MULTIPLE YIELD CURVE MODELLING WITH CBI PROCESSES

CLAUDIO FONTANA, ALESSANDRO GNOATTO, AND GUILLAUME SZULDA

Abstract. We develop a modelling framework for multiple yield curves driven by continuous-state branching processes with immigration (CBI processes). Exploiting the self-exciting behavior of CBI jump processes, this approach can reproduce the relevant empirical features of spreads between different interbank rates. We provide a complete analytical framework, including a detailed study of discounted exponential moments of CBI processes. The proposed framework yields explicit valuation formulae for all linear interest rate derivatives as well as semi-closed formulae for non-linear derivatives via Fourier techniques and quantization. We show that a simple specification of the model can be successfully calibrated to market data.

1. Introduction

The emergence of multiple yield curves can be rightfully regarded as the most relevant feature of interest rate markets over the last decade, starting from the 2007-2009 financial crisis. While pre-crisis interest rate markets were adequately described by a single yield curve and interbank rates (to which we generically refer as Ibor rates[1]) associated to different tenors were determined by simple no-arbitrage relations, this proved to be no longer valid in the post-crisis scenario, where yield curves associated to interbank rates of different tenors exhibit a distinct behavior. This is reflected by the presence of tenor-dependent spreads between different yield curves. In the midst of the financial crisis, such spreads reached their peak beyond 200 basis points and since then, and still nowadays, they continue to remain at non-negligible levels, as shown in Figure 1. The credit, liquidity and funding risks existing in the interbank market, which were deemed negligible before the crisis, are at the origin of this phenomenon (see [CD13, FT13] in this regard).

In this paper, we propose a novel modelling approach to multiple yield curves, ensuring analytical tractability as well as consistency with the most relevant empirical features. An inspection of Figure 1 reveals several important properties of spreads: first, spreads are typically greater than one and increasing with respect to the tenor; second, there are strong comovements (in particular, common upward jumps) among spreads associated to different tenors; third, there are periods of high volatility followed by more stable periods (volatility clustering); fourth, low values of the

Date: November 7, 2019.

2010 *Mathematics Subject Classification.* 60G51, 60J85, 91G20, 91G30, 91G60.

Key words and phrases. Branching process; self-exciting process; multi-curve model; interest rate; Libor rate; OIS rate; multiplicative spread; affine process.

The first author is grateful to the Europlace Institute of Finance for financial support to this work.

[1]The most relevant Ibor rates are represented by the Libor rates in the London interbank market and the Euribor rates in the Eurozone (see Section 6 for a brief discussion on the ongoing reforms of interbank benchmark rates).

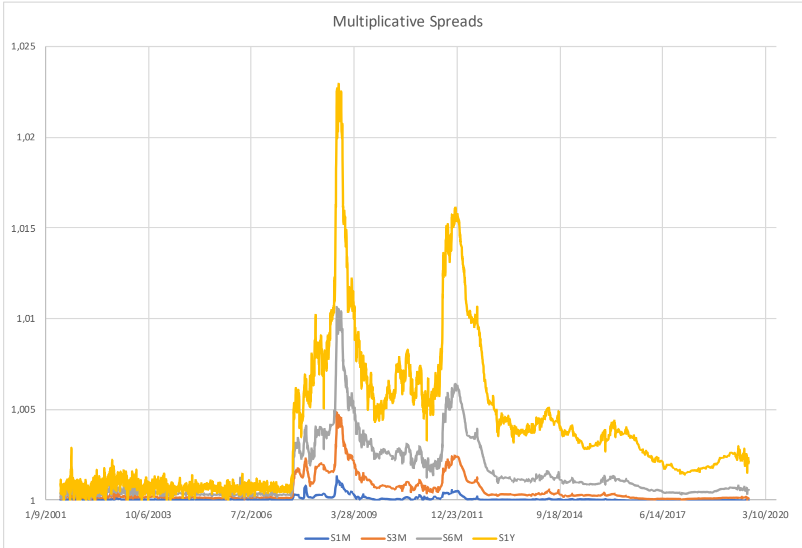

FIGURE 1. Euribor-OIS spreads from 06/2001 to 09/2019. Source: Bloomberg.

spreads can persist for prolonged periods of time. To the best of our knowledge, a model that can adequately reproduce all these features does not yet exist. We do not attempt here a review of the numerous approaches to multiple curve modelling, which has attracted huge attention in the literature, and we refer instead the reader to the volumes [BM13, Hen14, GR15] for detailed accounts on the topic (see Section 3.1 for additional references specific to our approach).

By relying on the theory of *continuous-state branching processes with immigration* (CBI processes), we develop a modelling framework that can capture all the empirical properties mentioned above and, at the same time, allows for an efficient valuation of interest rate derivatives written on Ibor rates. Exploiting the affine property of CBI processes, we design our modelling framework in the spirit of the affine multi-curve models recently studied in [CFG19b], taking multiplicative spreads and the OIS short rate as fundamental modelling objects. By construction, the model achieves a perfect fit to the initially observed term structures and can generate spreads greater than one and increasing with respect to the tenor (see Section 3). The construction of the model requires a detailed study of the finiteness of exponential moments of a CBI process. To this effect, we prove a general and explicit characterization of the time of explosion of (discounted) exponential moments of a CBI process (see Section 2.1), specializing to our context some techniques introduced in [KR11]. This result can be considered of independent interest in the theory of CBI processes.

By specializing our general modelling framework, we propose a tractable model driven by a flow of tempered alpha-stable CBI processes (see Sections 2.2 and 4). The adoption of a flow of CBI processes enables us to capture strong comovements among spreads, including common upwards jumps and jump clustering effects. The characteristic self-exciting behavior of CBI processes proves to be a key ingredient to generate these features. Moreover, the choice of a tempered alpha-stable jump measure presents a good balance between flexibility and analytical tractability and allows for an explicit characterization of several important properties of the model. All linear interest rate derivatives admit closed form pricing formulae, convexity adjustments can be explicitly computed and, by relying on Fourier techniques, we derive a semi-closed pricing formula for a caplet. In addition, we also develop an efficient valuation method based on quantization, which is here applied for the first time to an interest rate setting. A specification of this model with two tenors is then calibrated to market data, showing an excellent fit to market data (see Section 5).

We close this introduction by briefly discussing some related literature. After their original applications to population dynamics (see [Par16] for an overview), CBI processes have been recently adopted with success in finance, mainly due to their characteristic self-exciting behavior. Starting with the seminal work [Fil01], CBI processes have found a natural application in the context of interest rate modelling. In particular, in a single-curve interest rate model, [JMS17] have shown that an alpha-stable CBI process allows to reproduce short rates with persistently low values. The same stochastic process has been used in [JMSZ18] for stochastic volatility modelling, extending the classical model by Heston. CBI processes have been also applied to the modelling of forward prices in energy markets, where jump clustering phenomena are often observed, see [CMS19, JMSS19]. We also mention that, in a multiple curve setting, self-exciting features have been recently studied by [NLH19] in a reduced-form model of interbank credit risk.

The paper is structured as follows. In Section 2, we recall the definition and the basic properties of CBI processes, together with some probabilistic results that will be relevant in the following. Section 3 presents the general modelling approach, which is then specialized in Section 4 to a model driven by a flow of tempered alpha-stable CBI processes. Section 5 contains some numerical results, including model calibration to market data. We conclude in Section 6 by briefly commenting on multiple curves in light of the recent reforms of Ibor rates. For the sake of readability, we postpone to Appendix A the proofs of the results stated in Section 2. General pricing formulae for interest rate derivatives in a multi-curve setting are presented in Appendix B.

2. Some Results on CBI Processes

In this section, we recall the definition of a CBI process and establish some theoretical results which play a fundamental role in the construction of multi-curve models driven by CBI processes. For ease of exposition, all proofs are postponed to Appendix A. For comprehensive accounts on CBI processes we refer to [Li11], [Li20] and [Kyp06, Chapter 10].

2.1. **General properties of CBI processes.** We start by recalling the general definition of a (conservative, stochastically continuous) CBI process, which has been first introduced in [KW71]. To this effect, we define the functions $\phi : \mathbb{R}_+ \to \mathbb{R}$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(z) := bz + \frac{\sigma^2}{2}z^2 + \int_0^{+\infty}(e^{-zu} - 1 + zu)\pi(\mathrm{d}u), \tag{2.1}$$

$$\psi(z) := \beta z + \int_0^{+\infty}(1 - e^{-zu})\nu(\mathrm{d}u), \tag{2.2}$$

for all $z \geq 0$, where $(b, \sigma) \in \mathbb{R}^2$, $\beta \geq 0$ and π and ν are two sigma-finite measures on $(0, +\infty)$ such that $\int_0^{+\infty}(u \wedge u^2)\pi(\mathrm{d}u) < +\infty$ and $\int_0^{+\infty}(1 \wedge u)\nu(\mathrm{d}u) < +\infty$, respectively. For $p \geq 0$, we also define the function $v(\cdot,p,0):\mathbb{R}_+\to\mathbb{R}_+$ as the unique non-negative solution to the ODE

$$\frac{\partial}{\partial t}v(t,p,0)=-\phi\big(v(t,p,0)\big),\qquad v(0,p,0)=p.$$

Definition 2.1. A Markov process $X=(X_t)_{t\geq 0}$ with initial value $X_0=x$ and state space $[0,+\infty)$ is a *continuous-state branching process with immigration* (CBI process) with *branching mechanism* ϕ and *immigration rate* ψ, denoted as CBI(ϕ,ψ), if its transition semigroup $(P_t)_{t\geq 0}$ on $[0,+\infty)$ is defined by

$$\int_{[0,+\infty)}e^{-py}P_t(x,\mathrm{d}y)=\exp\left(-xv(t,p,0)-\int_0^t\psi\big(v(s,p,0)\big)\mathrm{d}s\right),\qquad \text{for all } t\geq 0.$$

CBI processes admit a representation as solutions to stochastic integral equations, which are especially useful for numerical simulation. To this effect, let $(\Omega,\mathcal{F},\mathbb{Q})$ be a probability space endowed with a right-continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$, with respect to which all processes introduced in the following are assumed to be adapted. Let $W(\mathrm{d}s,\mathrm{d}u)$ be a white noise on $(0,+\infty)^2$ with intensity $\mathrm{d}s\,\mathrm{d}u$ and $M(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ a Poisson time-space random measure on $(0,+\infty)^3$ with intensity $\mathrm{d}s\,\pi(\mathrm{d}z)\,\mathrm{d}u$. The associated compensated random measure is denoted by $\widetilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u):=M(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)-\mathrm{d}s\,\pi(\mathrm{d}z)\,\mathrm{d}u$. Let also $L=(L_t)_{t\geq 0}$ be an increasing Lévy process (subordinator) with $L_0=0$ and Laplace exponent ψ as given in (2.2). By the Lévy-Itô decomposition, there exists a Poisson random measure $N(\mathrm{d}s,\mathrm{d}z)$ on $(0,+\infty)^2$ with intensity $\mathrm{d}s\,\nu(\mathrm{d}z)$ such that $L_t=\beta t+\int_0^t\int_0^{+\infty}zN(\mathrm{d}s,\mathrm{d}z)$, for all $t\geq 0$. We assume that W, M and N are independent.

For $x\geq 0$, let us consider the following stochastic integral equation, referring to [Li11, Section 7.3] for a detailed account of time-space random measures and the corresponding stochastic integrals:

$$\begin{aligned}X_t=x&+\int_0^t(\beta-bX_{s-})\mathrm{d}s+\sigma\int_0^t\int_0^{X_{s-}}W(\mathrm{d}s,\mathrm{d}u)\\&+\int_0^t\int_0^{+\infty}\int_0^{X_{s-}}z\widetilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)+\int_0^t\int_0^{+\infty}zN(\mathrm{d}s,\mathrm{d}z),\qquad \text{for all } t\geq 0.\end{aligned}\tag{2.3}$$

The following result, which follows directly from [Li20, Theorems 8.3 and 8.5] and [DL12, Theorem 3.1], provides the connection between CBI processes and the stochastic integral equation (2.3).

Proposition 2.2. *A non-negative càdlàg process $X=(X_t)_{t\geq 0}$ with $X_0=x$ is a CBI(ϕ,ψ) process if and only if it is a weak solution to* (2.3). *Moreover, for every $x\geq 0$, equation* (2.3) *admits a unique strong solution $X=(X_t)_{t\geq 0}$ on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$ with $X_0=x$ taking values in $[0,+\infty)$.*

Remark 2.3. Along the lines of [Li11, Theorem 9.32], the process $B=(B_t)_{t\geq 0}$ constructed as

$$B_t:=\int_0^t\int_0^{X_{s-}}X_{s-}^{-1/2}\mathbf{1}_{\{X_{s-}>0\}}W(\mathrm{d}s,\mathrm{d}u)+\int_0^t\int_0^1\mathbf{1}_{\{X_{s-}=0\}}W(\mathrm{d}s,\mathrm{d}u),\qquad \text{for all } t\geq 0,$$

is a Brownian motion on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$. The stochastic integral equation (2.3) can be equivalently rewritten replacing the term $\int_0^t\int_0^{X_{s-}}W(\mathrm{d}s,\mathrm{d}u)$ with the usual stochastic integral $\int_0^t\sqrt{X_s}\mathrm{d}B_s$. This shows that CBI processes can be viewed as discontinuous extensions of the classical square-root process, widely adopted for interest rate modelling. However, the general representation (2.3) will turn out to be necessary when considering a *flow of CBI processes*, as in Section 4.1.

The stochastic integral equation (2.3) makes evident the self-exciting behavior of CBI processes. In particular, since the integral with respect to $\widetilde{M}$ depends on the value of the process itself, the jump frequency increases whenever a jump occurs, thereby generating jump clustering effects.

From the perspective of financial modelling, the analytical tractability of CBI processes is ensured by the fundamental link with affine processes (see [DFS03]). This is the content of the next theorem. As a preliminary, let us define the convex set

$$\mathcal{Y} := \left\{ y \in \mathbb{R} : \int_{[1,+\infty)} e^{-yz}(\pi+\nu)(\mathrm{d}z) < +\infty \right\} \supseteq \mathbb{R}_+. \tag{2.4}$$

Clearly, the functions ϕ and ψ given in (2.1)-(2.2) are well-defined on the extended domain $\mathcal{Y}$. For simplicity of presentation, we introduce the following mild technical assumption.

Assumption 2.4. For every $y \in \partial\mathcal{Y}$, it holds that $\phi'(y) > -\infty$.

It is well-known that ϕ is locally Lipschitz continuous on the interior $\mathcal{Y}^\circ$, but in general it may fail to be Lipschitz continuous at the boundary $\partial\mathcal{Y}$. Assumption 2.4 excludes this behavior and is always satisfied by tempered alpha-stable CBI processes, as considered in Section 2.2.

Theorem 2.5. *Let $X = (X_t)_{t\geq 0}$ be a CBI(ϕ, ψ) process with $X_0 = x$. Then X is a regular affine process. If Assumption 2.4 holds, then, for every $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$, the ODE*

$$\frac{\partial}{\partial t} v(t,p,q) = q - \phi\big(v(t,p,q)\big), \qquad v(0,p,q) = p, \tag{2.5}$$

admits a unique solution $v(\cdot,p,q) : [0, T^{(p,q)}) \to \mathcal{Y}$, where $T^{(p,q)} \in (0,+\infty]$, and it holds that

$$\mathbb{E}\left[\exp\left(-pX_t - q\int_0^t X_s\,\mathrm{d}s\right)\right] = \exp\left(-xv(t,p,q) - \int_0^t \psi\big(v(s,p,q)\big)\mathrm{d}s\right), \tag{2.6}$$

for all $t < T^{(p,q)}$, where ϕ and ψ are defined as in (2.1)-(2.2) *on the extended domain $\mathcal{Y}$.*

For $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$, the time $T^{(p,q)}$ appearing in Theorem 2.5 represents the maximum joint lifetime of $v(\cdot,p,q)$ and $\int_0^\cdot \psi(v(s,p,q))\mathrm{d}s$. The lifetime $T^{(p,q)}$ characterizes the finiteness of (discounted) exponential moments, a crucial technical requirement of our modelling framework introduced in Section 3. Indeed, [KRM15, Proposition 3.3] applied to the bi-dimensional affine process $(X, \int_0^\cdot X_s\,\mathrm{d}s)$ implies that

$$T^{(p,q)} = \sup\left\{ t \in \mathbb{R}_+ : \mathbb{E}\big[e^{-pX_t - q\int_0^t X_s\,\mathrm{d}s}\big] < +\infty \right\}. \tag{2.7}$$

In particular, we have that $\mathbb{E}[\exp(-pX_t - q\int_0^t X_s\,\mathrm{d}s)] < +\infty$, for all $t < T^{(p,q)}$. An explicit characterization of the lifetime $T^{(p,q)}$ is given in the next proposition, which refines the result of [KR11, Theorem 4.1] for the specific case of CBI processes.[2] As a preliminary, let us define

$$\ell := \inf\{y \in \mathbb{R} : \phi(y) < +\infty\} \qquad \text{and} \qquad \kappa := \inf\{y \in \mathbb{R} : \psi(y) > -\infty\}.$$

It can be easily verified that $\mathcal{Y} = [\ell\vee\kappa, +\infty)$ as long as $\phi(\ell\vee\kappa)\vee(-\psi(\ell\vee\kappa)) < +\infty$ (or, equivalently, $\int_{[1,+\infty)} e^{-(\ell\vee\kappa)z}(\pi+\nu)(\mathrm{d}z) < +\infty$, if $\ell\vee\kappa > -\infty$), while $\mathcal{Y} = (\ell\vee\kappa, +\infty)$ otherwise.

[2] More specifically, [KR11, Theorem 4.1] assumes the validity of additional hypotheses, which in particular only cover the case of CBI processes with a strictly subcritical branching mechanism ϕ (compare with Proposition 2.13).

Proposition 2.6. *Suppose that Assumption* (2.4) *holds and let $p \in \mathcal{Y}$. For $q \in \mathbb{R}_+$, define the quantity $p_q := \inf\{y \in \mathcal{Y} : q - \phi(y) \geq 0\}$. If $p \geq p_q$, then it holds that $T^{(p,q)} = +\infty$. Otherwise, if $p < p_q$, then*

$$T^{(p,q)} = \int_{\ell\vee\kappa}^{p} \frac{\mathrm{d}y}{\phi(y) - q}. \tag{2.8}$$

Suppose furthermore that $\psi(\ell\vee\kappa) > -\infty$. Then, $T^{(p,q)} = +\infty$ holds for all $(p,q) \in [\ell\vee\kappa, +\infty)\times\mathbb{R}_+$ if and only if $\phi(\ell\vee\kappa) \leq 0$.

Remark 2.7. To price non-linear products (see Section 4.2), an extension of the affine transform formula (2.6) to the complex domain is needed. To this effect, let $S(\mathcal{Y}^\circ) := \{p \in \mathbb{C} : \mathrm{Re}(p) \in \mathcal{Y}^\circ\}$, with $\mathrm{Re}(p)$ denoting the real part of p. For every $(p,q) \in S(\mathcal{Y}^\circ) \times \mathbb{R}_+$, Theorem 2.5 yields the existence of a unique solution $v(\cdot, \mathrm{Re}(p), q)$ to the ODE (2.5) with initial value $v(0, \mathrm{Re}(p), q) = \mathrm{Re}(p)$ up to a lifetime $T^{(\mathrm{Re}(p),q)}$. By [KRM15, Theorem 2.26], if $T \in \mathbb{R}_+$ is such that $T \leq T^{(\mathrm{Re}(p),q)}$ and $v(t, \mathrm{Re}(p), q) \in \mathcal{Y}^\circ$ for all $t \in [0,T]$, then the affine transform formula (2.6) holds for $p \in \mathbb{C}$ for all $t \in [0,T]$, replacing ϕ and ψ by their analytic extensions to the complex domain $S(\mathcal{Y}^\circ)$ (see [KRM15, Proposition 2.21]). In particular, as a consequence of Proposition 2.6, the affine transform formula (2.6) is always valid for all $p \in \mathbb{C}$ such that $\mathrm{Re}(p) \in [p_q, +\infty) \cap \mathcal{Y}^\circ$.

2.2. **Tempered alpha-stable CBI processes.** The specific properties of a CBI process are determined by the branching mechanism ϕ and the immigration rate ψ, notably by the measures π and ν appearing in (2.1)-(2.2). In the following definition, we introduce a tractable and flexible specification, which is particularly well-suited to the modelling of multiple curves.

Definition 2.8. A CBI(ϕ, ψ) process $X = (X_t)_{t\geq 0}$ with $X_0 = x$ is called a *tempered α-stable* CBI process if $\nu(\mathrm{d}u) = 0$ and

$$\pi(\mathrm{d}u) = C\frac{e^{-\theta u}}{u^{1+\alpha}}\mathbf{1}_{\{u>0\}}\mathrm{d}u, \tag{2.9}$$

where $\theta \geq 0$, $\alpha < 2$ (with $\alpha > 0$ if $\theta = 0$) and C is a suitable normalizing constant.

For a tempered α-stable CBI process, it holds that $\mathcal{Y} = [-\theta, +\infty)$. Indeed, since $\nu(\mathrm{d}u) = 0$, we have that $\int_1^{+\infty} e^{\theta u}\pi(\mathrm{d}u) = C/\alpha$, while $\int_1^{+\infty} e^{(\theta+\epsilon)u}\pi(\mathrm{d}u) = +\infty$ for every $\epsilon > 0$. The measure π given in (2.9) corresponds to the Lévy measure of a spectrally positive generalized tempered α-stable compensated Lévy process $Z = (Z_t)_{t\geq 0}$, whose characteristic function is given by

$$\mathbb{E}[e^{\mathrm{i}uZ_t}] = \exp\left(C\,\Gamma(-\alpha)\,\theta^\alpha\left(\left(1 - \frac{\mathrm{i}u}{\theta}\right)^\alpha - 1 + \alpha\frac{\mathrm{i}u}{\theta}\right)t\right),$$

for all $u \in \mathbb{R}$ and $t \geq 0$, as long as $\alpha \notin \{0, 1\}$, see [CT04, Proposition 4.2]. Depending on the choice of the parameter α, the process Z has different path properties:

- if $\alpha < 0$, then Z is a compensated compound Poisson process, since $\pi(\mathbb{R}_+) < +\infty$;
- if $\alpha \in [0, 1)$, then Z has infinite activity and finite variation, since $\int_0^1 u\,\pi(\mathrm{d}u) < +\infty$;
- if $\alpha \in [1, 2)$, then Z has infinite activity and infinite variation.

If $\theta > 0$, choosing $\alpha = 0$ yields the Lévy measure of a Gamma subordinator, while $\alpha = 1/2$ corresponds to an inverse Gaussian subordinator. For $\theta = 0$, the limit case $\alpha \to 2$ yields a Gaussian distribution, see [CT04, Section 3.7]. The tempering parameter θ determines the tail behavior of the jump measure π and will be important to ensure the finiteness of exponential moments (compare with Lemma 2.9 and Remark 2.12 below).

In Section 4, we shall focus on the case $\theta > 0$ and $\alpha \in (1, 2)$. In this case, the normalizing constant C can be chosen as follows, for some $\eta > 0$:

$$C(\alpha, \eta) := -\frac{\eta^\alpha}{\Gamma(-\alpha)\cos(\alpha\pi/2)}. \tag{2.10}$$

As will become clear in the sequel, in our model the constant η will play the role of a volatility parameter determining the impact of jumps in the dynamics of the model (see equation (4.1)).

Since $\nu(\mathrm{d}u) = 0$ (see Definition 2.8), the immigration rate ψ of a tempered α-stable CBI process reduces to $\psi(z) = \beta z$. The branching mechanism ϕ is described in the following lemma.

Lemma 2.9. *For $\theta \geq 0$, $\alpha \in (1, 2)$ and $C = C(\alpha, \eta)$, the branching mechanism ϕ of a tempered α-stable CBI process is a convex function on $[-\theta, +\infty)$ and is explicitly given by*

$$\phi(z) = bz + \frac{\sigma^2}{2}z^2 + \eta^\alpha \frac{\theta^\alpha + z\alpha\theta^{\alpha-1} - (z+\theta)^\alpha}{\cos(\pi\alpha/2)}, \qquad \text{for all } z \geq -\theta. \tag{2.11}$$

The branching mechanism ϕ is decreasing with respect to the tempering parameter θ. Moreover, Assumption 2.4 is satisfied.

The following proposition states two important properties of a tempered α-stable CBI process.

Proposition 2.10. *Let $X = (X_t)_{t\geq 0}$ be a tempered α-stable CBI process, with $X_0 = x > 0$, $\theta \geq 0$, $\alpha \in (1, 2)$ and $C = C(\alpha, \eta)$. Then the following hold:*

(i) $\mathbb{E}[e^{\gamma X_t}] < +\infty$ *for all $\gamma \leq \theta$ and $t \geq 0$ if and only if $\phi(-\theta) \leq 0$;*

(ii) 0 *is an inaccessible boundary for X if and only if $2\beta \geq \sigma^2$.*

Remark 2.11. In view of Remark 2.7, part (i) of Proposition 2.10 implies that, if $\phi(-\theta) \leq 0$, then for a tempered α-stable CBI process the affine transform formula (2.6) can be always extended to the complex domain for all $p \in \mathbb{C}$ such that $\mathrm{Re}(p) > -\theta$.

Remark 2.12 (The non-tempered case). In the case $\theta = 0$ and $\alpha \in (1, 2)$, an α-stable CBI process $X = (X_t)_{t\geq 0}$ with $X_0 = x$ can be represented as the solution to the following SDE:

$$X_t = x + \int_0^t (\beta - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \eta \int_0^t X_{s-}^{1/\alpha}\mathrm{d}Z_s, \qquad \text{for all } t \geq 0,$$

where $Z = (Z_t)_{t\geq 0}$ is a spectrally positive α-stable compensated Lévy process, see [Li11, Theorem 9.32]. This specification has been recently proposed as a model for the short interest rate in [JMS17]. Observe that for the limit case $\alpha = 2$ and $C = C(2, \eta)$, the process X reduces to a classical CIR process with volatility $\sqrt{\sigma^2 + 2\eta^2}$. A tempered α-stable CBI process can be constructed from a non-tempered α-stable CBI process by means of an equivalent change of measure (see [JMS17, Proposition 4.1]). It is important to note that, in the non-tempered case, for any $\alpha \in (1, 2)$, the process X does not admit finite exponential moments of any order, meaning that $\mathbb{E}[e^{\gamma X_t}] = +\infty$ for all $\gamma > 0$ and $t > 0$. The finiteness of exponential moments represents an indispensable requirement of the modelling framework introduced in the following section.

We conclude this section with the following result, which characterizes the ergodic distribution of a tempered α-stable CBI process. We recall that the branching mechanism ϕ is said to be *strictly subcritical* if $b > 0$ (see [Li11, Chapter 3]).

Proposition 2.13. *Let $X = (X_t)_{t\geq 0}$ be a tempered α-stable CBI process, with $X_0 = x$, $\theta \geq 0$, $\alpha \in (1,2)$ and $C = C(\alpha,\eta)$. If ϕ is strictly subcritical, then $(P_t(\cdot,x))_{t\geq 0}$ converges weakly to a stationary distribution ρ with Laplace transform*

$$\mathcal{L}_\rho(p) := \int_0^{+\infty} e^{-py}\rho(\mathrm{d}y) = \exp\left(-\beta\int_0^p \frac{z}{\phi(z)}\mathrm{d}z\right), \qquad \text{for } p > p_0, \tag{2.12}$$

where p_0 is defined as in Proposition 2.6 for $q = 0$. The first moment of ρ is given by

$$\int_0^{+\infty} y\rho(\mathrm{d}y) = \frac{\beta}{b}. \tag{2.13}$$

Moreover, the process X is exponentially ergodic, in the sense that

$$\|P_t(\cdot,x) - \rho(\cdot)\| \leq C\big(x + \beta/b\big)e^{-bt}, \qquad \text{for all } t \geq 1,$$

for a positive constant C and where $\|\cdot\|$ denotes the total variation norm.

3. General Modelling of Multiple Curves via CBI Processes

In this section, we develop a general modelling framework for multiple curve markets based on CBI processes. To this effect, we adapt the affine short rate multi-curve approach of [CFG19b], to which we refer for additional details on the general features of the post-crisis interest rate market. In this section, we focus on the construction and properties of the framework. A detailed analysis of a tractable specification will be proposed in Section 4.

3.1. **OIS rates, Ibor rates and multiplicative spreads.**

In fixed income markets, the reference rates for overnight transactions are the EONIA (Euro overnight index average) rate in the Eurozone and the Federal Funds rate in the US market. The Eonia and the Federal Funds rates are determined on the basis of overnight transactions and are the underlying of overnight indexed swaps (OIS) (see Appendix B). The short end of the swap rate of an OIS is referred to as *OIS rate*, here denoted as r_t. In market practice, the OIS rate is typically used as the collateral rate and proxies a risk-free rate. The term structure of OIS discount factors at date t is represented by the map $T \mapsto B(t,T)$, where $B(t,T)$ denotes the price at t of an OIS zero-coupon bond with maturity T. The simply compounded spot OIS rate for the period $[t,t+\delta]$ is defined as

$$L^{\mathrm{OIS}}(t,t,\delta) := \frac{1}{\delta}\left(\frac{1}{B(t,t+\delta)} - 1\right), \qquad \text{for } \delta \geq 0 \text{ and } t \geq 0. \tag{3.1}$$

Note that the right-hand side of (3.1) corresponds to the pre-crisis textbook definition of Ibor rate.

Ibor rates are the underlying rates of fixed-income derivatives and are determined by a panel of primary financial institutions for unsecured lending. We denote by $L(t,t,\delta)$ the (spot) Ibor rate for the time interval $[t,t+\delta]$ fixed at time t, where the tenor δ is typically one day (1D), one week (1W), or several months (1M, 2M, 3M, 6M, 12M). We consider Ibor rates for a generic set $\mathcal{D} := \{\delta_1,\ldots,\delta_m\}$ of tenors, with $0<\delta_1<\ldots<\delta_m$, for some $m\in\mathbb{N}$. In the post-crisis environment Ibor rates associated to different tenors exhibit a distinct behavior and are no longer determined by simple no-arbitrage relations. As mentioned in the introduction, this leads to non-negligible basis spreads and to the emergence of multiple curves.

Our main modelling quantities are represented by the *spot multiplicative spreads*

$$S^\delta(t,t) := \frac{1+\delta L(t,t,\delta)}{1+\delta L^{\mathrm{OIS}}(t,t,\delta)}, \qquad \text{for all } \delta\in\mathcal{D} \text{ and } t\geq 0, \tag{3.2}$$

together with the OIS short rate process $(r_t)_{t\geq 0}$. In the post-crisis environment, multiplicative spreads are usually greater than one and increasing with respect to the tenor. Neglecting liquidity and funding issues, this is due to the fact that Ibor rates embed the risk that the average credit quality of the panel deteriorates over the term of the loan, while OIS rates reflect the credit quality of a newly refreshed panel (see, e.g., [CDS01, FT13]). As will be shown in Section 3.3, a key feature of our approach is the facility of generating multiplicative spreads satisfying such requirements.

The idea of modelling multi-curve interest rate markets via multiplicative spreads is due to M. Henrard (see, e.g., [Hen14]) and has been recently pursued in [NS15, CFG16, CFG19b, EGG18]. In particular, one of the advantages of modelling multiplicative spreads is that they can be directly inferred from quoted Ibor and OIS rates. Multiplicative spreads admit a natural economic interpretation: indeed, $S^\delta(t,t)$ can be regarded as a market expectation (at date t) of the riskiness of the Ibor panel over the period $[t,t+\delta]$. As explained in [CFG16, Appendix B], this interpretation can be made rigorous via a foreign exchange analogy. Furthermore, in comparison to additive spreads (as considered for instance in [Mer13, MX12]), multiplicative spreads represent a particularly tractable modelling quantity in relation with CBI processes.

3.2. **The modelling framework.** In this section, we present a general modelling framework for the OIS short rate $(r_t)_{t\geq 0}$ and spot multiplicative spreads $\{(S^\delta(t,t))_{t\geq 0};\delta\in\mathcal{D}\}$ based on CBI processes. We adopt a martingale approach, in the spirit of the affine short rate multi-curve models introduced in [CFG19b, Section 3.3], and construct the model under a probability measure $\mathbb{Q}$ under which all traded assets are martingales when discounted by the OIS bank account $\exp(\int_0^\cdot r_s\mathrm{d}s)$.

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$ be a filtered probability space supporting a d-dimensional process $X=(X_t)_{t\geq 0}$ such that each component X^j is a CBI process with branching mechanism ϕ^j and immigration rate ψ^j, for $j=1,\ldots,d$. We assume that X^j and X^k are independent, for all $k\neq j$.

Besides the driving process X, we introduce the following modelling ingredients:

(i) a function $\ell:\mathbb{R}_+\to\mathbb{R}$ such that $\int_0^T|\ell(u)|\mathrm{d}u<+\infty$, for all $T>0$;
(ii) a vector $\lambda\in\mathbb{R}^d_+$;
(iii) a family of functions $\mathbf{c}=(c_1,\ldots,c_m)$, with $c_i:\mathbb{R}_+\to\mathbb{R}$ for all $i=1,\ldots,m$;
(iv) a family of vectors $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_m)$, with $\gamma_i\in\mathbb{R}^d$ for all $i=1,\ldots,m$.

Definition 3.1. The tuple $(X, \ell, \lambda, \mathbf{c}, \boldsymbol{\gamma})$ is said to generate a *CBI-driven multi-curve model* if

$$r_t = \ell(t) + \lambda^\top X_t, \tag{3.3}$$

$$\log S^{\delta_i}(t,t) = c_i(t) + \gamma_i^\top X_t. \tag{3.4}$$

for all $t \geq 0$ and $i = 1, \ldots, m$, and if the following conditions hold:

$$-\gamma_{i,j} \in \mathcal{Y}^j \qquad \text{and} \qquad T^{(-\gamma_{i,j},\lambda_j)} = +\infty, \qquad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, d, \tag{3.5}$$

where the set $\mathcal{Y}^j$ is defined as in (2.4) with respect to the CBI process X^j and $T^{(-\gamma_{i,j},\lambda_j)}$ denotes the lifetime as in Theorem 2.5 for the process X^j, with $p = -\gamma_{i,j}$ and $q = \lambda_j$.

Condition (3.5) serves to ensure that the model can be applied to arbitrarily large maturities, as will become clear from the proof of Proposition 3.2. The role of the time-dependent functions ℓ and $\mathbf{c}$ consists in allowing the model to perfectly fit the observed term structures, as shown in Proposition 3.5 below, in the spirit of the deterministic shift introduced in [BM01]. A multi-curve model constructed as in Definition 3.1 inherits the properties of the CBI process X, in particular its jump clustering behavior. We refer to Section 4 for a more specific discussion of the adequacy of this approach in reproducing the empirical features of spreads mentioned in the introduction. Let us also mention that models driven by a vector of independent CBI processes have been recently adopted for electricity spot prices in [JMSS19].

As shown in Appendix B, in the multi-curve setting the basic building blocks for the valuation of interest rate derivatives are represented by OIS zero-coupon bond prices and *forward multiplicative spreads* $S^\delta(t,T)$, defined as follows (see [CFG16] and compare with equation (3.2)):

$$S^\delta(t,T) := \frac{1 + \delta L(t,T,\delta)}{1 + \delta L^{\mathrm{OIS}}(t,T,\delta)}, \qquad \text{for } \delta \in \mathcal{D} \text{ and } 0 \leq t \leq T < +\infty, \tag{3.6}$$

where $L(t,T,\delta)$ denotes the *forward Ibor rate* and $L^{\mathrm{OIS}}(t,T,\delta)$ is the simply compounded OIS forward rate defined by $L^{\mathrm{OIS}}(t,T,\delta) := (B(t,T)/B(t,T+\delta)-1)/\delta$. The following proposition shows that in a CBI-driven multi-curve model OIS zero-coupon bond prices and forward multiplicative spreads can be computed in closed form. This represents a fundamental result in view of the practical applicability of the framework. For $j = 1, \ldots, d$ and $(p,q) \in \mathcal{Y}^j \times \mathbb{R}_+$, we denote by $v^j(t,p,q)$ the solution to the ODE (2.5) with branching mechanism ϕ^j.

Proposition 3.2. *Let $(X, \ell, \lambda, \mathbf{c}, \boldsymbol{\gamma})$ generate a CBI-driven multi-curve model. Then:*

(i) *for all $0 \leq t \leq T < +\infty$, the OIS zero-coupon bond price $B(t,T)$ is given by*

$$B(t,T) = \exp\Big(\mathcal{A}_0(t,T) + \mathcal{B}_0(T-t)^\top X_t\Big), \tag{3.7}$$

where the functions $\mathcal{A}_0(t,T)$ and $\mathcal{B}_0(T-t) = (\mathcal{B}_0^1(T-t), \ldots, \mathcal{B}_0^d(T-t))^\top$ are given by

$$\mathcal{A}_0(t,T) := -\int_0^{T-t}\Big(\ell(t+s) + \sum_{j=1}^d \psi^j\big(v^j(s,0,\lambda_j)\big)\Big)\mathrm{d}s,$$

$$\mathcal{B}_0^j(T-t) := -v^j(T-t,0,\lambda_j), \qquad \textit{for } j = 1, \ldots, d;$$

(ii) *for all* $0 \leq t \leq T < +\infty$ *and* $i = 1, \dots, m$, *the multiplicative spread* $S^{\delta_i}(t,T)$ *is given by*

$$S^{\delta_i}(t,T) = \exp\Big(\mathcal{A}_i(t,T) + \mathcal{B}_i(T-t)^\top X_t\Big), \tag{3.8}$$

where the functions $\mathcal{A}_i(t,T)$ *and* $\mathcal{B}_i(T-t) = (\mathcal{B}_i^1(T-t), \dots, \mathcal{B}_i^d(T-t))^\top$ *are given by*

$$\mathcal{A}_i(t,T) := c_i(T) + \sum_{j=1}^d \int_0^{T-t} \Big(\psi^j\big(v^j(s,0,\lambda_j)\big) - \psi^j\big(v^j(s,-\gamma_{i,j},\lambda_j)\big)\Big)\mathrm{d}s,$$

$$\mathcal{B}_i^j(T-t) := v^j(T-t,0,\lambda_j) - v^j(T-t,-\gamma_{i,j},\lambda_j), \qquad \textit{for } j = 1, \dots, d.$$

Proof. (i): Since the probability measure $\mathbb{Q}$ is a martingale measure with respect to the OIS bank account as numéraire, it follows that $B(t,T)$ can be computed as follows, for all $0 \leq t \leq T < +\infty$, using Theorem 2.5 and the independence of the processes $(X^1, \dots, X^d)$:

$$\begin{aligned} B(t,T) &= \mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s}\Big|\mathcal{F}_t\right] = e^{-\int_t^T \ell(s)\mathrm{d}s}\,\mathbb{E}\left[e^{-\sum_{j=1}^d \lambda_j \int_t^T X_s^j \mathrm{d}s}\Big|\mathcal{F}_t\right] \\ &= e^{-\int_t^T \ell(s)\mathrm{d}s} \prod_{j=1}^d e^{-X_t^j v^j(T-t,0,\lambda_j) - \int_0^{T-t} \psi^j(v^j(s,0,\lambda_j))\mathrm{d}s}. \end{aligned}$$

(ii): In view of [CFG16, Lemma 3.11], the fact that $\mathbb{Q}$ is a martingale measure implies that, for every $\delta \in \mathcal{D}$ and $T \in \mathbb{R}_+$, the forward multiplicative spread $(S^\delta(t,T))_{t\in[0,T]}$ introduced in (3.6) is a martingale under the T-forward probability measure $\mathbb{Q}^T$, defined by $\mathrm{d}\mathbb{Q}^T := e^{-\int_0^T r_s \mathrm{d}s} B(0,T)^{-1}\mathrm{d}\mathbb{Q}$. Therefore, denoting by $\mathbb{E}^T[\cdot]$ the expectation under the measure $\mathbb{Q}^T$, it holds that

$$\begin{aligned} S^{\delta_i}(t,T) &= \mathbb{E}^T[S^{\delta_i}(T,T)|\mathcal{F}_t] = \frac{1}{B(t,T)}\mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s} S^{\delta_i}(T,T)\Big|\mathcal{F}_t\right] \\ &= e^{-\mathcal{A}_0(t,T) - \mathcal{B}_0(T-t)^\top X_t - \int_t^T \ell(s)\mathrm{d}s + c_i(T)}\,\mathbb{E}\left[e^{-\sum_{j=1}^d \left(\lambda_j \int_t^T X_s^j \mathrm{d}s - \gamma_{i,j} X_T^j\right)}\Big|\mathcal{F}_t\right] \\ &= e^{c_i(T)} \prod_{j=1}^d e^{\left(v^j(T-t,0,\lambda_j) - v^j(T-t,-\gamma_{i,j},\lambda_j)\right)X_t^j + \int_0^{T-t}\left(\psi^j(v^j(s,0,\lambda_j)) - \psi^j(v^j(s,-\gamma_{i,j},\lambda_j))\right)\mathrm{d}s}, \end{aligned}$$

for $0 \leq t \leq T < +\infty$ and $i = 1, \dots, m$, where the last equality follows again from the independence of the processes $(X^1, \dots, X^d)$ together with Theorem 2.5. Note that, in view of (2.7), the finiteness of the conditional expectations, for all $T \in \mathbb{R}_+$, is ensured by condition (3.5). □

Linear fixed-income products, such as forward rate agreements, interest rate swaps, basis swaps, can be priced in closed form by relying on the explicit expressions for OIS zero-coupon bond prices and forward multiplicative spreads given in Proposition 3.2 together with the valuation formulae stated in Appendix B. Non-linear interest rate products such as caps, floors and swaptions can be efficiently priced via Fourier techniques, as illustrated in Sections 4.2-4.3 in the case of caps.

Remark 3.3 (Convexity adjustments). A further advantage of CBI-driven multi-curve models consists in the possibility of computing in closed form *convexity adjustments*. We recall that the convexity adjustment $C(t,T,\delta)$ is defined as the difference at time t between future and forward Ibor rates for the same reference period $[T, T+\delta]$ (see [GM10, Mer18]). More specifically,

$$C(t,T,\delta_i) := \mathbb{E}[L(T,T,\delta_i)|\mathcal{F}_t] - L(t,T,\delta_i),$$

for $i \in \{1,\dots,m\}$ and $0 \le t \le T < +\infty$. The forward Ibor rate $L(t,T,\delta_i)$ can be directly obtained from (3.6) together with Proposition 3.2. Furthermore, by applying the affine transform formula (2.6) again with Proposition 3.2, the future Ibor rate can be explicitly computed as

$$\mathbb{E}[L(T,T,\delta_i)|\mathcal{F}_t] = \frac{1}{\delta_i}\left(e^{c_i(T)-\mathcal{A}_0(T,T+\delta_i)-\sum_{j=1}^d\int_0^{T-t}v(s,\mathcal{B}_0^j(\delta_i)-\gamma_{i,j},0)\mathrm{d}s-\sum_{j=1}^d v(T-t,\mathcal{B}_0^j(\delta_i)-\gamma_{i,j},0)X_t^j}-1\right).$$

3.3. **General properties.** In typical market scenarios, multiplicative spreads are greater than one and increasing with respect to the tenor. As shown in the following proposition, these features can be easily obtained within the proposed framework.

Proposition 3.4. *Let $(X,\ell,\lambda,\mathbf{c},\boldsymbol{\gamma})$ generate a CBI-driven multi-curve model. Then, for every $i=1,\dots,m$, the following hold:*

(i) *if $\gamma_i \in \mathbb{R}_+^d$ and $c_i(t) \ge 0$, for all $t \ge 0$, then $S^{\delta_i}(t,T) \ge 1$ a.s. for all $0 \le t \le T < +\infty$;*
(ii) *if $\gamma_{i+1}-\gamma_i \in \mathbb{R}_+^d$ and $c_i(t) \le c_{i+1}(t)$, for all $t \ge 0$, then $S^{\delta_i}(t,T) \le S^{\delta_{i+1}}(t,T)$ a.s. for all $0 \le t \le T < +\infty$.*

Proof. Arguing similarly as in [Li11, Proposition 3.1], it can be shown that, for every $j=1,\dots,d$, $q \in \mathbb{R}_+$ and $t \ge 0$, the function $\mathcal{Y}^j \ni p \mapsto v^j(t,p,q)$ is strictly increasing. Moreover, by (2.2), each immigration rate ψ^j is an increasing function. By relying on these facts and since X takes values in $\mathbb{R}_+^d$, the result follows as a direct consequence of part (ii) of Proposition 3.2. □

An additional feature of our framework consists in the possibility of fitting the term structures implied by market data. We parametrize by $B^{\mathrm{mkt}}(0,T)$ and $S^{\mathrm{mkt},\delta}(0,T)$, for $\delta \in \mathcal{D}$ and $T \in \mathbb{R}_+$, the term structures of OIS rates and Ibor rates observed on the market at $t=0$. We say that the multi-curve model generated by $(X,\ell,\lambda,\mathbf{c},\boldsymbol{\gamma})$ achieves a *perfect fit to the initial term structures* if

$$B(0,T) = B^{\mathrm{mkt}}(0,T) \qquad \text{and} \qquad S^{\delta_i}(0,T) = S^{\mathrm{mkt},\delta_i}(0,T), \qquad \text{for all } i=1,\dots,m \text{ and } T \in \mathbb{R}_+.$$

The following result shows that there exists a unique specification of ℓ and $\mathbf{c}$ achieving a perfect fit. We recall that $f_t^l(T) := -\partial_T \log(B^l(t,T))$, for $l \in \{0,\mathrm{mkt}\}$, denotes the instantaneous OIS forward rate, and denote by the superscript 0 quantities computed using the model $(X,0,\lambda,0,\boldsymbol{\gamma})$ via formulae (3.7)-(3.8).

Proposition 3.5. *Let $(X,\ell,\lambda,\mathbf{c},\boldsymbol{\gamma})$ generate a CBI-driven multi-curve model. The model achieves a perfect fit to the initial term structures if and only if*

$$\begin{aligned} &\ell(t) = f_0^{\mathrm{mkt}}(t) - f_0^0(t), && \text{for all } t \ge 0,\\ &c_i(t) = \log S^{\mathrm{mkt},\delta_i}(0,t) - \log S^{0,\delta_i}(0,t), && \text{for all } i=1,\dots,m \text{ and } t \ge 0. \end{aligned}$$

Proof. Similarly as in [CFG19b, Proposition 3.8], the claim easily follows from the observation that

$$\begin{aligned} B(0,T) &= \mathbb{E}\big[e^{-\int_0^T r_s \mathrm{d}s}\big] = e^{-\int_0^T \ell(s)\mathrm{d}s}B^0(0,T),\\ S^{\delta_i}(0,T) &= \mathbb{E}^T[S^{\delta_i}(T,T)] = e^{c_i(T)}\mathbb{E}^T[S^{0,\delta_i}(T,T)] = e^{c_i(T)}S^{0,\delta_i}(0,T), \end{aligned}$$

using the $\mathbb{Q}^T$-martingale property of $(S^{\delta_i}(t,T))_{t\in[0,T]}$ and $(S^{0,\delta_i}(t,T))_{t\in[0,T]}$ together with the fact that the two models $(X,0,\lambda,0,\boldsymbol{\gamma})$ and $(X,\ell,\lambda,\mathbf{c},\boldsymbol{\gamma})$ yield the same T-forward measure $\mathbb{Q}^T$. □

Remark 3.6 (On the possibility of negative rates). In recent years, negative short rates have been observed to coexist with spreads which are greater than one. Since the function ℓ in (3.3) is allowed to take negative values, our framework does not exclude this possibility. Moreover, a slight extension of Definition 3.1 permits to generate OIS short rates which are not bounded from below by the deterministic function ℓ. Indeed, it suffices to replace X with a $(d+1)$-dimensional process $X' = (X, Y)$ such that X' is an affine process and $\mathbb{Q}(Y_t < 0) > 0$, for all $t \geq 0$. Specification (3.3) is then replaced by $r_t = \ell(t) + \lambda^\top X_t + Y_t$, while multiplicative spreads are given as in (3.4). Note that Y is not restricted to be independent of X. A simple extension of this type has been tested in our calibration to market data (see Section 5.2.3 below).

4. A Tractable Specification via a Flow of CBI Processes

In this section, we introduce a multi-curve model driven by a *flow of tempered α-stable CBI processes* (see Section 2.2). The proposed specification is parsimonious, captures the most relevant features of post-crisis interest rate markets and, at the same time, allows for efficient pricing of non-linear interest rate products via Fourier and quantization techniques. The empirical performance of this specification will be studied in Section 5 by calibration to market data.

4.1. Model specification.

As in Section 2.1, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space supporting a white noise $W(\mathrm{d}s, \mathrm{d}u)$ on $(0, +\infty)^2$ with intensity $\mathrm{d}s\,\mathrm{d}u$ and a Poisson time-space random measure $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $(0, +\infty)^3$ with intensity $\mathrm{d}s\,\pi(\mathrm{d}z)\,\mathrm{d}u$. As in Section 3, we consider a set of tenors $\mathcal{D} = \{\delta_1, \dots, \delta_m\}$. For each $i = 1, \dots, m$, let the process $Y^i = (Y^i_t)_{t \geq 0}$ be the unique strong solution to the following stochastic integral equation (see Proposition 2.2):

$$Y^i_t = y^i_0 + \int_0^t (\beta(i) - bY^i_{s-})\mathrm{d}s + \sigma \int_0^t \int_0^{Y^i_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \eta \int_0^t \int_0^{+\infty} \int_0^{Y^i_{s-}} z\widetilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \tag{4.1}$$

for all $t \geq 0$, with initial condition $y^i_0 \in \mathbb{R}_+$, for all $i = 1, \dots, m$, and where

- $\beta : \{1, \dots, m\} \to \mathbb{R}_+$, with $\beta(i) \leq \beta(i+1)$, for all $i = 1, \dots, m-1$;
- $(b, \sigma) \in \mathbb{R}^2$ and $\eta \geq 0$;
- $\pi(\mathrm{d}z)$ is as in (2.9), with $\theta > \eta$, $\alpha \in (1, 2)$ and $C = C(\alpha, 1)$ as in (2.10).

The family of processes $\{Y^i; i = 1, \dots, m\}$ is a simple instance of a *flow of CBI processes*, see [DL12, Section 3]. All components of the flow have a common branching mechanism ϕ, explicitly given in Lemma 2.9 (with the parameter θ in (2.11) being replaced by θ/η, due to the appearance of η in front of the last integral in (4.1)), while the immigration rate of Y^i is equal to $\psi^i(z) = \beta(i)z$, for each $i = 1, \dots, m$. In this section, we assume the validity of the following condition:

$$b \geq \frac{\sigma^2}{2}\frac{\theta}{\eta} + \eta\frac{(1-\alpha)\theta^{\alpha-1}}{\cos(\pi\alpha/2)}. \tag{4.2}$$

Condition (4.2) is equivalent to $\phi(-\theta/\eta) \leq 0$. In view of part (i) of Proposition 2.10, since $\theta > \eta$, this condition suffices to ensure that $\mathbb{E}[e^{Y^i_t}] < +\infty$ for all $i = 1, \dots, m$ and $t \geq 0$.

Defining the factor process $Y = (Y_t)_{t\geq 0}$ by $Y_t := (Y_t^1, \dots, Y_t^m)^\top$, for $t \geq 0$, we specify the OIS short rate and spot multiplicative spreads as follows, for all $t \geq 0$ and $i = 1, \dots, m$:

$$r_t = \ell(t) + \mu^\top Y_t, \tag{4.3}$$

$$\log S^{\delta_i}(t,t) = c_i(t) + Y_t^i, \tag{4.4}$$

where $\ell : \mathbb{R}_+ \to \mathbb{R}$, with $\int_0^T |\ell(u)|\mathrm{d}u < +\infty$ for all $T > 0$, $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mu \in \mathbb{R}_+^m$.

Under the specification (4.4), multiplicative spreads are by construction greater than one. Moreover, thanks to the properties of a flow of CBI processes, monotonicity of multiplicative spreads can be easily achieved, provided that initially observed spreads are increasing in the tenor.

Proposition 4.1. *Suppose that $y_0^i \leq y_0^{i+1}$ and $c_i(t) \leq c_{i+1}(t)$, for all $i = 1, \dots, m-1$ and $t \geq 0$. Then it holds that $S^{\delta_i}(t,T) \leq S^{\delta_{i+1}}(t,T)$ a.s., for all $i = 1, \dots, m-1$ and $0 \leq t \leq T < +\infty$.*

Proof. Since $\beta : \{1, \dots, m\} \to \mathbb{R}_+$ is increasing, [DL12, Theorem 3.2] implies that, if $y_0^i \leq y_0^{i+1}$, then $\mathbb{Q}(Y_t^i \leq Y_t^{i+1}$, for all $t \geq 0) = 1$. Therefore, if in addition $c_i(t) \leq c_{t+1}(t)$ for all $t \geq 0$, it follows that $S^{\delta_i}(t,t) \leq S^{\delta_{i+1}}(t,t)$ a.s. for all $t \geq 0$. The claim follows by the fact that $(S^{\delta_i}(t,T))_{t\in[0,T]}$ is a martingale under the T-forward probability measure $\mathbb{Q}^T$ (see the proof of Proposition 3.2). □

The factors $Y^1, \dots, Y^m$ possess the characteristic self-exciting behavior of CBI processes. This translates directly into a self-exciting property of spreads: for each $i = 1, \dots, m$, a large value of $S^{\delta_i}(t,t)$ increases the likelihood of upward jumps of the spread itself. Under the conditions of Proposition 4.1, there is a further self-exciting effect *among different spreads*: a large value of $S^{\delta_i}(t,t)$ increases the likelihood of common upward jumps of all other spreads with tenor δ_j, for $j > i$, reflecting the higher risk implicit in Ibor rates with longer tenors. As mentioned in the introduction (see in particular Figure 1), these features represent empirically relevant properties of post-crisis interest rate markets. Figure 2 provides an illustration of a sample trajectory for a model with $\mathcal{D} = \{3\mathrm{M}, 6\mathrm{M}\}$.

In the model (4.3)-(4.4), each factor Y^i drives the multiplicative spread with corresponding tenor δ_i, while all the factors can affect the OIS short rate. This generates a rich dependence between OIS rates and multiplicative spreads, as well as among the spreads themselves, in line with the dynamics observed on market data. In particular, under the conditions of Proposition 4.1, the quadratic covariation of spreads associated to tenors δ_i and δ_j, with $i < j$, is given by

$$\left[\log S^{\delta_i}(t,t), \log S^{\delta_j}(t,t)\right] = \sigma^2 \int_0^t Y_s^i \mathrm{d}s + \eta^2 \int_0^t \int_0^{+\infty} \int_0^{Y_{s-}^i} z^2 M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \qquad \text{for } t \geq 0.$$

The components of the flow of CBI processes $\{Y^i; 1, \dots, m\}$ are highly dependent. Therefore, the present specification apparently does not belong to the class of CBI-driven multi-curve models as introduced in Definition 3.1. However, an easy transformation allows to embed model (4.3)-(4.4) into the framework of Section 3.2. To this effect, we define the d-dimensional process $X = (X_t)_{t\geq 0}$ by

$$X_t^i := Y_t^i - Y_t^{i-1}, \qquad \text{for all } t \geq 0 \text{ and } i = 1, \dots, m, \tag{4.5}$$

with $Y^0 \equiv 0$, and $X_t := (X_t^1, \dots, X_t^m)^\top$. We are now in a position to state the following result.

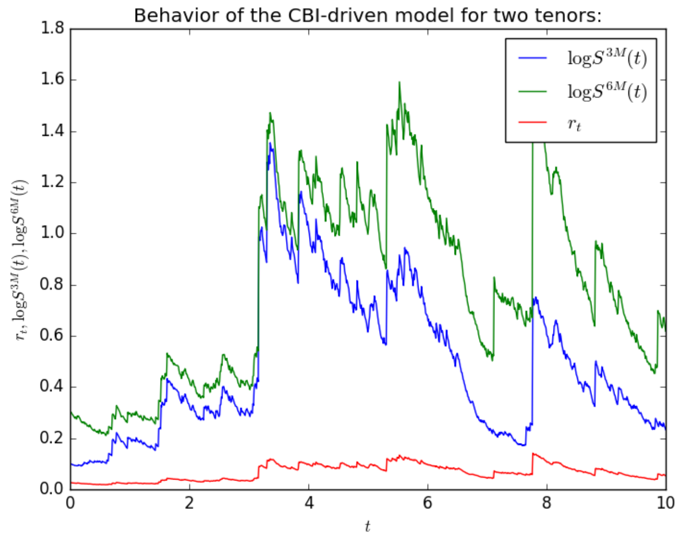

FIGURE 2. One sample paths of the short rate (red line) and multiplicative spreads for two tenors (3M in blue and 6M in green).

Proposition 4.2. *Suppose that $y_0^i \leq y_0^{i+1}$, for all $i = 1, \ldots, m-1$. Let us define $\lambda \in \mathbb{R}_+^m$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}_+^{m \times m}$ by*

$$\lambda_j := \sum_{k=j}^{m} \mu_k \qquad \text{and} \qquad \gamma_{i,j} := \mathbf{1}_{\{j \leq i\}}, \qquad \text{for all } i, j = 1, \ldots, m. \tag{4.6}$$

Then the tuple $(X, \ell, \lambda, \mathbf{c}, \boldsymbol{\gamma})$ generates a CBI-driven multi-curve model such that

(i) *for each $i = 1, \ldots, m$, the process $X^i = (X_t^i)_{t \geq 0}$ is a tempered α-stable CBI process with branching mechanism ϕ and immigration rate $\psi^i(z) = (\beta(i) - \beta(i-1))z$, with $\beta(0) := 0$;*
(ii) *the processes $(X^1, \ldots, X^m)$ are mutually independent;*
(iii) *the OIS short rate and multiplicative spreads are given by* (4.3)-(4.4).

Proof. Parts (i) and (ii) are a direct consequence of [DL12, Theorems 3.2 and 3.3]. To prove part (iii), it suffices to observe that, due to the definition of λ and $\boldsymbol{\gamma}$ in (4.6), it holds that

$$\mu^\top Y_t = \lambda^\top X_t \qquad \text{and} \qquad Y_t^i = \gamma_i^\top X_t, \qquad \text{for all } i = 1, \ldots, m \text{ and } t \geq 0.$$

Note that condition (3.5) is implied by condition (4.2), since $\mathcal{Y}^i = [-\theta/\eta, +\infty)$ (see Lemma 2.9), for all $i = 1, \ldots, m$, where $\theta > \eta$, and in view of part (i) of Proposition 2.10. $\square$

In view of the above proposition, the model (4.3)-(4.4) driven by the CBI flow $\{Y^i; i = 1, \ldots, m\}$ can be equivalently represented by a family of risk factors $(X^1, \ldots, X^m)$, where each factor X^i is affecting all spreads with tenor $\delta_j \geq \delta_i$ (and, possibly, the OIS short rate). We can explicitly compute OIS zero-coupon bond prices and forward multiplicative spreads by relying on Proposition

3.2. More specifically, it holds that

$$
\begin{aligned}
\mathcal{A}_0(t,T) &= -\int_0^{T-t}\ell(t+s)\mathrm{d}s - \sum_{j=1}^m\big(\beta(j)-\beta(j-1)\big)\int_0^{T-t} v\Big(s,0,\sum_{k=j}^m\mu_k\Big)\mathrm{d}s,\\
\mathcal{B}_0^j(T-t) &= -v\Big(T-t,0,\sum_{k=j}^m\mu_k\Big),\\
\mathcal{A}_i(t,T) &= c_i(T) + \sum_{j=1}^i\big(\beta(j)-\beta(j-1)\big)\int_0^{T-t}\bigg(v\Big(s,0,\sum_{k=j}^m\mu_k\Big) - v\Big(s,-1,\sum_{k=j}^m\mu_k\Big)\bigg)\mathrm{d}s,\\
\mathcal{B}_i^j(T-t) &= \bigg(v\Big(T-t,0,\sum_{k=j}^m\mu_k\Big) - v\Big(T-t,-1,\sum_{k=j}^m\mu_k\Big)\bigg)\mathbf{1}_{\{j\leq i\}}.
\end{aligned}
\tag{4.7}
$$

Observe that, unlike in the general framework of Section 3.2, the function v appearing in the above formulae is the same for all $i,j=1,\ldots,m$, due to the fact that the components of a flow of CBI processes share a common branching mechanism. We recall that the function v is given by the unique solution to the ODE (2.5) with ϕ given as in (2.11), with the parameter θ replaced by θ/η.

4.2. **Valuation of caplets via Fourier methods.** In this section, we provide a semi-closed formula for the price of a caplet. Let us consider a caplet written on the Ibor rate with tenor δ_i, strike $K>0$, maturity $T>0$ and settled in arrears at time $T+\delta_i$. For simplicity of presentation, we consider a unitary notional amount. By formula (B.1), the arbitrage-free price of such a caplet can be expressed as

$$
\Pi^{\mathrm{CPLT}}(t;T,\delta_i,K,1) = B(t,T+\delta_i)\mathbb{E}^{T+\delta_i}\Big[\big(e^{\mathcal{X}_T^i}-(1+\delta_iK)\big)^+\Big|\mathcal{F}_t\Big], \qquad \text{for } t\leq T, \tag{4.8}
$$

where $\mathbb{E}^{T+\delta_i}[\cdot]$ denotes the expectation under the $(T+\delta_i)$-forward measure $\mathbb{Q}^{T+\delta_i}$ and the process $\mathcal{X}^i=(\mathcal{X}_t^i)_{t\geq0}$ is defined by $\mathcal{X}_t^i := \log(S^{\delta_i}(t,t)/B(t,t+\delta_i))$, for all $t\geq0$. As a consequence of (4.4), Proposition 4.2 and Proposition 3.2, this process admits the explicit representation

$$
\mathcal{X}_t^i = c_i(t) - \mathcal{A}_0(t,t+\delta_i) + \big(\gamma_i - \mathcal{B}_0(\delta_i)\big)^\top X_t, \qquad \text{for all } t\geq0,
$$

where the d-dimensional process $(X_t)_{t\geq0}$ is defined in (4.5) and the functions $\mathcal{A}_0$ and $\mathcal{B}_0$ in (4.7). For $T\geq0$ and $i=1,\ldots,m$, let us introduce the set

$$
\Theta_i(T) := \Big\{u\in\mathbb{R} : \mathbb{E}^{T+\delta_i}\big[e^{u\mathcal{X}_T^i}\big]<+\infty\Big\}^\circ
$$

and the strip $\Lambda_i(T) := \{\zeta\in\mathbb{C} : -\mathrm{Im}(\zeta)\in\Theta_i(T)\}$. Using condition (4.2) and the facts that $-\mathcal{B}_0^j(\delta_i) = v(\delta_i,0,\lambda_j)\geq0$, for all $i,j=1,\ldots,m$, together with the increasingness of the map $\mathbb{R}_+\ni q\mapsto v(\delta_i,0,q)$, it can be checked that the condition

$$
u < \frac{\theta/\eta + v(\delta_i,0,\lambda_1)}{1+v(\delta_i,0,\lambda_1)} \tag{4.9}
$$

is sufficient to ensure that $u\in\Theta_i(T)$, for all $T>0$. In particular, it always holds that $(-\infty,+1]\subseteq\Theta_i(T)$. For $\zeta\in\Lambda_i(T)$, the *modified characteristic function* of $\mathcal{X}_T^i$ can be defined and explicitly computed as follows:

$$
\begin{aligned}
\Phi^i_{t,T}(\zeta) &:= B(t,T+\delta_i)\,\mathbb{E}^{T+\delta_i}\big[e^{\mathrm{i}\zeta\mathcal{X}^i_T}|\mathcal{F}_t\big] = \mathbb{E}\left[e^{-\int_t^T r_s\mathrm{d}s}B(T,T+\delta_i)e^{\mathrm{i}\zeta\mathcal{X}^i_T}\middle|\mathcal{F}_t\right] \\
&= \exp\Bigg(-\int_0^{T-t}\ell(t+s)\mathrm{d}s + (1-\mathrm{i}\zeta)\mathcal{A}_0(T,T+\delta_i) + \mathrm{i}\zeta c_i(T) \\
&\qquad -\sum_{j=1}^m\big(\beta(j)-\beta(j-1)\big)\int_0^{T-t} v\big(s,(\mathrm{i}\zeta-1)\mathcal{B}^j_0(\delta_i)-\mathrm{i}\zeta\gamma_{i,j},\lambda_j\big)\mathrm{d}s \\
&\qquad -\sum_{j=1}^m v\big(T-t,(\mathrm{i}\zeta-1)\mathcal{B}^j_0(\delta_i)-\mathrm{i}\zeta\gamma_{i,j},\lambda_j\big)X^j_t\Bigg).
\end{aligned}
\tag{4.10}
$$

We remark that, under condition (4.2), the above application of the affine transform formula (2.6) in the complex domain is justified by Remark 2.11. More specifically, $\zeta \in \Lambda_i(T)$ ensures that $\mathrm{Re}((\mathrm{i}\zeta-1)\mathcal{B}^j_0(\delta_i)-\mathrm{i}\zeta\gamma_{i,j}) > -\theta/\eta$. Therefore, by condition (4.2) together with Remark 2.11, it follows that $v(t,\mathrm{Re}((\mathrm{i}\zeta-1)\mathcal{B}^j_0(\delta_i)-\mathrm{i}\zeta\gamma_{i,j}),\lambda_j) > -\theta/\eta$, for all $t\geq 0$, meaning that the solution to the ODE (2.5) with $p=\mathrm{Re}((\mathrm{i}\zeta-1)\mathcal{B}^j_0(\delta_i)-\mathrm{i}\zeta\gamma_{i,j})$ stays in $\mathcal{Y}^\circ$ (compare with Remark 2.7).

We are now in a position to state the caplet valuation formula, which is a direct consequence of [Lee04, Theorem 5.1]. Note that, according to the notation of [Lee04], in our case we have that $G=G_1$ and $b_0=b_1=1\in\Theta_i(T)$ (due to the fact that $\theta>\eta$ and condition (4.2) holds).

Proposition 4.3. *Let $\bar{K}_i := 1+\delta_i K$ and $\epsilon\in\mathbb{R}$ such that $1+\epsilon\in\Theta_i(T)$. The arbitrage-free price at time $t\leq T$ of a caplet written on the Ibor rate with tenor δ_i, strike $K>0$, maturity $T>0$ and settled in arrears at time $T+\delta_i$, is given by*

$$
\Pi^{\mathrm{CPLT}}(t;T,\delta_i,K,1) = R^i_{t,T}\left(\bar{K}_i,\epsilon\right) + \frac{1}{\pi}\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\mathrm{Re}\left(e^{-\mathrm{i}\zeta\log(\bar{K}_i)}\frac{\Phi^i_{t,T}(\zeta-\mathrm{i})}{-\zeta(\zeta-\mathrm{i})}\right)\mathrm{d}\zeta,
$$

where $\Phi^i_{t,T}$ is given in (4.10) and $R^i_{t,T}(\bar{K}_i,\epsilon)$ is given by

$$
R^i_{t,T}\left(\bar{K}_i,\epsilon\right) = \begin{cases}
\Phi^i_{t,T}(-\mathrm{i}) - \bar{K}_i\Phi^i_{t,T}(0), & \text{if } \epsilon<-1,\\
\Phi^i_{t,T}(-\mathrm{i}) - \frac{\bar{K}_i}{2}\Phi^i_{t,T}(0), & \text{if } \epsilon=-1,\\
\Phi^i_{t,T}(-\mathrm{i}), & \text{if } -1<\epsilon<0,\\
\frac{1}{2}\Phi^i_{t,T}(-\mathrm{i}), & \text{if } \epsilon=0,\\
0, & \text{if } \epsilon>0.
\end{cases}
$$

4.3. **Valuation of caplets via quantization.** The analytical tractability of CBI processes allows for the development of a quantization-based pricing methodology, which is here proposed for the first time in an interest rate model. In this section, we show that the Fourier-based quantization technique recently introduced in [CFG19a] can be easily applied for the pricing of caplets.

The key ingredient of this approach is represented by the *quantization grid* $\Gamma^N=\{x_1,\dots,x_N\}$, with $x_1<\ldots<x_N$, for some chosen $N\in\mathbb{N}$ (see [GL00, Pag15] for details). Once the quantization grid Γ^N has been determined, the random variable $e^{\mathcal{X}^i_T}$ appearing in the caplet valuation formula

(4.8) is approximated by its *Voronoi* Γ^N*-quantization*, i.e., the nearest neighbour projection $\widehat{e^{\mathcal{X}_T^i}}$ of $e^{\mathcal{X}_T^i}$ onto Γ^N, given by the discrete random variable

$$\widehat{e^{\mathcal{X}_T^i}} = \sum_{j=1}^N x_j \mathbf{1}_{\{x_j^- \leq e^{\mathcal{X}_T^i} \leq x_j^+\}},$$

where $x_j^- = (x_{j-1} + x_j)/2$ and $x_j^+ = (x_{j+1} + x_j)/2$, for $j = 1, \ldots, N$, with $x_1^- = 0$ and $x_N^+ = +\infty$. The caplet valuation formula (4.8) can then be approximated as follows (considering $t = 0$ for simplicity of presentation):

$$\Pi^{\mathrm{CPLT}}(0; T, \delta_i, K, 1) \approx B(0, T+\delta_i) \sum_{j=1}^N \big(x_j - (1 + K\delta_i)\big)^+ \mathbb{Q}^{T+\delta_i}\big(\widehat{e^{\mathcal{X}_T^i}} = x_j\big),$$

where the *companion weights* $\mathbb{Q}^{T+\delta_i}(\widehat{e^{\mathcal{X}_T^i}} = x_j)$, for $j = 1, \ldots, N$, can be computed by

$$\mathbb{Q}^{T+\delta_i}\big(\widehat{e^{\mathcal{X}_T^i}} = x_j\big) = \mathbb{Q}^{T+\delta_i}\big(e^{\mathcal{X}_T^i} \leq x_j^+\big) - \mathbb{Q}^{T+\delta_i}\big(e^{\mathcal{X}_T^i} \leq x_j^-\big). \tag{4.11}$$

The core of quantization consists in optimally determining the quantization grid Γ^N in such a way that the discrete distribution of $\widehat{e^{\mathcal{X}_T^i}}$ over Γ^N is a good approximation of the continuous distribution of $e^{\mathcal{X}_T^i}$. This is achieved by choosing a grid Γ that minimizes the following L^p-distance:

$$D_p(\Gamma) = D_p(\{x_1, \ldots, x_N\}) := \big\| e^{\mathcal{X}_T^i} - \widehat{e^{\mathcal{X}_T^i}} \big\|_{L^p(\mathbb{Q}^{T+\delta_i})} = \mathbb{E}^{T+\delta_i}\left[\min_{j=1,\ldots,N} \big| e^{\mathcal{X}_T^i} - x_j \big|^p\right]^{1/p}. \tag{4.12}$$

In the present one-dimensional setting, it can be shown that this minimization problem admits a unique solution of full size N (see [Pag15, Proposition 1.1]). In practice, Γ^N is typically determined by searching the critical points of the map $\Gamma \mapsto D_p(\Gamma)$ (*sub-optimal* quantization grid). In view of [CFG19a, Theorem 2.5], a sub-optimal quantization grid Γ^N is given by the solution to the following equation:

$$\int_0^{+\infty} \mathrm{Re}\left[\Psi_T^i(u) e^{-\mathrm{i}u\log(x_j)} \left(\bar{\beta}\left(\frac{x_j^-}{x_j}, -\mathrm{i}u, p\right) - \bar{\beta}\left(\frac{x_j}{x_j^+}, 1 - p + \mathrm{i}u, p\right)\right)\right] \mathrm{d}u = 0, \tag{4.13}$$

where $\bar{\beta}$ is defined as

$$\bar{\beta}(x, a, b) = \int_x^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t, \qquad \text{for } a \in \mathbb{C},\ \mathrm{Re}(b) > 0 \text{ and } x \in (0,1),$$

and Ψ_T^i stands for the $(T + \delta_i)$-forward characteristic function of $\mathcal{X}_T^i$:

$$\Psi_T^i(u) := \mathbb{E}^{T+\delta_i}[e^{\mathrm{i}u\mathcal{X}_T^i}] = \frac{\Phi_{0,T}^i(u)}{B(0, T+\delta_i)}.$$

Equation (4.13) can be efficiently solved by relying on Newton-Raphson-type algorithms. Indeed, in the present framework, the gradient ∇D_p of the function D_p can be analytically computed and the associated Hessian matrix $H[D_p]$ turns out to be tridiagonal. To initialize the algorithm, the starting grid $\Gamma^N_{(0)}$ can be constructed by using a regular spacing around the expectation of the state variable $e^{\mathcal{X}^i_T}$, which is fully determined by market data:

$$\mathbb{E}^{T+\delta_i}\big[e^{\mathcal{X}^i_T}\big] = \mathbb{E}^{T+\delta_i}\left[\frac{S^{\delta_i}(T,T)}{B(T,T+\delta_i)}\right] = 1+\delta_i L(0,T,\delta_i).$$

Starting from $\Gamma^N_{(0)}$, a basic formulation of the Newton-Raphson algorithm for the determination of a sub-optimal quantization grid Γ^N is then based on the following iterations:

$$\Gamma^N_{(n+1)} = \Gamma^N_{(n)} - \Big(H[D_p](\Gamma^N_{(n)})\Big)^{-1}\nabla D_p(\Gamma^N_{(n)}), \quad \text{at each iteration } n\in\mathbb{N}.$$

Remark 4.4. It is important to emphasize that the companion weights $\mathbb{Q}^{T+\delta_i}(\widehat{e^{\mathcal{X}^i_T}} = x_j)$, for $j=1,\ldots,N$, and the density function of the random variable $e^{\mathcal{X}^i_T}$ needed for the computation of the function $D_p(\Gamma)$ in (4.12) can be recovered from the $(T+\delta_i)$-forward characteristic function Ψ^i_T. Indeed, it holds that

$$\begin{aligned}
\mathbb{Q}^{T+\delta_i}\big(e^{\mathcal{X}^i_T}\in dx\big) &= \left(\frac{1}{x\pi}\int_0^{+\infty}\mathrm{Re}\left(e^{-\mathrm{i}u\log(x)}\Psi^i_T(u)\right)du\right)dx,\\
\mathbb{Q}^{T+\delta_i}\big(e^{\mathcal{X}^i_T}\le x\big) &= \frac{1}{2}-\frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\frac{e^{-\mathrm{i}u\log(x)}\Psi^i_T(u)}{\mathrm{i}u}\right)du.
\end{aligned}$$

As shown above, Ψ^i_T can be analytically computed for a CBI-driven interest rate model.

5. Numerical Results and Model Calibration

In this section, we present some numerical results. In particular, we calibrate the model introduced in Section 4 to market data relative to the 3M and 6M tenors.

5.1. **Numerical comparison of caplet pricing methods.** We implemented the Fourier and the quantization-based pricing methodologies developed in Sections 4.2-4.3[3]. As a preliminary analysis to assess the reliability of both approaches, we compared them under different combinations of moneyness, maturity and model parameters, also with the help of Monte Carlo simulations. Table 1 provides an example of such comparisons, for a caplets with strikes 1% and 2% and maturities ranging from 1 up to 2 years. As a fundamental step in view of calibration to market data, this validation procedure enabled us to assess the numerical efficiency of the two pricing methodologies. In particular, we tested their stability with respect to different choices of the model parameters. Taking into account also the computational time, we found that the Fourier-based methodology performs better than the quantization. Hence, we adopted the Fourier approach for the following calibration analysis.

5.2. **Model calibration.** To illustrate the calibration of our model to market data, we start by describing the market data and the reconstruction of the term structures.

[3]The Java language has been used for the whole calibration procedure. The source code is available at `https://github.com/AlessandroGnoatto/CBIMultiCurve`.

	FFT 2%	Quant. 2%	Difference 2%	FFT 1%	Quant. 1%	Difference 1%
1	0.0044507	0.0038513	−13.468%	0.0045416	0.0043363	−4.5203%
1.1	0.0046833	0.0041762	−10.828%	0.0047786	0.0046674	−2.3272%
1.2	0.0049715	0.0045288	−8.9062%	0.0050723	0.0050283	−0.86746%
1.3	0.0053098	0.0049117	−7.4971%	0.0054170	0.0054217	0.085953%
1.4	0.0056936	0.0053270	−6.4398%	0.0058082	0.0058493	0.70745%
1.5	0.0061157	0.0057721	−5.6183%	0.0062384	0.0063083	1.1212%
1.6	0.0065746	0.0062481	−4.9657%	0.0067061	0.0067997	1.3960%
1.7	0.0070702	0.0067561	−4.4424%	0.0072114	0.0073247	1.5708%
1.8	0.0076044	0.0072981	−4.0282%	0.0077560	0.0078850	1.6639%
1.9	0.0081780	0.0078745	−3.7110%	0.0083407	0.0084812	1.6849%
2	0.0087918	0.0084857	−3.4813%	0.0089664	0.0091137	1.6417%

TABLE 1. Comparison of FFT and quantization prices for different maturities (strikes at 2% and 1%, differences in relative terms).

5.2.1. *Market data.* We consider market data for the EUR market as of 25 June 2018, consisting of both linear and non-linear interest rate derivatives. The set of tenors is $\mathcal{D} = \{3M, 6M\}$. Market data on linear products consist of OIS and interest rate swaps, from which the discount curve $T \mapsto B(0,T)$ and the forward curves $T \mapsto L(0,T,\delta_i)$, for $\delta_1 = 3M$ and $\delta_2 = 6M$, are constructed using the bootstrapping procedure from the Finmath Java library (see [Fri15, Fri16]). The OIS discount curve is bootstrapped from OIS swaps, using cubic spline interpolation on logarithmic discount factors with constant extrapolation. Similarly, the 3M and 6M forward curves are bootstrapped from market quotes of FRAs (for short maturities) and swaps (for maturities beyond two years), using cubic spline interpolation on forwards with constant extrapolation. Figure 3 reports the resulting discount and forward curves. We can observe that the spread between the 3M and the 6M curves is more pronounced below twelve years and decreases afterwards. We also notice that, for short maturities, discount factors are larger than one and forward rates are negative.

Concerning non-linear interest rate products, we focus on caplet market data, suitably bootstrapped from market cap volatilities. Consistently with the presence of negative interest rates, we have market quotes for caps having a negative strike rate. Therefore, the boostrapped caplet volatility surface refers to strike prices ranging between -0.13% and 2% and maturities between 6 months and 6 years. Caplets with maturity larger than two years are indexed to the 6-month forward rate while those with shorter expiry are linked to the 3-month curve. Market data are given in terms of normal implied volatilities. A normal implied volatility is obtained by numerically searching for the value of $\sigma_{\mathrm{mkt}}^{\mathrm{imp}}$ such that the Bachelier pricing formula for a caplet

$$\Pi_{Bac}^{\mathrm{CPLT}}(t;T_{i-1},\delta_i,K,1) := B(t,T_i)\delta\sigma_{\mathrm{mkt}}^{\mathrm{imp}}\sqrt{T_{i-1}-t}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} + zN(z)\right), \tag{5.1}$$

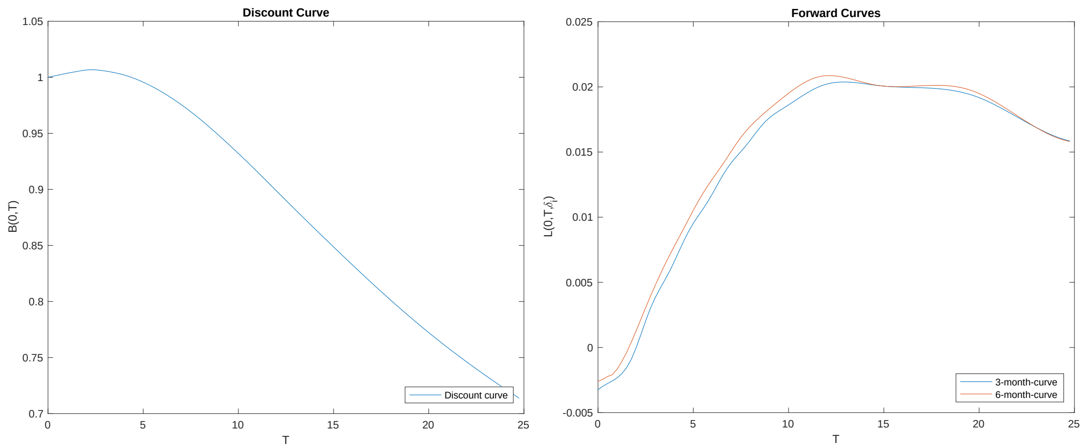

FIGURE 3. Discount and forward curves as of June 2018.

with

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \qquad \text{and} \qquad z = \frac{L(t, T_{i-1}, \delta_i) - K}{\sigma_N \sqrt{T_{i-1} - t}},$$

provides the best fit to the market price of a given caplet.

5.2.2. *Implementation.* For a vector p of model parameters belonging to the set $\mathcal{P}$ of admissible values, we compute model-implied caplet prices by means of the Fourier approach of Section 4.2 (see in particular Proposition 4.3). The numerical integration is performed by means of the FFT approach of [CM99], with 32768 points and integration mesh size 0.05. For a fixed maturity, a single execution of the FFT pricing routine yields a vector of model prices for several moneyness levels. Prices are then converted into normal implied volatilities by using formula (5.1). Repeating this procedure for different maturities, we generate a corresponding model-implied volatility $\sigma^{\text{imp}}_{\text{mod}}(K_k, T_j, p)$ for each strike K_k and maturity T_j present in our sample of market data.

The aim of the calibration procedure is to find the vector of parameters which solves the problem

$$\min_{p \in \mathcal{P}} \sum_{j,k} \left(\sigma^{\text{imp}}_{\text{mkt}}(K_k, T_j) - \sigma^{\text{imp}}_{\text{mod}}(K_k, T_j, p) \right)^2 . \tag{5.2}$$

5.2.3. *Calibration results.* We calibrated a two-dimensional version of the model of Section 4.1. To solve problem (5.2), we used the multi-threaded Levenberg-Marquardt optimizer of the Finmath Java library with 8 threads, imposing the parameter restrictions listed after equation (4.1) to ensure well-posedness. The calibrated values of the parameters are shown in Table 2. As illustrated by Figures 4, the model achieves a good fit to market data, across different strikes and maturities. We remark that, in terms of number of parameters, the model under consideration is even more parsimonious than the simple specifications calibrated in [CFG19b]. Motivated by the presence of negative forward rates, we also calibrated a version of the model where the OIS short rate is affected by an auxiliary Ornstein-Uhlenbeck process, in line with Remark 3.6. However, this alternative specification did not yield a significant improvement of the quality of the fit. This seems to indicate that the deterministic shift $\ell(t)$ introduced in (4.3) does suffice to capture the probability mass in the negative axis for the short rate. This is also in line with the widespread use of deterministic shift extensions in the financial industry.

b	0.05353	α	1.31753
σ	0.00582	y_0	$(0.00495, 0.00507)^\top$
η	0.04070	β	$(9.99999E-4, 0.00340)^\top$
θ	0.05070	μ	$(1.49999, 1.00000)^\top$

Table 2. Calibrated values of the parameters.

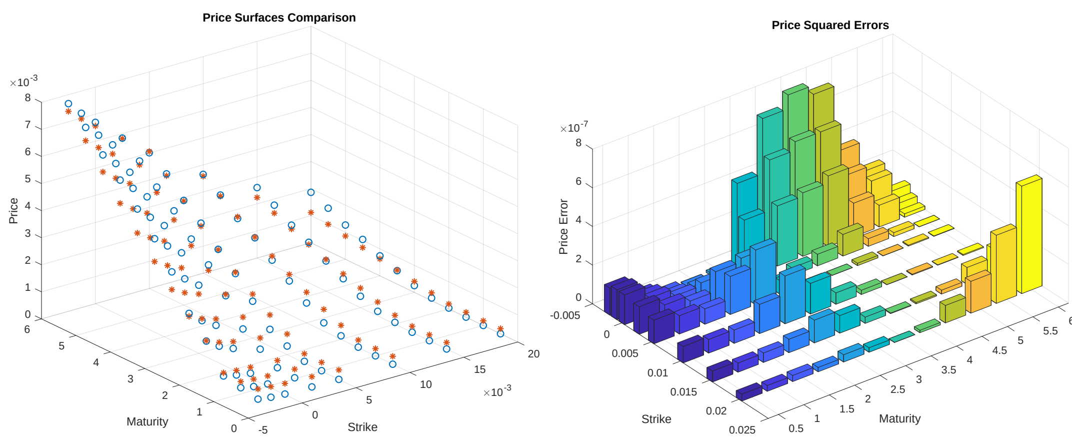

Figure 4. Model prices against market prices as of 25 June 2018. On the left panel, market prices are represented by blue circles while model prices by red stars.

6. Conclusions

In the present paper, we have proposed a modelling framework for multiple yield curves based on CBI processes. The self-exciting behavior of jump-type CBI processes enables us to capture most of the empirical features of spreads. At the same time, exploiting the fundamental link with affine processes, our setup allows for an efficient valuation of interest rate derivatives. Models driven by a flow of tempered alpha-stable CBI processes represent a parsimonious way of modelling spreads in a realistic way, with a natural interpretation of the stochastic drivers in terms of risk factors.

We conclude by commenting on the relevance of multiple curve modelling in view of the reforms of benchmark interbank rates. In all major economies, transaction-based backward-looking rates are being introduced as a replacement for Libor rates (e.g., SOFR in the US market, ESTER in the Eurozone, SONIA in the UK market), also as a response to the 2012 Libor manipulation scandal. At the time of the writing, definitive conclusions on the evolution of Ibor rates cannot be drawn. However, there seems to be a consensus on the fact that the multiple curve framework will remain relevant (and, actually, possibly even more relevant). Indeed, in line with [LM19], a complete disappearence of unsecured Ibor rates, reflecting the unsecured funding costs of banks, does not seem a realistic scenario. As an example, in the Eurozone the Euribor rate will not be abandoned, but only replaced by a reformed version in 2022. Moreover, Ibor proxies may arise to address the need for term rates containing systemic credit or liquidity risk premia (see again [LM19]).

Appendix A. Proofs of the Results on CBI Processes

In this appendix, we collect the proofs of the theoretical results stated in Section 2.

Proof of Theorem 2.5. The fact that X is a regular affine process follows from [DFS03, Corollary 2.10]. For $p \geq 0$ and $q = 0$, formula (2.6) simply follows from Definition 2.1 and $T^{(p,0)} = +\infty$. Under Assumption 2.4, ϕ is a locally Lipschitz continuous function on $\mathcal{Y}$. Therefore, for every $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$, standard existence and uniqueness results for solutions to first-order ODEs imply the existence of a maximal lifetime $T^{(p,q)} \in (0,+\infty]$ such that (2.5) admits a unique solution $v(\cdot,p,q) : [0,T^{(p,q)}) \to \mathcal{Y}$ and the integral $\int_0^t \psi(v(s,p,q))\mathrm{d}s$ is finite, for all $t < T^{(p,q)}$. Hence, part (b) of [KRM15, Theorem 2.14] applied to the bi-dimensional affine process $(X, \int_0^\cdot X_s\,\mathrm{d}s)$ implies that the affine transform formula (2.6) holds for every $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$ and $t < T^{(p,q)}$. □

Proof of Proposition 2.6. Let $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$. In the trivial case $\phi \equiv 0$, the ODE (2.5) is solved by the function $v(t,p,q) = p + qt$ and hence $T^{(p,q)} = +\infty$. In the rest of the proof, we shall assume that $\phi(y) \neq 0$ for some $y \in \mathcal{Y}$. Note that $\{y \in \mathcal{Y} : q - \phi(y) \geq 0\} \cap \mathbb{R}_- \neq \emptyset$, so that p_q is always well-defined with values in $[\ell \vee \kappa, 0]$ and, by continuity of ϕ, it satisfies $\phi(p_q) \leq q$. If $\phi(p_q) = q$, then the constant function $\tilde{v}(\cdot) := p_q$ is a solution to (2.5) with initial value $p = p_q$. Since the ODE (2.5) admits a unique solution for every $(p,q) \in \mathcal{Y} \times \mathbb{R}_+$ by Assumption 2.4, it holds that $v(t,p_q,q) = p_q$, for all $t \geq 0$. Since $p_q \in \mathcal{Y}$, it follows that $T^{(p_q,q)} = +\infty$. On the other hand, if $\phi(p_q) < q$, then $q - \phi(y) > 0$ for all $y \in \mathcal{Y} \cap (-\infty,0)$. By convexity of ϕ, the equation $\phi(y) = q$ admits a unique solution p_q^+ in $[0,+\infty)$. The ODE (2.5) implies that

$$\int_{p_q}^{v(t,p_q,q)} \frac{\mathrm{d}y}{q - \phi(y)} = t, \qquad \text{for all } t \geq 0.$$

Letting $t \to +\infty$ on both sides of the above equality, we get that $v(t,p_q,q) \to p_q^+$ as $t \to +\infty$, while $v(t,p_q,q) < p_q^+$ for all $t \geq 0$. Since ψ is increasing, it holds that $\psi(p_q) \leq \psi(v(t,p_q,q)) \leq \psi(p_q^+)$. This implies that $T^{(p_q,q)} = +\infty$. By [KRM15, Theorem 2.14] applied to the bi-dimensional affine process $(X, \int_0^\cdot X_s\,\mathrm{d}s)$, this means that $\mathbb{E}[\exp(-p_q X_t - q\int_0^t X_s\,\mathrm{d}s)] < +\infty$, for all $t \geq 0$. Therefore, it holds that $\mathbb{E}[\exp(-pX_t - q\int_0^t X_s\,\mathrm{d}s)] < +\infty$ for all $t \geq 0$ and $p \geq p_q$. In turn, by [KRM15, Proposition 3.3], this implies that $T^{(p,q)} = +\infty$, for all $p \geq p_q$. Let us now consider the case $p < p_q$ and suppose first that $\kappa \leq \ell$. In this case, due to the convexity of ϕ, it holds that $q - \phi(y) < 0$ for all $y \in [\ell, p_q)$. Arguing similarly as in [KR11, Theorem 4.1], the ODE (2.5) admits a unique solution $v(t,p,q)$ which is strictly decreasing in t, with values in $[\ell, p]$. This solution admits a maximal extension to an interval $[0,T^*)$ such that one of the following two situations occurs:

(i) $T^* = +\infty$;

(ii) $T^* < +\infty$ and $\lim_{t\to T^*} v(t,p,q) = \ell$.

In case (i), since $v(\cdot,p,q)$ is strictly decreasing, $\alpha := \lim_{t\to+\infty} v(t,p,q)$ is well-defined, with values in $\{-\infty\}\cup[\ell,p)$. If $\alpha > -\infty$, then α must be a stationary point, i.e., $q-\phi(\alpha)=0$. However, this contradicts the fact that $\alpha < p < p_q$. The case $\alpha = -\infty$ can only happen if $\ell = -\infty$ and in this case $\lim_{t\to T^*} v(t,p,q) = \ell$, exactly as in case (ii). In case (ii), let $(T_n)_{n\in\mathbb{N}}$ be an increasing sequence with $T_n < T^*$, for all $n\in\mathbb{N}$, such that $T_n \to T^*$ as $n\to+\infty$. Similarly as above, it holds that

$$\int_p^{v(T_n,p,q)} \frac{\mathrm{d}y}{q-\phi(y)} = T_n, \qquad \text{for each } n\in\mathbb{N}. \tag{A.1}$$

Letting $n\to+\infty$ on both sides of (A.1), we obtain that $T^* = \int_p^\ell (q-\phi(y))^{-1}\mathrm{d}y$. If $\kappa\le\ell$, then $\int_0^t \psi(v(s,p,q))\mathrm{d}s$ is always finite whenever $v(t,p,q)$ is finite, so that $T^{(p,q)} = T^*$, thus proving (2.8) in the case $\kappa\le\ell$. If $\kappa>\ell$, then the lifetime is given by $T^{(p,q)} = \inf\{t\in\mathbb{R}_+ : v(t,p,q)=\kappa\}$. Replacing T_n with $T^{(p,q)}$ into (A.1) yields (2.8), thus proving the first part of the proposition.

To prove the last statement of the proposition, suppose that $\psi(\ell\vee\kappa) > -\infty$. In this case, if $\phi(\ell\vee\kappa)\le 0$, then $\mathcal{Y} = [\ell\vee\kappa,+\infty)$ and $p_q = \ell\vee\kappa$, for every $q\in\mathbb{R}_+$. By the first part of the proposition, it follows that $T^{(p,q)}=+\infty$ for all $(p,q)\in[\ell\vee\kappa,+\infty)\times\mathbb{R}_+$. Conversely, if $T^{(p,q)}=+\infty$ for all $(p,q)\in[\ell\vee\kappa,+\infty)\times\mathbb{R}_+$, then in particular $\ell\vee\kappa\in\mathcal{Y}$ and $T^{(\ell\vee\kappa,0)}=+\infty$. This is only possible if $\phi(\ell\vee\kappa)\le 0$. Indeed, if $\kappa\le\ell$ and $\phi(\ell)>0$, then the solution $v(t,\ell,0)$ to the ODE (2.5) with $p=\ell$ would explode immediately (i.e., $T^{(\ell,0)}=0$). Similarly, if $\kappa>\ell$ and $\phi(\kappa)>0$, then the solution $v(t,\kappa,0)$ to (2.5) with $p=\kappa$ would be strictly decreasing in a neighborhood of zero and, therefore, the integral $\int_0^\cdot \psi(v(s,\kappa,0))\mathrm{d}s$ would immediately diverge to $-\infty$ (i.e., $T^{(\kappa,0)}=0$). □

Proof of Lemma 2.9. We only consider the case $\theta>0$, referring to [JMS17] for the case $\theta=0$. Note first that

$$\int_0^{+\infty} (e^{-zu}-1+zu)\frac{e^{-\theta u}}{u^{1+\alpha}}\mathrm{d}u = \int_0^{+\infty}\sum_{n=2}^{+\infty}\frac{(-zu)^n}{n!}u^{-1-\alpha}e^{-\theta u}\mathrm{d}u.$$

If $z>-\theta$, we can interchange the order of integration and summation, thus obtaining

$$\begin{aligned}\int_0^{+\infty} (e^{-zu}-1+zu)\frac{e^{-\theta u}}{u^{1+\alpha}}\mathrm{d}u &= \sum_{n=2}^{+\infty}\frac{(-z/\theta)^n}{n!}\theta^\alpha\Gamma(n-\alpha)\\ &= \theta^\alpha\Gamma(-\alpha)\left(\frac{\alpha(\alpha-1)}{2!}\left(\frac{z}{\theta}\right)^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(\frac{z}{\theta}\right)^3+\dots\right).\end{aligned}$$

The last line of the above expression is related to the power series

$$\left(1+\frac{z}{\theta}\right)^\alpha = 1+\alpha\frac{z}{\theta}+\frac{\alpha(\alpha-1)}{2!}\left(\frac{z}{\theta}\right)^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(\frac{z}{\theta}\right)^3+\dots,$$

which converges if and only if $z>-\theta$. Therefore, it holds that

$$\int_0^{+\infty} (e^{-zu}-1+zu)\frac{e^{-\theta u}}{u^{1+\alpha}}\mathrm{d}u = \theta^\alpha\Gamma(-\alpha)\left(\left(1+\frac{z}{\theta}\right)^\alpha - 1 - \alpha\frac{z}{\theta}\right),$$

from which (2.11) follows due to the definition of $C(\alpha,\eta)$ given in (2.10). By continuity, formula (2.11) can then be extended to $z=-\theta$. The convexity of ϕ follows by noting that

$$\phi''(z)=\sigma^2-\eta^\alpha\frac{\alpha(\alpha-1)(z+\theta)^{\alpha-2}}{\cos(\pi\alpha/2)}\geq 0,\qquad \text{for all } z\geq -\theta.$$

By computing $\partial\phi(z)/\partial\theta$ and using Bernoulli's inequality, it can be easily verified that ϕ is decreasing with respect to θ. Finally, since $\phi\in\mathcal{C}^1([-\theta,+\infty))$, Assumption 2.4 is satisfied. □

Proof of Proposition 2.10. (i): this is a direct consequence of the last part of Proposition 2.6 together with (2.7). (ii): as a consequence of (2.11), it holds that $\phi(z)\geq bz+\sigma^2z^2/2$, for all $z\geq 0$. Furthermore, if $2\beta\geq\sigma^2$, it can be checked that $\psi(z)/\phi(z)\geq z^{-1}(1+O(z^{\alpha-2}))$ for all sufficiently large z. The result follows by the same arguments used in the proof of [JMS17, Proposition 3.4]. □

Proof of Proposition 2.13. The fact that $(P_t(\cdot,x))_{t\geq 0}$ converges weakly to a stationary distribution ρ follows from [Li11, Corollary 3.21], while formula (2.12) for $p\geq 0$ corresponds to [Li11, Theorem 3.20]. Consider the case $p\in(p_0,0)$, with $p_0<0$. Since $\phi(z)<0$ for all $z\in(p_0,0)$, the solution $v(t,p,0)$ to the ODE (2.5) with $q=0$ is strictly increasing. Furthermore, (2.5) implies that

$$-\int_p^{v(t,p,0)}\frac{\mathrm{d}y}{\phi(y)}=t,\qquad \text{for all } t\geq 0.$$

Therefore, letting $t\to+\infty$ on both sides, it follows that $\lim_{t\to+\infty}v(t,p,0)=0$. In turn, as a consequence of (2.6) (with $q=0$), this implies that

$$\lim_{t\to+\infty}\mathbb{E}[e^{-pX_t}]=\exp\left(-\int_0^{+\infty}\psi(v(s,p,0))\mathrm{d}s\right)=\exp\left(-\beta\int_0^p\frac{z}{\phi(z)}\mathrm{d}z\right),$$

where the last equality follows by a change of variable together with equation (2.5). Formula (2.13) follows by differentiating (2.12) at $p=0$. Finally, to prove the exponential ergodicity of X, recall that $\phi(z)\geq bz+\sigma^2z^2/2$, for all $z\geq 0$ (see the proof of Proposition 2.10). Therefore, it holds that

$$\int_c^{+\infty}\frac{1}{\phi(z)}\mathrm{d}z\leq\int_c^{+\infty}\frac{1}{bz+\frac{\sigma^2z^2}{2}}\mathrm{d}z<+\infty,\qquad \text{for any } c>0.$$

In view of [LM15, Theorem 2.5], this suffices to prove the claim. □

Appendix B. Pricing Formulae for Fixed-Income Derivatives

In this appendix, we state general pricing formulas for fixed-income derivatives in a multi-curve context, expressed in terms of multiplicative spreads. These pricing formulae give *clean prices*, assuming perfect collateralization with a collateral rate equal to the OIS rate. We refer the reader to [CFG16, Section 5.2] and [GR15, Section 1.4] for more details on the derivation of the formulas.

B.1. **Linear products.**

Forward rate agreement. For $\delta \in \mathcal{D}$, a standard ("textbook") forward rate agreement (FRA) settled at T, with maturity $T+\delta$, rate K and notional N is a contract which delivers at $T+\delta$ the payoff

$$\Pi^{\mathrm{FRA}}(T+\delta; T, \delta, K, N) = N\delta\big(L(T,T,\delta) - K\big).$$

The arbitrage-free price of the FRA contract at time $t \leq T$ is given by

$$\Pi^{\mathrm{FRA}}(t; T, \delta, K, N) = N\big(B(t,T)S^{\delta}(t,T) - B(t,T+\delta)(1+\delta K)\big).$$

Remark B.1 (On market FRAs). In the market, traded FRA contracts are specified slightly differently, with a payoff of

$$\Pi^{\mathrm{mFRA}}(T; T, \delta, K, N) = N\frac{\delta(L(T,T,\delta) - K)}{1+\delta L(T,T,\delta)}$$

delivered at time T (see [GR15, Remark 1.3]). The corresponding price can be computed as

$$\Pi^{\mathrm{mFRA}}(t; T, \delta, K, N) = NB(t,T) - N(1+\delta K)\mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s}\frac{B(T,T+\delta)}{S^{\delta}(T,T)}\Big|\mathcal{F}_t\right], \qquad \text{for } t \leq T.$$

In the context of a CBI-driven multi-curve model, this conditional expectation can be explicitly computed by relying on the affine transform formula (2.6) together with Proposition 3.2.

Overnight indexed swap. An overnight indexed swap (OIS) is a contract where two cash flows are exchanged: the first one is computed with respect to a fixed rate K, whereas the second one is indexed by an overnight rate. Let us denote by $T_1, \ldots, T_n$ the payment dates, with $T_{i+1} - T_i = \delta \in \mathcal{D}$ for all $i = 1, \ldots, n-1$. The swap is initiated at $T_0 \in [0, T_1]$. The value at $t \leq T_0$ of an OIS with notional N can be expressed as follows:

$$\Pi^{\mathrm{OIS}}(t; T_0, T_1, n, \delta, K, N) = N\left(B(t,T_0) - B(t,T_n) - K\delta\sum_{i=1}^{n} B(t,T_i)\right).$$

The OIS rate K^{OIS}, defined as the rate K such that the OIS contract has zero value at t, is given by

$$K^{\mathrm{OIS}}(t, T_0, T_n) = \frac{B(t,T_0) - B(t,T_n)}{\delta\sum_{i=1}^{n} B(t,T_i)},$$

and note that $K^{\mathrm{OIS}}(t,t,t+\delta) = L^{\mathrm{OIS}}(t,t,\delta)$, where the latter is defined in Section 3.1.

Interest rate swap. In an interest rate swap (IRS), two cash flows are exchanged: the first one is computed with respect to a fixed rate K, whereas the second one is indexed by the Ibor rate with tenor $\delta \in \mathcal{D}$. The value of the IRS at time $t \leq T_0$, where T_0 denotes the inception time, is given by

$$\Pi^{\mathrm{IRS}}(t; T_1, n, \delta, K, N) = N\sum_{i=1}^{n}\Big(B(t,T_{i-1})S^{\delta}(t,T_{i-1}) - B(t,T_i)(1+\delta K)\Big).$$

The swap rate K^{IRS}, defined as the rate K such that the IRS contract has zero value at t, is given by

$$K^{\mathrm{IRS}}(t, T_0, T_n) = \frac{\sum_{i=1}^{n}\big(B(t,T_{i-1})S^{\delta}(t,T_{i-1}) - B(t,T_i)\big)}{\delta\sum_{i=1}^{n} B(t,T_i)} = \frac{\sum_{i=1}^{n} B(t,T_i)L(t,T_{i-1},\delta)}{\sum_{i=1}^{n} B(t,T_i)}.$$

Basis swap. A basis swap is a swap contract where two cash flows related to Ibor rates associated to different tenors are exchanged. For instance, a basis swap may involve the exchange of the 3-month against the 6-month Ibor rate. Following the standard convention in the Euro market (see [AB13]), the basis swap is equivalent to a long/short position on two interest rate swaps which share the same fixed leg. Let $\mathcal{T}^1 = \{T^1_0, \dots, T^1_{n_1}\}$, $\mathcal{T}^2 = \{T^2_0, \dots T^2_{n_2}\}$ and $\mathcal{T}^3 = \{T^3_0, \dots, T^3_{n_3}\}$, with $T^1_{n_1} = T^2_{n_2} = T^2_{n_3}$, $T^1_0 = T^2_0 = T^2_0$, $\mathcal{T}^1 \subset \mathcal{T}^2$, $n_1 < n_2$ and corresponding tenor lengths $\delta_1 > \delta_2$, with no constraints on δ_3. The tenor structures $\mathcal{T}^1$ and $\mathcal{T}^2$ correspond to floating legs, while $\mathcal{T}^3$ is associated to a fixed leg. The value at $t \leq T^1_0$ of a swap with notional N initiated at T^1_0 is given by

$$\begin{aligned}\Pi^{\mathrm{BSW}}(t;\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3,N) = N \Bigg(& \sum_{i=1}^{n_1} \big(B(t,T^1_{i-1})S^{\delta_1}(t,T^1_{i-1}) - B(t,T^1_i)\big) \\ & - \sum_{j=1}^{n_2} \big(B(t,T^2_{j-1})S^{\delta_2}(t,T^2_{j-1}) - B(t,T^2_j)\big) - K \sum_{\ell=1}^{n_3} \delta_3 B(t,T^3_\ell) \Bigg).\end{aligned}$$

The rate K^{BSW} (called *basis swap spread*) such that the value of the contract is zero is given by

$$K^{\mathrm{BSW}}(\mathcal{T}^1,\mathcal{T}^2,\mathcal{T}^3) = \frac{\sum_{i=1}^{n_1}\big(B(t,T^1_{i-1})S^{\delta_1}(t,T^1_{i-1}) - B(t,T^1_i)\big) - \sum_{j=1}^{n_2}\big(B(t,T^2_{j-1})S^{\delta_2}(t,T^2_{j-1}) - B(t,T^2_j)\big)}{\delta_3 \sum_{\ell=1}^{n_3} B(t,T^3_\ell)}.$$

Note that, in the pre-crisis single-curve setting, the value of K^{BSW} used to be approximately zero.

B.2. **Non-linear products.**

Caplet. A caplet can be regarded as a Call option with an Ibor rate as underlying. The price at time t of a caplet with strike price K, maturity T, settled in arrears at $T+\delta$, is given by

$$\begin{aligned}\Pi^{\mathrm{CPLT}}(t;T,\delta,K,N) &= N\delta\,\mathbb{E}\left[e^{-\int_t^{T+\delta} r_s \mathrm{d}s}\big(L(T,T,\delta) - K\big)^+ \Big| \mathcal{F}_t\right] \\ &= N\mathbb{E}\left[e^{-\int_t^{T} r_s \mathrm{d}s}\big(S^\delta(T,T) - (1+\delta K)B(T,T+\delta)\big)^+ \Big| \mathcal{F}_t\right]. \qquad \text{(B.1)}\end{aligned}$$

Remark B.2. In the pre-crisis single-curve setting (i.e., under the assumption that $S^\delta(T,T)$ is identically equal to one), formula (B.1) reduces to the classical relationship between a caplet and a put option written on a OIS zero-coupon bond with strike $1/(1+\delta K)$.

Swaption. We consider a standard European payer swaption with maturity T, written on a (payer) interest rate swap starting at $T_0 = T$ and payment dates $T_1, ..., T_n$, with $T_{i+1} - T_i = \delta \in \mathcal{D}$ for all $i = 1, \dots, n-1$, with notional N. The value of such a swaption at time $t \leq T_0$ is given by

$$\Pi^{\mathrm{SWPTN}}(t;T_1,T_n,K,N) = N\mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s}\left(\sum_{i=1}^n B(T,T_{i-1})S^\delta(T,T_{i-1}) - (1+\delta K)B(T,T_i)\right)^+ \Bigg| \mathcal{F}_t\right].$$

Due to the affine structure of our model, swaptions and basis swaptions can be efficiently priced by relying on a suitable approximation of the exercise region, following the methodology proposed in [CFG17] (compare also with [CFG19b, Section 4.2] in the specific context of a multi-curve model).

References

[AB13] F.M. Ametrano and M. Bianchetti. Everything you always wanted to know about multiple interest rate curve bootstrapping but were afraid to ask. Preprint (available at `http://ssrn.com/abstract=2219548`), 2013.

[BM01] D. Brigo and F. Mercurio. A deterministic-shift extension of analytically-tractable and time-homogeneous short-rate models. *Finance and Stochastics*, 5(3):369–387, 2001.

[BM13] M. Bianchetti and M. Morini, editors. *Interest Rate Modelling After the Financial Crisis*. Risk Books, 2013.

[CD13] S. Crépey and R. Douady. Lois: credit and liquidity. *Risk Magazine*, June:82–86, 2013.

[CDS01] P. Collin-Dufresne and B. Solnik. On the term structure of default premia in the swap and libor markets. *The Journal of Finance*, 56(3):1095–1115, 2001.

[CFG16] C. Cuchiero, C. Fontana, and A. Gnoatto. A general HJM framework for multiple yield curve modeling. *Finance and Stochastics*, 20(2):267–320, 2016.

[CFG17] R. Caldana, G. Fusai, and A.M. Gambaro. Approximate pricing of swaptions in affine and quadratic models. *Quantitative Finance*, 17(9):1325–1345, 2017.

[CFG19a] G. Callegaro, L. Fiorin, and M. Grasselli. Quantization meets Fourier: a new technology for pricing options. *Annals of Operations Research*, 282(1–2):59–86, 2019.

[CFG19b] C. Cuchiero, C. Fontana, and A. Gnoatto. Affine multiple yield curve models. *Mathematical Finance*, 29(2):568–611, 2019.

[CM99] P. Carr and D.B. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2(4):61–73, 1999.

[CMS19] G. Callegaro, A. Mazzoran, and C. Sgarra. A self-exciting modelling framework for forward prices in power markets. Preprint (available at `https://arxiv.org/abs/1910.13286`), 2019.

[CT04] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall CRC, London, 2004.

[DFS03] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Annals of Applied Probability*, 13(3):984–1053, 2003.

[DL12] D.A. Dawson and Z. Li. Stochastic equations, flows and measure-valued processes. *Annals of Probability*, 40(2):813–857, 2012.

[EGG18] E. Eberlein, C. Gerhart, and Z. Grbac. Multiple curve Lévy forward price model allowing for negative interest rates. *Mathematical Finance*, forthcoming, 2018.

[Fil01] D. Filipović. A general characterization of one factor affine term structure models. *Finance and Stochastics*, 5(3):389–412, 2001.

[Fri15] C. Fries. Finmath lib v1.3.0. `http://www.finmath.net`, 2015.

[Fri16] C. Fries. Multi-curve construction. In K. Glau, Z. Grbac, M. Scherer, and R. Zagst, editors, *Innovations in Derivatives Markets*, pages 227–250. Springer, Cham, 2016.

[FT13] D. Filipović and A.B. Trolle. The term structure of interbank risk. *Journal of Financial Economics*, 109(3):707–733, 2013.

[GL00] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*, volume 1730 of *Lecture Notes in Mathematics*. Springer, Berlin - Heidelberg, 2000.

[GM10] R.M. Gaspar and A. Murgoci. Convexity adjustments. In R. Cont, editor, *Encyclopedia of Quantitative Finance*, volume I. Wiley, 2010.

[GR15] Z. Grbac and W.J. Runggaldier. *Interest Rate Modeling: Post-Crisis Challenges and Approaches*. SpringerBriefs in Quantitative Finance. Springer, 2015.

[Hen14] M. Henrard. *Interest Rate Modelling in the Multi-Curve Framework*. Palgrave Macmillan, 2014.

[JMS17] Y. Jiao, C. Ma, and S. Scotti. Alpha-CIR model with branching processes in sovereign interest rate modeling. *Finance and Stochastics*, 21(3):789–813, 2017.

[JMSS19] Y. Jiao, C. Ma, S. Scotti, and C. Sgarra. A branching process approach to power markets. *Energy Economics*, 79:144–156, 2019.

[JMSZ18] Y. Jiao, C. Ma, S. Scotti, and C. Zhou. The alpha-Heston stochastic volatility model. Preprint (available at `https://arxiv.org/abs/1812.01914`), 2018.

[KR11] M. Keller-Ressel. Moment explosions and long-term behavior of affine stochastic volatility models. *Mathematical Finance*, 21(1):73–98, 2011.

[KRM15] M. Keller-Ressel and E. Mayerhofer. Exponential moments of affine processes. *Annals of Applied Probability*, 25(2):714–752, 2015.

[KW71] K. Kawazu and S. Watanabe. Branching processes with immigration and related limit theorems. *Theory of Probability and its Applications*, 16(1):36–54, 1971.

[Kyp06] A.E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin - Heidelberg, 2006.

[Lee04] R. Lee. Option pricing by transform methods: extensions, unification and error control. *Journal of Computational Finance*, 7(3):51–86, 2004.

[Li11] Z. Li. *Measure-Valued Branching Markov Processes*. Springer, Berlin - Heidelberg, 2011.

[Li20] Z. Li. Continuous-state branching processes with immigration. In Y. Jiao, editor, *From Probability to Finance - Lecture Notes of BICMR Summer School on Financial Mathematics*. Springer, Singapore, 2020.

[LM15] Z. Li and C. Ma. Asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model. *Stochastic Processes and their Applications*, 125(8):3196–3233, 2015.

[LM19] A. Lyashenko and F. Mercurio. Looking forward to backward-looking rates: A modeling framework for term rates replacing LIBOR. Preprint (available at `https://doi.org/10.2139/ssrn.3330240`), 2019.

[Mer13] F. Mercurio. A Libor market model with a stochastic basis. *Risk Magazine*, pages 96–101, December 2013.

[Mer18] F. Mercurio. The present of futures. *Risk Magazine*, pages 1–6, March 2018.

[MX12] F. Mercurio and Z. Xie. The basis goes stochastic. *Risk Magazine*, pages 78–83, December 2012.

[NLH19] C.G. Njike Leunga and D. Hainaut. Interbank credit risk modelling with self-exciting jump processes. ISBA discussion paper 2019/17, 2019.

[NS15] T.A. Nguyen and F.T. Seifried. The multi-curve potential model. *International Journal of Theoretical and Applied Finance*, 18(7):1550049, 2015.

[Pag15] G. Pagès. Introduction to vector quantization and its applications for numerics. *ESAIM: Proceedings and Surveys*, 48:29–79, 2015.

[Par16] E. Pardoux. *Probabilistic Models of Population Evolution*. Springer, Switzerland, 2016.

Claudio Fontana, Department of Mathematics "Tullio Levi Civita", University of Padova (Italy)
Email address: `fontana@math.unipd.it`

Alessandro Gnoatto, Department of Economics, University of Verona (Italy)
Email address: `alessandro.gnoatto@univr.it`

Guillaume Szulda, Laboratoire de Probabilités, Statistique et Modélisation (LPSM), Paris Diderot University (France)
Email address: `guillaume.szulda@upmc.fr`