance by the case of *incomplete information* on the state space, where we can model our knowledge of the state of the system by a probability measure and study the corresponding evolution. This may occur even when the evolution is purely deterministic (as in [12,14]), or when we consider games with incomplete information (see [11,12]) or repeated games with signals (see [28]).

In this paper we will consider a Bolza-type problem, i.e., the minimization of a functional $\boldsymbol{J}(\cdot)$

$$\boldsymbol{\mu} \mapsto J_{[s,T]}(\boldsymbol{\mu}) := \int_s^T \mathcal{L}(\mu_t)\, dt + \mathscr{G}(\mu_T) \in \mathbb{R} \cup \{+\infty\}, \tag{3}$$

on trajectories $\boldsymbol{\mu} = \{\mu_t\}_{t\in[s,T]}$ satisfying the continuity equation (1) with an initial datum μ, and subject to the constraint $v_t(x) \in F(\mu_t, x)$ for a.e. $t \in [s, T]$, μ_t-a.e. $x \in \mathbb{R}^d$.

A relevant class of bounded uniformly continuous functional $\mathcal{L}$ which are interesting for the applications is provided by

$$\mathcal{L}(\mu) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} K(x, y)\, d(\mu \otimes \mu)(x, y),$$

where $K \in C_c^0(\mathbb{R}^d \times \mathbb{R}^d)$. In terms of multi-agent systems, $K(x, y)$ describes the cost of the interactions between an agent located at the point x and an agent located at the point y. In its simplest form, it can be expressed by $K(x, y) = k(|x - y|)$, where $k : [0, +\infty[\to [0, +\infty[$ is continuous with compact support. The boundedness of the support of k express in this case the fact that each agent is not influenced by the agents located too far away from him/her.

For the problem (3), a notion of value function can be given in analogy to classical Bolza problem in optimal control, and our main goal is to characterize it as the unique solution of a first-order Hamilton–Jacobi–Bellman equation (HJB in short) in the space of probability measures. To this aim, we will use a convenient notion of viscosity sub/superdifferential, and prove a comparison principle for first-order HJB equations.

The theory of HJB equation in the space of measures could be considered as a part of a more general theory in metric spaces (see, e.g. [2,22]), but, since the space of measures enjoys a much richer structure, specific tools were later developed in [12,14,21]. Using the representation of the space of probability measures as a subspace of L^2 function on a sufficiently "rich" probability space (see [10,24]), it was also developed a theory of generalized differentiation and viscosity solution in the space of measures by adapting the concepts of viscosity theory in infinite-dimensional spaces (see [17]).

In this paper the main result consists in proving that the value function is the unique viscosity solution of a HJB equation in the Wasserstein space. For this task, we introduce a suitable notion of sub/super differential in the Wasserstein space (which is very much inspired from [12,25]) which leads to a definition of viscosity solution. Then we prove a comparison principle for viscosity solution of first-order HJB equations, by adapting a doubling of variables argument used also in [12,25], extending the previous results to cover Hamiltonian function arising in the study of multi agent type. We also give several equivalent formulations of sub/super differential which give equivalent definitions of viscosity solutions. It is worth pointing out that here our intent is not to give abstract results comparing subdifferentials in Wasserstein space (for this, the reader can refer to the recent paper [23]), but only to define and study a subdifferential well-adapted to obtain a comparison result of the HJB associated to our multiagent control problem. Compared with [12], the comparison principle in this paper requires milder regularity assumptions on the Hamiltonian (just a sort of uniform continuity), while in [12] was asked a much stronger positive homogeneity in the second variable and a Lipschitz conditions. This is reflected by the fact that the comparison principle in [12] (and of [25], which was an extension of [12]) provides uniqueness only in the class of Lipschitz continuous function, while the comparison principle of the present paper leads to uniqueness in the bounded and uniformly continuous case. Anyway, the regularity of the value function in our case is enough to authomatically guarantee the consistency between the multiagent system and the mean field formulation. Furthermore, for such kind of result we do not need the regularity assumptions of [9] or [8].

The paper is structured as follows: in Sect. 2 we introduce the basic notation and background, in Sect. 3 we describe the properties of the set of admissible trajectories, establishing some results of existence and representation formulas, in Sect. 4 we analyze the optimal control problem in the Wasserstein space, studying the regularity property of its value function, and prove a dynamic programming principle, and finally in Sect. 5 we provide the main results of the paper, namely a comparison principle for viscosity solution of first-order HJB equation, and the characterization of the value function of the Bolza problem as the unique viscosity solution of a suitable HJB equation. At the end of the section, we also discuss several equivalent formulations of the definition of viscosity solution in this context.

2 Preliminaries and notations

We will use the following notation.

$B(x,r)$	the open ball of radius r of a metric space (X,d_X), i.e., $B(x,r) := \{y \in X : d_X(y,x) < r\}$;
$\overline{K}$	the closure of a subset K of a topological space X;
$I_K(\cdot)$	the indicator function of K, i.e. $I_K(x) = 0$ if $x \in K$, $I_K(x) = +\infty$ if $x \notin K$;
$d_K(\cdot)$	the distance function from a subset K of a metric space (X,d), i.e. $d_K(x) := \inf\{d(x,y) : y \in K\}$;
$C^0_b(X;Y)$	the set of continuous bounded function from a Banach space X to Y, endowed with $\\|f\\|_\infty = \sup_{x\in X} \|f(x)\|$ (if $Y = \mathbb{R}$, Y will be omitted);
$C^0_c(X;Y)$	the set of compactly supported functions of $C^0_b(X;Y)$, with the topology induced by $C^0_b(X;Y)$;
$BUC(X;\mathbb{R})$	the space of bounded real-valued uniformly continuous functions defined on X
Γ_I	the set of continuous curves from a real interval I to $\mathbb{R}^d$;
Γ_T	the set of continuous curves from $[0,T]$ to $\mathbb{R}^d$;
e_t	the evaluation operator $e_t : \mathbb{R}^d \times \Gamma_I$ defined by $e_t(x,\gamma) = \gamma(t)$ for all $t \in I$;
$\mathscr{P}(X)$	the set of Borel probability measures on a Banach space X, endowed with the weak* topology induced from $C^0_b(X)$;

$\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$	the set of vector-valued Borel measures on $\mathbb{R}^d$ with values in $\mathbb{R}^d$, endowed with the weak* topology induced from $C^0_c(\mathbb{R}^d;\mathbb{R}^d)$;
$\lvert\nu\rvert$	the total variation of a measure $\nu\in\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$;
$\ll$	the absolutely continuity relation between measures defined on the same σ-algebra;,
$\mathrm{m}_2(\mu)$	the second moment of a probability measure $\mu\in\mathscr{P}(X)$;
$r\sharp\mu$	the push-forward of the measure μ by the Borel map r;
$\mu\otimes\pi_x$	the product measure of $\mu\in\mathscr{P}(X)$ with the Borel family of measures $\{\pi_x\}_{x\in X}\subseteq\mathscr{P}(Y)$ (see Definition 6);
pr_i	the i-th projection map $\mathrm{pr}_i(x_1,\dots,x_N)=x_i$;
$\varPi(\mu,\nu)$	the set of admissible transport plans from μ to ν;
$\varPi_o(\mu,\nu)$	the set of optimal transport plans from μ to ν;
$W_2(\mu,\nu)$	the 2-Wasserstein distance between μ and ν;
$\mathscr{P}_2(X)$	the subset of the elements $\mathscr{P}(X)$ with finite second moment, endowed with the 2-Wasserstein distance;
$\mathscr{L}^d$	the Lebesgue measure on $\mathbb{R}^d$;
$\dfrac{\nu}{\mu}$	the Radon-Nikodym derivative of the measure ν w.r.t. the measure μ;
$\mathrm{Lip}(f)$	the Lipschitz constant of a function f.

Now we give some preliminaries and fix the notation.

Given two nonempty sets $\varDelta$, S, we will denote by $\{s_\delta\}_{\delta\in\varDelta}\subseteq S$ the images of a map $\delta\mapsto s_\delta$ defined from $\varDelta$ to S, seen as a subset of S indexed by the elements of $\varDelta$. In particular, when $\varDelta=\mathbb{N}$, $\{s_n\}_{n\in\mathbb{N}}\subseteq S$ will denote a *sequence* of elements in S. When the set $\varDelta$, S have more structure, we will refer to *regularity properties of* $\{s_\delta\}_{\delta\in\varDelta}\subseteq S$ meaning the regularity properties of the underlying map $\delta\mapsto s_\delta$.

Given Banach spaces X, Y, we denote by $\mathscr{P}(X)$ the set of Borel probability measures on X endowed with the weak* topology induced by the duality with the Banach space $C^0_b(X)$ of the real-valued continuous bounded functions on X with the uniform convergence norm. The second moment of $\mu\in\mathscr{P}(X)$ is defined by $\mathrm{m}_2(\mu)=\int_X\|x\|^2_X\,d\mu(x)$, and we set $\mathscr{P}_2(X)=\{\mu\in\mathscr{P}(X):\ \mathrm{m}_2(\mathbb{R}^d)<+\infty\}$. For any Borel map $r:X\to Y$ and $\mu\in\mathscr{P}(X)$, we define the *push forward measure* $r\sharp\mu\in\mathscr{P}(Y)$ by setting $r\sharp\mu(B)=\mu(r^{-1}(B))$ for any Borel set B of Y.

We denote by $\mathscr{M}(X;Y)$ the set of Y-valued Borel measures defined on X. The total variation measure of $\nu\in\mathscr{M}(X;Y)$ is defined for every Borel set $B\subseteq X$ as

$$|\nu|(B):=\sup_{\{B_i\}_{i\in\mathbb{N}}}\left\{\sum\|\nu(B_i)\|_Y\right\},$$

where the sup ranges on countable Borel partitions of B.

We now recall the definitions of transport plans and Wasserstein distance (cf for instance [31]). Let X be a complete separable Banach space, $\mu_1,\mu_2\in\mathscr{P}(X)$. The set of *admissible transport plans* between μ_1 and μ_2 is

$$\varPi(\mu_1,\mu_2)=\{\boldsymbol{\pi}\in\mathscr{P}(X\times X):\mathrm{pr}_i\sharp\boldsymbol{\pi}=\mu_i,\ i=1,2\},$$

where for $i=1,2$, $\mathrm{pr}_i:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is a projection $\mathrm{pr}_i(x_1,x_2)=x_i$. The *inverse* $\boldsymbol{\pi}^{-1}$ of a transport plan $\boldsymbol{\pi}\in\varPi(\mu,\nu)$ is defined by $\boldsymbol{\pi}^{-1}=i\sharp\boldsymbol{\pi}\in\varPi(\nu,\mu)$, where $i(x,y)=(y,x)$ for all $x,y\in X$. The *Wasserstein distance* between μ_1 and μ_2 is

$$W_2^2(\mu_1,\mu_2)=\inf_{\boldsymbol{\pi}\in\varPi(\mu_1,\mu_2)}\int_{X\times X}|x_1-x_2|^2\,d\boldsymbol{\pi}(x_1,x_2).$$

If $\mu_1, \mu_2 \in \mathscr{P}_2(X)$ then the above infimum is actually a minimum, and the set of minima is denoted by

$$\Pi_o(\mu_1, \mu_2) := \left\{ \boldsymbol{\pi} \in \Pi(\mu_1, \mu_2) : W_2^2(\mu_1, \mu_2) = \int_{X \times X} |x_1 - x_2|^p \, d\boldsymbol{\pi}(x_1, x_2) \right\}.$$

Recall that $\mathscr{P}_2(X)$ endowed with the W_2-Wasserstein distance is a complete separable metric space, moreover for all $\mu \in \mathscr{P}_2(X)$ there exists a sequence $\{\mu^N\}_{N \in \mathbb{N}} \subseteq \mathrm{co}\{\delta_x : x \in \operatorname{supp} \mu\}$ such that $W_2(\mu^N, \mu) \to 0$ as $N \to +\infty$.

To maintain the flow of the paper we postpone to an appendix the statement of the Disintegration Theorem and of the Superposition principle which will largely used throughout the article.

3 The set of admissible trajectories and its properties

Here the admissible trajectories are the solutions of a continuity equation with constraints in the flux. From a multi-agent system point of view, we have the following properties

- during the evolution, the total mass of the agents is preserved: we have neither creation nor loss of agents;
- the dynamic of each agent is subject to non holonomic and possibly nonlocal constraints
- the macroscopic evolution will be the result of the superposition (average) of the microscopic evolution of the agents.

We will focus now on the properties of the set of admissible trajectories.

Definition 1 *(Admissible trajectories)* Let $I = [a, b]$ be a compact real interval, $\boldsymbol{\mu} = \{\mu_t\}_{t \in I} \subseteq \mathscr{P}_2(\mathbb{R}^d)$, $\boldsymbol{\nu} = \{\nu_t\}_{t \in I} \subseteq \mathscr{M}(\mathbb{R}^d; \mathbb{R}^d)$, $F : \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a set-valued map. We say that $\boldsymbol{\mu}$ is an *admissible trajectory driven by* $\boldsymbol{\nu}$ defined on I with underlying dynamics F if

- the map $t \mapsto \mu_t$ is Borel (see Definition 6);
- $|\nu_t| \ll \mu_t$ for a.e. $t \in I$;
- $v_t(x) := \dfrac{\nu_t}{\mu_t}(x) \in F(\mu_t, x)$ for a.e. $t \in I$ and μ_t-a.e. $x \in \mathbb{R}^d$;
- the map $(t, x) \mapsto v_t(x)$ is Borel and

$$\int_I \|v_t\|_{L^2_{\mu_t}} \, dt < +\infty;$$

- $\partial_t \mu_t + \operatorname{div} \nu_t = 0$ in the sense of distributions on $]0, T[\times \mathbb{R}^d$, equivalently

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t(x) = \int_{\mathbb{R}^d} \langle \nabla \varphi(x), v_t(x) \rangle \, d\mu_t(x), \text{ for all } \varphi \in C_c^1(\mathbb{R}^d)$$

  in the sense of distributions in $]0, T[$ (see (8.1.3) in [3]).

Given $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, we define the set

$$\begin{aligned} \mathscr{A}_I^F(\mu) := \Big\{ \boldsymbol{\mu} = &\{\mu_t\}_{t \in I} \subseteq \mathscr{P}_2(\mathbb{R}^d) : \text{ there exists } \boldsymbol{\nu} = \{\nu_t\}_{t \in I} \subseteq \mathscr{M}(\mathbb{R}^d; \mathbb{R}^d) \\ &\text{such that } \boldsymbol{\mu} \text{ is an admissible traj. driven by } \boldsymbol{\nu}, \\ &\text{defined on } I \text{ with underlying dynamics } F \text{ and } \mu_a = \mu \Big\}. \end{aligned}$$

We recall that, by Lemma 8.1.2 of [3], by possibly changing the family $\{\mu_t\}_{t\in I}$ on a Lebesgue negligible subset of I, we may always assume that $t\mapsto \mu_t$ is narrowly continuous. Therefore given $\{\mu_t\}_{t\in I}\in \mathscr{A}_I^F(\mu)$ we will assume always that it is continuous without loss of generality. Moreover, by Theorem 8.3.1 in [3], the map $t\mapsto\mu_t$ is actually absolutely continuous from I to $\mathscr{P}_2(\mathbb{R}^d)$ endowed with the Wasserstein metric.

Conversely, for any absolutely continuous curve $t\mapsto\mu_t$ defined on $I=[a,b]$ with values in $\mathscr{P}_2(\mathbb{R}^d)$ endowed with the Wasserstein metric there exists a Borel vector field $(t,x)\mapsto v_t(x)$ such that

$$\int_I \|v_t\|_{L^2_{\mu_t}}\,dt<+\infty,$$

and $\partial_t\mu_t+\mathrm{div}(v_t\mu_t)=0$ holds in the sense of distributions on $]a,b[\times\mathbb{R}^d$.

An alternative characterization is of the admissible trajectories is given by the following

Remark 1 Define $\mathscr{I}_F:\mathscr{P}_2(\mathbb{R}^d)\times\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)\to[0,+\infty]$ and $\mathscr{F}:\mathscr{P}_2(\mathbb{R}^d)\rightrightarrows\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)\to\mathbb{R}\cup\{+\infty\}$ by

$$\mathscr{I}_F(\mu,\nu):=\begin{cases}\displaystyle\int_{\mathbb{R}^d} I_{F(\mu,x)}\left(\frac{\nu}{\mu}(x)\right)d\mu(x), & \text{if } |\nu|\ll\mu,\\[2ex] +\infty, & \text{otherwise.}\end{cases}$$

$$\mathscr{F}(\mu):=\{\nu\in\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d):\ \mathscr{I}_F(\mu,\nu)<+\infty\},$$

we have that $\boldsymbol{\mu}\in\mathscr{A}_I^F(\mu)$ if and only if there exists a Borel family $\boldsymbol{\nu}=\{\nu_t\}_{t\in I}$ such that $\partial_t\mu_t+\mathrm{div}\,\nu_t=0$ in the sense of distributions, $\mu_a=\mu$, and $\nu_t\in\mathscr{F}(\mu_t)$ for a.e. $t\in I$. Given $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, we say that $\nu\in\mathscr{F}(\mu)$ is an *admissible measure-valued velocity* at μ.

We first show a result about the closedness of set of trajectories. To this aim, we consider the following property of the dynamics

$(\boldsymbol{F}_1)$ $F:\mathscr{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\rightrightarrows\mathbb{R}^d$ is continuous with convex, compact and nonempty images, where on $\mathscr{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$ we consider the metric

$$d_{\mathscr{P}_2(\mathbb{R}^d)\times\mathbb{R}^d}((\mu_1,x_1),(\mu_2,x_2))=|x_1-x_2|+W_2(\mu_1,\mu_2).$$

We obtain the following result

Proposition 1 *Assume that F satisfies $(\boldsymbol{F}_1)$. Let $\{\boldsymbol{\mu}^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of admissible trajectories defined on I such that $\boldsymbol{\mu}^{(n)}=\{\mu_t^{(n)}\}_{t\in I}$ for all $n\in\mathbb{N}$, and let $\boldsymbol{\mu}=\{\mu_t\}_{t\in I}\subseteq\mathscr{P}_2(\mathbb{R}^d)$, $\boldsymbol{\nu}=\{\nu_t\}_{t\in I}\subseteq\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ be Borel curves.*

Suppose that

- *for all $n\in\mathbb{N}$ we have that $\boldsymbol{\mu}^{(n)}$ is driven by $\boldsymbol{\nu}^{(n)}=\{\nu_t^{(n)}=v_t^{(n)}\mu_t^{(n)}\}_{t\in I}$, where $v_t^{(n)}(x)\in F(\mu_t^{(n)},x)$ for a.e. $t\in I$ and μ_t-a.e. $x\in\mathbb{R}^d$;*
- $\displaystyle\liminf_{n\to+\infty}\int_I\|v_t^{(n)}\|_{L^2_{\mu_t^{(n)}}}<+\infty$*;*
- *$\boldsymbol{\mu}^{(n)}\to\boldsymbol{\mu}$ in the sense of distributions on $I\times\mathbb{R}^d$, and for a.e. $t\in I$ we have $W_2(\mu_t^{(n)},\mu_t)\to 0$ as $n\to+\infty$;*
- *$\boldsymbol{\nu}^{(n)}\to\boldsymbol{\nu}$ in the sense of distributions on $I\times\mathbb{R}^d$, and for a.e. $t\in I$ we have $\nu_t^{(n)}\rightharpoonup^*\nu_t$ as $n\to+\infty$;*

Then $\boldsymbol{\mu}$ is an admissible trajectory driven by $\boldsymbol{\nu}$.

Before proving the above Proposition we state a Lemma which is a consequence of well-known results of lower semicontinuity for functional depending on measures.

Lemma 1 *Assume* ($\boldsymbol{F_1}$). *Let* $\{\mu^{(n)}\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathscr{P}_2(\mathbb{R}^d)$ *converging in* W_2 *to* $\bar{\mu}$ *and* $\{\nu^{(n)}\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ w^**-converging to* $\bar{\nu}$. *Then*

$$\mathscr{I}_F(\bar{\mu},\bar{\nu}) \le \liminf_{n\to+\infty} \mathscr{I}_F(\mu^{(n)},\nu^{(n)}).$$

Proof We have $\mathscr{I}_F(\mu^{(n)},\nu^{(n)}) \in \{0,+\infty\}$. In the case $\mathscr{I}_F(\mu^{(n)},\nu^{(n)}) = +\infty$ for all but a finite number of indexes n, there is nothing to prove. Thus without loss of generality, we may assume that $\mathscr{I}_F(\mu^{(n)},\nu^{(n)}) = 0$ for all $n \in \mathbb{N}$.

Let $(\bar{\mu}, x) \in \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. By the upper semicontinuity property of F, for all $\varepsilon > 0$ there exists $\delta_{\varepsilon,\mu,x} > 0$ such that if

$$d_{\mathscr{P}_2(\mathbb{R}^d)\times\mathbb{R}^d}((\theta, y), (\bar{\mu}, x)) \le \delta_{\varepsilon,\mu,x},$$

then

$$F(\theta, y) \subseteq F(\bar{\mu}, x) + \varepsilon\overline{B(0,1)}.$$

Let $\{x_i\}_{i>0}$ be a countably dense sequence in $\mathbb{R}^d$. We set $\delta_i = \delta_{\varepsilon,\bar{\mu},x_i}$ and $B_i = B(x_i, \min\{\delta_i/2, 1/i\})$. Clearly, we have

$$\int_{\mathbb{R}^d} I_{F(\bar{\mu},x)}\left(\frac{\nu}{\bar{\mu}}(x)\right) d\bar{\mu}(x) = \sup_{i\in\mathbb{N}} \int_{\overline{B_i}} I_{F(\bar{\mu},x)}\left(\frac{\nu}{\bar{\mu}}(x)\right) d\bar{\mu}(x)$$

There exists $\bar{n} > 0$ such that for all $n > \bar{n}$ we have $W_2(\bar{\mu}, \mu^{(n)}) < \delta_i/2$, in particular for any $i \in \mathbb{N}$ we have

$$\begin{aligned} 0 &= \int_{\overline{B_i}} I_{F(\mu^{(n)},x)}\left(\frac{\nu^{(n)}}{\mu^{(n)}}(x)\right) d\mu^{(n)}(x) \\ &\ge \int_{\overline{B_i}} I_{F(\bar{\mu},x_i)+\varepsilon\overline{B(0,1)}}\left(\frac{\nu^{(n)}}{\mu^{(n)}}(x)\right) d\mu^{(n)}(x) \end{aligned}$$

According to e.g. Theorem 2.34 in [4], we have that for all $i \in I$

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\overline{B_i}} I_{F(\bar{\mu},x_i)+\varepsilon\overline{B(0,1)}}\left(\frac{\nu^{(n)}}{\mu^{(n)}}(x)\right) d\mu^{(n)}(x) \\ &\ge \int_{\overline{B_i}} I_{F(\bar{\mu},x_i)+\varepsilon\overline{B(0,1)}}\left(\frac{\bar{\nu}}{\bar{\mu}}(x)\right) d\bar{\mu}(x), \end{aligned}$$

and so for $\bar{\mu}$-a.e. $x \in \overline{B_i}$ we have

$$\frac{\bar{\nu}}{\bar{\mu}}(x) \in F(\bar{\mu}, x_i) + \varepsilon\overline{B(0,1)}.$$

Fix now a density point $\bar{x}$ for $\bar{\mu}$. By density of the sequence $\{x_i\}_{i\in\mathbb{N}}$, there exists a subsequence x_{i_k} such that $\bar{x} \in \overline{B_{i_k}}$ for all k, thus for k large enough we have

$$\frac{\bar{\nu}}{\bar{\mu}}(\bar{x}) \in F(\bar{\mu}, x_{i_k}) + \varepsilon\overline{B}(0,1) \subseteq F(\bar{\mu}, \bar{x}) + 2\varepsilon\overline{B}(0,1).$$

by letting $\varepsilon \to 0^+$ and recalling the arbitrariness of the density point $\bar{x}$, we conclude that $\frac{\bar{\nu}}{\bar{\mu}}(x) \in F(\bar{\mu}, x)$ for a.e. $x \in \mathbb{R}^d$, so $\mathscr{I}_F(\bar{\mu},\bar{\nu}) = 0$. The proof is complete. □

Proof of Proposition 1 **1.** Since for all $n \in \mathbb{N}$ the trajectory $\boldsymbol{\mu}^{(n)}$ is an admissible trajectory driven by $\boldsymbol{\nu}^{(n)}$, for all $n \in \mathbb{N}$ we have

$$\partial_t \mu_t^{(n)} + \operatorname{div} \nu_t^{(n)} = 0.$$

Recalling that by assumption $\boldsymbol{\mu}^{(n)}$ and $\boldsymbol{\nu}^{(n)}$ converges in the sense of distributions to $\boldsymbol{\mu}^{(n)}$ and $\boldsymbol{\nu}^{(n)}$, respectively, by passing to the limit in the sense of distributions we have

$$\partial_t \mu_t + \operatorname{div} \nu_t = 0.$$

2. We denote by $\mathscr{N} \subseteq I$ the set of $t \in I$ where $\mu_t^{(n)}$ does not W_2-converge to μ_t or $\nu_t^{(n)}$ does not w^*-converge to ν_t. By definition, we have that $\mathscr{N}$ is negligible. Since the trajectories of the sequence are admissible, we have for all $n \in \mathbb{N}$

$$\int_I \mathscr{I}_F\left(\mu_t^{(n)}, \nu_t^{(n)}\right) dt = 0,$$

thus, by Fatou Lemma and Lemma 3.4 we have

$$\begin{aligned} 0 &= \liminf_{n\to+\infty} \int_{I\setminus\mathscr{N}} \mathscr{I}_F\left(\mu_t^{(n)}, \nu_t^{(n)}\right) dt \geq \int_{I\setminus\mathscr{N}} \liminf_{n\to+\infty} \mathscr{I}_F(\mu_t^{(n)}, \nu_t^{(n)})\, dt \\ &\geq \int_{I\setminus\mathscr{N}} \mathscr{I}_F(\mu_t, \nu_t)\, dt = \int_I \mathscr{I}_F(\mu_t, \nu_t)\, dt \geq 0. \end{aligned}$$

Thus we have $|\nu_t| \ll \mu_t$ and $v_t(x) := \dfrac{\nu_t}{\mu_t}(x) \in F(\mu_t, x)$ for a.e. $t \in I$ and μ_t-a.e. $x \in \mathbb{R}^d$. Moreover, since $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are Borel maps, we have that $(t, x) \mapsto v_t(x)$ is Borel.

3. We recall that the functional

$$(\mu, \nu) \mapsto \int_{\mathbb{R}^d} \left|\frac{\nu}{\mu}(x)\right|^2 d\mu(x)$$

is l.s.c. w.r.t. the weak* convergence of measures (see e.g. Theorem 2.34 in [4]). In particular, for all $t \notin \mathscr{N}$ we have by Fatou's lemma

$$\begin{aligned} \int_{\mathbb{R}^d} |v_t(x)|^2\, d\mu_t(x) &= \int_{\mathbb{R}^d} \left|\frac{\nu_t}{\mu_t}(x)\right|^2 d\mu_t(x) \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \left|\frac{\nu_t^{(n)}}{\mu_t^{(n)}}(x)\right|^2 d\mu_t^{(n)}(x) \\ &= \liminf_{n\to+\infty} \|v_t^{(n)}\|^2_{L^2_{\mu_t^{(n)}}}. \end{aligned}$$

Taking the square root and integrating on I we have

$$\int_I \|v_t\|_{L^2_{\mu_t}}\, dt \leq \int_I \liminf_{n\to+\infty} \|v_t^{(n)}\|_{L^2_{\mu_t^{(n)}}}\, dt \leq \liminf_{n\to+\infty} \int_I \|v_t^{(n)}\|_{L^2_{\mu_t^{(n)}}}\, dt < +\infty.$$

According to the previous steps, we obtain that

- $\boldsymbol{\mu}$ is a narrowly continuous curve, satisfying the continuity equation $\partial\mu_t + \operatorname{div} \nu_t = 0$ in the sense of distributions;
- $|\nu_t| \ll \mu_t$ for a.e. $t \in I$, and $v_t(x) = \dfrac{\nu_t}{\mu_t}(x) \in F(\mu_t, x)$ for μ_t a.e. $x \in \mathbb{R}^d$ and a.e. $t \in I$;
- it holds

$$\int_I \|v_t\|_{L^2_{\mu_t}}\, dt < +\infty.$$

Thus $\boldsymbol{\mu}$ is an admissible trajectory driven by $\boldsymbol{\nu}$. This ends the proof. □

Before stating our existence result for admissible trajectories we first need some more assumptions on the set-valued map F

($\boldsymbol{F}_2$) there exists a continuous increasing function $\Upsilon : [1, +\infty[\to]0, +\infty[$ and $\theta_0 > 0$ such that

- the Cauchy problem

$$\begin{cases} \dot{\theta}(s) = \Upsilon(\theta(s))\theta(s), \text{ for } s > 0, \\ \\ \theta(0) = 1 + \theta_0. \end{cases} \tag{4}$$

  has a solution $\theta(\cdot)$ defined on $[0, T]$.
- $F(\mu, x) \subseteq \Upsilon(1 + \mathrm{m}_2^{1/2}(\mu))(1 + |x|)\overline{B(0, 1)}$.

($\boldsymbol{F}_3$) there exists $L > 0$, a compact metric space U and a continuous map $f : \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times U \to \mathbb{R}^d$ satisfying

$$|f(\mu_1, x_1, u) - f(\mu_2, x_2, u)| \le L(W_2(\mu_1, \mu_2) + |x_1 - x_2|),$$

for all $\mu_i \in \mathscr{P}_2(\mathbb{R}^d)$, $x_i \in \mathbb{R}^d$, $i = 1, 2$, $u \in U$, such that the set-valued map F can be represented as

$$F(\mu, x) = \{f(\mu, x, u) : u \in U\}.$$

Assumption ($\boldsymbol{F}_2$) is strictly related to the construction of an a priori upper bound on the second order moment of the time-evolving measure $t \mapsto \mu_t$. Indeed, in order to prove the existence, we aim to construct a relatively compact invariant domain and to apply a fixed-point iterative procedure to build a sequence of curves in the space of probability measure converging to an admissible trajectory.

Remark 2 We notice that actually **($\boldsymbol{F}_3$)** implies **($\boldsymbol{F}_2$)**. Indeed, assume **($\boldsymbol{F}_3$)**. Then for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, set

$$C := \max\{1, L \cdot \max\{|y| : y \in F(\delta_0, 0)\}\},$$

we have

$$\begin{aligned} F(\mu, x) \subseteq & F(\delta_0, 0) + L(W_2(\mu, \delta_0) + |x|)\overline{B(0, 1)} \subseteq C(1 + m_2^{1/2}(\mu) + |x|)\overline{B(0, 1)}, \\ \subseteq & C(1 + m_2^{1/2}(\mu))(1 + |x|)\overline{B(0, 1)}, \end{aligned}$$

hence we can take $\Upsilon(r) = Cr$, leading to existence of a solution to (4) in $[0, T]$ with $T < \dfrac{1}{C(1 + \theta_0)}$.

Now we state the main result of this section

Theorem 1 (Existence and representation of solutions) *Let $T > 0$ and assume ($\boldsymbol{F_1} - \boldsymbol{F_2}$). Then for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ with $\mathrm{m}_2(\mu) < \theta_0^2$, where θ_0 is as in (4), we have that there exist $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \subseteq \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,T]} \subseteq \mathscr{M}(\mathbb{R}^d; \mathbb{R}^d)$ such that $\boldsymbol{\mu} \in \mathscr{A}^F_{[0,T]}(\mu)$ is an admissible trajectory driven by $\boldsymbol{\nu}$. Moreover, there exists $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that*

1. *$\mu_t = e_t \sharp \boldsymbol{\eta}$ for all $t \in [0, T]$;*

2. *for $\boldsymbol{\eta}$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$, we have*

$$\begin{cases}\dot\gamma(t)\in F(e_t\sharp\boldsymbol{\eta},\gamma(t)), \text{ for a.e. } t\in[0,T];\\ \gamma(0)=x.\end{cases}$$

Conversely, if $\boldsymbol{\eta}\in\mathscr{P}(\mathbb{R}^d\times\Gamma_T)$ satisfies (2) above, we have that $\boldsymbol{\mu}:=\{\mu_t:=e_t\sharp\boldsymbol{\eta}\}_{t\in[0,T]}\in\mathscr{A}^F_{[0,T]}(\mu)$ is an admissible trajectory driven by $\boldsymbol{v}=\{v_t\mu_t\}_{t\in[0,T]}$, where for a.e. $t\in[0,T]$ and μ_t-a.e. $y\in\mathbb{R}^d$

$$v_t(y)=\int_{e_t^{-1}(y)}\dot\gamma(t)\,d\eta_{t,y}(x,\gamma),$$

and $\eta_{t,y}$ is given by the disintegration $\boldsymbol{\eta}=\mu_t\otimes\eta_{t,y}$.

The measure $\boldsymbol{\eta}\in\mathscr{P}(\mathbb{R}^d\times\Gamma_T)$ can be identified with a measure on the space of continuous paths in $\mathbb{R}^d$. In analogy to Theorem 5, the macroscopical behaviour of the system is reconstructed as a (weighted) superposition of paths.

Before proving this theorem we need two Lemmas

Lemma 2 *Let $C,T>0$ and $w:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ be a map satisfying*

1. *$x\mapsto w_t(x)$ continuous for all $t\in[0,T]$,*
2. *$t\mapsto w_t(x)$ measurable for all $x\in\mathbb{R}^d$,*
3. *$|w_t(x)|\le C(1+|x|)$ for all $t\in[0,T]$, $x\in\mathbb{R}^d$.*

Then

- *there is a Borel map $x\mapsto\gamma_x$ from $\mathbb{R}^d$ to Γ_T such that for all $x\in\mathbb{R}^d$ we have $\gamma_x(0)=x$ and $\dot\gamma_x(t)=w_t(\gamma_x(t))$ for a.e. $t\in[0,T]$.*
- *for every $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, set $\boldsymbol{\eta}=\mu\otimes\delta_{\gamma_x}$ and $\boldsymbol{\mu}=\{\mu_t\}_{t\in[0,T]}$ with $\mu_t=e_t\sharp\boldsymbol{\eta}$, we have that $\partial_t\mu_t+\operatorname{div}(w_t\mu_t)=0$ and $\mu_0=\mu$.*

Proof Assumption (3) yields the existence of solutions of the Cauchy problem $\dot\gamma(t)=w_t(\gamma(t))$ with $\gamma(0)=x$ defined in $[0,T]$ for all $x\in\mathbb{R}^d$.

We notice that if $\dot\gamma(t)=w_t(\gamma(t))$ with $\gamma(0)=x$, then

$$|\gamma(t)|\le|x|+\int_0^t|w_s(\gamma(s))|\,ds\le|x|+C\int_0^t(1+|\gamma(s)|)\,ds,$$

and so, by Grönwall's inequality,

$$(1+|\gamma(t)|)\le(1+|x|)e^{Ct}\le(1+|x|)e^{CT}.$$

We define the following map $g:\mathbb{R}^d\times\Gamma_T\to\Gamma_T$

$$g(x,\gamma)(t):=x+\int_0^t w_s\circ\gamma(s)\,ds-\gamma(t).$$

Notice that g is continuous. Consider the set-valued map $H:\mathbb{R}^d\to\Gamma_T$ defined by

$$H(x):=\{\gamma\in\overline{B_\infty(0,(1+|x|)e^{CT})}:\ g(x,\gamma)\equiv0\},$$

where, given $r\ge0$,

$$\overline{B_\infty(0,r)}=\{\gamma\in AC([0,T];\mathbb{R}^d):\ |\gamma(t)|\le r\text{ for all }t\in[0,T]\},$$

i.e., the r-ball of the sup norm centered at the origin. The first assertion of the thesis now follows from Theorem 8.2.9 in [5], while the second one is trivial.

Lemma 3 *Let $T > 0$ and assume $(\boldsymbol{F_1} - \boldsymbol{F_2})$. Let $\theta(\cdot)$ be a solution of (4) fulfilling all the properties in $(\boldsymbol{F_2})$, and set $\zeta(\mu) = 1 + \mathrm{m}_2^{1/2}(\mu)$ for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Define*

$$\begin{aligned}\mathcal{D} := \Big\{\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in AC([0,T];\mathscr{P}_2(\mathbb{R}^d)) : \zeta(\mu_0) < \theta(0),\\ \text{and } \zeta(\mu_t) \le \theta(t) \text{ for all } t \in]0,T]\Big\}.\end{aligned} \tag{5}$$

For all $\hat{\boldsymbol{\mu}} = \{\hat{\mu}_t\}_{t\in[0,T]} \in \mathcal{D}$ we set

$$\begin{aligned}\mathcal{Q}(\hat{\boldsymbol{\mu}}) := \Big\{\boldsymbol{\mu} =\{\mu_t\}_{t\in[0,T]} :\ & \text{there exists a Borel map } (t,x) \mapsto v_t(x)\\ & \text{such that } \partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0,\ \mu_0 = \hat{\mu}_0,\ \text{and}\\ & v_t(x) \in F(\hat{\mu}_t, x) \text{ for a.e. } t \in [0,T] \text{ and } \mu_t\text{ -a.e. } x \in \mathbb{R}^d\Big\}.\end{aligned} \tag{6}$$

Then we have $\emptyset \neq \mathcal{Q}(\hat{\boldsymbol{\mu}}) \subseteq \mathcal{D}$.

In particular, given $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathcal{Q}(\hat{\boldsymbol{\mu}})$, the map $(t,x) \mapsto v_t(x)$ associated to $\boldsymbol{\mu}$ can be chosen also satisfying

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2\, d\mu_t\, dt < +\infty.$$

Proof We first prove that $\mathcal{Q}(\hat{\boldsymbol{\mu}}) \neq \emptyset$. Since the set-valued map $(t,x) \mapsto F(\hat{\mu}_t, x)$ is continuous with convex closed values, it possesses a continuous selection $v_t(x)$. By assumption, we have

$$|v_t(x)| \le \Upsilon(1 + \mathrm{m}_2^{1/2}(\hat{\mu}_t))(1 + |x|) \le \Upsilon(\theta(t))(1 + |x|) \le \Upsilon(\theta(T))(1 + |x|),$$

recalling that $\theta(\cdot)$ is increasing since $\Upsilon(\cdot)$ is nonnegative. In particular, we have that every integral solution of $\dot{\gamma}(t) = v_t(\gamma(t))$ is defined on $[0,T]$. By Lemma 2, there exists a Borel map $x \mapsto \gamma_x$ such that for all $x \in \mathbb{R}^d$ we have $\dot{\gamma}_x(t) = v_t(\gamma_x(t))$ in $]0,T]$ and $\gamma_x(0) = x$. Then, set $\bar{\boldsymbol{\eta}} = \hat{\mu}_0 \otimes \delta_{\gamma_x}$, $\bar{\mu}_t = e_t\sharp\bar{\boldsymbol{\eta}}$, $\bar{v}_t = v_t\bar{\mu}_t$, we have that $\bar{\boldsymbol{\mu}} = \{\bar{\mu}_t\}_{t\in[0,T]} \in \mathcal{Q}(\hat{\boldsymbol{\mu}})$ thanks to $\bar{\boldsymbol{\nu}} = \{\bar{v}_t\}_{t\in[0,T]}$.

We consider now any $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathcal{Q}(\hat{\boldsymbol{\mu}})$. Since $\hat{\mu}_0 = \mu_0$, we have $\mathrm{m}_2(\mu_0) = \mathrm{m}_2(\hat{\mu}_0) < \theta_0^2$. Moreover, there exists $\boldsymbol{\nu} = \{v_t\mu_t\}_{t\in[0,T]}$ such that

$$\partial_t\mu_t + \mathrm{div}\, v_t\mu_t = 0,$$

and $v_t(x) \in F(\hat{\mu}_t, x)$ for a.e. $t \in [0,T]$ and for μ_t-a.e. $x \in \mathbb{R}^d$. In particular, we have

$$\|v_t\|_{L^2_{\mu_t}} \le \Upsilon(\theta(t))(1 + \mathrm{m}_2(\mu_t)) \tag{7}$$

According to Proposition 5, there exists $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ concentrated on the pairs (x,γ) where γ is an integral solutions of $\dot{\gamma}(s) = v_s(\gamma(s))$ satisfying $\gamma(0) = x$, such that $\mu_t = e_t\sharp\boldsymbol{\eta}$ for all $t \in [0,T]$.

For $\boldsymbol{\eta}$-a.e. $(\gamma(0),\gamma) \in \mathbb{R}^d \times \Gamma_T$, and for $0 \le t \le s \le T$ we have

$$|\gamma(s)| \le |\gamma(t)| + \int_t^s \Upsilon(1 + \mathrm{m}_2^{1/2}(\mu_\tau))(1 + |\gamma(\tau)|)\, d\tau$$

By taking the $L^2_{\boldsymbol{\eta}}$ norm, and applying Jensen's inequality, we have

$$\mathrm{m}_2^{1/2}(\mu_s) \le \mathrm{m}_2^{1/2}(\mu_t) + \int_t^s \Upsilon(1 + \mathrm{m}_2^{1/2}(\mu_\tau))(1 + \mathrm{m}_2^{1/2}(\mu_\tau))\, d\tau,$$

so if we set $z(s) = 1 + \mathrm{m}_2^{1/2}(\mu_s)$, we obtain that $z(\cdot)$ is a continuous function satisfying

$$\begin{cases} z(s) \le z(t) + \displaystyle\int_t^s \Upsilon(z(\tau))z(\tau)\,d\tau, \\ \\ z(0) < \theta(0). \end{cases}$$

Given $0 \le t < s \le [0, T]$, we have

$$\theta(s) - z(s) \ge \theta(t) - z(t) + \int_t^s [\Upsilon(\theta(\tau))\theta(\tau) - \Upsilon(z(\tau))z(\tau)]\,d\tau.$$

Since $\theta(0) > z(0)$, we can define

$$\bar{s} = \sup\{\tau \in [0, T] :\ \theta(t) > z(t) \text{ for all } t \in [0, \tau]\} > 0.$$

We want to prove that $\bar{s} = T$. Assume by contradiction that $\bar{s} < T$. According to the above relation, we have

$$\theta(\bar{s}) - z(\bar{s}) \ge \theta(0) - z(0) + \int_0^{\bar{s}} [\Upsilon(\theta(\tau))\theta(\tau) - \Upsilon(z(\tau))z(\tau)]\,d\tau > \theta(0) - z(0) > 0.$$

in particular, by continuity, there exists $\varepsilon > 0$ such that $\bar{s} + \varepsilon < T$ and $\theta(\tau) > z(\tau)$ for all $\tau \in [\bar{s}, \bar{s} + \varepsilon[$, thus contradicting the maximality of $\bar{s}$. Thus, we have that $z(s) \le \theta(s)$ for all $s \in [0, T]$, and the inequality is strict at $s = 0$, hence $\boldsymbol{\mu} \in \mathcal{D}$. Recalling (7), the increasing character of $\Upsilon(\cdot)$ and $\theta(\cdot)$, and the definition of $\mathcal{D}$, we have also

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2\,d\mu_t\,dt \le \Upsilon(\theta(T)) \int_0^T (1 + \mathrm{m}_2(\mu_t))\,dt < +\infty.$$

□

Proof of Theorem 1 The existence will be proved by a fixed point argument. We define by induction a sequence $\boldsymbol{\mu}^{(n)} = \{\mu_t^{(n)}\}_{t\in[0,T]} \subseteq \mathcal{D}$ as follows.

- We set $\mu_t^{(0)} \equiv \mu$ and $\nu_t^{(0)} \equiv 0$ for all $t \in [0, T]$. By assumption, we have that $\boldsymbol{\mu}^{(0)} = \{\mu_t^{(0)}\}_{t\in[0,T]} \in \mathcal{D}$.
- Given $\boldsymbol{\mu}^{(n)} = \{\mu_t^{(n)}\}_{t\in[0,T]} \in \mathcal{D}$, we choose $\boldsymbol{\mu}^{(n+1)} \in \mathcal{Q}(\boldsymbol{\mu}^{(n)}) \subseteq \mathcal{D}$. The choice is possible thanks to Lemma 3.

We notice that, by definition of $\mathcal{Q}(\cdot)$ and by Lemma 3, for all $n \in \mathbb{N}$ it exists a Borel map $(t, x) \mapsto w_t^{(n)}(x)$ such that

- $\partial_t \mu_t^{(n+1)} + \mathrm{div}(w_t^{(n)} \mu_t^{(n)}) = 0$, $\mu_0^{(n)} = \mu$,
- $w_t^{(n+1)}(x) \in F(\mu_t^{(n)}, x)$ for μ_t^{n+1}-a.e. $x \in \mathbb{R}^d$ and a.e. $t \in [0, T]$,
- $\displaystyle\int_0^T \int_{\mathbb{R}^d} |w_t^{(n)}(x)|^2\,d\mu_t^n\,dt < +\infty.$

Thus, recalling $(\boldsymbol{F_2})$,

$$|w_t^{(n)}(x)| \le \Upsilon(1 + \mathrm{m}_2^{1/2}(\mu_t^{(n+1)}))(1 + |x|) \le \Upsilon(\theta(t))(1 + |x|) \le \Upsilon(\theta(T))(1 + |x|),$$

hence, by applying Theorem 5, we define a sequence $\{\boldsymbol{\eta}^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ satisfying $\mu_t^{(n)} = e_t \sharp \boldsymbol{\eta}^{(\boldsymbol{n})}$ for all $n \in \mathbb{N}$, $t \in [0, T]$. Moreover, for $\boldsymbol{\eta}^{(\boldsymbol{n})}$-a.e. $(x, \gamma) \in \mathbb{R}^d \times \Gamma_T$, we have $\dot{\gamma}(t) = w_t^{(n)}(\gamma(t))$ and $\gamma(0) = x$, then

$$|\gamma(t)| \le |x| + \int_0^t |w_s^{(n)}(\gamma(s))|\,ds \le |x| + \Upsilon(\theta(T)) \int_0^t (1 + |\gamma(s)|)\,ds,$$

and so, by Grönwall's inequality, for all $t \in [0, T]$

$$(1 + |\gamma(t)|) \leq (1 + |x|)e^{\Upsilon(\theta(T))t} \leq (1 + |x|)e^{\Upsilon(\theta(T))T},$$

and for a.e. $t \in [0, T]$

$$|\dot{\gamma}(t)| = |w_t^{(n)}(\gamma(t))| \leq \Upsilon(\theta(T))(1 + |\gamma(t)|) \leq \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1 + |x|).$$

This yields

$$\begin{aligned} \mathrm{m}_2^{1/2}(\boldsymbol{\eta}^{(\boldsymbol{n})}) =& \left(\int_{\mathbb{R}^d \times \Gamma_T} (|x|^2 + \|\gamma\|_\infty^2)\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma)\right)^{1/2} \\ \leq& \mathrm{m}_2^{1/2}(\mu) + (1 + \mathrm{m}_2^{1/2}(\mu))e^{\Upsilon(\theta(T))T} < +\infty, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^d \times \Gamma_T} \|\dot{\gamma}\|_{L^\infty}\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma) \leq& \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T} \left(\int_{\mathbb{R} \times \Gamma_T} (1 + |x|)\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma)\right) \\ =& \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1 + \mathrm{m}_2^{1/2}(\mu_0^{(n)})) \\ \leq& \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1 + \mathrm{m}_2^{1/2}(\mu)), \end{aligned}$$

recalling that $\mu_0^{(n)} = \mu$ for all $n \in \mathbb{N}$.

Defined the functional $\mathcal{E} : \mathbb{R}^d \times \Gamma_T \to [0, +\infty]$ by setting

$$\mathcal{E}(x, \gamma) = \begin{cases} |x|^2 + \|\gamma\|_\infty^2 + \|\dot{\gamma}\|_{L^\infty}, & \text{if } \gamma \in AC(I) \text{ and } \dot{\gamma} \in L^\infty(I), \\ +\infty, & \text{otherwise,} \end{cases}$$

we have that $\mathcal{E}$ has compact sublevels in $\mathbb{R}^d \times \Gamma_T$ and

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d \times \Gamma_T} \mathcal{E}(x, \gamma)\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma) < +\infty.$$

By Remark 5.1.5 in [3], the sequence $\{\boldsymbol{\eta}^{(\boldsymbol{n})}\}_{n \in \mathbb{N}}$ is tight. In particular, up to a subsequence, there exists $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that $\boldsymbol{\eta}^{(\boldsymbol{n})} \rightharpoonup^* \boldsymbol{\eta}$. By the continuity of e_t, we have that $\mu_t^{(n)} \rightharpoonup^* \mu_t := e_t \sharp \boldsymbol{\eta}$ for all $t \in [0, T]$.

Indeed, given $0 \leq s \leq t \leq T$ we have

$$\begin{aligned} W_2(\mu_s^{(n)}, \mu_t^{(n)}) \leq& \left[\int_{\mathbb{R}^d \times \Gamma_T} |e_t(x, \gamma) - e_s(x, \gamma)|^2\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma)\right]^{1/2} \\ =& \left[\int_{\mathbb{R}^d \times \Gamma_T} |\gamma(t) - \gamma(s)|^2\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma)\right]^{1/2} \\ \leq& |t - s| \cdot \left[\int_{\mathbb{R}^d \times \Gamma_T} \|\dot{\gamma}\|_{L^\infty}^2\, d\boldsymbol{\eta}^{(\boldsymbol{n})}(x, \gamma)\right]^{1/2} \\ \leq& |t - s| \cdot \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T} \cdot (1 + \mathrm{m}_2(\mu_0^{(n)})) \\ =& |t - s| \cdot \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T} \cdot (1 + \mathrm{m}_2(\mu)), \end{aligned}$$

recalling that $\mu_0^{(n)} = \mu$ for all $n \in \mathbb{N}$. Therefore we have that $\{\boldsymbol{\mu}^{(n)}\}_{n \in \mathbb{N}}$ is a sequence of equiLipschitz continuous curves in $\mathscr{P}_2(\mathbb{R}^d)$ w.r.t. W_2-distance. Since they satisfy also $\mu_0^{(n)} = \mu$ for all $n \in \mathbb{N}$, the sequence is also equibounded. By Ascoli-Arzelà Theorem, up to a (non relabeled) subsequence, it converges uniformly to a Lipschitz continuous curve. By the uniqueness of the limit, we have that $\boldsymbol{\mu}^{(n)}$ converges uniformly to $\boldsymbol{\mu}$.

By Proposition 5.1.8 in [3], we have that for all $(x,\gamma)\in\operatorname{supp}\boldsymbol{\eta}$ there exists a sequence $\{(x_n,\gamma_n)\}_{n\in\mathbb{N}}$ such that $(x_n,\gamma_n)\in\operatorname{supp}\boldsymbol{\eta}^{(n)}$, $x_n\to x$ and $\|\gamma_n-\gamma\|_\infty\to 0$. By the estimates on $w_t^{(n)}(x)$, we have for n sufficiently large and for a.e. $t\in[0,T]$

$$|\dot\gamma_n(t)|\le \Upsilon(\theta(T))(1+|x_n|)\le \Upsilon(\theta(T))(2+|x|),$$

so $\{\gamma_n\}_{n\in\mathbb{N}}$ having uniformely bounded Lipschitz constants, thus γ is Lipschitz continuous. Moreover, for a.e. $t\in[0,T]$ we have

$$\dot\gamma(t)\in\bigcap_{m\in\mathbb{N}}\overline{\operatorname{co}}\{\dot\gamma_n(t)\,:\,n\ge m\}\subseteq\bigcap_{m\in\mathbb{N}}\overline{\operatorname{co}}\bigcup_{n\ge m}F(\mu_t^{(n)},\gamma_n(t)).$$

Recalling the continuity of F and the fact that F is convex valued, for any $\varepsilon>0$ there exists n_ε sufficiently large such that if $n\ge m>n_\varepsilon$ we have $F(\mu_t^{(n)},\gamma_n(t))\subseteq F(\mu_t,\gamma(t))+\varepsilon\overline{B(0,1)}$, thus

$$\dot\gamma(t)\in\overline{\operatorname{co}}\bigcup_{n\ge m}F(\mu^{(n)},\gamma_n(t))\subseteq F(\mu_t,\gamma(t))+\varepsilon\overline{B(0,1)}.$$

In particular, by letting $\varepsilon\to 0^+$, we have that $\boldsymbol{\eta}$ is supported on the set of $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$, where $\gamma\in AC([0,T];\mathbb{R}^d)$ such that $\dot\gamma(t)\in F(\mu_t,\gamma(t))$ for a.e. $t\in[0,T]$. Define now $\boldsymbol{\nu}=\{\nu_t\}_{t\in[0,T]}$ by $\nu_t=v_t\mu_t$ with

$$v_t(x)=\int_{e_t^{-1}(x)}\dot\gamma(t)\,d\eta_{t,x}(y,\gamma),$$

where $\{\eta_{t,x}\}_{\substack{t\in[0,T]\\ x\in\mathbb{R}^d}}$ is the Borel family of probability measures obtained disintegrating $\boldsymbol{\eta}$ w.r.t. e_t, i.e., $\boldsymbol{\eta}=\mu_t\otimes\eta_{t,x}$. Notice that $v_t(x)\in F(\mu_t,x)$ by the convexity assumption on $F(\mu_t,x)$, thus $\boldsymbol{\mu}$ is an admissible trajectory driven by $\boldsymbol{\nu}$.

Conversely, if $\boldsymbol{\eta}$ is supported on $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$, where $\gamma\in AC([0,T];\mathbb{R}^d)$ such that $\dot\gamma(t)\in F(\mu_t,\gamma(t))$ for a.e. $t\in[0,T]$, we define $\boldsymbol{\mu}=\{\mu_t\}_{t\in[0,T]}$ and $\boldsymbol{\nu}=\{\nu_t\}_{t\in[0,T]}$ by setting $\mu_t=e_t\sharp\boldsymbol{\eta}$, and $\nu_t=v_t\mu_t$ with

$$v_t(x)=\int_{e_t^{-1}(x)}\dot\gamma(t)\,d\eta_{t,x}(y,\gamma),$$

where $\boldsymbol{\eta}=\mu_t\otimes\eta_{t,x}$. As before, $v_t(x)\in F(\mu_t,x)$ by the convexity assumption on $F(\mu_t,x)$, thus $\boldsymbol{\mu}$ is an admissible trajectory driven by $\boldsymbol{\nu}$. The proof is complete. □

Remark 3 The convexity of the images of F is essential in the proof of Theorem 1. Roughly speaking, from a multi-agent point of view it means that the *macroscopical* mass diplacement can be *faithfully* represented by the mass transported by the agents at the *micoscopical* level. Indeed, when the convexity assumption on the images of F fails we have two main consequences:

– at the *microscopical level* the trajectories of $\dot\gamma(t)\in F(\mu_t,\gamma(t))$ are *dense* in the set of the trajectories of the relaxed differential inclusion $\dot\gamma(t)\in\overline{\operatorname{co}}\,F(\mu_t,\gamma(t))$ for the metric of uniform convergence by Filippov - Ważewski Relaxation Theorem (see e.g. Theorem 10.4.4 in [5]), provided that F is Borel and Lipschitz w.r.t. x. In particular, at the microscopical level, the difference between working with F or $\overline{\operatorname{co}}\,F$ can be made arbitrary small, if no derivatives of the trajectories are involved.

– at the *macroscopical level* we loose the link between the function $v_t(\cdot)$ and the set-valued map regulating the dynamics of the agents, since the formula providing $v_t(\cdot)$ as weighted average of the velocities of the concurrent characteristics induces *intrinsecally* a convexification for the macroscopical flux $v_t\mu_t$.

In this sense, an example of such a situation was already provided in Example 1 of [16], for an F independent on μ, where the velocities of a nonnegligible set of microscopical trajectories were different from the mean field v_t for a nonnegligible amount of time. Therefore in order to face meaningfully problems where the images of F are not necessarily convex, it is necessary to distinguish between the microscopical dynamics (governed by F) and the macroscopical vector field which in any case must be allowed to belong to $\overline{\text{co}}\, F$.

Combining the above Theorem 1 with Lemma 1, we have the following compactness result.

Corollary 1 *Assume* $(\boldsymbol{F_1}) - (\boldsymbol{F_2})$. *Let* $\mathcal{S} \subseteq \bigcup_{\substack{\mu\in\mathscr{P}_2(\mathbb{R}^d)\\ \mathrm{m}_2(\mu)<\theta_0^2}} \mathscr{A}^F_{[0,T]}(\mu)$, *where* θ_0 *is as in* (4). *Then* $\mathcal{S}$ *is relatively compact in* $C^0([0,T];\mathscr{P}_2(\mathbb{R}^d))$ *w.r.t. the uniform norm (endowing* $\mathscr{P}_2(\mathbb{R}^d)$ *with the* W_2*-distance).*

Proof Let $\{\boldsymbol{\mu}^{(n)}\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$. In particular, there exists a sequence $\{\boldsymbol{\nu}^{(n)}\}_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$ it holds $\boldsymbol{\mu}^{(n)} = \{\mu_t^{(n)}\}_{t\in[0,T]}$, $\boldsymbol{\nu}^{(n)} = \{\nu_t^{(n)} = v_t^{(n)}\mu_t^{(n)}\}_{t\in[0,T]} \subseteq \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ with $\partial_t\mu_t^{(n)} + \mathrm{div}(v_t^{(n)}\mu_t^{(n)}) = 0$ and $v_t^{(n)}(x) \in F(\mu_t^{(n)}, x)$ for a.e. $t\in[0,T]$ and μ_t-a.e. $x\in\mathbb{R}^d$. Moreover, $\mathrm{m}_2(\mu_0^{(n)}) < \theta_0^2$. Thus $\boldsymbol{\mu}^{(n)} \in \mathcal{Q}(\boldsymbol{\mu})$. According to Lemma 3, we have that $\boldsymbol{\mu}^{(n)} \in \mathcal{D}$. In particular, there exists $\boldsymbol{\eta}^{(n)} \in \mathscr{P}(\mathbb{R}^d\times\Gamma_T)$ such that $\mu_t^{(n)} = e_t\sharp\boldsymbol{\eta}^{(n)}$ for all $t\in[0,T]$ and for $\boldsymbol{\eta}^{(n)}$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$ it holds $\gamma(0)=x$ and $\dot\gamma(t) = v_t^{(n)}\circ\gamma(t) \in F(e_t\sharp\mu_t^{(n)}, \gamma(t))$ for a.e. $t\in[0,T]$. Thus, recalling $(\boldsymbol{F_2})$,

$$|v_t^{(n)}(x)| \le \Upsilon(1+\mathrm{m}_2^{1/2}(\mu_t^{(n)}))(1+|x|) \le \Upsilon(\theta(t))(1+|x|) \le \Upsilon(\theta(T))(1+|x|),$$

As done in the proof of Theorem 1,

$$|\gamma(t)| \le |x| + \int_0^t |v_s^{(n)}(\gamma(s))|\,ds \le |x| + \Upsilon(\theta(T))\int_0^t (1+|\gamma(s)|)\,ds,$$

and so, by Grönwall's inequality, for all $t\in[0,T]$

$$(1+|\gamma(t)|) \le (1+|x|)e^{\Upsilon(\theta(T))t} \le (1+|x|)e^{\Upsilon(\theta(T))T},$$

and for a.e. $t\in[0,T]$

$$|\dot\gamma(t)| = |v_t^{(n)}(\gamma(t))| \le \Upsilon(\theta(T))(1+|\gamma(t)|) \le \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1+|x|).$$

This yields

$$\begin{aligned}\mathrm{m}_2^{1/2}(\boldsymbol{\eta}^{(n)}) = &\left(\int_{\mathbb{R}^d\times\Gamma_T} (|x|^2 + \|\gamma\|_\infty^2)\,d\boldsymbol{\eta}^{(n)}(x,\gamma)\right)^{1/2}\\ \le &\mathrm{m}_2^{1/2}(\mu_0^{(n)}) + (1+\mathrm{m}_2^{1/2}(\mu_0^{(n)}))e^{\Upsilon(\theta(T))T}\\ < &\theta_0 + (1+\theta_0)e^{\Upsilon(\theta(T))T},\end{aligned}$$

and

$$\int_{\mathbb{R}^d\times\Gamma_T} \|\dot\gamma\|_{L^\infty}\,d\boldsymbol{\eta}^{(n)}(x,\gamma) \le \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}\left(\int_{\mathbb{R}\times\Gamma_T}(1+|x|)\,d\boldsymbol{\eta}^{(n)}(x,\gamma)\right)$$

$$\leq \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1 + \mathrm{m}_2^{1/2}(\mu_0^{(n)}))$$
$$\leq \Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}\theta_0.$$

Considering again the functional $\mathcal{E} : \mathbb{R}^d \times \Gamma_T \to [0, +\infty]$ defined by setting

$$\mathcal{E}(x, \gamma) = \begin{cases} |x|^2 + \|\gamma\|_\infty^2 + \|\dot{\gamma}\|_{L^\infty}, & \text{if } \gamma \in AC(I) \text{ and } \dot{\gamma} \in L^\infty(I), \\ \\ +\infty, & \text{otherwise,} \end{cases}$$

we have that $\mathcal{E}$ has compact sublevels in $\mathbb{R}^d \times \Gamma_T$ and

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d\times\Gamma_T} \mathcal{E}(x, \gamma)\, d\boldsymbol{\eta}^{(n)}(x, \gamma) < +\infty.$$

By Remark 5.1.5 in [3], the sequence $\{\boldsymbol{\eta}^{(n)}\}_{n\in\mathbb{N}}$ is tight. In particular, up to a subsequence, there exists $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that $\boldsymbol{\eta}^{(n)} \rightharpoonup^* \boldsymbol{\eta}$. By the continuity of e_t, we have that $\mu_t^{(n)} \rightharpoonup^* \mu_t := e_t \sharp \boldsymbol{\eta}$ for all $t \in [0, T]$. Furthermore, we have

$$\begin{aligned} W_2(\mu_t^{(n)}, \mu_s^{(n)}) &\leq \left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |e_t(x, \gamma) - e_s(x, \gamma)|^2\, d\boldsymbol{\eta}^{(n)}(x, \gamma) \right]^{\frac{1}{2}} \\ &= \left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |\gamma(t) - \gamma(s)|^2\, d\boldsymbol{\eta}^{(n)}(x, \gamma) \right]^{\frac{1}{2}} \\ &\leq |t - s| \left[\int_{\mathbb{R}^d\times\Gamma_T} \|\dot{\gamma}\|_{L^\infty}^2\, d\boldsymbol{\eta}^{(n)}(x, \gamma) \right]^{\frac{1}{2}} \\ &\leq |t - s| \left[\int_{\mathbb{R}^d\times\Gamma_T} \Upsilon^2(\theta(T))e^{2\Upsilon(\theta(T))T}(1 + |x|)^2\, d\boldsymbol{\eta}^{(n)}(x, \gamma) \right]^{\frac{1}{2}} \\ &\leq |t - s|\Upsilon(\theta(T))e^{\Upsilon(\theta(T))T}(1 + \theta_0). \end{aligned}$$

So the sequence $\{\boldsymbol{\mu}^{(n)}\}_{n\in\mathbb{N}}$ is equibounded and equiLipschitz continuous, hence $\boldsymbol{\mu}$ is Lipschitz continuous in $\mathscr{P}_2(\mathbb{R}^d)$. Arguing as in the proof of Theorem 1, we have that $\boldsymbol{\eta}$ is supported on pairs $(x, \gamma) \in \mathbb{R}^d \times \Gamma_T$ such that $\gamma(0) = x$ and $\dot{\gamma}(t) \in F(e_t\sharp\boldsymbol{\eta}, \gamma(t))$ for a.e. $t \in [0, T]$, thus $\boldsymbol{\mu}$ is an admissible trajectory. □

Remark 4 Assume $(\boldsymbol{F_1}) - (\boldsymbol{F_3})$. Given any continuous curve $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$, we set $g_{\boldsymbol{\mu}}(t, x, u) := f(\mu_t, x, u)$ and let

$$\mathcal{U} := \{u(\cdot) : u(\cdot) \text{ measurable and } u([0, T]) \subseteq U\}.$$

Recalling e.g. Lemma 7.3 in [18],

- the map $\mathbb{R}^d \times \mathcal{U} \to \Gamma_T$ associating to $(x, u(\cdot))$ the unique solution $\gamma_{x,u(\cdot)}^{\boldsymbol{\mu}}$ of $\dot{\gamma}(t) = g(t, x, u(t)), \gamma(0) = x$ is continuous w.r.t. both the variable when on $\mathcal{U}$ we put the metric of the convergence in measure;
- the set $D_{\boldsymbol{\mu}} := \{\gamma_{x,u(\cdot)}^{\boldsymbol{\mu}} : u(\cdot) \in \mathcal{U},\ x \in \mathbb{R}^d\}$ is closed.

If we define the set-valued map $G : D_{\boldsymbol{\mu}} \rightrightarrows \mathcal{U}$

$$G_{\boldsymbol{\mu}}(\gamma) := \{u \in \mathcal{U} : \gamma_{\gamma(0),u(\cdot)}^{\boldsymbol{\mu}}(t) = \gamma(t) \text{ for all } t \in [0, T]\},$$

we have that G admits a Borel selection $\gamma \mapsto u_\gamma$ (see e.g. Theorem 8.2.9 in [5]). In particular, for all $\gamma \in D_{\boldsymbol{\mu}}$ we have $\gamma = \gamma_{\gamma(0),u_\gamma(\cdot)}^{\boldsymbol{\mu}}$.

As in classical control, it is crucial to be able to construct an approximation of a given trajectory starting from an initial data by another trajectory starting from another initial data.

Proposition 2 (Grönwall-Filippov type estimate) *Assume* $(\boldsymbol{F_1}) - (\boldsymbol{F_3})$. *Let* $\mu_0, \mu_0^{(G)} \in \mathscr{P}_2(\mathbb{R}^d)$*, and* $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathscr{A}^F_{[0,T]}(\mu_0)$ *be an admissible trajectory. Then there exists an admissible trajectory* $\boldsymbol{\mu}^{(G)} = \{\mu_t^{(G)}\}_{t\in[0,T]} \in \mathscr{A}^F_{[0,T]}(\mu_0^{(G)})$ *such that for all* $t \in [0,T]$ *we have*

$$W_2(\mu_t, \mu_t^{(G)}) \le e^{LT+Te^{LT}} \cdot W_2(\mu_0, \mu_0^{(G)}),$$

where L *is as in* $(\boldsymbol{F_3})$.

Proof We will proceed by defining by recurrence a sequence converging to the desired trajectory. Set $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu} = \{\mu_t^{(0)}\}_{t\in[0,T]}$ and $\boldsymbol{\pi} \in \Pi_o(\mu_0^{(0)}, \mu_0^{(G)})$. By assumption on F and since $\boldsymbol{\mu}^{(0)}$ is admissible, there exists $\boldsymbol{\eta}^{(0)} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that $\mu_t^{(0)} = e_t\sharp\boldsymbol{\eta}^{(0)}$ for all $t \in [0,T]$ and for $\boldsymbol{\eta}^{(0)}$-a.e. $(x,\gamma) \in \mathbb{R}^d \times \Gamma_T$ there exists a Borel map $u_\gamma : [0,T] \to U$ such that $\gamma \mapsto u_\gamma$ is also Borel and

$$\begin{cases} \dot{\gamma}(t) = f(\mu_t^{(0)}, \gamma(t), u_\gamma(t)), \text{ for a.e. } t \in [0,T], \\ \gamma(0) = x. \end{cases}$$

Given $y \in \mathbb{R}^d$, we define a map $\tau_{y,\boldsymbol{\mu}} : \Gamma_T \to \Gamma_T$ by letting $\tau_{y,\boldsymbol{\mu}}(\gamma)$ be the solution of

$$\begin{cases} \dot{\tilde{\gamma}}(t) = f(\mu_t^{(0)}, \tilde{\gamma}(t), u_\gamma(t)), \text{ for a.e. } t \in [0,T], \\ \tilde{\gamma}(0) = y. \end{cases}$$

for $\boldsymbol{\eta}^{(0)}$-a.e. $(\gamma(0),\gamma) \in \mathbb{R}^d \times \Gamma_T$.

In other words, given γ such that $(\gamma(0),\gamma) \in \operatorname{supp}\boldsymbol{\eta}^{(0)}$ we consider the control strategy $u_\gamma(\cdot)$ generating it, and use the same control strategy to construct a curve $\tau_{y,\boldsymbol{\mu}}$ starting from y.

Define two maps $\psi_{\boldsymbol{\mu}^{(0)}}, \phi : \mathbb{R}^d \times \mathbb{R}^d \times \Gamma_T \to \mathbb{R}^d \times \Gamma_T$ by setting $\phi(x,y,\gamma) = (x,\gamma)$ and $\psi_{\boldsymbol{\mu}^{(0)}}(x,y,\gamma) = (y, \tau_{y,\boldsymbol{\mu}^{(0)}}(\gamma))$. Notice that $\psi_{\boldsymbol{\mu}^{(0)}}(x,y,\gamma)$ is well-defined only for $\boldsymbol{\eta}^{(0)}$-a.e. $(\gamma(0),\gamma) \in \mathbb{R}^d \times \Gamma_T$ and all $y \in \mathbb{R}^d$.

Written $\boldsymbol{\eta}^{(0)} = \mu_0^{(0)} \otimes \eta_x^{(0)}$ for a Borel family $\{\eta_x^{(0)}\}_{x\in\mathbb{R}^d}$, set

$$\boldsymbol{\eta}^{(1)} := \psi_{\boldsymbol{\mu}^{(0)}}\sharp\left(\boldsymbol{\pi} \otimes \eta_x^{(0)}\right),$$

and let $\boldsymbol{\mu}^{(1)} = \{\mu_t^{(1)}\}_{t\in[0,T]}$ be defined by $\mu_t^{(1)} := e_t\sharp\boldsymbol{\eta}^{(1)}$. Notice that, by construction, we have $\boldsymbol{\eta}^{(0)} = \phi\sharp\left(\boldsymbol{\pi} \otimes \eta_x^{(0)}\right)$. Thus (see e.g. formula (7.1.6) in [3])

$$W_2(\mu_t^{(0)}, \mu_t^{(1)}) \le \left\| e_t \circ \phi - e_t \circ \psi_{\boldsymbol{\mu}^{(0)}} \right\|_{L^2_{\boldsymbol{\pi}\otimes\eta_x}}. \tag{8}$$

For $\boldsymbol{\pi} \otimes \eta_x$-a.e. $(x,y,\gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times \Gamma_T$, recalling $(\boldsymbol{F_3})$, we have

$$\begin{aligned} &|e_t \circ \phi(x,y,\gamma) - e_t \circ \psi_{\boldsymbol{\mu}^{(0)}}(x,y,\gamma)| \\ &\quad = \left| x - y + \int_0^t \left[f(\mu_s, \gamma(s), u_\gamma(s)) - f(\mu_s, \tau_{y,\boldsymbol{\mu}^{(0)}}(\gamma)(s), u_\gamma(s)) \right] ds \right| \end{aligned}$$

$$\leq |x-y| + \int_0^t \left| f(\mu_s, \gamma(s), u_\gamma(s)) - f(\mu_s, \tau_{y,\boldsymbol{\mu^{(0)}}}(\gamma)(s), u_\gamma(s)) \right| ds$$
$$\leq |x-y| + L\int_0^t \left| \gamma(s) - \tau_{y,\boldsymbol{\mu^{(0)}}}(\gamma)(s) \right| ds$$
$$= |x-y| + L\int_0^t |e_s \circ \phi(x,y,\gamma) - e_s \circ \psi_{\boldsymbol{\mu^{(0)}}}(x,y,\gamma)|\, ds \tag{9}$$

Recalling that, by the optimality of $\boldsymbol{\pi}$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\Gamma_T} |x-y|^2\, d(\boldsymbol{\pi}\otimes\eta_x^{(0)})(x,y,\gamma) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\, d\boldsymbol{\pi}(x,y)$$
$$= W_2^2(\mu_0^{(0)}, \mu_0^{(G)}),$$

taking the L^2 norm of (9) w.r.t. $\boldsymbol{\pi}\otimes\eta_x$ and using Jensen's inequality yields

$$\left\| e_t \circ \phi - e_t \circ \psi_{\boldsymbol{\mu^{(0)}}} \right\|_{L^2_{\boldsymbol{\pi}\otimes\eta_x^{(0)}}}$$
$$\leq W_2(\mu_0^{(0)}, \mu_0^{(G)}) + L\int_0^t \left\| e_s \circ \phi - e_s \circ \psi_{\boldsymbol{\mu^{(0)}}} \right\|_{L^2_{\boldsymbol{\pi}\otimes\eta_x^{(0)}}} ds$$

By Grönwall inequality, and recalling (8), we obtain

$$W_2(\mu_t^{(0)}, \mu_t^{(1)}) \leq e^{Lt}\cdot W_2(\mu_0^{(0)}, \mu_0^{(G)}) \leq e^{LT}\cdot W_2(\mu_0^{(0)}, \mu_0^{(G)}).$$

We construct now a sequence by induction.

Assume to have defined $\boldsymbol{\eta^{(k)}} \in \mathscr{P}(\mathbb{R}^d\times\Gamma_T)$, $k=1,\dots,n-1$, in such a way that for $\boldsymbol{\eta^{(k)}}$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$ there exists $u_\gamma:[0,T]\to U$ such that

$$\begin{cases} \dot{\gamma}(t) = f(e_t\sharp\boldsymbol{\eta^{(k-1)}}, \gamma(t), u_\gamma(t)), \text{ for a.e. } t\in[0,T], \\ \gamma(0) = x, \end{cases}$$

and satisfying

$$W_2(e_t\sharp\boldsymbol{\eta^{(k-1)}}, e_t\sharp\boldsymbol{\eta^{(k)}}) \leq e^{kLT}\cdot W_2(\mu_0^{(0)}, \mu_0^{(G)})\frac{t^{k-1}}{(k-1)!},$$

for all $k=1,\dots,n-1$ and $t\in[0,T]$ (recall that $0!=1$). Then for $\boldsymbol{\eta^{(n-1)}}$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\Gamma_T$ we define $\phi_n(x,\gamma) = (x,\tau_n(\gamma))$ where $\tau_n:\Gamma_T\to\Gamma_T$, and for $\boldsymbol{\eta}^{(n-1)}$-a.e. $(\gamma(0),\gamma)\in\mathbb{R}^d\times\Gamma_T$, we have that $\tau_n(\gamma)$ is the solution of

$$\begin{cases} \dot{\tilde{\gamma}}(t) = f(e_t\sharp\boldsymbol{\eta^{(n-1)}}, \tilde{\gamma}(t), u_\gamma(t)), \text{ for a.e. } t\in[0,T], \\ \gamma(0) = x, \end{cases}$$

and we set $\boldsymbol{\eta}^{(n)} := \phi_n\sharp\boldsymbol{\eta}^{(n-1)}$. We have

$$W_2(e_t\sharp\boldsymbol{\eta^{(n-1)}}, e_t\sharp\boldsymbol{\eta^{(n)}}) \leq \|e_t - e_t\circ\phi_n\|_{L^2_{\boldsymbol{\eta^{(n-1)}}}}.$$

Since

$$|e_t(x,\gamma) - e_t\circ\phi_n(x,\gamma)|$$
$$\leq \int_0^t \left| f(e_s\sharp\boldsymbol{\eta^{(n-2)}}, \gamma(s), u_\gamma(s)) - f(e_s\sharp\boldsymbol{\eta^{(n-1)}}, \tau_n(\gamma)(s), u_\gamma(s)) \right| ds$$

$$
\begin{aligned}
&\leq L \int_0^t W_2(e_s \sharp \boldsymbol{\eta}^{(n-2)}, e_s \sharp \boldsymbol{\eta}^{(n-1)})\, ds + L \int_0^t |\gamma(s) - \tau_n(\gamma)(s)|\, ds \\
&= L \int_0^t W_2(e_s \sharp \boldsymbol{\eta}^{(n-2)}, e_s \sharp \boldsymbol{\eta}^{(n-1)})\, ds + L \int_0^t |e_s(x,\gamma) - e_s \circ \phi_n(x,\gamma)|\, ds,
\end{aligned}
$$

taking the L^2 norm w.r.t. $\boldsymbol{\eta}^{(n-1)}$ and using Jensen's inequality, we have

$$
\begin{aligned}
&\|e_t - e_t \circ \phi_n\|_{L^2_{\boldsymbol{\eta}^{(n-1)}}} \\
&\qquad \leq L \int_0^t W_2(e_s \sharp \boldsymbol{\eta}^{(n-2)}, e_s \sharp \boldsymbol{\eta}^{(n-1)})\, ds + L \int_0^t \|e_s - e_s \circ \phi_n\|_{L^2_{\eta^{(n-1)}}}\, ds
\end{aligned}
$$

By Grönwall inequality, we obtain

$$
\begin{aligned}
W_2(e_t \sharp \boldsymbol{\eta}^{(n-1)}, e_t \sharp \boldsymbol{\eta}^{(n)}) &\leq \|e_t - e_t \circ \phi_n\|_{L^2_{\boldsymbol{\eta}^{(n-1)}}} \\
&\leq L e^{LT} \int_0^t W_2(e_s \sharp \boldsymbol{\eta}^{(n-2)}, e_s \sharp \boldsymbol{\eta}^{(n-1)})\, ds \\
&\leq e^{LT} \int_0^t e^{(n-1)LT} \cdot W_2(\mu_0^{(0)}, \mu_0^{(G)}) \frac{s^{n-2}}{(n-2)!}\, ds \\
&= e^{nLT} \cdot W_2(\mu_0^{(0)}, \mu_0^{(G)}) \frac{t^{n-1}}{(n-1)!}.
\end{aligned}
$$

In particular, since

$$
\begin{aligned}
\sum_{n=1}^{+\infty} \max_{t\in[0,T]} W_2(e_t \sharp \boldsymbol{\eta}^{(n-1)}, e_t \sharp \boldsymbol{\eta}^{(n)}) &\leq e^{LT} W_2(\mu_0^{(0)}, \mu_0^{(G)}) \sum_{n=1}^{\infty} \frac{[Te^{LT}]^{n-1}}{(n-1)!} \\
&= e^{LT + Te^{LT}} \cdot W_2(\mu_0^{(0)}, \mu_0^{(G)}),
\end{aligned}
$$

the sequence of continuous curves $\boldsymbol{\mu}^{(n)} = \{\mu_t^{(n)}\}_{t\in[0,T]}$ is a Cauchy sequence in $C^0([0,T]; \mathscr{P}_2(\mathbb{R}^d))$, hence it converges uniformly to a continuous curve $\boldsymbol{\mu}^{\infty} = \{\mu_t^{\infty}\}_{t\in[0,T]}$, where $\mu_0^{\infty} = \mu_0^{(G)}$. In particular, a measurable selection theorem (see e.g. Theorem 8.2.11 in [5]) yields

$$
\begin{aligned}
\mathrm{m}_2(\boldsymbol{\eta}^{(n)}) &= \iint_{\mathbb{R}^d \times \Gamma_T} \sup_{t\in[0,T]} \left[|x|^2 + |e_t(\gamma)|^2\right] d\boldsymbol{\eta}^{(n)}(x,\gamma) \\
&= \sup_{t\in[0,T]} \iint_{\mathbb{R}^d \times \Gamma_T} |x|^2 + |e_t(\gamma)|^2\, d\boldsymbol{\eta}^{(n)}(x,\gamma) \\
&\leq \sup_{t\in[0,T]} 2\mathrm{m}_2(e_t \sharp \boldsymbol{\eta}^{(n)}) = 2 \sup_{t\in[0,T]} \mathrm{m}_2(\mu_t^{(n)}).
\end{aligned}
$$

We recall that

$$
\mathrm{m}_2^{1/2}(\mu_t^{(n)}) = W_2(\delta_0, \mu_t^{(n)}),
$$

and so the sequence of continuous maps $\{t \mapsto \mathrm{m}_2^{1/2}(\mu_t^{(n)})\}_{n\in\mathbb{N}}$ uniformly converges to the continuous map $t \mapsto \mathrm{m}_2^{1/2}(\mu_t^{\infty})$ (recalling also the Lipschitz continuity of $W_2(\delta_0, \cdot)$). For n sufficiently large, we then have

$$
\mathrm{m}_2(\boldsymbol{\eta}^{(n)}) \leq 4 \sup_{t\in[0,T]} \mathrm{m}_2(\mu_t^{\infty}) < +\infty,
$$

where the finiteness of the right hand side is ensured by the continuity of $\boldsymbol{\mu}^\infty$ on the compact $[0, T]$.

Since

$$
\begin{aligned}
F(\mu, x) \subseteq & F(\delta_0, 0) + L(W_2(\delta_0, \mu) + |x|) \cdot \overline{B(0,1)} \\
= & F(\delta_0, 0) + (L\mathrm{m}_2(\mu) + L|x|) \cdot \overline{B(0,1)},
\end{aligned}
$$

we have also that for $\boldsymbol{\eta}^{(n)}$-a.e. $(\gamma(0), \gamma) \in \mathbb{R}^d \times \Gamma_T$

$$
|\dot{\gamma}(t)| \le \max_{w \in F(\delta_0, 0)} |w| + L\mathrm{m}_2(\mu_t^{(n)}) + L|\gamma(t)|.
$$

Taking the $L^2_{\boldsymbol{\eta}^{(n)}}$ norm yields

$$
\begin{aligned}
\|\dot{\gamma}(t)\|_{L^2_{\boldsymbol{\eta}^{(n)}}} & \le \max_{w \in F(\delta_0, 0)} |w| + 2L(1 + \mathrm{m}_2(\mu_t^{(n)})) \\
& \le \max_{w \in F(\delta_0, 0)} |w| + 2L(1 + \sup_{s \in [0,T]} \mathrm{m}_2(\mu_s^\infty)) \\
\int_0^T \|\dot{\gamma}(t)\|^2_{L^2_{\boldsymbol{\eta}^{(n)}}} \, dt & \le T \left(\max_{w \in F(\delta_0, 0)} |w| + 8L(1 + \sup_{s \in [0,T]} \mathrm{m}_2(\mu_s^\infty)) \right)^2 < +\infty.
\end{aligned}
$$

We have

$$
\begin{aligned}
& \int_{\mathbb{R}^d \times \Gamma_T} \left[|x|^2 + \|\gamma\|_\infty^2 + \|\dot{\gamma}\|^2_{L^2} \right] d\boldsymbol{\eta}^{(n)}(x, \gamma) \\
& = \mathrm{m}_2(\boldsymbol{\eta}^{(n)}) + \int_{\mathbb{R}^d \times \Gamma_T} \int_0^T |\dot{\gamma}(t)|^2 \, dt \, d\boldsymbol{\eta}^{(n)}(x, \gamma) \\
& = \mathrm{m}_2(\boldsymbol{\eta}^{(n)}) + \int_0^T \int_{\mathbb{R}^d \times \Gamma_T} |\dot{\gamma}(t)|^2 \, d\boldsymbol{\eta}^{(n)}(x, \gamma) \, dt \\
& = \mathrm{m}_2(\boldsymbol{\eta}^{(n)}) + \int_0^T \|\dot{\gamma}(t)\|^2_{L^2_{\boldsymbol{\eta}^{(n)}}} \, dt \\
& \le 4 \sup_{t \in [0,T]} \mathrm{m}_2(\mu_t^\infty) + T \left(\max_{w \in F(\delta_0, 0)} |w| + 8L(1 + \sup_{s \in [0,T]} \mathrm{m}_2(\mu_s^\infty)) \right)^2 < +\infty.
\end{aligned}
$$

Hence, again by Remark 5.1.5 in [3], there exists $\boldsymbol{\eta}^\infty \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that, up to a subsequence $\boldsymbol{\eta}^{(n)} \rightharpoonup^* \boldsymbol{\eta}^\infty$. By the continuity of the operator e_t on $\mathbb{R}^d \times \Gamma_T$, and recalling that $\mu_t^{(n)} = e_t \sharp \boldsymbol{\eta}^{(n)}$, we obtain $\mu_t^\infty = e_t \sharp \boldsymbol{\eta}^\infty$. Moreover, by Proposition 5.1.8 in [3], for $\boldsymbol{\eta}^\infty$-a.e. $(x, \gamma) \in \mathbb{R}^d \times \Gamma_T$ there exists a sequence $\{(x_n, \gamma_n)\}_{n \in \mathbb{N}}$ such that $(x_n, \gamma_n) \in \operatorname{supp} \boldsymbol{\eta}^{(n)}$, $x_n \to x$ and $\|\gamma_n - \gamma\|_\infty \to 0$. For a.e. $t \in [0, T]$ we have

$$
\dot{\gamma}_n(t) \in F(\mu_t^{n-1}, \gamma_n(t)) \subseteq F(\mu_t^\infty, 0) + L(|\gamma_n(t)| + W_2(\mu_t^{(n)}, \mu_t^\infty) \overline{B(0,1)}). \tag{10}
$$

Thus, for n sufficiently large, we have $W_2(\mu_t^{(n)}, \mu_t^\infty) \le 1$ and $|\gamma_n(t) - \gamma(t)| \le 1$ for all $t \in [0, T]$, and so

$$
|\dot{\gamma}_n(t)| \le C + L|\gamma_n(t)| \le C + L(\|\gamma\|_\infty + 1),
$$

where $C := L + \max\{|v| : v \in F(\mu_t^\infty, 0),\ t \in [0, T]\}$. So γ is Lipschitz continuous. By passing to the limit in the equation (10) , for a.e. $t \in [0, T]$ we obtain

$$
\dot{\gamma}(t) = f(\mu_t^\infty, \gamma(t), u_\gamma(t)), \text{ for a.e. } t \in [0, T], \text{ and } \gamma(0) = x,
$$

In particular, set $\boldsymbol{\mu}^{(G)} := \boldsymbol{\mu}^{\infty}$, we have that $\boldsymbol{\mu}^{(G)}$ is an admissible trajectory satisfying all the requested properties. □

Example 1 Possible choices for $\Upsilon(\cdot)$ are

- $\Upsilon(r) = C \cdot \log r$ for any $C > 0$ and $T > 0$. In this case we can also choose $\theta_0 > 0$ arbitrarily, and $\theta(t) = (1+\theta_0)^{e^{Ct}}$.
- $\Upsilon(r) = C \cdot r^{\alpha}$ for $\alpha > 0$. In this case we have
$$\theta(t) = \frac{1+\theta_0}{(1-\alpha C t(1+\theta_0)^{\alpha})^{1/\alpha}},$$
thus we require $\alpha C T(1+\theta_0)^{\alpha} < 1$.

Remark 5 A possible variant of $(\boldsymbol{F_2})$, is to consider instead of (4), the existence of a solution $\theta(\cdot)$ defined in $[0,T]$ to

$$\begin{cases} \dot\theta(s) = \Xi(\theta(s), \theta(s)), \text{ for } s > 0, \\ \theta(0) = 1 + \theta_0, \end{cases} \tag{11}$$

where $\Xi : [1,+\infty[\times[1,+\infty[\to]0,+\infty[$ is a continuous function such that $r \mapsto \Xi(r,r)$ is increasing, and $r_2 \mapsto \Xi(r_1, r_2)$ is convex for all r_1. In this case, if $\theta(\cdot)$ is such a solution, we have also to assume that

$$\begin{cases} \dot\gamma(s) = \Xi(\theta(s), \gamma(s)), \\ \gamma(0) = x \end{cases}$$

has a solution defined on $[0,T]$ for all $x \in \mathbb{R}^d$, while in the previous setting this was granted by the sublinear growth of $\Xi(r_1,r_2) = \Upsilon(r_1)r_2$ w.r.t. the second variable. The proof of Lemma 3 in this new setting requires only a straighforward adaption of the previous proof. Moreover, in the case of $(\boldsymbol{F_3})$, this yields to existence for all initial conditions θ_0 and all times T.

4 The value function and its properties

Let $T > 0$, $\mathcal{L} : \mathscr{P}_2(\mathbb{R}^d) \times \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R}\cup\{+\infty\}$ and $\mathscr{G} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Given $\boldsymbol{\mu} = \{\mu_t\}_{t\in[s,T]} \in \mathscr{A}^F_{[s,T]}(\mu)$ driven by $\boldsymbol{\nu} = \{\nu_t\}_{t\in[s,T]} \subseteq \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ we define the functional $J(\cdot)$ by

$$J_{[s,T]}(\boldsymbol{\mu},\boldsymbol{\nu}) := \int_s^T \mathcal{L}(\mu_t,\nu_t)\,dt + \mathscr{G}(\mu_T).$$

In particular, we say that $\hat{\boldsymbol{\mu}} = \{\hat\mu_t\}_{t\in[s,T]} \in \mathscr{A}^F_{[s,T]}(\mu)$ is an *optimal trajectory* starting from μ at time s if there exists $\hat{\boldsymbol{\nu}} = \{\hat\nu_t\}_{t\in[s,T]} \subseteq \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\hat{\boldsymbol{\mu}}$ is driven by $\hat{\boldsymbol{\nu}}$ and

$$J_{[s,T]}(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\nu}}) = \inf\left\{J_{[s,T]}(\boldsymbol{\mu},\boldsymbol{\nu}) : \boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu) \text{ driven by } \boldsymbol{\nu}\right\}.$$

This enables us to define the *value function* $V : [0,T]\times\mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}\cup\{+\infty\}$ by

$$V(s,\mu) := \inf\left\{J_{[s,T]}(\boldsymbol{\mu},\boldsymbol{\nu}) : \boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu) \text{ driven by } \boldsymbol{\nu}\right\}. \tag{12}$$

Recall that one can concatenate admissible trajectories as follows: Let $I_1 = [a,b]$, $I_2 = [b,c]$ with $a \le b \le c$. Given $\boldsymbol{\mu}^{(i)} \in \mathscr{A}^F_{I_i}(\mu^{(i)}$ driven by $\boldsymbol{\nu}^{(i)}$, $i = 1,2$ with $\mu^{(1)} \in \mathscr{P}_2(\mathbb{R}^d)$ and

$\mu^{(2)} = \mu^{(1)}_b$, set $(\mu_t, \nu_t) = (\mu^{(i)}_t, \nu^{(i)}_t)$ for $t \in I_i \setminus \{b\}$, $i = 1, 2$, and $(\mu_b, \nu_b) = (\mu^{(1)}_b, \nu^{(1)}_b)$. Then $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,c]} \in \mathscr{A}^F_{[a,c]}(\mu^{(1)})$ is an admissible trajectory driven by $\boldsymbol{\nu} = \{\nu_t\}_{t\in]b,c]}$, that will be called the *concatenation* of $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\nu}^{(1)})$ and $(\boldsymbol{\mu}^{(2)}, \boldsymbol{\nu}^{(2)})$. We will denote $\boldsymbol{\mu}$ by $\boldsymbol{\mu}^{(1)} \odot \boldsymbol{\mu}^{(2)}$ and $\boldsymbol{\nu}$ by $\boldsymbol{\nu}^{(1)} \odot \boldsymbol{\nu}^{(2)}$.

Proposition 3 (Dynamic Programming Principle) *We have for every $\tau \in [s, T]$,*

$$V(s, \mu) = \inf \left\{ \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau) : (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathscr{A}^F_{[s,T]}(\mu) \right\}. \tag{13}$$

In particular, given any $\boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu)$ driven by $\boldsymbol{\nu}$, we have that the map

$$\tau \mapsto h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\tau) := \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau)$$

is nondecreasing on $[s, T]$, and it is constant if and only if $V(s, \mu) = J_{[s,T]}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

Proof Denote by $W(s, \mu)$ the right-hand side of formula (13).
Take $\varepsilon > 0$, $s \le \tau \le T$, $\boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu)$ driven by $\boldsymbol{\nu}$. Let $\tilde{\boldsymbol{\mu}} \in \mathscr{A}^F_{[\tau,T]}(\mu_\tau)$ driven by $\tilde{\boldsymbol{\nu}}$ be such that

$$J_{[\tau,T]}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\nu}}) - \varepsilon \le V(\tau, \mu_\tau).$$

We set $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_{|[s,\tau]} \odot \tilde{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}} = \boldsymbol{\nu}_{|[s,\tau]} \odot \tilde{\boldsymbol{\nu}}$, and notice that $\hat{\boldsymbol{\mu}} \in \mathscr{A}^F_{[s,T]}(\mu)$ is driven by $\hat{\boldsymbol{\nu}}$.

With this choice we have

$$V(s, \mu) \le J(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}) = \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + J(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\nu}}) \le \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau) + \varepsilon.$$

By letting $\varepsilon \to 0^+$, we obtain

$$V(s, \mu) \le \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau),$$

for all $s \le \tau \le T$. By the arbitrariness of $\boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu)$ driven by $\boldsymbol{\nu}$, we obtain $V(s, \mu) \le W(s, \mu)$.

Fix now $\varepsilon > 0$, and let $\tau \in [s, T]$ and $\boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu)$ driven by $\boldsymbol{\nu}$ be such that

$$W(s, \mu) + \varepsilon \ge \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau).$$

As before, let $\tilde{\boldsymbol{\mu}} \in \mathscr{A}^F_{[\tau,T]}(\mu_\tau)$ driven by $\tilde{\boldsymbol{\nu}}$ be such that

$$J_{[\tau,T]}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\nu}}) - \varepsilon \le V(\tau, \mu_\tau).$$

Define $\hat{\boldsymbol{\mu}} \in \mathscr{A}^F_{[s,T]}(\mu_s)$ and $\hat{\boldsymbol{\nu}}$ by setting $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_{|[s,\tau]} \odot \tilde{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}} = \boldsymbol{\nu}_{|[s,\tau]} \odot \tilde{\boldsymbol{\nu}}$, and notice that $\hat{\boldsymbol{\mu}}$ is driven by $\hat{\boldsymbol{\nu}}$. This leads to

$$\begin{aligned} W(s, \mu) + 2\varepsilon &\ge \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau, \mu_\tau) + \varepsilon \ge \int_s^\tau \mathcal{L}(\mu_t, \nu_t)\, dt + J_{[\tau,T]}(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\nu}}) \\ &= J_{[s,T]}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}) \ge V(s, \mu), \end{aligned}$$

and so $V(s, \mu) = W(s, \mu)$.

We prove now the assertions on the map $h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\cdot)$. Let $s \le \tau_1 \le \tau_2 \le T$. We have

$$h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\tau_1) = \int_s^{\tau_1} \mathcal{L}(\mu_t, \nu_t)\, dt + V(\tau_1, \mu_{\tau_1})$$

$$\leq \int_s^{\tau_1} \mathcal{L}(\mu_t, \nu_t)\,dt + \int_{\tau_1}^{\tau_2} \mathcal{L}(\mu_t, \nu_t)\,dt + V(\tau_2, \mu_{\tau_2}) = h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\tau_2),$$

where we used the fact that $V = W$ and $\boldsymbol{\mu}_{|[\tau_1,T]} \in \mathscr{A}^F_{[\tau_1,T]}(\mu_{\tau_1})$ is driven by $\boldsymbol{\nu}_{|[\tau_1,T]}$. This proves that $\tau \mapsto h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\tau)$ is nondecreasing.

To conclude the proof, we notice that

$$h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(s) = V(s, \mu),$$
$$h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(T) = \int_s^T \mathcal{L}(\mu_t, \nu_t)\,dt + V(T, \mu_T) = \int_s^T \mathcal{L}(\mu_t, \nu_t)\,dt + g(\mu_T) = J(\boldsymbol{\mu}, \boldsymbol{\nu}),$$

Thus $V(s, \mu) = J(\boldsymbol{\mu}, \boldsymbol{\nu})$ if and only if $h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(s) = h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(T)$ which, recalling the monotonicity property of $h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\cdot)$, is equivalent to say that $h_{(\boldsymbol{\mu},\boldsymbol{\nu})}(\cdot)$ is constant. □

We will now focus our attention on the special case where $\mathcal{L}$ depends only on μ.

From Corollary 1, we may deduce the existence of optimal trajectories

Corollary 2 *Assume $\boldsymbol{F}_1$–$\boldsymbol{F}_2$, that $\mathcal{L}$ depends only on μ, and that $\mathcal{L}, \mathscr{G}$ are lower semicontinuous. Then we have that the infimum (12) is actually a minimum, i.e., for all (s, μ) such that $1 + \mathrm{m}_2^{1/2}(\mu) < \theta(s)$ there exists an optimal trajectory starting from μ at time s.*

Proof Since $\mathcal{L}$ depends only on μ, we have that the functional $J_{[s,T]}$ defining $V(\cdot)$ reduces to

$$J_{[s,T]}(\boldsymbol{\mu}) = \int_s^T \mathcal{L}(\mu_t)\,dt + \mathscr{G}(\mu_T),$$

and it is l.s.c. By Corollary 1, we have that $\mathscr{A}^F_{[s,T]}(\mu)$ is compact, therefore $J_{[s,T]}(\cdot)$ admits a minimizer in $\mathscr{A}^F_{[s,T]}(\mu)$, i.e., there exists an optimal trajectory. □

We also obtain the following regularity property of the value function

Proposition 4 *Assume $\boldsymbol{F}_1$ - $\boldsymbol{F}_3$, that $\mathcal{L}$ depends only on μ and that $\mathcal{L}, \mathscr{G}$ are bounded and uniformly continuous with modulus ω. Then the value function is bounded and uniformly continuous.*

Proof Let $\mu_0, \theta_0 \in \mathscr{P}_2(\mathbb{R}^d)$. Given an optimal trajectory $\boldsymbol{\mu} \in \mathscr{A}^F_{[s,T]}(\mu_0)$, by Proposition 2 there exists admissible trajectory $\boldsymbol{\theta} \in \mathscr{A}^F_{[s,T]}(\theta_0)$ such that for all $t \in [s, T]$

$$W_2(\mu_t, \theta_t) \leq e^{LT + Te^{LT}} \cdot W_2(\mu_0, \theta_0).$$

Thus we have

$$V(s, \theta_0) - V(s, \mu_0) \leq \int_s^T [\mathcal{L}(\theta_t) - \mathcal{L}(\mu_t)]\,dt + \mathscr{G}(\theta_T) - \mathscr{G}(\mu_T)$$

By the uniform continuity of $\mathcal{L}$ and $\mathscr{G}$, we have

$$V(s, \theta_0) - V(s, \mu_0) \leq (T - s)\omega(e^{LT+Te^{LT}} \cdot W_2(\mu_0, \theta_0)) + \omega(e^{LT+Te^{LT}} \cdot W_2(\mu_0, \theta_0)).$$

Switching the roles of θ_0 and μ_0, we obtain a similar estimate, yielding

$$|V(s, \theta_0) - V(s, \mu_0)|$$
$$\leq (T - s)\omega(e^{LT+Te^{LT}} \cdot W_2(\mu_0, \theta_0)) + \omega(e^{LT+Te^{LT}} \cdot W_2(\mu_0, \theta_0)),$$

i.e., the continuity w.r.t. the μ-variable. We prove the continuity with respect to t. Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Assume that $0 \le s_1 \le s_2 \le T$. By taking an optimal trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[s_1,T]} \in \mathscr{A}^F_{[s_1,T]}(\mu)$, the dynamic programming principle yields

$$\begin{aligned} V(s_1,\mu) - V(s_2,\mu) &= \left[\int_{s_1}^{s_2} \mathcal{L}(\mu_t)\,dt + V(s_2,\mu_{s_2})\right] - V(s_2,\mu) \\ &\le \int_{s_1}^{s_2} \mathcal{L}(\mu_t)\,dt + (T-s_2)\omega(e^{LT+Te^{LT}} \cdot W_2(\mu,\mu_{s_2})) \\ &\quad + \omega(e^{LT+Te^{LT}} \cdot W_2(\mu,\mu_{s_2})), \end{aligned}$$

and the right hand side tends to 0 as $|s_1 - s_2| \to 0$ by the continuity of $\mathcal{L}$ and μ_t. If instead $0 \le s_2 \le s_1 \le T$, by taking an optimal trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[s_2,T]} \in \mathscr{A}^F_{[s_2,T]}(\mu)$, we have

$$\begin{aligned} V(s_1,\mu) - V(s_2,\mu) =& V(s_1,\mu) - \left[\int_{s_2}^{s_1} \mathcal{L}(\mu_t)\,dt + V(s_1,\mu_{s_1})\right] \\ \le& -\int_{s_2}^{s_1} \mathcal{L}(\mu_t)\,dt + (T-s_1)\omega(e^{LT+Te^{LT}} \cdot W_2(\mu,\mu_{s_1})) \\ &+ \omega(e^{LT+Te^{LT}} \cdot W_2(\mu,\mu_{s_1})), \end{aligned}$$

and the right hand side tends to 0 as $|s_1 - s_2| \to 0$ by the continuity of $\mathcal{L}$ and μ_t. In particular, we obtain that for any $s_1, s_2 \in [0,T]$ the difference $V(s_1,\mu) - V(s_2,\mu)$ is bounded from above by a quantity which tends to zero as $|s_1 - s_2| \to 0$. By switching the roles of s_1 and s_2, we obtain the desired continuity. □

5 Characterization of the value function

We first introduce a notion of super/sub differential and associated viscosity solutions for HJB equation. The relevance of this notion is demonstrated later by obtaining a comparison result and the characterization of the value as the unique solution of the HJB equation.

5.1 Viscosity solutions of Hamilton Jacobi Bellman equation

We will first proceed the construction of the sub/superdifferential. Before this we introduce the following definition (Section 8.5 in [3])

Definition 2 *(Optimal displacements)* Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. A function $p \in L^2_\mu(\mathbb{R}^d)$ is called an *optimal displacement from* μ if $p = \mathrm{Id}_{\mathbb{R}^d} - T$ where T is an optimal transport map between μ and $T\sharp\mu$. In particular, we have

$$W_2^2(\mu, (\mathrm{Id}_{\mathbb{R}^d} - p)\sharp\mu) = W_2^2(\mu, T\sharp\mu) = \int_{\mathbb{R}^d} |p(x)|^2\,d\mu(x).$$

We will use extensively the following characterization of optimal displacements.

Lemma 4 (Characterization of optimal displacements) *Let* $\mu_1 \in \mathscr{P}_2(\mathbb{R}^d)$*,* $p \in L^2_{\mu_1}(\mathbb{R}^d)$*. The following are equivalent:*

i. *p is an optimal displacement from μ_1;*

ii. *there exists* $\mu_2 \in \mathscr{P}_2(\mathbb{R}^d)$ *and* $\boldsymbol{\pi} \in \Pi_o(\mu_1, \mu_2)$ *such that*

$$p(x) = x - \int_{\mathbb{R}^d} y \, d\pi_x(y),$$

where $\{\pi_x\}_{x\in\mathbb{R}^d}$ *is the family obtained by the disintegration* $\boldsymbol{\pi} = \mu_1 \otimes \pi_x$*;*

iii. *there exists* $\mu_2 \in \mathscr{P}_2(\mathbb{R}^d)$ *and* $\boldsymbol{\pi} \in \Pi_o(\mu_1, \mu_2)$ *such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \phi(x), x - y\rangle \, d\boldsymbol{\pi}(x, y) = \int_{\mathbb{R}^d} \langle \phi(x), p(x)\rangle \, d\mu_1(x),$$

for all $\phi \in L^2_{\mu_1}(\mathbb{R}^d)$.

Proof $(i.) \implies (iii.)$. We easily get $(iii.)$ by setting $\boldsymbol{\pi} = (\mathrm{Id}_{\mathbb{R}^d} \times (\mathrm{Id}_{\mathbb{R}^d} - p))\sharp\mu_1$ and $\mu_2 = (\mathrm{Id}_{\mathbb{R}^d} - p)\sharp\mu_1$.

$(iii.) \implies (ii.)$. We have

$$\int_{\mathbb{R}^d} \langle \phi(x), p(x)\rangle \, d\mu_1(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \phi(x), x - y\rangle \, d\boldsymbol{\pi}(x, y)$$

$$= \int_{\mathbb{R}^d} \langle \phi(x), x - y\rangle \, d\pi_x(y) d\mu_1(x) = \int_{\mathbb{R}^d} \langle \phi(x), x - \int_{\mathbb{R}^d} y \, d\pi_x(y)\rangle \, d\mu_1(x)$$

for all $\phi \in L^2_{\mu_1}(\mathbb{R}^d)$. Thus $p(x) = x - \int_{\mathbb{R}^d} y \, d\pi_x(y)$ for μ_1-a.e. $x \in \mathbb{R}^d$.

$(ii.) \implies (i.)$. Let $\mu_2 \in \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\pi} \in \Pi_o(\mu_1, \mu_2)$ such that

$$p(x) = x - \int_{\mathbb{R}^d} y \, d\pi_x(y), \text{ for } \mu_1\text{-a.e. } x \in \mathbb{R}^d,$$

with $\boldsymbol{\pi} = \mu_1 \otimes \pi_x \in \Pi_o(\mu_1, \mu_2)$, and set

$$T(x) := \int_{\mathbb{R}^d} y \, d\pi_x(y) \text{ for } \mu_1\text{-a.e. } x \in \mathbb{R}^d,$$

we have to prove that T is an optimal transport map between μ_1 and $T\sharp\mu_1$. By Section 6.3.2 in [3], we have that $\operatorname{supp} \boldsymbol{\pi}$ is cyclically monotone, i.e.,

$$\sum_{i=1}^{N} \langle x_i, y_i\rangle \leq \sum_{i=1}^{N} \langle x_i, y_{i+1}\rangle, \tag{14}$$

for all finite family $\{(x_i, y_i)\}_{i=1,\dots,N} \subseteq \operatorname{supp} \boldsymbol{\pi}$ with $x_{N+1} = x_1$ and $y_{N+1} = y_1$. Let $\mathcal{N}$ be a μ-negligible set such that π_x is defined for all $x \notin \mathcal{N}$. Let $\{x_i\}_{i=1,\dots,N}$ a family such that $x_i \notin \mathcal{N}$ for all $i = 1, \dots, N$ then for $\pi_{x_1} - a.e. y_1, \dots \pi_{x_n} - a.e. y_n$ the inequality (14) holds. Thus, by integration, taking all $x_i \notin \mathcal{N}$:

$$\begin{aligned}\sum_{i=1}^{N} \langle x_i, T(x_i)\rangle &= \sum_{i=1}^{N} \langle x_i, \int_{\mathbb{R}^d} y_i \, d\pi_{x_i}(y_i)\rangle \\ &= \int_{\mathbb{R}^{Nd}} \left(\sum_{i=1}^{N} \langle x_i, y_i\rangle\right) d\pi_{x_1}(y_1) \dots d\pi_{x_N}(y_N) \\ &\leq \int_{\mathbb{R}^{Nd}} \left(\sum_{i=1}^{N} \langle x_i, y_{i+1}\rangle\right) d\pi_{x_1}(y_1) \dots d\pi_{x_N}(y_N)\end{aligned}$$

$$= \sum_{i=1}^{N} \langle x_i, \int_{\mathbb{R}^d} y_{i+1}\, d\pi_{x_{i+1}}(y_{i+1})\rangle = \sum_{i=1}^{N} \langle x_i, T(x_{i+1})\rangle,$$

hence graph T is cyclically monotone outside $\mathcal{N}$, so T is an optimal transport map. □

Lemma 5 (Optimal displacements and W_2) *Let $\mu, \bar{\mu}_1, \bar{\mu}_2 \in \mathscr{P}_2(\mathbb{R}^d)$ and $\bar{\boldsymbol{\pi}} \in \Pi_o(\bar{\mu}_1, \bar{\mu}_2)$. Then, considered the disintegration $\bar{\boldsymbol{\pi}} = \bar{\mu}_1 \otimes \bar{\pi}^x$ of $\bar{\boldsymbol{\pi}}$ w.r.t. the first marginal, and defined*

$$p(x) = x - \int_{\mathbb{R}^d} y\, d\bar{\pi}^x(y)$$

we have that p is an optimal displacement from $\bar{\mu}_1$ and for all $\boldsymbol{\pi} \in \Pi(\bar{\mu}_1, \mu)$ it holds

$$\begin{aligned} &\frac{1}{2}W_2^2(\mu, \bar{\mu}_2) - \frac{1}{2}W_2^2(\bar{\mu}_1, \bar{\mu}_2) \\ &\le \int \langle p(x), y - x\rangle\, d\boldsymbol{\pi}(x, y) + o\left(\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2\, d\boldsymbol{\pi}(x, y)\right)^{1/2}\right). \end{aligned}$$

Proof $p(\cdot)$ is an optimal displacement by Lemma 4 (ii). By disintegration of $\boldsymbol{\pi}$ write:

$$\boldsymbol{\pi} = \bar{\mu}_1 \otimes \pi^x.$$

Then build a transport plan $\tilde{\boldsymbol{\pi}} \in \Pi(\mu, \bar{\mu}_2)$ by setting for all $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^d)$:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(y, z)\, d\tilde{\boldsymbol{\pi}}(y, z) := \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \varphi(y, z)\, d\pi^x(y) d\bar{\pi}^x(z) d\bar{\mu}(x).$$

Then, it holds:

$$\begin{aligned} &\frac{1}{2}W_2^2(\mu, \bar{\mu}_2) - \frac{1}{2}W_2^2(\bar{\mu}_1, \bar{\mu}_2) \\ &\le \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - z|^2 d\tilde{\boldsymbol{\pi}}(y, z) - \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - z|^2 d\bar{\boldsymbol{\pi}}(y, z) \\ &= \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |y - x + x - z|^2\, d\pi^x(y) d\bar{\pi}^x(z) d\bar{\mu}_1(x) - \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - z|^2 d\bar{\boldsymbol{\pi}}(y, z) \\ &= \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 d\boldsymbol{\pi}(x, y) + \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \langle x - z, y - x\rangle\, d\bar{\pi}^x(z) d\boldsymbol{\pi}(x, y). \end{aligned}$$

The conclusion follows by moving the integral in z inside the scalar product. □

We now define the following notions of generalized gradients

Definition 3 (Super/sub differentials in $\mathbb{R} \times \mathscr{P}_2$) Let $w : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a map. Given $(\bar{t}, \bar{\mu}) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$ and $\delta \ge 0$, we say that $(p_{\bar{t}}, p_{\bar{\mu}}) \in \mathbb{R} \times L^2_{\bar{\mu}}(\mathbb{R}^d)$ is a δ-*viscosity superdifferential* of w at $(\bar{t}, \bar{\mu})$ if

1. $p_{\bar{\mu}}$ is an optimal displacement from $\bar{\mu}$;
2. for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $t \in [0, T]$ and $\boldsymbol{\pi} \in \Pi(\bar{\mu}, \mu)$,

$$\begin{aligned} w(t, \mu) - w(\bar{t}, \bar{\mu}) \le p_{\bar{t}}(t - \bar{t}) + \int_{\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y - x\rangle\, d\boldsymbol{\pi}(x, y) \\ +\delta\sqrt{|t - \bar{t}|^2 + \int_{\mathbb{R}^d} |x - y|^2\, d\boldsymbol{\pi}(x, y)} \end{aligned}$$

$$+o\left(|t-\bar{t}|+\left(\int_{\mathbb{R}^d}|x-y|^2\,d\boldsymbol{\pi}(x,y)\right)^{1/2}\right).$$

We denote by $D^+_\delta w(\bar{t},\bar{\mu})$ the set of such δ-superdifferential $(p_{\bar{t}},p_{\bar{\mu}})$. Similarly the set of *$\delta$-viscosity subdifferentials* $D^-_\delta w(\bar{t},\bar{\mu})$ is given by $D^-_\delta w(\bar{t},\bar{\mu})=-D^+_\delta(-w)(\bar{t},\bar{\mu})$.

We consider an equation in the form

$$\partial_t w(t,\mu)+\mathscr{H}(\mu,Dw(t,\mu))=0, \tag{15}$$

where $\mathscr{H}(\mu,p)$ is defined for any $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ and $p\in L^2_\mu(\mathbb{R}^d)$.

Definition 4 *(Viscosity Solutions)* A function $w:[0,T]\times\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is

- a *subsolution* of (15) if w is upper semicontinuous and there exists a map $C:\mathscr{P}_2(\mathbb{R}^d)\to]0,+\infty[$, $C(\cdot)$ bounded on bounded sets, such that
$$p_t+\mathscr{H}(\mu,p_\mu)\geq -C(\mu)\delta,$$
for all $(t,\mu)\in]0,T[\times\mathscr{P}_2(\mathbb{R}^d)$, $(p_t,p_\mu)\in D^+_\delta w(t,\mu)$, and $\delta>0$.
- a *supersolution* of (15) if w is lower semicontinuous and there exists a map $C:\mathscr{P}_2(\mathbb{R}^d)\to]0,+\infty[$, $C(\cdot)$ bounded on bounded sets, such that
$$p_t+\mathscr{H}(\mu,p_\mu)\leq C(\mu)\delta,$$
for all $(t,\mu)\in]0,T[\times\mathscr{P}_2(\mathbb{R}^d)$, $(p_t,p_\mu)\in D^-_\delta w(t,\mu)$, and $\delta>0$.
- a *solution* of (15) if w is both a supersolution and a subsolution.

Given $\mu_1,\mu_2\in\mathscr{P}_2(\mathbb{R}^d)$, and denoted by $\bar{\boldsymbol{\pi}}\in\Pi_o(\mu_1,\mu_2)$ the unique solution of the minimization problem

$$\min\left\{\int_{\mathbb{R}^d}\left|x-\int_{\mathbb{R}^d}y\,d\pi_x(y)\right|^2\,d\mu_1(x):\boldsymbol{\pi}=\mu_1\otimes\pi_x\in\Pi_o(\mu_1,\mu_2)\right\},$$

we disintegrate $\bar{\boldsymbol{\pi}}=\mu_1\otimes\bar{\pi}_x$ (see Theorem 4) and define $p_{\mu_1,\mu_2}\in L^2_{\mu_1}$, $q_{\mu_1,\mu_2}\in L^2_{\mu_2}$ by

$$\begin{cases} p_{\mu_1,\mu_2}(x)= & x-\displaystyle\int_{\mathbb{R}^d}y\,d\bar{\pi}_x(y),\\ q_{\mu_1,\mu_2}(y)= & y-\displaystyle\int_{\mathbb{R}^d}x\,d\bar{\pi}^{-1}_y(x),\end{cases} \tag{16}$$

where $\bar{\boldsymbol{\pi}}^{-1}\in\Pi_o(\mu_2,\mu_1)$ is the inverse of the transport plan $\bar{\boldsymbol{\pi}}$, disintegrated as $\bar{\boldsymbol{\pi}}^{-1}=\mu_2\otimes\bar{\pi}^{-1}_y$. Notice that, according to Lemma 4, p_{μ_1,μ_2} is the optimal displacement from μ_1 to μ_2 of minimal norm.

From now on, the space $X:=[0,T]\times\mathscr{P}_2(\mathbb{R}^d)$ is endowed with the metric

$$d_X((s_1,\mu_1),(s_2,\mu_2))=\sqrt{(s_1-s_2)^2+W_2^2(\mu_1,\mu_2)},$$

and we notice that (X,d_X) is a complete metric space. We endow $X\times X$ with the metric

$$d_{X\times X}(z_1,z_2)=d_X((s_1,\mu_1),(s_2,\mu_2))+d_X\left((t_1,\hat{\mu}_1),(t_2,\hat{\mu}_2)\right),$$

for all $z_i=(s_i,\mu_i,t_i,\hat{\mu}_i)\in X\times X$, $i=1,2$. Again, we have that $(X\times X,d_{X\times X})$ is a complete metric space.

We now state and prove the following comparison result

Theorem 2 (Comparison) *Let $w_1, w_2 \in BUC(X;\mathbb{R})$ be a viscosity subsolution and supersolution of the equation (15). Assume that there exists a continuous nondecreasing map $\omega_{\mathscr{H}} : \mathbb{R}^2 \to [0,+\infty[$ such that $\omega_{\mathscr{H}}(0,0) = 0$ and*

$$|\mathscr{H}(\mu^{(1)}, \lambda p_{\mu^{(1)},\mu^{(2)}}) - \mathscr{H}(\mu^{(2)}, \lambda q_{\mu^{(1)},\mu^{(2)}})| \\ \leq \omega_{\mathscr{H}}\left(W_2(\mu^{(1)},\mu^{(2)}), \lambda W_2^2(\mu^{(1)},\mu^{(2)})\right),$$

for all $\lambda > 0$, $\mu^{(1)}, \mu^{(2)} \in \mathscr{P}_2(\mathbb{R}^d)$. Then

$$\inf_{(t,\mu)\in X} \{w_2(t,\mu) - w_1(t,\mu)\} = \inf_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} \{w_2(T,\mu) - w_1(T,\mu)\}.$$

In particular, the equation (15) coupled with a terminal condition $w(T,\mu) = g(\mu)$, admits at most one continuous and bounded solution.

We will need the following Lemma, of independent interest.

Lemma 6 *Let $w_1, w_2 \in BUC(X;\mathbb{R})$. Given $\varepsilon, \eta, \sigma > 0$, we define the functional $\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ by setting*

$$\Phi(s,\mu_1,t,\mu_2) = \begin{cases} w_2(t,\mu_2) - w_1(s,\mu_1) + \dfrac{1}{2\varepsilon} d_X^2\left((s,\mu_1),(t,\mu_2)\right) \\ -\eta s + \dfrac{\sigma}{s} + \dfrac{\sigma}{t}, & \text{if } st \neq 0, \\ \\ +\infty, & \text{if } st = 0. \end{cases}$$

Let $\delta > 0$ and assume that $\bar{z} = (\bar{s},\bar{\mu}_1,\bar{t},\bar{\mu}_2) \in X \times X$ with $\bar{s},\bar{t} \in]0,T[$ satisfies

$$\Phi(\bar{z}) \leq \Phi(z) + \delta d_{X\times X}(z,\bar{z}) \text{ for all } z = ((s,\mu_1),(t,\mu_2)) \in X \times X. \tag{17}$$

Then

$$\begin{cases} \left(\dfrac{\bar{s}-\bar{t}}{\varepsilon} - \eta - \dfrac{\sigma}{\bar{s}^2}, \dfrac{\bar{p}}{\varepsilon}\right) \in D_\delta^+ w_1(\bar{s},\bar{\mu}_1), \\ \left(\dfrac{\bar{s}-\bar{t}}{\varepsilon}, \dfrac{\bar{q}}{\varepsilon}\right) \in D_\delta^- w_2(\bar{t},\bar{\mu}_2), \end{cases} \tag{18}$$

where $\bar{p} = p_{\bar{\mu}_1,\bar{\mu}_2}$ and $\bar{q} = q_{\bar{\mu}_1,\bar{\mu}_2}$ are as in (16).

Proof The proof follows from the same argument of Claim 1 of Theorem 3.4 in [25], we repeat it for sake of completeness. By taking $(t,\mu_2) = (\bar{t},\bar{\mu}_2)$ in (17), we have

$$\Phi(\bar{z}) \leq \Phi(s,\mu_1,\bar{t},\bar{\mu}_2) + \delta d_X((s,\mu_1),(\bar{s},\bar{\mu}_1)), \text{ for all } (s,\mu_1) \in X,$$

which, recalling the definition of Φ, yields

$$\begin{aligned} w_1(s,\mu_1) - w_1(\bar{s},\bar{\mu}_1) \leq \\ \leq \frac{1}{2\varepsilon}\left[W_2^2(\mu_1,\bar{\mu}_2) - W_2^2(\bar{\mu}_1,\bar{\mu}_2) + (s-\bar{t})^2 - (\bar{s}-\bar{t})^2\right] + \\ + \delta\sqrt{W_2^2(\mu_1,\bar{\mu}_1) + |s-\bar{s}|^2} - \eta(s-\bar{s}) + \sigma\left(\frac{1}{s} - \frac{1}{\bar{s}}\right). \end{aligned} \tag{19}$$

Observing that

$$\frac{1}{s} - \frac{1}{\bar{s}} = -\frac{1}{\bar{s}^2}(s-\bar{s}) + o(|s-\bar{s}|)$$

and using Lemma 5 we obtain formula (18). The proof of the second relation of (18) follows a symmetric argument. □

Proof of Theorem 2 Set

$$A := \inf_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} \{w_2(T,\mu) - w_1(T,\mu)\},$$

and notice that since $\mathscr{H}$ does not involve w, we have that $w_1 - A$ is still a subsolution. Thus without loss of generality we can assume $A = 0$. Assume by contradiction that

$$-\xi := \inf_{(s,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)} \{w_2(s,\mu) - w_1(s,\mu)\} < 0,$$

and choose $(t_0, \mu_0) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$ such that

$$w_2(t_0,\mu_0) - w_1(t_0,\mu_0) < -\frac{\xi}{2}.$$

We notice that, by continuity of w_1 and w_2, we can always assume that $t_0 \neq 0$, moreover we fix $\sigma > 0$ such that $\dfrac{2\sigma}{t_0} \le \dfrac{\xi}{8}$. Let $R > 0$.

Given $\varepsilon, \eta > 0$, we define the functional $\Phi_{\varepsilon\eta} : X \times X \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\Phi_{\varepsilon\eta}(s,\mu^{(1)},t,\mu^{(2)}) = w_2(t,\mu^{(2)}) - w_1(s,\mu^{(1)}) + \frac{1}{2\varepsilon} d_X^2\left((s,\mu^{(1)}),(t,\mu^{(2)})\right) - \eta s + \frac{\sigma}{s} + \frac{\sigma}{t},$$

if $st \neq 0$ and $W_2(\mu_0,\mu^{(i)}) \le R$, $i = 1, 2$, while $\Phi_{\varepsilon\eta}(s,\mu^{(1)},t,\mu^{(2)}) = +\infty$ otherwise. Define $z_0 = (t_0,\mu_0,t_0,\mu_0) \in X \times X$. Since $\Phi_{\varepsilon\eta}$ is lower semicontinuous and bounded from below and $(X \times X, d_{X\times X})$ is complete, by Ekeland Variational Principle, for any $\delta > 0$ there exists $z_{\varepsilon\eta\delta} = (s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}, t_{\varepsilon\eta\delta}, \mu^{(2)}_{\varepsilon\eta\delta}) \in X \times X$ such that for any $z = (s,\mu^{(1)},t,\mu^{(2)}) \in X \times X$ we have

$$\begin{cases} \Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) \le \Phi_{\varepsilon\eta}(z_0), \\ \\ \Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) \le \Phi_{\varepsilon\eta}(z) + \delta d_{X\times X}(z, z_{\varepsilon\eta\delta}), \end{cases} \tag{20}$$

moreover we set $\rho_{\varepsilon\eta\delta} = d_X\left((s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta}),(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta})\right)$, and notice that $s_{\varepsilon\eta\delta} \neq 0$, and $W_2(\mu_0,\mu^{(i)}_{\varepsilon\eta\delta}) \le R$, $i = 1, 2$.

Claim 1 For all $0 < \eta < 1$ we have $\lim_{\varepsilon,\delta\to0^+} \dfrac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta} = 0$.

Proof (of Claim 1) In the second inequality of (20), we choose

$$z_1 = (s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}, s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}), \quad z_2 = (t_{\varepsilon\eta\delta}, \mu^{(2)}_{\varepsilon\eta\delta}, t_{\varepsilon\eta\delta}, \mu^{(2)}_{\varepsilon\eta\delta}),$$

thus obtaining

$$\begin{cases} \Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) - \Phi_{\varepsilon\eta}(z_1) \le \delta d_{X\times X}(z_1, z_{\varepsilon\eta\delta}), \\ \Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) - \Phi_{\varepsilon\eta}(z_2) \le \delta d_{X\times X}(z_2, z_{\varepsilon\eta\delta}), \end{cases}$$

and so

$$2\Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) - \Phi_{\varepsilon\eta}(z_1) - \Phi_{\varepsilon\eta}(z_2) \le \delta d_{X\times X}(z_1, z_{\varepsilon\eta\delta}) + \delta d_{X\times X}(z_2, z_{\varepsilon\eta\delta})$$

Recalling the definition of $\Phi_{\varepsilon\eta}$, this implies

$$\left[w_2(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta})-w_2(s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta})\right]+\left[w_1(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta})-w_1(s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta})\right]+ \\ +\frac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta}\leq 2\delta\rho_{\varepsilon\eta\delta}+\eta(s_{\varepsilon\eta\delta}-t_{\varepsilon\eta\delta})\leq\rho_{\varepsilon\eta\delta}(2\delta+\eta). \tag{21}$$

We prove first that $\lim_{\varepsilon,\delta\to0^+}\rho_{\varepsilon\eta\delta}=0$ for all $\eta>0$. To this aim, we fix $\eta>0$ and distinguish two cases:

- assume that there exist $\alpha>0$ and sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\delta_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n,\delta_n\to0^+$ such that $\lim_{n\to+\infty}\rho_{\varepsilon_n\eta\delta_n}=2\alpha$. Then there exists $\bar n>0$ such that for all $n\geq\bar n$ sufficiently large, we have $\alpha<\rho_{\varepsilon_n\eta\delta_n}<3\alpha$, and so

$$-2\left(\|w_1\|_\infty+\|w_2\|_\infty\right)+\frac{\alpha^2}{\varepsilon_n}\leq 3\alpha(2\delta_n+\eta),$$

  leading to a contradiction since the left hand side tends to $+\infty$, while the right hand side is bounded.
- assume that there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\delta_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n,\delta_n\to0^+$ such that $\lim_{n\to+\infty}\rho_{\varepsilon_n\eta\delta_n}=+\infty$. Then there exists $\bar n>0$ such that for all $n\geq\bar n$ such that $\varepsilon_n,\delta_n\leq1/2$

$$-2\left(\|w_1\|_\infty+\|w_2\|_\infty\right)+2\rho^2_{\varepsilon_n\eta\delta_n}\leq 2\rho_{\varepsilon_n\eta\delta_n},$$

  leading to a contradiction.

Thus for all $\eta>0$ we have $\limsup_{\varepsilon,\delta\to0^+}\rho_{\varepsilon\eta\delta}\leq0$, and so $\lim_{\varepsilon,\delta\to0^+}\rho_{\varepsilon\eta\delta}=0$ for all $\eta>0$.

We conclude now the proof of the Claim noticing that (21) implies

$$\frac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta}\leq(2\delta+\eta)\rho_{\varepsilon\eta\delta}\quad+|w_2(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta})-w_2(s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta})| \\ +|w_1(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta})-w_1(s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta})|.$$

Since $\rho_{\varepsilon\eta\delta}\to0$ as $\varepsilon,\delta\to0^+$, by the continuity of w_1,w_2 we conclude that the right hand side tends to 0, thus proving Claim 1. □

Claim 2 *Claim 2:* For $\varepsilon,\delta,\eta>0$ sufficiently small, we have $s_{\varepsilon\eta\delta},t_{\varepsilon\eta\delta}\notin]0,T[$.

Proof (of Claim 2) We argue by contradiction, assuming that $s_{\varepsilon\eta\delta},t_{\varepsilon\eta\delta}\in]0,T[$. By Lemma 6 we have

$$\begin{cases}\left(\dfrac{s_{\varepsilon\eta\delta}-t_{\varepsilon\eta\delta}}{\varepsilon}-\eta-\dfrac{\sigma}{s^2_{\varepsilon\eta\delta}},\ \dfrac{p_{\varepsilon\eta\delta}}{\varepsilon}\right)\in D^+_\delta w_1\left(s_{\varepsilon\eta\delta},\mu^{(1)}_{\varepsilon\eta\delta}\right),\\ \left(\dfrac{s_{\varepsilon\eta\delta}-t_{\varepsilon\eta\delta}}{\varepsilon},\ \dfrac{q_{\varepsilon\eta\delta}}{\varepsilon}\right)\in D^-_\delta w_2\left(t_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta}\right),\end{cases} \tag{22}$$

where $p_{\varepsilon\eta\delta}=p_{\mu^{(1)}_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta}}$ and $q_{\varepsilon\eta\delta}=q_{\mu^{(1)}_{\varepsilon\eta\delta},\mu^{(2)}_{\varepsilon\eta\delta}}$.

Since w_1 and w_2 are a sub- and super-solution, respectively, and noticing that $W_2(\mu_0,\mu^{(i)}_{\varepsilon\eta\delta})\leq R$ for all $\varepsilon,\eta,\delta>0$, $i=1,2$, we have

$$\begin{cases}-C\delta\leq & \dfrac{s_{\varepsilon\eta\delta}-t_{\varepsilon\eta\delta}}{\varepsilon}-\eta-\dfrac{\sigma}{s^2_{\varepsilon\eta\delta}}+\mathscr{H}\left(\mu^{(1)}_{\varepsilon\eta\delta},\dfrac{p_{\varepsilon\eta\delta}}{\varepsilon}\right),\\ C\delta\geq & \dfrac{s_{\varepsilon\eta\delta}-t_{\varepsilon\eta\delta}}{\varepsilon}+\mathscr{H}\left(\mu^{(2)}_{\varepsilon\eta\delta},\dfrac{q_{\varepsilon\eta\delta}}{\varepsilon}\right),\end{cases}$$

where $C = \sup\{C(\mu) : W_2(\mu, \mu_0) \le R\}$, and $C(\mu)$ is as in the definition of viscosity superdifferential (Definition 4). By combining the above relations, we have

$$\mathscr{H}\left(\mu^{(2)}_{\varepsilon\eta\delta}, \frac{q_{\varepsilon\eta\delta}}{\varepsilon}\right) - \mathscr{H}\left(\mu^{(1)}_{\varepsilon\eta\delta}, \frac{p_{\varepsilon\eta\delta}}{\varepsilon}\right) \le 2C\delta - \eta - \frac{\sigma}{s^2_{\varepsilon\eta\delta}} \le 2C\delta - \eta.$$

By assumption, we have

$$\mathscr{H}\left(\mu^{(2)}_{\varepsilon\eta\delta}, \frac{q_{\varepsilon\eta\delta}}{\varepsilon}\right) - \mathscr{H}\left(\mu^{(1)}_{\varepsilon\eta\delta}, \frac{p_{\varepsilon\eta\delta}}{\varepsilon}\right) \ge -\omega_{\mathscr{H}}\left(\rho_{\varepsilon\eta\delta}, \frac{\rho^2_{\varepsilon\eta\delta}}{\varepsilon}\right),$$

and so

$$-\omega_{\mathscr{H}}\left(\rho_{\varepsilon\eta\delta}, \frac{\rho^2_{\varepsilon\eta\delta}}{\varepsilon}\right) \le 2C\delta - \eta,$$

leading to a contradiction, since - recalling Claim 1 - the limit for $\varepsilon, \delta \to 0^+$ of the left hand side is 0 for all $\eta > 0$, while the limit of the right hand side is strictly negative. □

Claim 3 For $\varepsilon, \delta, \eta > 0$ sufficiently small, we have $s_{\varepsilon\eta\delta} \neq T$ and $t_{\varepsilon\eta\delta} \neq T$.

Proof (of Claim 3) We notice that, by definition of ξ and recalling (20),

$$\begin{aligned}
-\frac{\xi}{2} + \frac{2\sigma}{t_0} \ge & w_2(t_0, \mu_0) - w_1(t_0, \mu_0) - \eta t_0 + \frac{2\sigma}{t_0} = \Phi_{\varepsilon\eta}(z_0) \\
\ge & \Phi_{\varepsilon\eta}(z_{\varepsilon\eta\delta}) = w_2(t_{\varepsilon\eta\delta}, \mu^{(2)}_{\varepsilon\eta\delta}) - w_1(s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}) \\
& + \frac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta} - \eta s_{\varepsilon\eta\delta} + \frac{\sigma}{s_{\varepsilon\eta\delta}} + \frac{\sigma}{t_{\varepsilon\eta\delta}} \\
\ge & -\omega_2(\rho_{\varepsilon\eta\delta}) + w_2(s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}) - w_1(s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}) + \frac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta} - \eta T,
\end{aligned}$$

where $\omega_2(\cdot)$ is the continuity modulus of $w_2(\cdot)$. Given $0 < \eta < \xi/(8T)$, we can choose $\varepsilon, \delta > 0$ such that

$$\omega_2(\rho_{\varepsilon\eta\delta}) - \frac{1}{\varepsilon}\rho^2_{\varepsilon\eta\delta} + \eta T + \frac{2\sigma}{t_0} \le \frac{\xi}{4}, \tag{23}$$

and so

$$-\frac{\xi}{2} \ge -\frac{\xi}{4} + w_2(s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}) - w_1(s_{\varepsilon\eta\delta}, \mu^{(1)}_{\varepsilon\eta\delta}). \tag{24}$$

We prove the assertion by contradiction, assuming first $s_{\varepsilon\eta\delta} = T$. In this case, since we have assumed $A = 0$, we have $w_2(T, \mu^{(1)}_{\varepsilon\eta\delta}) - w_1(T, \mu^{(1)}_{\varepsilon\eta\delta}) \ge 0$, leading to a contradiction with (24), thus $s_{\varepsilon\eta\delta} \neq T$. The proof of the case $t_{\varepsilon\eta\delta} = T$ can be done in the same way.

By Claim 2 and Claim 3 and the choice of σ, we have $s_{\varepsilon\eta\delta}, t_{\varepsilon\eta\delta} \notin [0, T]$, against the definition of ξ. Thus we have $\xi = 0$ and the proof is complete. □

5.2 Main result

Now we characterize the value function as unique viscosity solution of a suitable HJB equation.

We consider a set-valued map F satisfying $\boldsymbol{(F_1) - (F_3)}$, and assume that $\mathcal{L}$ and $\mathscr{G}$ are bounded and uniformly continuous, and define the following Hamiltonian function for all

$\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p_\mu \in L^2_\mu(\mathbb{R}^d)$,

$$\mathscr{H}(\mu, p_\mu) := \mathcal{L}(\mu) + \inf_{\substack{v(\cdot)\in L^2_\mu(\mathbb{R}^d) \\ v(x)\in F(\mu,x)\ \mu\text{-a.e.}x}} \int_{\mathbb{R}^d} \langle p_\mu(x), v(x)\rangle \, d\mu(x). \tag{25}$$

We recall that, as observed in Remark 4.2 in [25], from Theorem 8.2.11 in [5] we have indeed

$$\mathscr{H}(\mu, p_\mu) = \mathcal{L}(\mu) + \mathscr{H}_F(\mu, p_\mu),$$

where

$$\mathscr{H}_F(\mu, p_\mu) = \int_{\mathbb{R}^d} \inf_{v\in F(\mu,x)} \langle p_\mu(x), v\rangle \, d\mu(x).$$

Theorem 3 *Consider a set-valued map F satisfying* $(\boldsymbol{F_1}) - (\boldsymbol{F_3})$*, and assume that $\mathcal{L}$ and $\mathscr{G}$ are bounded and uniformly continuous. Then the value function $V : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ of the Bolza problem is the unique bounded and uniformly continuous viscosity solution of*

$$\begin{cases} \partial_t u(t, \mu) + \mathscr{H}(\mu, Du(t, \mu)) = 0, \\ \\ u(T, \mu) = \mathscr{G}(\mu), \end{cases} \tag{26}$$

where the Hamitonian $\mathscr{H}$ is defined by (25).

Proof We will proceed in several steps.

Step 1. The Hamiltonian $\mathscr{H}$ of Definition (25) *satisfies the assumptions of Theorem* 2.

Let $\mu_1, \mu_2 \in \mathscr{P}_2(\mathbb{R}^d)$ be given, and $p_{\mu_1,\mu_2}, q_{\mu_1,\mu_2}$ as in the statement of Theorem 2 defined by $\boldsymbol{\pi} \in \Pi_o(\mu_1, \mu_2)$, L as in $(\boldsymbol{F_3})$. From a measurable selection theorem (see e.g. Theorem 8.2.11 in [5]), we have

$$\begin{aligned}
&\mathscr{H}_F(\mu_1, p_{\mu_1,\mu_2}) \\
&\quad = \inf \left\{ \int_{\mathbb{R}^d} \langle v(x), x - y\rangle \, d\boldsymbol{\pi}(x, y) : \ v(\cdot) \text{ Borel selection of } F(\mu_1, \cdot) \right\} \\
&\quad = \int_{\mathbb{R}^d} \inf_{v_1\in F(\mu_1,x)} \langle v_1, x - y\rangle \, d\boldsymbol{\pi}(x, y), \\
&\mathscr{H}_F(\mu_2, q_{\mu_1,\mu_2}) \\
&\quad = \inf \left\{ \int_{\mathbb{R}^d} \langle v(y), y - x\rangle \, d\boldsymbol{\pi}^{-1}(x, y) : \ v(\cdot) \text{ Borel selection of } F(\mu_2, \cdot) \right\} \\
&\quad = \inf \left\{ \int_{\mathbb{R}^d} \langle v(x), x - y\rangle \, d\boldsymbol{\pi}(x, y) : \ v(\cdot) \text{ Borel selection of } F(\mu_2, \cdot) \right\} \\
&\quad = \int_{\mathbb{R}^d} \inf_{v_2\in F(\mu_2,x)} \langle v_2, x - y\rangle \, d\boldsymbol{\pi}(x, y).
\end{aligned}$$

Set $\delta = LW_2(\mu_1, \mu_2)$. Given any $\varepsilon > 0$ let $w_\varepsilon \in F(\mu_1, x) + \delta\overline{B(0, 1)}$ be such that

$$\begin{aligned}
\inf_{v_2\in F(\mu_2,x)} \langle v_2, x - y\rangle &\geq \inf \left\{ \langle w, x - y\rangle : \ w \in F(\mu_1, x) + \delta\overline{B(0, 1)} \right\} \\
&\geq \langle w^\varepsilon, x - y\rangle - \varepsilon.
\end{aligned}$$

In particular, there are $v_1^\varepsilon \in F(\mu_1, x)$ and $v^\varepsilon \in \overline{B(0, 1)}$ such that $w_\varepsilon = v_1^\varepsilon + \delta v^\varepsilon$ and so

$$\inf_{v_2\in F(\mu_2,x)} \langle v_2, x - y\rangle \geq \langle v_1^\varepsilon, x - y\rangle + \delta\langle v^\varepsilon, x - y\rangle - \varepsilon$$

$$\geq \langle v_1^\varepsilon, x-y\rangle - \delta|x-y| - \varepsilon.$$

We then have

$$\begin{aligned}\inf_{v_1\in F(\mu_1,x)}\langle v_1, x-y\rangle - \inf_{v_2\in F(\mu_2,x)}\langle v_2, x-y\rangle \leq & L W_2(\mu_1,\mu_2)\cdot|x-y| + \varepsilon\\ \leq & \frac{L}{2}\left(W_2^2(\mu_1,\mu_2)+|x-y|^2\right)+\varepsilon.\end{aligned}$$

By integrating w.r.t. $\boldsymbol{\pi}$ (and recalling that $\boldsymbol{\pi}$ is optimal),

$$\begin{aligned}&\mathscr{H}_F(\mu_1, p_{\mu_1,\mu_2}) - \mathscr{H}_F(\mu_2, q_{\mu_1,\mu_2})\\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}\left[\inf_{v_1\in F(\mu_1,x)}\langle v_1, x-y\rangle - \inf_{v_2\in F(\mu_2,x)}\langle v_2, x-y\rangle\right] d\boldsymbol{\pi}(x,y)\\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}\left[\frac{L}{2}\left(W_2^2(\mu_1,\mu_2)+|x-y|^2\right)+\varepsilon\right] d\boldsymbol{\pi}(x,y)\\ &= L\cdot W_2^2(\mu_1,\mu_2)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0^+$ and switching the roles of μ_1,μ_2 yields

$$|\mathscr{H}_F(\mu_1, p_{\mu_1,\mu_2}) - \mathscr{H}_F(\mu_2, q_{\mu_1,\mu_2})| \leq L\cdot W_2^2(\mu_1,\mu_2).$$

Since $\mathcal{L}:\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is uniformly continuous with modulus $\omega_{\mathcal{L}}$, and recalling that $\mathscr{H}_F(\mu,\lambda p_\mu)=\lambda\mathscr{H}_F(\mu,p_\mu)$ for all $\lambda\geq 0$, we have

$$\left|\mathscr{H}(\mu_1,\lambda p_{\mu_1,\mu_2}) - \mathscr{H}(\mu_2,\lambda q_{\mu_1,\mu_2})\right| \leq \omega_{\mathcal{L}}(W_2(\mu_1,\mu_2)) + L\lambda\cdot W_2^2(\mu_1,\mu_2),$$

hence the assumptions of Theorem 2 are satisfied by taking $\omega_{\mathscr{H}}(r,s)=\omega_{\mathcal{L}}(r)+Ls$. This proves the statement of Step 1. □

Step 2. The value function V is a viscosity solution of (26).

Claim 1 V is a subsolution of (25).

Proof (of Claim 1) Take $(\bar{t},\bar{\mu})\in]0,T[\times\mathscr{P}_2(\mathbb{R}^d)$, $\delta>0$, $(p_{\bar{t}},p_{\bar{\mu}})\in D_\delta^+V(\bar{t},\bar{\mu})$. Given any admissible trajectory $\{\mu_t\}_{t\in[\bar{t},T]}\in\mathscr{A}^F_{[\bar{t},T]}(\bar{\mu})$, and $\boldsymbol{\pi_t}\in\Pi(\bar{\mu},\mu_t)$, set

$$\Delta_t := \sqrt{(t-\bar{t})^2+\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,d\boldsymbol{\pi_t}(x,y)}.$$

By the Dynamic Programming Principle in Proposition 3, we have for any $\boldsymbol{\pi_t}\in\Pi(\bar{\mu},\mu_t)$

$$\begin{aligned}0\leq & V(t,\mu_t)-V(\bar{t},\bar{\mu})+\int_{\bar{t}}^t\mathcal{L}(\mu_s)\,ds\\ \leq & p_{\bar{t}}(t-\bar{t})+\int_{\mathbb{R}^d\times\mathbb{R}^d}\langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y)+\delta\cdot\Delta_t+o(\Delta_t)+\int_{\bar{t}}^t\mathcal{L}(\mu_s)\,ds\end{aligned}$$

Hence, dividing by $t-\bar{t}>0$,

$$\begin{aligned}-\frac{\Delta_t}{t-\bar{t}}\cdot\delta\leq p_{\bar{t}}+\frac{1}{t-\bar{t}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y)+\\ +\frac{1}{t-\bar{t}}\int_{\bar{t}}^t\mathcal{L}(\mu_s)\,ds+\frac{o(\Delta_t)}{\Delta_t}\frac{\Delta_t}{t-\bar{t}}.\end{aligned}\tag{27}$$

Fix $\varepsilon > 0$ and let $v_0^\varepsilon(\cdot)$ be a Borel selection of $F(\bar{\mu}, \cdot)$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), v_0^\varepsilon(x)\rangle \, d\bar{\mu}(x) \le \inf_v \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), v(x)\rangle \, d\bar{\mu}(x) + \varepsilon,$$

where the inf is taken on all the Borel selections of $F(\bar{\mu}, \cdot)$. By Filippov's Theorem (see e.g. Theorem 8.2.10 in [5]), we can find a Borel map $u^\varepsilon : \mathbb{R}^d \to U$ such that $v_0^\varepsilon(x) = f(\bar{\mu}, x, u_x^\varepsilon)$. Choosing $\{\mu_t\}_{t\in[\bar{t},T]}$ as $\mu_t = e_t\sharp\boldsymbol{\eta}$ with $\boldsymbol{\eta}$ supported on

$$\begin{cases} \dot{\gamma}(t) = f(\mu_t, \gamma(t), u_x^\varepsilon), \\ \gamma(\bar{t}) = x, \end{cases}$$

and $\boldsymbol{\pi_t} = (e_t, e_{\bar{t}})\sharp\boldsymbol{\eta}$, leads to

$$\begin{aligned} \lim_{t\to\bar{t}^+} \frac{1}{t-\bar{t}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y - x\rangle \, d\boldsymbol{\pi_t}(x, y) \\ &= \lim_{t\to\bar{t}^+} \int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \langle p_{\bar{\mu}}(x), \frac{\gamma(t)-\gamma(\bar{t})}{t-\bar{t}}\rangle \, d\boldsymbol{\eta}(x, \gamma) \\ &= \int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \langle p_{\bar{\mu}}(x), v_0^\varepsilon(x)\rangle \, d\boldsymbol{\eta}(x, \gamma) \\ &\le \inf_v \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), v(x)\rangle \, d\bar{\mu}(x) + \varepsilon, \end{aligned}$$

where we used the fact that for $\boldsymbol{\eta}$-a.e. (x, γ), we have that $\gamma \in C^1([\bar{t}, T])$ and $\dot{\gamma}(\bar{t}) = f(\bar{\mu}, x, u_x^\varepsilon) = v_0^\varepsilon(x)$.

Similarly,

$$\begin{aligned} \lim_{t\to\bar{t}} \frac{1}{(t-\bar{t})^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, d\boldsymbol{\pi_t}(x, y) &= \lim_{t\to\bar{t}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left| \frac{\gamma(t)-\gamma(\bar{t})}{t-\bar{t}} \right|^2 d\boldsymbol{\eta}(x, \gamma) \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} |v_0^\varepsilon(x)|^2 \, d\boldsymbol{\eta}(x, \gamma) \\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\max_{w\in F(\delta_0,0)} |w| + LW_2(\delta_0, \bar{\mu}) + L|x| \right)^2 d\boldsymbol{\eta}(x, \gamma) \\ &\le 3 \cdot \left(\max_{w\in F(\delta_0,0)} |w|^2 + 2L^2 \mathrm{m}_2(\bar{\mu}) \right), \end{aligned}$$

leading to

$$\lim_{t\to\bar{t}} \frac{\Delta_t}{t-\bar{t}} \le \sqrt{1 + 3 \cdot \left(\max_{w\in F(\delta_0,0)} |w|^2 + 2L^2 \mathrm{m}_2(\bar{\mu}) \right)} =: C(\bar{\mu}).$$

Notice that the function $C(\cdot)$ defined above is bounded on every bounded set. In particular, by taking the limit as $t \to \bar{t}^+$ in (27), and recalling the continuity of $t \mapsto \mu_t$ and of $\mathcal{L}$, we have

$$-C(\bar{\mu})\delta \le p_{\bar{t}} + \inf_v \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), v(x)\rangle \, d\bar{\mu}(x) + \varepsilon + \mathcal{L}(\bar{\mu}).$$

By letting $\varepsilon \to 0^+$, we have $p_{\bar{t}} + \mathscr{H}(\bar{\mu}, p_{\bar{\mu}}) \ge -C(\bar{\mu})\delta$, which ends the proof of Claim 1.

Claim 2 V is a supersolution of (25).

Proof (of Claim 2) Take $(\bar{t},\bar{\mu}) \in]0,T[\times\mathscr{P}_2(\mathbb{R}^d)$, $\delta > 0$, $(p_{\bar{t}},p_{\bar{\mu}}) \in D^-_\delta V(\bar{t},\bar{\mu})$. By the Dynamic Programming Principle in Proposition 3, given an optimal trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[\bar{t},T]} \in \mathscr{A}^F_{[\bar{t},T]}(\bar{\mu})$ we have for all $\boldsymbol{\pi_t} \in \Pi(\bar{\mu},\mu_t)$

$$\begin{aligned}0 =&V(t,\mu_t) - V(\bar{t},\bar{\mu}) + \int_{\bar{t}}^t \mathcal{L}(\mu_s)\,ds\\ \geq& p_{\bar{t}}(t-\bar{t}) + \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y) - \delta\cdot\Delta_t + o(\Delta_t) + \int_{\bar{t}}^t \mathcal{L}(\mu_s)\,ds\end{aligned}$$

where Δ_t is defined as in Claim 1. In particular, we have

$$\begin{aligned}&\frac{\Delta_t}{t-\bar{t}}\cdot\left[\delta - \frac{o(\Delta_t)}{\Delta_t}\right]\\ &\geq p_{\bar{t}} + \frac{1}{t-\bar{t}}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y) + \frac{1}{t-\bar{t}}\int_{\bar{t}}^t \mathcal{L}(\mu_s)\,ds.\end{aligned}$$

By taking the lim inf for $t \to \bar{t}^+$, we have

$$C(\bar{\mu})\cdot\delta \geq p_{\bar{t}} + \liminf_{t\to\bar{t}^+} \frac{1}{t-\bar{t}}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y) + \mathcal{L}(\bar{\mu}),$$

where $C(\bar{\mu})$ is defined as in Claim 1. Let $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_{[\bar{t},T]})$ be such that $\mu_t = e_t\sharp\boldsymbol{\eta}$ and that for $\boldsymbol{\eta}$-a.e. $(x,\gamma) \in \mathbb{R}^d \times \Gamma_{[\bar{t},T]}$ we have $\gamma(\bar{t}) = x$ and $\dot{\gamma}(t) = f(\mu_t,\gamma(t),u_x(t))$, where $u_x(t)$ is a suitable Borel selection with values in U. By choosing $\boldsymbol{\pi_t} = (e_t,e_{\bar{t}})\sharp\boldsymbol{\eta}$ and $\boldsymbol{\eta} = \bar{\mu}\otimes\eta_x$, we have

$$\begin{aligned}\frac{1}{t-\bar{t}}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y) &= \int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \langle p_{\bar{\mu}}(x), \frac{\gamma(t)-\gamma(\bar{t})}{t-\bar{t}}\rangle\,d\boldsymbol{\eta}(x,\gamma)\\ &=\int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \langle p_{\bar{\mu}}(x), \frac{1}{t-\bar{t}}\int_{\bar{t}}^t \dot{\gamma}(s)\,ds\rangle\,d\boldsymbol{\eta}(x,\gamma)\\ &\geq\frac{1}{t-\bar{t}}\int_{\bar{t}}^t\int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \inf_{v\in F(\mu_s,\gamma(s))}\langle p_{\bar{\mu}}(x), v\rangle\,d\boldsymbol{\eta}(x,\gamma)\,ds\\ &\geq\frac{1}{t-\bar{t}}\int_{\bar{t}}^t\int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \left[\inf_{v\in F(\bar{\mu},x)}\langle p_{\bar{\mu}}(x), v\rangle - \delta_s|p_{\bar{\mu}}(x)|\right]d\boldsymbol{\eta}(x,\gamma)\,ds,\end{aligned}$$

where $\delta_s = L(W_2(\mu_s,\bar{\mu}) + |\gamma(s)-x|)$. By inverting the order of integrals and applying Fatou's Lemma, we have

$$\begin{aligned}&\liminf_{t\to\bar{t}^+}\frac{1}{t-\bar{t}}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y-x\rangle\,d\boldsymbol{\pi_t}(x,y) \geq\\ &\geq\int_{\mathbb{R}^d\times\Gamma_{[\bar{t},T]}} \liminf_{t\to\bar{t}^+}\frac{1}{t-\bar{t}}\int_{\bar{t}}^t\left[\inf_{v\in F(\bar{\mu},x)}\langle p_{\bar{\mu}}(x), v\rangle - \delta_s|p_{\bar{\mu}}(x)|\right]d\boldsymbol{\eta}(x,\gamma)\,ds\\ &=\int_{\mathbb{R}^d}\inf_{v\in F(\bar{\mu},x)}\langle p_{\bar{\mu}}(x), v\rangle\,d\bar{\mu}(x),\end{aligned}$$

by the continuity of $s \mapsto \delta_s$. Thus we have obtained

$$C(\bar{\mu})\delta \geq p_{\bar{t}} + \int_{\mathbb{R}^d}\inf_{v\in F(\bar{\mu},x)}\langle p_{\bar{\mu}}(x), v\rangle\,d\bar{\mu}(x) + \mathcal{L}(\bar{\mu}) = p_{\bar{t}} + \mathscr{H}(\bar{\mu},p_{\bar{\mu}}),$$

and the proof of Claim 2 is ended. □

Step 3 Since V is a viscosity solution to (26), in view of Step 1 we deduce from Theorem 2 that V is the unique BUC solution to (26). The proof is complete.

5.3 Equivalent formulations of viscosity solutions

In this section we discuss several equivalent definitions of sub/superdifferentials which leads to equivalent definitions of viscosity solutions.

The following Lemma shows that in Definition 3 we can restrict to $\boldsymbol{\pi} \in \Pi_o(\bar{\mu}, \nu)$.

Lemma 7 *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $\delta \geq 0$, and let $p \in L^2_{\bar{\mu}}(\mathbb{R}^d; \mathbb{R}^d)$ be an optimal displacement from $\bar{\mu} \in \mathscr{P}_2(\mathbb{R}^d)$ such that for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\gamma} \in \Pi_o(\bar{\mu}, \nu)$ we have*

$$
\begin{aligned}
& u(\nu) - u(\bar{\mu}) \\
& \leq \int_{\mathbb{R}^d} \langle p(x), y - x \rangle \, d\boldsymbol{\gamma}(x, y) + \delta \sqrt{\int_{\mathbb{R}^d} |x - y|^2 \, d\boldsymbol{\gamma}(x, y)} + o\left(W_2(\bar{\mu}, \nu)\right).
\end{aligned}
$$

Then $p \in D^+_\delta u(\bar{\mu})$.

Proof By Theorem 8.5.1 in [3], there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C^2_c(\mathbb{R}^d)$ such that $\nabla\varphi_n \to p$ in $L^2_{\bar{\mu}}(\mathbb{R}^d; \mathbb{R}^d)$. For any $x, y \in \mathbb{R}^d$, there exists $\theta \in \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ with

$$
\varphi_n(y) - \varphi_n(x) = \langle \nabla\varphi_n(x), y - x \rangle + \langle D^2\varphi_n(\theta)(y - x), y - x \rangle
$$

and

$$
\begin{aligned}
\varphi_n(y) - \varphi_n(x) - \|D^2\varphi_n\|_\infty \cdot |y - x|^2 \leq \langle \nabla\varphi_n(x), y - x \rangle \leq \\
\leq \varphi_n(y) - \varphi_n(x) + \|D^2\varphi_n\|_\infty \cdot |y - x|^2.
\end{aligned}
$$

Given $\boldsymbol{\pi} \in \Pi(\bar{\mu}, \nu)$ and $\boldsymbol{\gamma} \in \Pi_o(\bar{\mu}, \nu)$ we have

$$
\begin{aligned}
& \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla\varphi_n(x), y - x \rangle \, d\boldsymbol{\gamma} - \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla\varphi_n(x), y - x \rangle \, d\boldsymbol{\pi} \\
& \quad \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_n(y) \, d\boldsymbol{\gamma} - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_n(x) \, d\boldsymbol{\gamma} - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_n(y) \, d\boldsymbol{\pi} \\
& \qquad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_n(x) \, d\boldsymbol{\pi} + + \|D^2\varphi_n\|_\infty \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |y - x|^2 \, d\boldsymbol{\pi} + W_2^2(\bar{\mu}, \nu) \right) \\
& \quad = \|D^2\varphi_n\|_\infty \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |y - x|^2 \, d\boldsymbol{\pi} + W_2^2(\bar{\mu}, \nu) \right).
\end{aligned}
$$

For all $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \Pi(\bar{\mu}, \nu)$, and $\psi \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ set

$$
\Delta_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(\psi) := \frac{u(\nu) - u(\bar{\mu}) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \psi(x), y - x \rangle \, d\boldsymbol{\theta}_1(x, y)}{\left(\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} |y - x|^2 \, d\boldsymbol{\theta}_2(x, y) \right)^{1/2}}
$$

Set $r := \int_{\mathbb{R}^d\times\mathbb{R}^d} |y - x|^2 \, d\boldsymbol{\pi}$, we have:

$$
0 \leq W_2(\bar{\mu}, \nu) \leq r,
$$

then

$$\begin{aligned}\Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(\nabla\varphi_n) \leq & \Delta_{\boldsymbol{\gamma},\boldsymbol{\pi}}(\nabla\varphi_n) + \frac{\|D^2\varphi_n\|_\infty \left(\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^2\,d\boldsymbol{\pi} + W_2^2(\bar{\mu},\nu)\right)}{\left(\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^2\,d\boldsymbol{\pi}\right)^{1/2}} \\ \leq & \Delta_{\boldsymbol{\gamma},\boldsymbol{\pi}}(\nabla\varphi_n) + 2r\|D^2\varphi_n\|_\infty.\end{aligned}$$

This implies

$$\begin{aligned}\Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(p) \leq & \Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(\nabla\varphi_n) + \|\nabla\varphi_n - p\|_{L^2_{\bar{\mu}}} \\ \leq & \Delta_{\boldsymbol{\gamma},\boldsymbol{\pi}}(p) + 2r\|D^2\varphi_n\|_\infty + 2\|\nabla\varphi_n - p\|_{L^2_{\bar{\mu}}}\end{aligned}$$

We recall that by assumption $\limsup_{\nu\to\bar{\mu}} \Delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}}(p) \leq \delta$. Thus

$$\begin{aligned}\Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(p) \leq & \frac{\left(\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^2\,d\boldsymbol{\pi}(x,y)\right)^{1/2}}{W_2(\bar{\mu},\nu)} \cdot \max\left\{0, \Delta_{\boldsymbol{\gamma},\boldsymbol{\pi}}(p)\right\} \\ & + 2r\|D^2\varphi_n\|_\infty + 2\|\nabla\varphi_n - p\|_{L^2_{\bar{\mu}}} \\ = & \max\{0, \Delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}}(p)\} + 2r\|D^2\varphi_n\|_\infty + 2\|\nabla\varphi_n - p\|_{L^2_{\bar{\mu}}}\end{aligned}$$

In particular, we have

$$\limsup_{\nu\to\bar{\mu}} \Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(p) \leq \max\{0, \limsup_{\nu\to\bar{\mu}} \Delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}}(p)\} + 2r\|D^2\varphi_n\|_\infty + 2\|\nabla\varphi_n - p\|_{L^2_{\bar{\mu}}},$$

and by letting $r \to 0$ and $n \to +\infty$, we deduce

$$\limsup_{\nu\to\bar{\mu}} \Delta_{\boldsymbol{\pi},\boldsymbol{\pi}}(p) \leq \delta,$$

i.e. $p \in D^+_\delta u(\bar{\mu})$. □

We present here another approach in the computation of generalized gradient in the Wasserstein space, which is frequently used in Mean Field Game theory. Following [10] and [23], the main idea is to represent the Wasserstein space as the space of the law of random variable of a certain probability space, and to use the linear structure of the space of random variables in order to define derivatives. We want to perform a comparison between these two approaches.

Consider $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, where Ω is a complete separable metric space, $\mathcal{B}$ is the Borel σ-algebra, and $\mathbb{P}$ an atomless Borel probability measure on $(\Omega, \mathcal{B})$.[1] Given a random variable $X : \Omega \to \mathbb{R}^d$ on $(\Omega, \mathcal{B}, \mathbb{P})$, we denote by $X\sharp\mathbb{P} \in \mathscr{P}(\mathbb{R}^d)$ its law, i.e., $X\sharp\mathbb{P}(B) = \mathbb{P}\left(X^{-1}(B)\right)$ for every Borel set $B \subseteq \mathbb{R}^d$. We recall (see e.g. [29]) that, by the assumptions on $\mathbb{P}$, for every $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists $X \in L^2(\Omega; \mathbb{R}^d)$ such that $\mu = X\sharp\mathbb{P}$. Conversely, for every $X \in L^2(\Omega; \mathbb{R}^d)$ we have $X\sharp\mathbb{P} \in \mathscr{P}_2(\mathbb{R}^d)$. Moreover

$$W_2(\mu_1, \mu_2) = \inf\left\{\|X_1 - X_2\|_{L^2_{\mathbb{P}}} : X_i\sharp\mathbb{P} = \mu_i,\ i = 1, 2\right\}.$$

[1] For instance, $(\Omega, \mathcal{B}, \mathbb{P}) = (\mathbb{R}^d, \mathrm{Bor}(\mathbb{R}^d), \mathscr{L}^d_{|[0,1]^d})$, where $\mathscr{L}^d_{|[0,1]^d}$ denotes the restriction of the Lebesgue measure on $[0, 1]^d$.

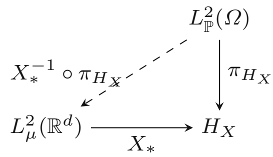

Fig. 1 The map X_* is a linear isometry from $L^2_\mu(\mathbb{R}^d)$ to H_X. The map π_{H_X} denotes the projection on H_X in the Hilbert space $L^2_{\mathbb{P}}(\Omega)$

Given $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we recall (see [10]) that its *lift* $U : L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is defined by $U(X) = u(X\sharp\mathbb{P})$ for all $X \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$. We also recall that a function $U : L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ is called *law-dependent* if $U(X_1) = U(X_2)$ for all $X_1, X_2 \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$ s.t. $X_1\sharp\mathbb{P} = X_2\sharp\mathbb{P}$.

Clearly, every lift of functions defined on $\mathscr{P}_2(\mathbb{R}^d)$ is law-dependent.

Lemma 8 *Let $X \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$, and define*

$$H_X := \{\phi \circ X \in L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d) : \phi \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d)\}.$$

Then H_X is a closed linear subspace of $L^2_{\mathbb{P}}(\Omega; \mathbb{R}^d)$. Moreover, the map

$$X_* : L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d) \to H_X$$

defined as $X_*(\phi) = \phi \circ X$ is a linear isometry.

Proof Let $\xi \in \overline{H_X}$. Then there exists a sequence $\{p_n \circ X\}_{n\in\mathbb{N}} \subseteq H_X$ such that

$$\lim_{n\to+\infty} \|p_n \circ X - \xi\|_{L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)} = 0.$$

In particular, we have that there exists $C > 0$ such that

$$\|p_n\|_{L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d;\mathbb{R}^d)} = \|p_n \circ X\|_{L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)} \le C,$$

thus, up to subsequences, we may assume $p_n \rightharpoonup p$ weakly in $L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d)$ for a certain $p \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d)$. Given any $\phi \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d)$, we have

$$0 = \lim_{n\to+\infty} \langle \phi, p_n - p\rangle_{L^2_{X\sharp\mathbb{P}}} = \lim_{n\to+\infty} \langle \phi \circ X, p \circ X - p_n \circ X\rangle_{L^2_{\mathbb{P}}} = \langle \phi \circ X, p \circ X - \xi\rangle_{L^2_{\mathbb{P}}}.$$

Thus for all $\phi \circ X \in H_X$ we have $\langle \phi \circ X, p \circ X - \xi\rangle_{L^2_{\mathbb{P}}} = 0$, hence $\langle \Phi, p \circ X - \xi\rangle_{L^2_{\mathbb{P}}} = 0$ for all $\Phi \in \overline{H}_X$. Hence $p \circ X - \xi \in H_X^\perp$. But since $p \circ X - \xi \in \overline{H}_X$ and because $\overline{H}_X \cap H_X^\perp = \{0\}$ we deduce that $\xi = p \circ X \in H_X$. The last assertion follows from the fact that $\phi \mapsto \phi \circ X$ is obviously linear, moreover, set $\mu = X\sharp\mathbb{P}$, we have (Fig. 1)

$$\|\phi \circ X\|^2_{L^2_{\mathbb{P}}(\Omega)} = \int_\Omega |\phi \circ X(\omega)|^2 \, d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} |\phi(x)|^2 \, d\mu(x) = \|\phi\|^2_{L^2_\mu}.$$

□

Our aim is to find a convenient representation of the sub/super differentials of Definition 3 by using the set H_X defined in Lemma 8. We state it only for the superdifferential, for the subdifferential the argument is symmetric.

Proposition 5 *Let $U : [0, T] \times L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$ be a map, $\bar{t} \in [0, T]$, $X \in L^2_{\mathbb{P}}(\Omega)$, $\delta > 0$, $(p_{\bar{t}}, \xi) \in \mathbb{R} \times L^2_{\mathbb{P}}(\Omega)$. Assume that $U(\cdot)$ and $\xi(\cdot)$ satisfy the following properties:*

1. *$U(t, \cdot)$ is law-dependent;*

2. *there exists* $Y \in H_X$ *such that* $\xi = X - Y$ *and* $W_2(X\sharp\mathbb{P}, Y\sharp\mathbb{P}) = \|\xi\|_{L^2_{\mathbb{P}}}$.
3. *for all* $Z \in L^2_{\mathbb{P}}(\Omega)$ *we have*

$$U(t, Z) - U(\bar{t}, X) \leq p_{\bar{t}}(t - \bar{t}) + \int_{\mathbb{R}^d} \langle \xi(\omega), Z(\omega) - X(\omega)\rangle \, d\mathbb{P}(\omega)$$
$$+\delta\sqrt{|t - \bar{t}|^2 + \|X - Z\|^2_{L^2_{\mathbb{P}}}} + o\left(|t - \bar{t}| + \|X - Z\|_{L^2_{\mathbb{P}}}\right).$$

Then, defining $u : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *by* $u(t, \mu) = U(t, Z)$ *for all* $Z \in L^2_{\mathbb{P}}(\Omega)$ *such that* $Z\sharp\mathbb{P} = \mu$ *and* $t \in [0, T]$*, and setting* $\bar{\mu} := X\sharp\mathbb{P}$*, we have that there exist* $p_{\bar{\mu}} \in L^2_{\bar{\mu}}(\mathbb{R}^d)$ *such that* $\xi = \pi_{H_X}(\xi) = p_{\bar{\mu}} \circ X$ *and* $(p_{\bar{t}}, p_{\bar{\mu}}) \in D^+_\delta u(\bar{t}, \bar{\mu})$.

Conversely, given $u : [0, T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$*, its lift* $U : [0, T] \times L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$*,* $(\bar{t}, \bar{\mu}) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$*,* $(p_{\bar{t}}, p_{\bar{\mu}}) \in D^+_\delta u(\bar{t}, \bar{\mu})$*, there exist* $X, Y \in L^2_{\mathbb{P}}(\Omega)$ *such that* $(X, Y)\sharp\mathbb{P} \in \Pi_o(\bar{\mu}, (\mathrm{Id} - p_{\bar{\mu}})\sharp\bar{\mu})$*, moreover* $\xi = p_{\bar{\mu}} \circ X$ *and* $U(\cdot)$ *satisfy all the properties (1–2–3) above.*

Proof Property (2) implies that $\xi = \pi_{H_X}(\xi) = p \circ X$ for a certain optimal displacement $p \in L^2_{\bar{\mu}}(\mathbb{R}^d)$ from $\bar{\mu}$, since $Y \in H_X$. Exploiting properties (1) and (3), given $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and chosen $Z \in L^2_{\mathbb{P}}(\Omega)$ such that $U(t, Z) = u(t, \nu)$, we obtain

$$u(t, \nu) - u(\bar{t}, \bar{\mu}) \leq p_{\bar{t}}(t - \bar{t}) + \int_{\mathbb{R}^d} \langle p_{\bar{\mu}}(x), y - x\rangle \, d\boldsymbol{\pi}(x, y)$$
$$+\delta\sqrt{|t - \bar{t}|^2 + \int_{\mathbb{R}^d} |x - y|^2 \, d\boldsymbol{\pi}(x, y)}$$
$$+o\left(|t - \bar{t}| + \left(\int_{\mathbb{R}^d} |x - y|^2 \, d\boldsymbol{\pi}(x, y)\right)^{1/2}\right).$$

by setting $\boldsymbol{\pi} = (X, Z)\sharp\mathbb{P}$. The converse is trivial.

We conclude this section by giving a characterization of superdifferentials with *specific* test functions from $L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$ whose gradients belong to the superdifferential. For sake of simplicity we omit here the t variable.

Definition 5 (Quadratic test functions) Given $Y \in L^2_{\mathbb{P}}(\Omega)$, we define the smooth map $Q_Y : L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$ by setting for all $Z \in L^2_{\mathbb{P}}(\Omega)$

$$Q_Y(Z) = \frac{1}{2}\|Z - Y\|^2_{L^2_{\mathbb{P}}}.$$

For all $X \in L^2_{\mathbb{P}}(\Omega)$ we consider the set $T(X)$ of all maps Q_Y such that

$$Y \in H_X \text{ and } W_2(X\sharp\mathbb{P}, Y\sharp\mathbb{P}) = \|X - Y\|_{L^2_{\mathbb{P}}}. \tag{28}$$

Proposition 6 (Superdifferentials with test functions) *Let* $U : L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$ *be a law-dependent map,* $\delta > 0$*,* $X \in L^2_{\mathbb{P}}(\Omega)$*,* $Y \in H_X$ *such that* (28) *holds true, and* $Q_Y \in T(X)$ *such that for any* $Z \in L^2_{\mathbb{P}}(\Omega)$

$$U(Z) - Q_Y(Z) \leq U(X) - Q_Y(X) + \delta\|Z - X\|,$$

i.e. $U - Q_Y$ *has a local* δ*-maximum at* X*. Then, denoting by* $\xi \in L^2_{\mathbb{P}}(\Omega)$ *the gradient in* $L^2_{\mathbb{P}}(\Omega)$ *of* Q_Y *at* X*, and defining* $u(S\sharp\mathbb{P}) = U(S)$ *for all* $S \in L^2_{\mathbb{P}}(\Omega)$*, we have that* $\xi = \pi_{H_X}(\xi) = p \circ X$ *with* $p \in D^+_\delta u(\mu)$*.*

Conversely, given $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *and denoted by* $U : L^2_{\mathbb{P}}(\Omega) \to \mathbb{R}$ *its lift, given* $p \in D^+_\delta u(\mu)$*, set* $\xi = p \circ X$*, there exists* $Q \in T(X)$ *such that for all* Z *we have that* $U - Q$ *has a local* δ*-maximum at* X *and* $DQ(X) = \xi \in L^2_{\mathbb{P}}(\Omega)$*.*

Proof Assume that $U - Q_Y$ has a local δ-maximum at X. Thus, for every $Z \in L^2_{\mathbb{P}}(\Omega)$ we have

$$\begin{aligned} U(Z) - U(X) \leq & Q_Y(Z) - Q_Y(X) + \delta\|Z - X\|_{L^2_{\mathbb{P}}} + o(\|Z - X\|_{L^2_{\mathbb{P}}}) \\ = & \langle \xi, Z - X\rangle + \delta\|Z - X\|_{L^2_{\mathbb{P}}} + o(\|Z - X\|_{L^2_{\mathbb{P}}}), \end{aligned}$$

where we used the smoothness of Q_Y. The first assertion now follows from Proposition 5.

We prove now the second assertion. According to the last part of Proposition 5, it is possible to find Y such that

$$\begin{aligned} U(Z) - U(X) \leq & \langle \xi, Z - X\rangle + \delta\|Z - X\|_{L^2_{\mathbb{P}}} + o(\|Z - X\|_{L^2_{\mathbb{P}}}) \\ = & \langle X - Y, Z - X\rangle + \delta \cdot \|Z - X\|_{L^2_{\mathbb{P}}} + o(\|Z - X\|_{L^2_{\mathbb{P}}}) \\ = & \frac{1}{2}\|Z - Y\|^2 - \frac{1}{2}\|X - Y\|^2 + \delta \cdot \|Z - X\|_{L^2_{\mathbb{P}}} + o(\|Z - X\|_{L^2_{\mathbb{P}}}), \end{aligned}$$

thus it is enough to take $Q = Q_Y(\cdot)$ to conclude. □

Remark 6 Let U be a bounded upper semicontinuous law-dependent map. Given any $Y \in L^2_{\mathbb{P}}$, if we fix $C = \overline{B_{L^2_{\mathbb{P}}}(X, r)}$, we have that $f := Q_Y - U$ is a lower semicontinuous function on C bounded from below. Since $L^2_{\mathbb{P}}(\Omega)$ is an Hilbert space, we can apply Stegall's variational principle (see [10]) obtaining for all $\delta > 0$ an element $X^*_\delta \in (L^2_{\mathbb{P}}(\Omega))'$ such that $f + X^*_\delta$ has a (strong) minimum in C and $\|X^*_\delta\| \leq \delta$. In particular, there exists $X_\delta \in C$ such that for all $Z \in C$

$$Q_Y(X_\delta) - U(X_\delta) + \langle X^*_\delta, X_\delta\rangle \leq Q_Y(Z) - U(Z) + \langle X^*_\delta, Z\rangle.$$

Rearranging the terms, we obtain for all $Z \in C$

$$U(Z) - Q_Y(Z) \leq U(X_\delta) - Q_Y(X_\delta) + \langle X^*_\delta, Z - X_\delta\rangle.$$

We can extend this inequality to the whole of $L^2_{\mathbb{P}}$ by adding a term on the right hand side which vanishes as $Z \to X$, thus for all $Z \in L^2_{\mathbb{P}}$ we have

$$\begin{aligned} U(Z) - Q_Y(Z) \leq & \quad U(X_\delta) - Q_Y(X_\delta) + \langle X^*_\delta, Z - X_\delta\rangle + o(\|Z - X\|), \\ \leq & \quad U(X_\delta) - Q_Y(X_\delta) + \delta\|Z - X_\delta\| + o(\|Z - X\|) \end{aligned}$$

i.e., $U - Q_Y$ has a local δ-maximum at X_δ.

Remark 7 We notice that in the definition of $T(X)$ it is required, beside the optimality condition $W_2(X\sharp\mathbb{P}, Y\sharp\mathbb{P}) = \|X - Y\|_{L^2_{\mathbb{P}}}$, also that $Y \in H_X$. In this way, we have that the L^2-gradient at X of any $Q \in T(X)$ is a law-dependent function, which is coherent with the fact that, to have a suitable notion of super/sub-tangent test function at $X \in L^2_{\mathbb{P}}$ for a law-dependent function, their gradient at X must actually define univocally an element of L^2_μ where $\mu = X\sharp\mathbb{P}$.

On the other hand, to restrict $Z \in H_X$ in the lifted function (i.e., considering less possible variations from the point of interest X) is equivalent to consider in the original function only measure ν which can be reached from μ by transport maps. In this case, even if $Y \notin H_X$, the projection $\pi_{H_X}(\xi)$, where ξ is defined as in Proposition 5, define an element of the δ-superdifferential (restricted in this sense). This was essentially the case considered in [12].

Remark 8 It is worth pointing out that even if the equivalent definitions of section 5.3 are given in $L^2_{\mathbb{P}}(\Omega)$, they do not reduce to the classical definition of viscosity solution [17] in the Hilbert space $L^2_{\mathbb{P}}(\Omega)$. In particular the comparison theorem of [17] does not apply. This is why we needed to state a new comparison theorem (Theorem 2) in the context of our optimal control problem. For other definitions of viscosity solutions where the uniqueness and comparison results of [17] may be used, we refer the reader to [23].

Remark 9 Several different notions of the sub/superdifferentials have been introduced and studied in the space of probability measures (see for instance [2,3,12,14,21–23,25]). Our goal is not to give a comparison between these sub/superdifferentials and the sub/superdifferential introduced in the present paper which is well adapted to obtain a comparison result for Hamilton Jacobi equation and thus to obtain a characterization of the value of the studied optimal control problem which is the aim of the article. A more detailed comparison between existing sub/superdifferentials and its relevance for Hamilton Jacobi equations will be discussed in a forthcoming paper.

A Some results on measure theory

We refer to Section 5.3 in [3] for the following preliminaries of measure theory.

Definition 6 *(Borel families of measures and generalized product)* Let X, Y be separable metric spaces and let $X \ni x \mapsto \pi_x \in \mathscr{P}(Y)$ be a measure-valued map. We say that $x \mapsto \pi_x$ is a Borel map (equivalently, that $\{\pi_x\}_{x\in X}$ is a Borel family) if $x \mapsto \pi_x(B)$ is a Borel map from X to $\mathbb{R}$ for any Borel set $B \subseteq Y$, or equivalently if this property holds for any open set $A \subseteq Y$. This implies also that for every bounded (or nonnegative) Borel function $f : X \times Y \to \mathbb{R}$. the function defined by

$$x \mapsto \int_Y f(x, y)\, d\pi_x(y)$$

is Borel. Thus given any Borel probability measure $\mu \in \mathscr{P}(X)$, we can define uniquely a measure $\mu \otimes \pi_x \in \mathscr{P}(X \times Y)$, called the generalized product between μ and the family $\{\pi_x\}_{x\in X}$ by setting

$$\int_{X\times Y} \varphi(x, y)\, d(\mu \otimes \pi_x)(x, y) = \int_X \left[\int_Y \varphi(x, y)\, d\pi_x(y)\right] d\mu(x)$$

for all $\varphi \in C^0_b(X \times Y)$. Notice that the first marginal of $\mu \otimes \pi_x$ is μ.

The following result is Theorem 5.3.1 in [3].

Theorem 4 (Disintegration) *Given a measure $\mu \in \mathscr{P}(\mathbb{X})$ and a Borel map $r : \mathbb{X} \to X$, there exists a family of probability measures $\{\mu_x\}_{x\in X} \subseteq \mathscr{P}(\mathbb{X})$, uniquely defined for $r\sharp\mu$-a.e. $x \in X$, such that $\mu_x(\mathbb{X}\backslash r^{-1}(x)) = 0$ for $r\sharp\mu$-a.e. $x \in X$, and for any Borel map $\varphi : X \times Y \to [0, +\infty]$ we have*

$$\int_{\mathbb{X}} \varphi(z)\, d\mu(z) = \int_X \left[\int_{r^{-1}(x)} \varphi(z)\, d\mu_x(z) \right] d(r\sharp\mu)(x).$$

We will write $\mu = (r\sharp\mu) \otimes \mu_x$. *If* $\mathbb{X} = X \times Y$ *and* $r^{-1}(x) \subseteq \{x\} \times Y$ *for all* $x \in X$, *we can identify each measure* $\mu_x \in \mathscr{P}(X \times Y)$ *with a measure on* Y.

We also recall an adapted version of Theorem 8.2.1 in [3].

Theorem 5 (Superposition principle) *Let* $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ *be a solution of the continuity equation* $\partial_t \mu_t + \mathrm{div}(v_t\mu_t) = 0$ *for a suitable Borel vector field* $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfying*

$$\int_0^T \int_{\mathbb{R}^d} \frac{|v_t(x)|}{1 + |x|}\, d\mu_t(x)\, dt < +\infty .$$

Then there exists a probability measure $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$, *with* $\Gamma_T = C^0([0, T]; \mathbb{R}^d)$ *endowed with the* sup *norm, such that*

(i) $\boldsymbol{\eta}$ *is concentrated on the pairs* $(x, \gamma) \in \mathbb{R}^d \times \Gamma_T$ *such that* γ *is an absolutely continuous solution of*

$$\begin{cases} \dot\gamma(t) = v_t(\gamma(t)), & \text{for } \mathscr{L}^1\text{-a.e } t \in (0, T) \\ \gamma(0) = x, \end{cases}$$

(ii) $\mu_t = e_t\sharp\boldsymbol{\eta}$ *for all* $t \in [0, T]$.

Conversely, given any $\boldsymbol{\eta}$ *satisfying* (i) *above and defined* $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ *as in* (ii) *above, we have that* $\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0$ *and* $\mu_{|t=0} = \gamma(0)\sharp\boldsymbol{\eta}$.

References

1. Albi, G., Bellomo, N., Fermo, L., Ha, S.-Y., Kim, J., Pareschi, L., Poyato, D., Soler, J.: Vehicular traffic, crowds, and swarms: from kinetic theory and multiscale methods to applications and research perspectives. Math. Models Methods Appl. Sci. **29**(10), 1901–2005 (2019). https://doi.org/10.1142/S0218202519500374
2. Ambrosio, L., Gangbo, W.: Hamiltonian ODEs in the Wasserstein space of probability measures. Commun. Pure Appl. Math. **61**(1), 18–53 (2008). https://doi.org/10.1002/cpa.20188
3. Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and in the space of probability measures. In: Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel (2008)
4. Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. In: Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York (2000)
5. Aubin, J.-P., Frankowska, H.: Set-valued analysis, Modern Birkhäuser Classics, Reprint of the 1990 edition. Birkhäuser Boston Inc., Boston MA (2009)
6. Bouchitté, G., Buttazzo, G.: New lower semicontinuity results for nonconvex functionals defined on measures. Nonlinear Anal. **15**(7), 679–692 (1990). https://doi.org/10.1016/0362-546X(90)90007-4
7. Bensoussan, A., Frehse, J., Yam, P.: Mean Field Games and Mean Field Type Control Theory, SpringerBriefs in Mathematics. Springer, New York (2013)
8. Bongini, M., Fornasier, M., Rossi, F., Solombrino, F.: Mean-field Pontryagin maximum principle. J. Optim. Theory Appl. **175**(1), 1–38 (2017). https://doi.org/10.1007/s10957-017-1149-5
9. Bonnet, B., Rossi, F.: The Pontryagin maximum principle in the Wasserstein space. Calc. Var. Partial Differ. Equ. **58**(1), Art. 11, 36 (2019). https://doi.org/10.1007/s00526-018-1447-2
10. Cardaliaguet, P.: Notes on Mean Field Games (unpublished notes) (2013)
11. Cardaliaguet, P., Jimenez, C., Quincampoix, M.: Pure and random strategies in differential game with incomplete informations. J. Dyn. Games **1**(3), 363–375 (2014). https://doi.org/10.3934/jdg.2014.1.363
12. Cardaliaguet, P., Quincampoix, M.: Deterministic differential games under probability knowledge of initial condition. Int. Game Theory Rev. **10**(1), 1–16 (2008)

13. Carrillo, J.A., Fornasier, M., Toscani, G., Vecil, F.: Particle, kinetic, and hydrodynamic models of swarming. In: Naldi, G., Pareschi, L., Toscani, G. (eds.) Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences. Modeling and Simulation in Science, Engineering and Technology. Birkhuser Boston, Boston (2010)
14. Cavagnari, G., Marigonda, A., Nguyen, K.T., Fabio, S.P.: Generalized control systems in the space of probability measures. Set Valued Variat. Anal. **26**(3), 663–691 (2018). https://doi.org/10.1007/s11228-017-0414-y
15. Cavagnari, G., Piccoli, B., Marigonda, A.: Optimal synchronization problem for a multi-agent system. Netw. Heterog. Media **12**(2), 277–295 (2017). https://doi.org/10.3934/nhm.2017012
16. Cavagnari, G., Marigonda, A., Piccoli, B.: Superposition principle for differential inclusions. In: Large-Scale Scientific Computing, Lecture Notes in Comput. Sci., vol. 10665, Springer, Cham (2018)
17. Crandall, M.G., Lions, P.-L.: Hamilton–Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions. J. Funct. Anal. **62**(3), 379–396 (1985). https://doi.org/10.1016/0022-1236(85)90011-4
18. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974). https://doi.org/10.1016/0022-247X(74)90025-0
19. Fornasier, M., Lisini, S., Orrieri, C., Savaré, G.: Mean-field optimal control as Gamma-limit of finite agent controls. Eur. J. Appl. Math. **30**(6), 1153–1186 (2019). https://doi.org/10.1017/S0956792519000044
20. Fornasier, M., Solombrino, F.: Mean-field optimal control. ESAIM Control Optim. Calc. Var. **20**(4), 1123–1152 (2014). https://doi.org/10.1051/cocv/2014009
21. Gangbo, W., Nguyen, T., Tudorascu, A.: Hamilton–Jacobi equations in the Wasserstein space. Methods Appl. Anal. **15**(2), 155–183 (2008). https://doi.org/10.4310/MAA.2008.v15.n2.a4
22. Gangbo, W., Swiech, A.: Metric viscosity solutions of Hamilton–Jacobi equations depending on local slopes. Calc. Var. Partial Differ. Equ. **54**(1), 1183–1218 (2015). https://doi.org/10.1007/s00526-015-0822-5
23. Gangbo, W., Tudorascu, A.: On differentiability in the Wasserstein space and well-posedness for Hamilton–Jacobi equations. Journal de Mathématiques Pures et Appliquées **125**, 119–174 (2019). https://doi.org/10.1016/j.matpur.2018.09.003
24. Lasry, J.-M., Lions, P.-L.: Mean field games. Jpn. J. Math. **2**(1), 229–260 (2007). https://doi.org/10.1007/s11537-007-0657-8
25. Marigonda, A., Quincampoix, M.: Mayer control problem with probabilistic uncertainty on initial positions. J. Differ. Equ. **264**(5), 3212–3252 (2018). https://doi.org/10.1016/j.jde.2017.11.014
26. Méléard, S.: Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models, Lecture Notes in Math., vol. 1627, Springer, Berlin, pp. 42–95 (1996). https://doi.org/10.1007/BFb0093177.
27. McKean, H.P., Jr.: Propagation of chaos for a class of non-linear parabolic equations. In: Stochastic Differential Equations, Lecture Series in Differential Equations, Session 7, Catholic Univ., Air Force Office Sci. Res. (1967), Arlington, VA., pp. 41–57
28. Perchet, V., Quincampoix, M.: A differential game on Wasserstein space. Application to weak approachability with partial monitoring. J. Dyn. Games **6**(1), 65–85 (2019)
29. Pratelli, A.: On the equality between Monge's infimum and Kantorovich's minimum in optimal mass transportation. Ann. Inst. H. Poincaré Probab. Statist. **43**(1), 1–13 (2007). https://doi.org/10.1016/j.anihpb.2005.12.001
30. Santambrogio, F.: Optimal transport for applied mathematicians. In: Progress in Nonlinear Differential Equations and Their Applications, vol. 87, Birkhäuser/Springer, Cham (2015)
31. Villani, C.: Topics in optimal transportation. Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI (2003)