PRÜFER MODULES OVER LEAVITT PATH ALGEBRAS

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Abstract. Let $L_K(E)$ denote the Leavitt path algebra associated to the finite graph E and field K. For any closed path c in E, we define and investigate the uniserial, artinian, non-noetherian left $L_K(E)$-module $U_{E,c-1}$. The unique simple factor of each proper submodule of $U_{E,c-1}$ is isomorphic to the Chen simple module $V_{[c^\infty]}$. In our main result, we classify those closed paths c for which $U_{E,c-1}$ is injective. In this situation, $U_{E,c-1}$ is the injective hull of $V_{[c^\infty]}$.

Keywords: Injective module; Leavitt path algebra; Chen simple module; Prüfer module.

2010 Mathematics Subject Classification: 16S99

1. Introduction

Leavitt path algebras have a well-studied, extremely tight relationship with their *pro*jective modules. On the other hand, very little is heretofore known about the structure of the *in*jective modules over $L_K(E)$. While the self-injective Leavitt path algebras have been identified in [7], we know of no study of the structure of injective modules over Leavitt path algebras (other than those arising as left ideals).

We initiate such a study in this article. For each closed path c in E we construct the *Prüfer module* $U_{E,c-1}$, recalling the classical construction of Prüfer abelian groups. These modules $U_{E,c-1}$ are Prüfer also in the sense of Ringel [13]; indeed, they admit a surjective locally nilpotent endomorphism (see Remark 2.5). In our main result (Theorem 6.4), we give necessary and sufficient conditions for the injectivity of $U_{E,c-1}$. In this case, $U_{E,c-1}$ is precisely the injective hull of the Chen simple module $V_{[c^\infty]}$. Our construction is similar to that established by Matlis [10] for modules over various commutative noetherian rings, but in a highly noncommutative, non-noetherian setting.

Perhaps surprisingly, achieving Theorem 6.4 relies on a set of highly nontrivial tools, including: some general results about uniserial modules over arbitrary associative unital rings; an explicit description of a projective resolution for $V_{[c^\infty]}$; a Division Algorithm in $L_K(E)$ with respect to the element $c-1$; the fact that every Leavitt path algebra is Bézout (i.e., that every finitely generated one-sided ideal is principal); and two types of Morita equivalences for Leavitt path algebras (one of which relates each graph having a source cycle to a graph having a source loop, the other of which eliminates source vertices).

The article is organized as follows. In Section 2 we construct what we call "Prüfer-like modules" over arbitrary unital rings. In Section 3 we remind the reader of the construction of the Leavitt path algebra $L_K(E)$ for a directed graph E and field K, and describe the Chen simple $L_K(E)$-module $V_{[p]}$ corresponding to an infinite path p arising from E. Specifically, if c is a closed path in E, we may build the Chen simple module $V_{[c^\infty]}$. Continuing our focus on closed paths c in E, in Section 4 we describe a Division Algorithm for arbitrary elements of $L_K(E)$ by the specific element $c-1$. With the discussion from these three sections in hand, we are then in position in Section 5 to construct the Prüfer-like $L_K(E)$-module $U_{E,c-1}$ corresponding to $c-1$. This sets the stage for our aforementioned main result (Theorem 6.4), which we present in Section 6. While one direction of the proof of Theorem 6.4 is not difficult, establishing the converse is a much heavier lift; we complete the proof in Sections 7 and 8. Along the way, we will establish in Section 8 that the endomorphism ring of $U_{E,c-1}$ is isomorphic to the ring $K[[c-1]]$ of formal power series in $c-1$ with coefficients in K, exactly as the ring of p-adic integers is isomorphic to the endomorphism ring of $\mathbb{Z}(p^\infty)$.

Unless otherwise stated, all modules are left modules. The symbol $\mathbb{N}$ denotes the set $\{0, 1, 2, \dots\}$.

2. Prüfer-like modules

In this section we develop a general ring-theoretic framework for the well-known Prüfer abelian groups $\mathbb{Z}(p^\infty)$. This framework will provide us with the appropriate context in which to construct the $L_K(E)$-modules $U_{E,c-1}$.

Let R be an associative ring with $1 \neq 0$ and $a \in R$. *For the remainder of the section we assume that a is not a right zero divisor* (i.e., that right multiplication $\rho_a : R \to R$ via $r \mapsto ra$ is a monomorphism of left R-modules), *and that a is not left invertible* (i.e., that $Ra \neq R$). For each integer $n \in \mathbb{N}_{\geq 1}$ we define the left R-module

$$M_{R,n,a} := R/Ra^n,$$

and we denote by $\eta_{n,a}$ the canonical projection $R \to M_{R,n,a}$. By the standing assumptions on a, each $M_{R,n,a}$ is a nonzero cyclic left R-module generated by $1 + Ra^n$. Moreover, for each $1 \leq i < \ell$ we have the following monomorphism of left R-modules

$$\psi_{R,i,\ell} : M_{R,i,a} \to M_{R,\ell,a}, \quad \text{via} \quad 1 + Ra^i \mapsto a^{\ell-i} + Ra^\ell.$$

The cokernel of $\psi_{R,i,\ell}$ is equal to $M_{R,\ell,a}/R(a^{\ell-i} + Ra^\ell) = (R/Ra^\ell)/(Ra^{\ell-i}/Ra^\ell) \cong M_{R,\ell-i,a}$.

The left R-modules $M_{R,n,a}$ can be recursively characterized in a categorical way.

Proposition 2.1. *For each $n \geq 2$ the following diagram of left R-modules is a pushout.*

$$\begin{array}{ccc} R & \xrightarrow{\rho_a} & R \\ {\scriptstyle \eta_{n-1,a}}\downarrow & & \downarrow{\scriptstyle \eta_{n,a}} \\ M_{R,n-1,a} & \xrightarrow{\psi_{R,n-1,n}} & M_{R,n,a}. \end{array}$$

Proof. Clearly we have $\eta_{n,a} \circ \rho_a = \psi_{R,n-1,n} \circ \eta_{n-1,a}$. Let $f : R \to X$ and $g : M_{R,n-1,a} \to X$ be two homomorphisms of left R-modules, with $g \circ \eta_{n-1,a} = f \circ \rho_a$.

It is easy to check that setting $h(1 + Ra^n) = f(1)$ defines a left R-homomorphism $h : M_{R,n,a} \to X$ such that $h \circ \eta_{n,a} = f$ and $h \circ \psi_{R,n-1,n} = g$. □

For any $1 \leq i < \ell$, using the monomorphism $\psi_{R,i,\ell}$ allows us to identify $M_{R,i,a}$ with its image submodule inside $M_{R,\ell,a}$.

Proposition 2.2. *Suppose $a \in R$ has these two properties:*

(1) $M_{R,1,a}$ is a simple left R-module, and

(2) the equation $a\mathbb{X} = 1 + Ra^i$ has no solution in $M_{R,i,a}$ for each $1 \leq i < n$.

Then the left R-module $M_{R,n,a}$ is uniserial of length n. Specifically, $M_{R,n,a}$ has the unique composition series

$$0 < \operatorname{Im} \psi_{R,1,n} < \cdots < \operatorname{Im} \psi_{R,n-1,n} < M_{R,n,a}\ ,$$

with all the composition factors isomorphic to $M_{R,1,a}$.

Proof. By induction on n.
Let $n = 1$. By hypothesis, $M_{R,1,a}$ is simple and hence is uniserial of length 1, and the only composition series is

$$0 < M_{R,1,a}.$$

Now assume that $n > 1$. By induction, $M_{R,1,a}\ , \ldots,\ M_{R,n-1,a}$ are uniserial,

$$0 < \operatorname{Im} \psi_{R,1,n-1} < \cdots < \operatorname{Im} \psi_{R,n-2,n-1} < M_{R,n-1,a}$$

is the only composition series of $M_{R,n-1,a}$, and all composition factors are isomorphic to $M_{R,1,a}$. For clarity, in the sequel we denote by H_i the submodule $\operatorname{Im} \psi_{R,i,n}$ of $M_{R,n,a}$ for each $1 \leq i < n$. Since $H_i \cong \operatorname{Im} \psi_{R,i,n-1}$ for each $1 \leq i < n - 1$ and $H_{n-1} \cong M_{R,n-1,a}$, then

$$0 < H_1 < H_2 < \cdots < H_{n-1}$$

is the unique composition series of H_{n-1}, and all the composition factors are isomorphic to $M_{R,1,a}$. To conclude the proof, we show that if $0 \neq L$ is a submodule of $M_{R,n,a}$, then either $L = M_{R,n,a}$, or otherwise $L \leq H_{n-1}$, so that $L = H_i$ for a suitable $1 \leq i \leq n - 1$. Assume on the contrary that both $L \neq M_{R,n,a}$ and $L \nleq H_{n-1}$. Since then $H_{n-1} \lneqq H_{n-1} + L$, and the quotient $M_{R,n,a}/H_{n-1} \cong M_{R,1,a}$ is simple, we have $H_{n-1} + L = M_{R,n,a}$ and H_{n-1} is not contained in L. Therefore

$$M_{R,n,a}/(L \cap H_{n-1}) = (H_{n-1} + L)/(L \cap H_{n-1}) = H_{n-1}/(L \cap H_{n-1}) \oplus L/(L \cap H_{n-1}).$$

The left R-module $L \cap H_{n-1}$ is properly contained in H_{n-1} and hence equal to some H_j for a suitable $0 \leq j < n - 1$. Then

$$M_{R,n-j,a} \cong M_{R,n,a}/H_j = L/H_j \oplus H_{n-1}/H_j \cong L/H_j \oplus M_{R,n-1-j,a}.$$

Since the direct summands L/H_j and $M_{R,n-1-j,a}$ are not zero for each $0 \leq j < n - 1$, then $M_{R,n-j,a}$ is not indecomposable and hence not uniserial. Therefore, by the induction hypothesis, necessarily $j = 0$ and we get $M_{R,n,a} \cong L \oplus H_{n-1}$ with

$L \cong M_{R,1,a}$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\rho_a} & R & & & & \\
 & & \downarrow{\scriptstyle \eta_{n-1,a}} & \overset{\varphi}{\swarrow} & \downarrow{\scriptstyle \eta_{n,a}} & & & & \\
0 & \longrightarrow & M_{R,n-1,a} & \xrightarrow{\psi_{R,n-1,n}} & M_{R,n,a} & \longrightarrow & L \cong M_{R,1,a} & \longrightarrow & 0 \quad ;
\end{array}$$

since the last row splits, there exists the dotted arrow φ such that $\varphi \circ \rho_a = \eta_{n-1,a}$. Therefore $X = \varphi(1)$ is a solution of the equation $a\mathbb{X} = 1 + Ra^{n-1}$ in $M_{R,n-1,a}$, a contradiction to the hypothesis. □

The maps $\psi_{R,i,j} : M_{R,i,a} \to M_{R,j,a}$, $1 + Ra^i \mapsto a^{j-i} + Ra^j$, $1 \leq i \leq j$, define a direct system of monomorphisms $\{M_{R,i,a}, \psi_{R,i,j}\}_{i \leq j}$. (Here we define $a^0 = 1$.)

Definition 2.3. The *a-Prüfer module* $U_{R,a}$ is the direct limit

$$U_{R,a} = \varinjlim \{M_{R,i,a}, \psi_{R,i,j}\}_{i \leq j}.$$

We denote by $\psi_{R,i} : M_{R,i,a} \to U_{R,a}$, $i \geq 1$, the induced monomorphisms.

Under the assumptions of Proposition 2.2 the, module $U_{R,a}$ is generated by the elements $\alpha_i := \psi_{R,i}(1 + Ra^i)$, $i \geq 1$. Clearly, $M_{R,i,a} \cong R\alpha_i \leq U_{R,a}$, and

$$a\alpha_i = \begin{cases} 0 & \text{if } i = 1, \\ \alpha_{i-1} & \text{if } i > 1. \end{cases}$$

Proposition 2.4. *If $M_{R,n,a}$ is uniserial of length n for each $n \geq 1$, then the module $U_{R,a}$ is uniserial and artinian (and not noetherian).*

Proof. We show that, if $0 < N \leq U_{R,a}$, then either $N = R\alpha_j$ for a suitable $j \in \mathbb{N}_{\geq 1}$, or $N = U_{R,a}$. If N is finitely generated, since $U_{R,a} = \bigcup_i R\alpha_i$ there exists a minimal integer $j \geq 1$ such that $N \leq R\alpha_j < U_{R,a}$: in particular $U_{R,a}$ is not finitely generated and hence not noetherian. Since, by Proposition 2.2, $R\alpha_j$ is uniserial and its non-zero submodules are the $R\alpha_\ell$ for $1 \leq \ell \leq j$, we conclude $N = R\alpha_j$.
If N is not finitely generated, write $N = \varinjlim N_\lambda$, where the N_λ are the finitely generated submodules of N. For any λ, by the previous paragraph, there exists j_λ such that $N_\lambda = R\alpha_{j_\lambda}$. Since $N \neq N_\lambda$ for any λ, the sequence $(j_\lambda)_\lambda$ is unbounded, so that N contains $R\alpha_\ell$ for every $\ell \in \mathbb{N}$, and so $N = U_{R,a}$.
Hence $\{R\alpha_i \mid i \in \mathbb{N}_{\geq 1}\}$ is the lattice of the proper submodules of $U_{R,a}$. It is totally ordered and so $U_{R,a}$ is uniserial. Since any $R\alpha_i$ is of finite length, we conclude that $U_{R,a}$ is artinian. □

Remark 2.5. *Considering the direct limit of the sequences*

$$0 \longrightarrow M_{R,i,a} \xrightarrow{\psi_{R,i,\ell}} M_{R,\ell,a} \longrightarrow M_{R,\ell-i,a} \longrightarrow 0\ , \quad \ell \in \mathbb{N}_{\geq i}$$

we get the short exact sequence

$$0 \longrightarrow M_{R,i,a} \xrightarrow{\psi_{R,i}} U_{R,a} \xrightarrow{\phi_{R,i}} U_{R,a} \longrightarrow 0.$$

Therefore all the proper quotients of $U_{R,a}$ are isomorphic to $U_{R,a}$. Each $\phi_{R,i}$ is a surjective, locally nilpotent endomorphism with kernel of finite length: therefore $U_{R,a}$ is a Prüfer module also in the sense of Ringel [13].

Example 2.6. If $R = \mathbb{Z}$ and $a = p$ is a prime number, then $M_{\mathbb{Z},i,p} = \mathbb{Z}/p^i\mathbb{Z}$, and $U_{\mathbb{Z},p}$ is the standard Prüfer abelian group $\mathbb{Z}(p^\infty)$.

Let $\varepsilon \in R$ be an idempotent such that $R = R\varepsilon R$. Then [4, Section 22] the rings R and $S := \varepsilon R\varepsilon$ are Morita equivalent; the Morita equivalence is induced by the functors:

$$\mathrm{Hom}_R(R\varepsilon, -) : R\text{-Mod} \rightleftarrows S\text{-Mod} : R\varepsilon \otimes_S -.$$

It is well known (and easy to verify) that, for each left R-module M, the map $\varphi \mapsto \varphi(\varepsilon)$ defines a natural isomorphism between the left S-modules $\mathrm{Hom}_R(R\varepsilon, M)$ and εM.

Proposition 2.7. *Let $\varepsilon \in R$ with $\varepsilon^2 = \varepsilon$ and $R = R\varepsilon R$. Set $S = \varepsilon R\varepsilon$. Assume $a \in R$ has these two properties:*

(1) $\varepsilon a = a\varepsilon$, and
(2) $a(1-\varepsilon) = u(1-\varepsilon)$ for some invertible central element u of R.

Then $\varepsilon a = \varepsilon a\varepsilon$ is neither a right zero divisor nor left invertible in S. Moreover, the Morita equivalence between the rings R and S sends the direct system of monomorphisms $\{M_{R,i,a}, \psi_{R,i,j}\}$ to the direct system of monomorphisms $\{M_{S,i,\varepsilon a}, \psi_{S,i,j}\}$, and sends the Prüfer module $U_{R,a}$ to the Prüfer module $U_{S,\varepsilon a}$.

Proof. By (1), $\varepsilon a = \varepsilon^2 a = \varepsilon a\varepsilon$ belongs to S. If εa were a right zero divisor in S there would exist $r \in R$ such that $0 = \varepsilon a\varepsilon r\varepsilon = a\big(\varepsilon r\varepsilon\big)$, contradicting the standing assumption that a is not a right zero divisor in R. If εa were left invertible in S, there would exist $r_1 \in R$ such that $\varepsilon r_1\varepsilon a = \varepsilon$; then by (1) and (2)

$$1 = \varepsilon + (1-\varepsilon) = \varepsilon r_1\varepsilon a + u^{-1}u(1-\varepsilon) = \varepsilon r_1\varepsilon a + u^{-1}a(1-\varepsilon) = \big(\varepsilon r_1\varepsilon + u^{-1}(1-\varepsilon)\big)a,$$

contradicting the standing assumption that a is not left invertible in R.
By (1), $Sa^n = S\varepsilon a^n = S(\varepsilon a)^n$ is a left S-ideal for each $n \in \mathbb{N}$. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ra^n\varepsilon = R\varepsilon a^n & \longrightarrow & R\varepsilon & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Ra^n & \longrightarrow & R & \longrightarrow & M_{R,n,a} & \longrightarrow & 0.
\end{array}$$

Applying the functor $\mathrm{Hom}_R(R\varepsilon, -)$ we get the following commutative diagram of left S-modules with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varepsilon Ra^n\varepsilon = \varepsilon R\varepsilon a^n = Sa^n = S(\varepsilon a)^n & \longrightarrow & S & \longrightarrow & M_{S,n,\varepsilon a} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \nu_n} & & \\
0 & \longrightarrow & \varepsilon Ra^n & \longrightarrow & \varepsilon R & \longrightarrow & \varepsilon M_{R,n,a} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varepsilon Ra^n(1-\varepsilon) & \xrightarrow{\xi} & \varepsilon R(1-\varepsilon) & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

where ν_n sends $\varepsilon r\varepsilon + S(\varepsilon a)^n$ to $\varepsilon r\varepsilon + Ra^n$. By (2) $Ra^n(1-\varepsilon) = Ru^n(1-\varepsilon) = R(1-\varepsilon)$; therefore the map ξ is surjective and hence $Q = 0$. Therefore ν_n is an isomorphism and $\varepsilon r\varepsilon + Ra^n = \varepsilon r + Ra^n$: indeed

$$\varepsilon r - \varepsilon r\varepsilon = \varepsilon r(1-\varepsilon) \in Ru^n(1-\varepsilon) = Ra^n(1-\varepsilon) = R(1-\varepsilon)a^n \subseteq Ra^n.$$

We now show that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} M_{S,i,\varepsilon a} & \xrightarrow{\psi_{S,i,j}} & M_{S,j,\varepsilon a} \\ \cong \downarrow \nu_i & & \cong \downarrow \nu_j \\ \varepsilon M_{R,i,a} & \xrightarrow{\mathrm{Hom}_R(R\varepsilon, \psi_{R,i,j})} & \varepsilon M_{R,j,a}. \end{array}$$

We have:

$$\begin{aligned} \mathrm{Hom}_R(R\varepsilon, \psi_{R,i,j})\big(\nu_i(\varepsilon r\varepsilon + S(\varepsilon a)^i)\big) &= \mathrm{Hom}_R(R\varepsilon, \psi_{R,i,j})(\varepsilon r\varepsilon + Ra^i) \\ &= \varepsilon r\varepsilon a^{j-i} + Ra^j \\ &= \varepsilon r\varepsilon(\varepsilon a)^{j-i} + Ra^j \\ &= \nu_j\big(\varepsilon r\varepsilon(\varepsilon a)^{j-i} + S(\varepsilon a)^j\big) \\ &= \nu_j\big(\psi_{S,i,j}(\varepsilon r\varepsilon + S(\varepsilon a)^i\big). \end{aligned}$$

Therefore the Morita equivalence between R and S sends the direct system of monomorphisms $\{M_{R,i,a}, \psi_{R,i,j}\}$ to the direct system of monomorphisms $\{M_{S,i,\varepsilon a}, \psi_{S,i,j}\}$. Since Morita equivalences commute with direct limits, we get also that the Prüfer module $U_{R,a}$ is sent to the Prüfer module $U_{S,\varepsilon a}$. □

3. Chen simple modules over Leavitt path algebras

In this section we give a (minimalist) review of the germane notation, first about directed graphs, then about Leavitt path algebras, and finally about Chen simple modules.

A (directed) graph $E = (E^0, E^1, s, r)$ consists of a *vertex set* E^0, an *edge set* E^1, and *source* and *range* functions $s, r : E^1 \to E^0$. For $v \in E^0$, the set of edges $\{e \in E^1 \mid s(e) = v\}$ is denoted $s^{-1}(v)$. E is called *finite* in case both E^0 and E^1 are finite sets. A *path* α in E is a sequence $e_1e_2\cdots e_n$ of edges in E for which $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$. We say that such α has *length* n, and we write $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We view each vertex $v \in E^0$ as a path of length 0, and denote $v = s(v) = r(v)$. We denote the set of paths in E by $\mathrm{Path}(E)$. We say a vertex v *connects to* a vertex w in case $v = w$, or there exists a path α in E for which $s(\alpha) = v$ and $r(\alpha) = w$. A path $\gamma = e_1e_2\cdots e_n$ $(n \geq 1)$ in E is *closed* in case $r(e_n) = s(e_1)$.

Unfortunately, the phrase "simple closed path" has come to be defined as two distinct concepts in the literature. We choose in the current article to follow what now seems to be the more common usage. Specifically, for a closed path $\gamma = e_1e_2\cdots e_n$, we call γ *simple* in case $s(e_i) \neq s(e_1)$ for all $1 < i \leq n$, and we call γ *basic* in case $\gamma \neq \delta^k$ for any closed path δ and positive integer k. (In our previous article [2] we followed Chen's usage of this phrase given in [8]; in those two places, "simple closed path" means what we are now calling a "basic closed path".)

Some additional properties of closed paths will play a role in the sequel. If $\gamma = e_1 e_2 \cdots e_n$ is a closed path in E, then a path γ' of the form $e_i e_{i+1} \cdots e_n e_1 \cdots e_{i-1}$ (for any $1 \leq i \leq n$ is called a *cyclic shift* of γ. The closed path $\gamma = e_1 e_2 \cdots e_n$ in E is called a *cycle* if $s(e_i) \neq s(e_j)$ for each $i \neq j$; a *loop* if $n = 1$; a *maximal cycle* if γ is a cycle, and there are no cycles in E other than cyclic shifts of γ which connect to $s(\gamma) = s(e_1)$; and a *source cycle* (resp., *source loop*) if γ is a cycle (resp., loop), and there are no edges $e \neq e_i$ in E such that $r(e) = r(e_i)$, for $1 \leq i \leq n$. Less formally, a source cycle is a cycle for which no vertices in the graph connect to the cycle, other than those vertices which are already in the cycle.

For any field K and graph E the Leavitt path algebra $L_K(E)$ has been the focus of sustained investigation since 2004. We give here a basic description of $L_K(E)$; for additional information, see [1]. Let K be a field, and let $E = (E^0, E^1, s, r)$ be a directed graph with vertex set E^0 and edge set E^1. The *Leavitt path K-algebra $L_K(E)$ of E with coefficients in K* is the K-algebra generated by a set $\{v \mid v \in E^0\}$, together with a set of symbols $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(V) $vu = \delta_{v,u} v$ for all $v, u \in E^0$,

(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,

(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$,

(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$, and

(CK2) $v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^*$ for every $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

It is easy to show that $L_K(E)$ is unital if and only if $|E^0|$ is finite; in this case, $1_{L_K(E)} = \sum_{v \in E^0} v$. Every element of $L_K(E)$ may be written as $\sum_{i=1}^n k_i \alpha_i \beta_i^*$, where k_i is a nonzero element of K, and each of the α_i and β_i are paths in E. If $\alpha \in \text{Path}(E)$ then we may view $\alpha \in L_K(E)$, and will often refer to such α as a *real path* in $L_K(E)$; analogously, for $\beta = e_1 e_2 \cdots e_n \in \text{Path}(E)$ we often refer to the element $\beta^* = e_n^* \cdots e_2^* e_1^*$ of $L_K(E)$ as a *ghost path* in $L_K(E)$.

We assume throughout the article that E is finite. In particular, we assume that $L_K(E)$ is unital. The multiplicative identity of a ring R will be denoted by 1_R, or more simply by 1 if the context is clear.

The ideas presented in the following few paragraphs come from [8]; however, some of the notation we use here differs from that used in [8], in order to make our presentation more notationally consistent with the general body of literature regarding Leavitt path algebras.

Let p be an *infinite path in E*; that is, p is a sequence $e_1 e_2 e_3 \cdots$, where $e_i \in E^1$ for all $i \in \mathbb{N}$, and for which $s(e_{i+1}) = r(e_i)$ for all $i \in \mathbb{N}$. We emphasize that while the phrase *infinite path in E* might seem to suggest otherwise, an infinite path in E is not an element of $\text{Path}(E)$, nor may it be interpreted as an element of the path algebra KE nor of the Leavitt path algebra $L_K(E)$. (Such a path is sometimes called a *left*-infinite path in the literature.) We denote the set of infinite paths in E by E^∞.

Let c be a closed path in E. Then the path $ccc\cdots$ is an infinite path in E, which we denote by c^∞; we call such a *cyclic infinite* path. For c a closed path in

E let d be the basic closed path in E for which $c = d^n$. Then $c^\infty = d^\infty$ as elements of E^∞.

For $p = e_1e_2e_3\cdots \in E^\infty$ we denote by $\tau_{>n}(p)$ the infinite path $e_{n+1}e_{n+2}\cdots$. If p and q are infinite paths in E, we say that p and q are *tail equivalent* (written $p \sim q$) in case there exist integers m, n for which $\tau_{>m}(p) = \tau_{>n}(q)$; intuitively, $p \sim q$ in case p and q eventually become the same infinite path. Clearly $\sim$ is an equivalence relation on E^∞, and we let $[p]$ denote the $\sim$ equivalence class of the infinite path p.

The infinite path p is called *rational* in case $p \sim c^\infty$ for some closed path c. By a previous observation, we may assume without loss of generality that such c is a basic closed path. In particular, for any closed path c we have that c^∞ is rational.

Let M be a left $L_K(E)$-module. For each $m \in M$ we define the $L_K(E)$-homomorphism $\rho_m : L_K(E) \to M$, given by $\rho_m(r) = rm$. The restriction of the right-multiplication map ρ_m may also be viewed as an $L_K(E)$-homomorphism from any left ideal I of $L_K(E)$ into M.

Following [8], for any infinite path p in E we construct a simple left $L_K(E)$-module $V_{[p]}$, as follows.

Definition 3.1. Let p be an infinite path in the graph E, and let K be any field. Let $V_{[p]}$ denote the K-vector space having basis $[p]$, that is, having basis consisting of distinct elements of E^∞ which are tail-equivalent to p. For any $v \in E^0$, $e \in E^1$, and $q = f_1f_2f_3\cdots \in [p]$, define

$$v\cdot q = \begin{cases} q & \text{if } v = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \quad e\cdot q = \begin{cases} eq & \text{if } r(e) = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } e^*\cdot q = \begin{cases} \tau_{>1}(q) & \text{if } e = f_1 \\ 0 & \text{otherwise.} \end{cases}$$

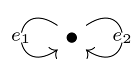
Then the K-linear extension to all of $V_{[p]}$ of this action endows $V_{[p]}$ with the structure of a left $L_K(E)$-module.

Theorem 3.2. *(*[8, Theorem 3.3]*). Let E be any directed graph and K any field. Let $p \in E^\infty$. Then the left $L_K(E)$-module $V_{[p]}$ described in Definition 3.1 is simple. Moreover, if $p, q \in E^\infty$, then $V_{[p]} \cong V_{[q]}$ as left $L_K(E)$-modules if and only if $p \sim q$, which happens precisely when $V_{[p]} = V_{[q]}$.*

We will refer to a module of the form $V_{[p]}$ as presented in Theorem 3.2 as a *Chen simple module.*

Because $V_{[c^\infty]} = V_{[(c^2)^\infty]}$ for any closed path c in E, when analyzing Chen simple modules $V_{[c^\infty]}$ we can without loss of generality assume that c is a basic closed path. Observe that if $c = e_1\cdots e_n$ and d are two basic closed paths, then $[c^\infty] = [d^\infty]$ if and only if $d = e_ie_{i+1}\cdots e_ne_1\cdots e_{i-1}$ for a suitable $1 \le i \le n$.

Example 3.3. *Let $E = R_2$ be the rose with two petals:*



Then, for example, the infinite paths $p = e_1e_2^2e_1e_2^2e_1e_2^2\cdots$ and $q = e_1e_2e_1e_2e_1e_2\cdots$ are rational paths which are not tail equivalent.

For the sake of completeness and reader convenience, we state and briefly sketch proofs of the following two lemmas. These include, in the case of a finite graph, some slight generalization of the results obtained in [2, Lemma 2.5, Proposition 2.6, Lemma 2.7, Theorem 2.8].

Lemma 3.4. *Let E be a finite graph.*

(1) Let c be a closed path in E, and $r \in L_K(E)$. Then $r(c-1)=0$ in $L_K(E)$ if and only if $r=0$.

(2) Let c be a basic closed path in E. Let $\alpha, \beta \in \mathrm{Path}(E)$ for which $0 \neq \alpha c^\infty = \beta c^\infty$ in $V_{[c^\infty]}$. Suppose also that $\alpha \neq \gamma c^N$ and $\beta \neq \delta c^{N'}$ for any $\gamma, \delta \in \mathrm{Path}(E)$ and positive integers N, N'. Then $\alpha = \beta$.

(3) Let c be a basic closed path in E. Given edges $f_1, ..., f_\ell, g_1, ..., g_m$ in E, if $0 \neq f_1 \cdots f_\ell c^\infty = g_1 \cdots g_m c^\infty$ in $V_{[c^\infty]}$, then $f_1 \cdots f_\ell - g_1 \cdots g_m \in L_K(E)(c-1)$.

Proof. (1) If $r(c-1)=0$, then $r=rc$ and hence $r=rc^m$ for each $m \geq 0$. Let $r=\sum_{i=1}^t k_i\alpha_i\beta_i^*$, with α_i, β_i real paths and $k_i \in K$. Denoting by N the maximum length of the β_i's, we have that $r=rc^N$ can be written as a K-linear combination $\sum_{i=1}^t k_i\gamma_i$ of real paths γ_i's. Then, by a degree argument, from $r=rc$ we get $r=0$.

(2) and (3) Write $c=e_1e_2\cdots e_n$. Assume

$$0 \neq f_1 \cdots f_\ell c^\infty = g_1 \cdots g_m c^\infty$$

for some edges $f_1, ..., f_\ell, g_1, ..., g_m$. If $\ell = m$ then $f_i = g_i$ for each $1 \leq i \leq \ell = m$. If $m > \ell$, then there exists $j \in \mathbb{N}$ and $1 \leq k \leq n$ such that

$$f_1 \cdots f_\ell c^\infty = f_1 \cdots f_\ell c^j e_1 \cdots e_k c^\infty = g_1 \cdots g_m c^\infty$$

with $m = \ell + j \times n + k$ and $1 \leq k \leq n$, $j \geq 0$. Then by the first equality we get $c^\infty = c^j e_1 \cdots e_k c^\infty$ and so $c^\infty = e_1 \cdots e_k c^\infty$; hence $e_1 \cdots e_k = c$ since c is basic. Therefore $k=n$ and $g_1 \cdots g_m = f_1 \cdots f_\ell c^{j+1}$. This contradicts the hypotheses in (2), so we have $m=\ell$ and $f_i = g_i$ for all $1 \leq i \leq m$ in that case. Further, this yields

$$g_1 \cdots g_m - f_1 \cdots f_\ell = f_1 \cdots f_\ell(c^{j+1}-1) = f_1 \cdots f_\ell(c^j + \cdots + c + 1)(c-1) \in L_K(E)(c-1),$$

which gives (3). □

Lemma 3.5. *Let E be a finite graph, and $c = e_1 \cdots e_n$ a basic closed path in E. Denoting by $\rho_{c^\infty} : L_K(E) \to V_{[c^\infty]}$ the map $r \mapsto rc^\infty$ and by $\rho_{c-1} : L_K(E) \to L_K(E)$ the right multiplication by $c-1$, we have the following short exact sequence of left $L_K(E)$-modules:*

$$0 \longrightarrow L_K(E) \xrightarrow{\rho_{c-1}} L_K(E) \xrightarrow{\rho_{c^\infty}} V_{[c^\infty]} \longrightarrow 0.$$

Proof. The map ρ_{c-1} is a monomorphism by Lemma 3.4(1), and ρ_{c^∞} is an epimorphism by construction. Clearly $\mathrm{Im}\,\rho_{c-1} = L_K(E)(c-1) \subseteq \mathrm{Ker}\,\rho_{c^\infty}$. Assume now $r = \sum_{i=1}^t k_i\alpha_i\beta_i^*$ belongs to $\mathrm{Ker}\,\rho_{c^\infty}$, with α_i, β_i real paths and $k_i \in K$. Our aim is to prove that $\overline{r} = r + L_K(E)(c-1) = 0$ and hence $r \in L_K(E)(c-1)$. If $\alpha_i\beta_i^* c^m = 0$ for a suitable $m \geq 1$, then $\alpha_i\beta_i^* = -\alpha_i\beta_i^*(c^m-1) = -\alpha_i\beta_i^*(1+\cdots+c^{m-1})(c-1)$ and hence $\overline{\alpha_i\beta_i^*} = 0$. Therefore we can assume $\alpha_i\beta_i^* c^m \neq 0$ for all $m \geq 0$ and $1 \leq i \leq t$. It follows that $\beta_i^* = e_{j_i}^* \cdots e_2^* e_1^* (c^{m_j})^*$ for suitable $1 \leq j_i \leq t$ and $m_j \geq 0$. Since

$$e_{j_i+1} \cdots e_n - e_{j_i}^* \cdots e_2^* e_1^* (c^{m_j})^* = e_{j_i}^* \cdots e_2^* e_1^* (c^{m_j})^* (c^{m_j+1} - 1) =$$

$$= e_{j_i}^* \cdots e_2^* e_1^* (c^{m_j})^* (c^{m_j} + c^{m_j-1} + \cdots + 1)(c-1) \in L_K(E)(c-1),$$

we have

$$\overline{r} = \sum_{i=1}^{t} k_i \alpha_i \overline{e_{j_i}^* \cdots e_2^* e_1^* (c^{m_j})^*} = \sum_{i=1}^{t} k_i \overline{\alpha_i e_{j_i+1} \cdots e_n} =$$

$$= \sum_{i=1}^{s} h_i \overline{f_1 \cdots f_{j_i}}$$

where the h_i's belongs to K and the $\overline{f_1 \cdots f_{j_i}}$ are distinct elements modulo $L_K(E)(c-1)$. Therefore by Lemma 3.4(3) the $f_1 \cdots f_{j_i} c^\infty$ expressions are distinct infinite paths which are tail equivalent to c^∞, and hence linearly independent. Since $0 = rc^\infty = \sum_{i=1}^{s} h_i f_1 \cdots f_{j_i} c^\infty$, we get $h_i = 0$ for $1 \leq i \leq s$ and hence

$$\overline{r} = \sum_{i=1}^{s} h_i \overline{f_1 \cdots f_{j_i}} = 0,$$

as desired. □

The short exact sequence established in Lemma 3.5 provides a projective resolution for the Chen simple module $V_{[c^\infty]}$. In particular, we get

Corollary 3.6. *Let c be a basic closed path in E. Then $L_K(E)/L_K(E)(c-1)$ is isomorphic to the Chen simple $L_K(E)$-module $V_{[c^\infty]}$.*

4. A division algorithm in $L_K(E)$

Let c be a basic closed path in E. In this section we show how any element of $L_K(E)$ may be "divided by" $c-1$, in an analogous manner to the standard division algorithm in $\mathbb{Z}$.

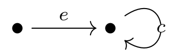
Definition 4.1. Let E be any finite graph, and c any basic closed path in E of length > 0 with $v = s(c)$. We denote by A_c the set of all non-vertex real paths α in E which are not divisible by c either on the left or on the right, but are non trivially composable with c on the right. Formally:

$$A_c = \{\alpha \in \text{Path}(E) : |\alpha| \geq 1; \alpha \neq \beta c; \text{ and } \alpha \neq c\gamma \text{ for any real paths } \beta, \gamma, \text{and } r(\alpha) = v\}.$$

For each $i \in \mathbb{N}_{\geq 1}$ we denote by $c^i A_c$ the subset $\{c^i \alpha : \alpha \in A_c\}$ of elements of $L_K(E)$. We understand $c^i A_c = \emptyset$ whenever $A_c = \emptyset$. We denote by G the K-vector subspace of $L_K(E)$ generated by $1_{L_K(E)}$, the elements in A_c and the elements in $c^i A_c$, $i \in \mathbb{N}_{\geq 1}$. That is,

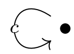
$$G := K[1_{L_K(E)}, A_c, \bigcup_{i \in \mathbb{N}_{\geq 1}} c^i A_c].$$

Example 4.2. *(1) Let E be the graph*



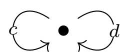
Then $A_c = \{e\}$ and $c^n A_c = \{0\}$ for each $n \geq 1$. Then G is the two dimensional vector space generated by 1 and e.

(2) Let $E = R_1$, the rose with one petal:

R_1 : 

Then $A_c = \emptyset$ (and so also $c^n A_c = \emptyset$ for each $n \geq 1$). Then G is the one dimensional vector space generated by 1.

(3) Let $E = R_2$, the rose with two petals:

R_2 : 

Then $A_c = \{d^i c^j d^k : i, k \in \mathbb{N}_{\geq 1}, j \in \mathbb{N}\}$ and $c^n A_c = \{c^n d^i c^j d^k : i, k \in \mathbb{N}_{\geq 1}, j \in \mathbb{N}\}$ for each $n \geq 1$. Then G is a countable dimensional vector space.

Remark 4.3. *Clearly the non-zero elements in $\{1_{L_K(E)}\} \cup A_c \cup \bigcup_{i \in \mathbb{N}_{\geq 1}} c^i A_c$ form a K-basis for G. Therefore a generic element g in G is of the form*

$$g = k1_{L_K(E)} + t_1 + ct_2 + c^2 t_3 + \cdots + c^{s-1} t_s$$

where $k \in K$ and t_i are K-linear combinations in $L_K(E)$ of elements in A_c. It is convenient to refer to $k1_{L_K(E)}$ as the scalar part of g: the latter commutes with any element in $L_K(E)$.
If c is a source loop, then $A_c = \emptyset$ and $c^i A_c = \emptyset$ for all $i \geq 1$: therefore G is the one-dimensional K-vector subspace of $L_K(E)$ generated by $1_{L_K(E)}$.
If $c = e_1 \cdots e_n$ is a source cycle, then $A_c = \{e_n, e_{n-1}e_n, ..., e_2 e_3 \cdots e_n\}$ and $c^i A_c = \{0\}$ for each $i \geq 1$. Therefore G is the K-vector subspace of $L_K(E)$ of dimension n generated by $1_{L_K(E)}$, and the paths $e_n, e_{n-1}e_n, ..., e_2 e_3 \cdots e_n$.
In general G is a finite dimensional space if and only if A_c is finite and cA_c is zero or empty. This happens if and only if there are no cycles different from c connected to $s(c)$, i.e., when c is a maximal cycle.

Definition 4.4. Let c be a basic closed path in E. As above, we denote by $\rho_{c^\infty} : L_K(E) \to V_{[c^\infty]}$ the right multiplication by c^∞ homomorphism. By Lemma 3.4(2), each infinite path p tail equivalent to c^∞ uniquely determines an element of $\{1_{L_K(E)}\} \cup A_c \cup \left(\bigcup_{i \in \mathbb{N}_{\geq 1}} c^i A_c \right)$, which we denote by $\sigma(p)$. Specifically, $\sigma(p)$ has the property that

$$p = \sigma(p) c^\infty = \rho_{c^\infty}(\sigma(p)).$$

Extending σ by linearity, the maps

$$\sigma : V_{[c^\infty]} \to G \quad \text{and} \quad \rho_{c^\infty|G} : G \to V_{[c^\infty]}$$

are then easily seen to be inverse isomorphisms of K-vector spaces.

Lemma 4.5. *Let c be a basic closed path in E. Then $L_K(E)(c-1) \cap G = \{0\}$.*

Proof. If $\ell = \ell_0(c-1) \in L_K(E)(c-1) \cap G$, then $\ell = \sigma(\rho_{c^\infty}(\ell))$ (by the previous observation, as $\ell \in G$), which in turn equals $\sigma(\rho_{c^\infty}(\ell_0(c-1))) = \sigma(\ell_0(c-1)c^\infty) = \sigma(\ell_0(c^\infty - c^\infty)) = \sigma(0) = 0$. □

Theorem 4.6 (Division Algorithm by $c-1$)**.** *Let E be any finite graph and K any field. Let c be a basic closed path in E. Then for any $\beta \in L_K(E)$ there exist unique $q_\beta \in L_K(E)$ and $r_\beta \in G$ such that*

$$\beta = q_\beta(c-1) + r_\beta.$$

Proof. Consider the element $r_\beta := \sigma(\rho_{c^\infty}(\beta))$; clearly r_β belongs to $G \subseteq L_K(E)$. The difference $\beta - r_\beta$ belongs to $\operatorname{Ker} \rho_{c^\infty}$, as

$$\rho_{c^\infty}(\beta - r_\beta) = \beta c^\infty - r_\beta c^\infty = \beta c^\infty - \sigma(\rho_{c^\infty}(\beta))\, c^\infty = \beta c^\infty - \beta c^\infty = 0.$$

Since $\operatorname{Ker} \rho_{c^\infty} = L_K(E)(c-1)$ by Lemma 3.5, we have $\beta - r_\beta = q_\beta(c-1)$ for a suitable $q_\beta \in L_K(E)$. Let us prove that $q_\beta \in L_K(E)$ and $r_\beta \in G$ are uniquely determined. Assume

$$\beta = q_1(c-1) + r_1 = q_2(c-1) + r_2;$$

then we have $r_1 - r_2 = (q_2 - q_1)(c-1) \in L_K(E)(c-1) \cap G$, which is 0 by Lemma 4.5. Therefore $r_1 = r_2$ and $\rho_{c-1}(q_2 - q_1) = (q_2 - q_1)(c-1) = r_1 - r_2 = 0$; since ρ_{c-1} is a monomorphism by Lemma 3.5, we have $q_1 = q_2$. □

Here are two specific applications of the Division Algorithm by $c-1$, both of which will be quite useful in the sequel.

Example 4.7. Since

$$c^n = (1 + (c-1))^n = \sum_{j=0}^{n} \binom{n}{j} (c-1)^j,$$

by Theorem 4.6 we deduce $q_{c^n} = \sum_{j=1}^{n} \binom{n}{j}(c-1)^j$, and $r_{c^n} = 1$.

Example 4.8. We will have need to multiply various elements of $L_K(E)$ on the left by $c-1$. Let $g = k1_{L_K(E)} + t_1 + ct_2 + c^2t_3 + \cdots + c^{s-1}t_s$ be an arbitrary element of G. Then multiplying and collecting appropriate terms yields

$$(c-1)g = k(c-1) - t_1 + c(t_1 - t_2) + c^2(t_2 - t_3) + \cdots + c^{s-1}(t_{s-1} - t_s) + c^s t_s.$$

So by the uniqueness part of the Division Algorithm, we get

$q_{(c-1)g} = k1_{L_K(E)}$, and $r_{(c-1)g} = -t_1 + c(t_1 - t_2) + c^2(t_2 - t_3) + \cdots + c^{s-1}(t_{s-1} - t_s) + c^s t_s$.

Note in particular that the scalar part of $r_{(c-1)g}$ is 0.

Remark 4.9. *If $E = R_1$ is the rose with one petal c, then $L_K(E) \cong K[x, x^{-1}]$ via $c \mapsto x$. In such a case the above Division Algorithm with respect to $c-1$ corresponds to the usual division by $x-1$.*

5. The Prüfer modules $U_{L_K(E),c-1}$

Let c be a basic closed path in E. By Lemmas 3.4(1) and 3.5, the element $c-1$ is neither a right zero divisor, nor left invertible. Therefore we can apply the construction of the Prüfer module given in Section 2 for $R = L_K(E)$ and $a = c-1$. For efficiency, in the sequel we use the following notation.

$$M_{E,n,c-1} := M_{L_K(E),n,c-1}; \quad \psi_{E,i,j} := \psi_{L_K(E),i,j};$$
$$U_{E,c-1} := U_{L_K(E),c-1}; \quad \text{and } \psi_{E,i} := \psi_{L_K(E),i}.$$

Most importantly for us, by Corollary 3.6 $M_{E,1,c-1} = L_K(E)/L_K(E)(c-1)$ is simple, indeed, is isomorphic to the Chen simple module $V_{[c^\infty]}$.

For the sequel, it is useful to have a canonical representation of the elements of the uniserial modules $M_{E,n,c-1}$, $n \geq 1$.

Proposition 5.1. *Let c be a basic closed path in E, $n \in \mathbb{N}$ and $x \in M_{E,n,c-1}$. Then x can be written in a unique way as*

$$x = g_1 + g_2(c-1) + \cdots + g_n(c-1)^{n-1} + L_K(E)(c-1)^n$$

*with the g_i's belonging to G. We call the displayed expression the G-*representation *of x.*

Proof. Assume $x \in M_{E,n,c-1}$, and write $x = y + L_K(E)(c-1)^n$ with $y \in L_K(E)$. Then invoking Theorem 4.6 n times we have $y = q_1(c-1)+g_1$, $q_1 = q_2(c-1)+g_2$,..., and $q_{n-1} = q_n(c-1) + g_n$. Therefore

$$x = y + L_K(E)(c-1)^n = g_1 + g_2(c-1) + \cdots + g_n(c-1)^{n-1} + L_K(E)(c-1)^n,$$

where the elements g_i, $i = 1, ..., n$, belong to G. Assume now $x = g'_1 + g'_2(c-1) + \cdots + g'_n(c-1)^{n-1} + L_K(E)(c-1)^n$, where g'_i, $i = 1, ..., n$, belong to G. Then $(g_1 - g'_1) + (g_2 - g'_2)(c-1) + \cdots + (g_n - g'_n)(c-1)^{n-1}$ belongs to $L_K(E)(c-1)^n$. Therefore $g_1 - g'_1$ belongs to $L_K(E)(c-1) \cap G = 0$ (by Lemma 4.5), and hence $g_1 = g'_1$. Since multiplication by $c-1$ on the right is a monomorphism, we get that $(g_2 - g'_2) + (g_3 - g'_3)(c-1) + \cdots + (g_n - g'_n)(c-1)^{n-2}$ belongs to $L_K(E)(c-1)^{n-1}$; therefore also $g_2 - g'_2$ belongs to $L_K(E)(c-1) \cap G = 0$, and hence $g_2 = g'_2$. Repeating the same argument we get $g_i = g'_i$ for $i = 1, ..., n$. ☐

Example 5.2. *If $E = R_1$ and hence $L_K(E) \cong K[x, x^{-1}]$, then $M_{R_1,n,c-1} \cong K[x, x^{-1}]/\langle (x-1)^n \rangle$. For instance, the G representation of*

$$x^{-4} + 2 + x + K[x, x^{-1}](x-1)^3$$

can easily be shown to be

$$4 - 3(x-1) + 10(x-1)^2 + K[x, x^{-1}](x-1)^3.$$

We are now in position to show that the modules $M_{E,i,c-1}$, $i \geq 1$, satisfy the hypotheses of Propositions 2.2 and 2.4.

Proposition 5.3. *For any basic closed path c in E, the equation*

$$(c-1)\mathbb{X} = 1 + L_K(E)(c-1)^n$$

has no solution in $M_{E,n,c-1}$.

Proof. By Proposition 5.1, we have to verify that the following equation in the n variables X_1,..., X_n does not admit solutions in G^n (the direct product of n copies of G):

$$(c-1)(X_1 + X_2(c-1) + \cdots + X_n(c-1)^{n-1} + L_K(E)(c-1)^n) = 1 + L_K(E)(c-1)^n.$$

Assume on the contrary that $X_i = g_i$ (for $i = 1, ..., n$) is a solution. Let $k_i 1_{L_K(E)}$ be the scalar part of g_i. Since $(c-1)g_i = k_i(c-1) + g'_i$ for a suitable $g'_i \in G$ whose scalar part is zero (see Lemma 4.8), we would have

$$\begin{aligned}1 + L_K(E)(c-1)^n &= (c-1)(g_1 + g_2(c-1) + \cdots + g_n(c-1)^{n-1} + L_K(E)(c-1)^n)\\ &= g_1' + (g_2' + k_1)(c-1) + \cdots + (g_n' + k_{n-1})(c-1)^{n-1} + L_K(E)(c-1)^n.\end{aligned}$$

By Proposition 5.1 the G-representation of each element of $M_{E,n,c-1}$ is unique. Therefore we would have that $g_1' = 1_{L_K(E)}$ has nonzero scalar part, which yields a contradiction. □

So Corollary 3.6 and Proposition 5.3 combine with Propositions 2.2 and 2.4 to immediately yield the following key result.

Theorem 5.4. *Let c be a basic closed path in E.*

1) For each $n \in \mathbb{N}$, the $L_K(E)$-module $M_{E,n,c-1}$ has a unique composition series, with all composition factors isomorphic to $V_{[c^\infty]}$.

2) The Prüfer $L_K(E)$-module $U_{E,c-1}$ is uniserial and artinian (and not noetherian).

The left $L_K(E)$-module $U_{E,c-1}$ is generated by the elements

$$\alpha_i := \psi_{E,i}(1 + L_K(E)(c-1)^i),$$

which satisfy

$$(c-1)\alpha_i = \begin{cases} 0 & \text{if } i = 1, \\ \alpha_{i-1} & \text{if } i > 1. \end{cases}$$

Remark 5.5. *By Proposition 5.3, the equation*

$$(c-1)\mathbb{X} = 1 + L_K(E)(c-1)^n$$

has no solution in $M_{E,n,c-1}$. But identifying $M_{E,n,c-1}$ with $\psi_{E,n,n+1}(M_{E,n,c-1})$ in $M_{E,n+1,c-1}$, the same equation has the form

$$(c-1)\mathbb{X} = (c-1) + L_K(E)(c-1)^{n+1},$$

which clearly admits the solution $\mathbb{X} = 1 + L_K(E)(c-1)^{n+1}$. This observation will be crucial to study the injectivity of the Prüfer modules discussed in the following section.

If c' is a cyclic shift of the basic closed path c, then it is clear that $V_{[c^\infty]} = V_{[c'^\infty]}$. We conclude the section with a perhaps-not-surprising result which shows that the cyclic shift of a basic closed path does not affect the isomorphism class of the corresponding Prüfer module.

Proposition 5.6. *Let $c = e_1e_2\cdots e_n$ with $n \geq 2$ be a basic closed path. Denote by c_i the basic closed path $e_i \cdots e_n e_1 \cdots e_{i-1}$. Then the modules $M_{E,m,c-1}$ and $M_{E,m,c_\ell-1}$ are isomorphic for all $1 \leq \ell \leq n$ and $m \in \mathbb{N}_{\geq 1}$. In addition, the corresponding Prüfer modules $U_{E,c-1}$ and $U_{E,c_\ell-1}$ are isomorphic for all $1 \leq \ell \leq n$.*

Proof. It is easy to verify that $(c_1-1)e_1\cdots e_{\ell-1} = e_1\cdots e_{\ell-1}(c_\ell-1)$, and that $(c_\ell-1)e_\ell\cdots e_n = e_\ell\cdots e_n(c_1-1)$. So the maps $\varphi_{1,\ell} : M_{E,m,c_1-1} \to M_{E,m,c_\ell-1}$ and $\varphi_{\ell,1} : M_{E,m,c_\ell-1} \to M_{E,m,c_1-1}$ given by

$$\varphi_{1,\ell} : 1 + L_K(E)(c_1-1)^m \mapsto e_1\cdots e_{\ell-1} + L_K(E)(c_\ell-1)^m, \quad \text{and}$$

$$\varphi_{\ell,1}: 1+L_K(E)(c_\ell-1)^m \mapsto (-1)^{m-1}e_\ell\cdots e_n\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_1^{i-1}+L_K(E)(c_1-1)^m$$

are well defined. Let us prove that they are inverse isomorphisms. Denote by $\overline{r}$ both the cosets $r + L_K(E)(c_1 - 1)^m$ and $r + L_K(E)(c_\ell - 1)^m$. Then

$$\begin{aligned}\varphi_{\ell,1}\circ\varphi_{1,\ell}(\overline{1}) &= \varphi_{\ell,1}(\overline{e_1\cdots e_{\ell-1}})\\ &= \overline{(-1)^{m-1}c_1\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_1^{i-1}}\\ &= \overline{(-1)^{m-1}\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_1^{i}}\\ &= \overline{(-1)^{m-1}\big((c_1-1)^m-(-1)^m\big)}\\ &= \overline{(-1)^{m-1}(-(-1)^m)}\\ &= \overline{1}.\end{aligned}$$

Analogously

$$\begin{aligned}\varphi_{1,\ell}\circ\varphi_{\ell,1}(\overline{1}) &= \varphi_{1,\ell}\big(\overline{(-1)^{m-1}e_\ell\cdots e_n\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_1^{i-1}}\big)\\ &= \overline{(-1)^{m-1}e_\ell\cdots e_n\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_1^{i-1}e_1\cdots e_{\ell-1}}\\ &= \overline{(-1)^{m-1}e_\ell\cdots e_ne_1\cdots e_{\ell-1}\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_\ell^{i-1}}\\ &= \overline{(-1)^{m-1}\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}c_\ell^{i}}\\ &= \overline{(-1)^{m-1}\big((c_\ell-1)^m-(-1)^m\big)}\\ &= \overline{(-1)^{m-1}(-(-1)^m)}\\ &= \overline{1}.\end{aligned}$$

Again using the initial observation, it is straightforward to check the commutativity of the appropriate diagrams, which gives the second statement. □

6. Conditions for injectivity of the Prüfer modules $U_{E,c-1}$

Let E be any finite graph, and let c denote a basic closed path in E.

Of course the module $U_{E,c-1}$ mimics in many ways the classical, well-studied Prüfer groups from abelian group theory (see Example 2.6). It is well-known that the Prüfer groups are divisible $\mathbb{Z}$-modules, hence injective. With that observation as motivation, we study in the sequel the question of whether the Prüfer modules $U_{E,c-1}$ for various basic closed paths c are injective $L_K(E)$-modules. The discussion will culminate in Theorem 6.4, characterizing the injectivity solely in terms of how the basic closed path c sits in the graph E.

Proposition 6.1. *Let E be a finite graph, let c be a basic closed path in E based at $s(c) = v$, and let $U_{E,c-1}$ be the Prüfer module associated to c. Suppose that there exists a cycle $d \neq c$ which connects to v. Then $U_{E,c-1}$ is not injective.*

Proof. The set of those vertices of E which are connected to v contains the source of d. Therefore by [2, Theorem 3.10], $\mathrm{Ext}^1(V_{[d^\infty]}, V_{[c^\infty]}) \neq 0$. Utilizing Remark 2.5, we get the exact sequence

$$0 \longrightarrow V_{[c^\infty]} \cong L_K(E)\alpha_1 \hookrightarrow U_{E,c-1} \longrightarrow\!\!\!\!\rightarrow U_{E,c-1}/L_K(E)\alpha_1 \cong U_{E,c-1} \longrightarrow 0.$$

We have $\mathrm{Hom}(V_{[d^\infty]}, U_{E,c-1}) = 0$, because the only simple submodule of $U_{E,c-1}$ is isomorphic to $V_{[c^\infty]} \ncong V_{[d^\infty]}$ (see Section 3). Consequently, the standard long exact sequence for Ext^1 gives that $\mathrm{Ext}^1(V_{[d^\infty]}, U_{E,c-1}) \neq 0$, so that $U_{E,c-1}$ is not injective, as claimed. □

Example 6.2. (1) Let $E = R_n$ be the graph with one vertex and n loops. If $n \geq 2$, then for any basic closed path c the Prüfer module $U_{E,c-1}$ is not injective. Indeed we can always find a loop different from c which connects to $s(c)$.

(2) If c is a basic closed path which is not a cycle, then the Prüfer module $U_{E,c-1}$ is not injective. Indeed there exists a cycle d such that $c = \alpha d \beta$ with α, $\beta \in \mathrm{Path}(E)$, and at least one of α, β is not a vertex. Clearly d is connected to $s(c)$.

By (2) of the previous example, it remains to study the injectivity of the Prüfer modules associated to cycles. Suggested by notation used in [5], we give the following.

Definition 6.3. Let E be a finite directed graph. A cycle c based at $s(c) = v$ is said to be *maximal* if there are no cycles in E other than cyclic shifts of c which connect to v.

In particular any source cycle is maximal. We are now in position to state the main result of the article, which characterizes when the Prüfer module $U_{E,c-1}$ is injective solely in terms of how the cycle c sits in the graph E.

Theorem 6.4. *Let E be a finite graph and let c be a basic closed path in E. Let $U_{E,c-1}$ be the Prüfer module associated to c. Then $U_{E,c-1}$ is injective if and only if c is a maximal cycle.*

In case $U_{E,c-1}$ is injective, then

(1) $U_{E,c-1}$ is the injective envelope of the Chen simple module $V_{[c^\infty]}$, and

(2) $\mathrm{End}_{L_K(E)}(U_{E,c-1})$ is isomorphic to the ring $K[[x]]$ of formal power series in x.

The proof of one direction of Theorem 6.4 has already been established: if c is not a maximal cycle then $U_{E,c-1}$ is not injective by Proposition 6.1 (see also Example 6.2(2)). Establishing the converse implication will be a more difficult task, and will take up the remainder of this article. The strategy is to start by reducing to the case when c is a source loop, and then subsequently prove the result in this somewhat more manageable configuration.

7. Reduction from the general case to the source loop case

We assume now that $L_K(E)$ is the Leavitt path algebra of a finite graph E which contains a maximal cycle c based at v. Then, as noted in Remark 4.3, A_c is a finite set. We show that we can reduce to the case where c is a source cycle (i.e., c is a cycle for which $A_c = \emptyset$).

Let $z \in E^0$ be a source vertex which is the source of a path entering on the cycle c; set $\varepsilon := 1 - z$. By [5, Lemma 4.3], the Leavitt path algebras $L_K(E)$ and $S = \varepsilon L_K(E)\varepsilon \cong L_K(E \setminus z)$ are Morita equivalent. Note that c is a cycle in the graph $E \setminus z$. Since

(1) $c - 1$ is neither a right zero divisor nor left invertible in $L_K(E \setminus z)$,
(2) $(c-1)\varepsilon = \varepsilon(c-1)$, and
(3) $(c-1)(1-\varepsilon) = -(1-\varepsilon)$,

we can apply Proposition 2.7 to yield that the Prüfer $L_K(E)$-module $U_{E,c-1} = \varinjlim_n M_{E,n,c-1}$ corresponds under the equivalence to the Prüfer $L_K(E \setminus z)$-module $U_{E\setminus z,c-1} = \varinjlim_n M_{E\setminus z,n,c-1}$. Moreover, the original Prüfer $L_K(E)$-module $U_{E,c-1}$ is the injective envelope of the Chen simple $L_K(E)$-module $V_{[c^\infty]}$ if and only if the Prüfer $L_K(E \setminus z)$-module $U_{E\setminus z,c-1}$ is the injective envelope of the Chen simple $L_K(E \setminus z)$-module $V_{[c^\infty]}$.

Thus by means of a finite number of "source eliminations" we then may reduce E to a subgraph which contains c, and in which c is a source cycle, for which the Prüfer modules correspond.

The second step is to show that we can indeed further reduce to the case in which c is a source loop. Assume $L_K(E)$ is a Leavitt path algebra with a source cycle c based on the vertex v. Assume c has length > 1 (i.e., that c is not a source loop). Let $v := v_1, v_2, ..., v_n$ be the vertices of the cycle c and $U = E^0 \setminus \{v_2, ..., v_n\}$. Consider the idempotent $\varepsilon := \sum_{u \in U} u$. As proved in [5, Lemma 4.4], $L_K(E)\varepsilon L_K(E) = L_K(E)$ and therefore $L_K(E)$ is Morita equivalent to $S := \varepsilon L_K(E)\varepsilon$. Since

(1) $c - 1$ is neither a right zero divisor nor left invertible,
(2) $(c-1)\varepsilon = \varepsilon(c-1)$, and
(3) $(c-1)(1-\varepsilon) = -(1-\varepsilon)$,

by Proposition 2.7 the uniserial left $L_K(E)$-module $M_{E,n,c-1}$ corresponds in the Morita equivalence to $M_{S,n,\varepsilon(c-1)}$.

Let F be the graph (F^0, F^1) defined by:

- $F^0 = E^0 \setminus \{v_2, ..., v_n\}$;
- $s_F^{-1}(w) = s_E^{-1}(w)$ for each $w \neq v$;
- $s_F^{-1}(v) = \{d\} \cup \bigcup_{i=1}^{n} \{f_g : g \in s_E^{-1}(v_i), r(g) \notin \{v_1, ..., v_n\}\}$ where d is a loop with $r(d) = v$ and the f_g's are new edges with $r(f_g) = r(g)$.

Then, as described in [5], the map $\theta : L_K(F) \to L_K(E)$ defined by

- $\theta(w) = w$ for each $w \in F^0$,
- $\theta(e) = e$ for all e with $s(e) \in F^0 \setminus \{v\}$,

- $\theta(f_g) = e_1 \cdots e_{i-1} g$ for each $g \in s_E^{-1}(v_i)$,
- $\theta(d) = e_1 \cdots e_n = c$,

defines an isomorphism between $L_K(F)$ and the corner $S = \varepsilon L_K(E)\varepsilon$.

We now show that the left $L_K(F)$-modules $M_{S,n,\varepsilon(c-1)}$ and $M_{F,n,(d-1)}$ are isomorphic. Indeed, by Remark 4.3 and Proposition 5.1, any element x in $M_{E,n,c-1}$ can be written in a unique way as

$$x = g_1 + g_2(c-1) + \cdots + g_n(c-1)^{n-1} + L_K(E)(c-1)^n,$$

with $g_j = k_j 1_{L_K(E)} + t_{j,1}$ where $k_i \in K$ and $t_{j,1}$ is a K-linear combination of the paths $e_2 \cdots e_n, ..., e_{n-1}e_n, e_n$. Therefore, since $\varepsilon e_i e_{i+1} \cdots e_n = 0$ for each $i > 1$, the elements of $\mathrm{Hom}_{L_K(E)}(L_K(E)\varepsilon, M_{E,n,c-1}) = \varepsilon M_{E,n,c-1} \cong M_{S,n,\varepsilon(c-1)}$ are of the type

$$k_1\varepsilon + k_2\varepsilon(c-1) + \cdots + k_n\varepsilon(c-1)^{n-1} + \varepsilon L_K(E)(c-1)^n$$

with $k_1, ..., k_n \in K$. Since

$$k_1\varepsilon + k_2\varepsilon(c-1) + \cdots + k_n\varepsilon(c-1)^n = \theta\big(k_1 1_{L_K(F)} + k_2(d-1_{L_K(F)}) + \cdots + k_n(d-1_{L_K(F)})^n\big),$$

the $L_K(F)$-module $M_{S,n,\varepsilon(c-1)}$ coincides with $M_{F,n,d-1}$. Since Morita equivalence respects direct limits, the Prüfer module $U_{E,c-1} = \varinjlim_n M_{E,n,c-1}$ corresponds to the Prüfer module $U_{F,d-1} = \varinjlim_n M_{F,n,d-1}$. Moreover, the Prüfer $L_K(E)$-module $U_{E,c-1}$ is the injective envelope of the Chen simple $L_K(E)$-module $V_{[c^\infty]}$ if and only if the Prüfer $L_K(F)$-module $U_{F,d-1}$ is the injective envelope of the Chen simple $L_K(F)$-module $V_{[d^\infty]}$.

Finally, since corresponding modules in a Morita equivalence have the same endomorphism ring, summarizing the discussion of this section, we have obtained the following.

Proposition 7.1. *In order to establish Theorem 6.4, it suffices to prove that, whenever c is a source loop in E, then*

(1) $U_{E,c-1}$ *is injective, and*
(2) $\mathrm{End}_{L_K(E)}(U_{E,c-1}) \cong K[[x]]$.

8. Establishing the main result: the case when c is a source loop

Having in the previous section reduced the verification of Theorem 6.4 to the case where c is a source loop, our aim in this section is to establish precisely that.

So suppose E is a graph in which there is a source loop c based at the vertex $v = s(c)$. In this case the Chen simple module $V_{[c^\infty]}$ has K-dimension 1, i.e $V_{[c^\infty]} = \{kc^\infty \mid k \in K\}$; moreover $A_c = \emptyset$, and hence G is the K-vector subspace of $L_K(E)$ generated by $1_{L_K(E)}$ (recall Definition 4.1). By Proposition 5.1 every element of $M_{E,n,c-1}$ can be written in a unique way as

$$x = k_1 + k_2(c-1) + \cdots + k_n(c-1)^{n-1} + L_K(E)(c-1)^n$$

with the k_i's belonging to K. Therefore the elements of $U_{E,c-1}$ can be written in a unique way as K-linear combinations of the $\alpha_i = \psi_{E,i}\big(1 + L_K(E)(c-1)^i\big)$, $i \geq 1$.

Intuitively, the reason that reduction to the source loop case will provide a more manageable situation than the general case is because the coefficients on each of the $(c-1)^i$ terms in the previous display come from K (since $G = K$ in this case), and as such these coefficients are central in $L_K(E)$.

Proposition 8.1. *Let c be a source loop in E. Then*

(1) $L_K(E)(c-1)^n$ is the two-sided ideal $\mathrm{Ann}_{L_K(E)}(M_{E,n,c-1})$.
(2) The left $L_K(E)$-module $M_{E,n,c-1}$ is also a right $L_K(E)$-module, and

$$rm = mr \quad \forall r \in L_K(E), m \in M_{E,n,c-1}.$$

Thus the maps $\psi_{E,i,j} : M_{E,i,c-1} \to M_{E,j,c-1}$ are also right $L_K(E)$-module monomorphisms for any $1 \le i \le j$.
(3) $\mathrm{Ann}_{L_K(E)}(U_{E,c-1}) = \bigcap_{n\ge1} L_K(E)(c-1)^n$. and it coincides with the two-sided ideal $\langle E^0 \setminus \{s(c)\}\rangle$;
(4) The left $L_K(E)$-module $U_{E,c-1}$ is also a right $L_K(E)$-module and $\psi_{E,n} : M_{E,n,c-1} \to U_{E,c-1}$ is a right $L_K(E)$-module monomorphism. Moreover $r\alpha_i = \alpha_i r$ for each $r \in L_K(E)$ and $i \ge 1$.
(5) $u \in U_{E,c-1}$ belongs to $L_K(E)\alpha_i$ if and only if $(c-1)^i u = 0$.

Proof. (1) If $r \in \mathrm{Ann}_{L_K(E)}(M_{E,n,c-1})$, then $r(1 + L_K(E)(c-1)^n) = 0$ in $M_{E,n,c-1}$ and hence r belongs to $L_K(E)(c-1)^n$. Conversely, let $r \in L_K(E)$ and $m \in M_{E,n,c-1}$. Since $m = h_1 + h_2(c-1) + \cdots + h_n(c-1)^{n-1} + L_K(E)(c-1)^n$ where each $h_i \in K$ (using that c is a source loop; see the previous observation), we get

$$\begin{aligned} r(c-1)^n m &= r(c-1)^n(h_1 + h_2(c-1) + \cdots + h_n(c-1)^{n-1} + L_K(E)(c-1)^n) \\ &= h_1 r(c-1)^n + h_2 r(c-1)^{n+1} + \cdots + h_n r(c-1)^{2n-1} + L_K(E)(c-1)^n \\ &= 0 \end{aligned}$$

in $M_{E,n,c-1} = L_K(E)/L_K(E)(c-1)^n$. (The point here is that each h_i commutes with expressions of the form $r(c-1)^j$ because $h_i \in K$.) Hence $L_K(E)(c-1)^n \le \mathrm{Ann}_{L_K(E)}(M_{E,n,c-1})$.

(2) Since $L_K(E)(c-1)^n$ is a two-sided ideal by point (1), then $M_{E,n,c-1}$ is also a right $L_K(E)$-module via the usual action. Let $r \in L_K(E)$ and $m \in M_{E,n,c-1}$; then

$$r = k_1 + k_2(c-1) + \cdots + k_n(c-1)^{n-1} + r'(c-1)^n \quad \text{and}$$
$$m = h_1 + \cdots + h_n(c-1)^{n-1} + L_K(E)(c-1)^n,$$

where $h_1, ..., h_n, k_1, ..., k_n \in K$ and $r' \in L_K(E)$. Since $L_K(E)(c-1)^n$ is a two-sided ideal we get $rm = mr$. The right $L_K(E)$-linearity of the maps $\psi_{E,i,j}$ for each $1 \le i \le j$ follows easily.

(3) Since $U_{E,c-1} = \bigcup_{n\ge1} L_K(E)\alpha_n$ and $L_K(E)\alpha_n \cong M_{E,n,c-1}$, the first equality follows from (1). For the second, we start by noting that $E^0 \setminus \{s(c)\}$ is the set of the vertices contained in $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. Indeed, $s(c) = 1 + (1 - s(c))(c-1)$ does not belong to $L_K(E)(c-1)$, and hence neither to $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. On the other hand, any vertex $w \ne s(c)$ belongs to $\bigcap_{n\ge1} L_K(E)(c-1)^n$, because the equality $w = -w(c-1)$ can be iterated to produce the sequence

$$w = -w(c-1) = w(c-1)^2 = \cdots = (-1)^n w(c-1)^n = \cdots .$$

In [12, Theorem 4], Rangaswamy proved that an arbitrary nonzero two sided ideal I in $L_K(E)$ (for E a finite graph) is generated by the union of two sets:

(i) $I \cap E^0$ (i.e., the vertices in I), together with
(ii) a (possibly empty) set of mutually orthogonal elements of I of the form $u + \sum_{i=1}^n k_i g^i$ where $u \in E^0 \setminus I \cap E^0$, $k_1, ..., k_n$ belong to K with $k_n \neq 0$, and g is a cycle without exits in $E^0 \setminus I \cap E^0$ based at the vertex u.

In our case we have

- $\mathrm{Ann}_{L_K(E)}(U_{E,c-1}) \cap E^0 = E^0 \setminus \{s(c)\}$, and
- c is a cycle in $E^0 \setminus (I \cap E^0) = \{s(c)\}$, and is the only cycle in the only cycle based in $s(c)$ (because c is a source loop), and c has no exits in $\{s(c)\}$ (because such an exit in $\{s(c)\}$ would necessarily be a second loop at $s(c)$, contrary to c being a source loop).

Therefore $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ is generated by $E^0 \setminus \{s(c)\}$ and possibly a single element of the form $s(c) + \sum_{i=1}^n k_i c^i$ with $k_n \neq 0$. Assume that $s(c) + \sum_{i=1}^n k_i c^i \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ where $k_n \neq 0$. We have

$$s(c) = 1 + (-1)^{n-1}(1 - s(c))(c - 1)^n$$

and, by applying Lemma 4.7 to each c^i and then collecting like powers of $c - 1$, we see

$$\sum_{i=1}^n k_i c^i = \sum_{i=1}^n k_i + (\sum_{i=1}^n \binom{i}{1} k_i)(c-1) + \cdots + (\sum_{i=j}^n \binom{i}{j} k_i)(c-1)^j + \cdots + k_n(c-1)^n.$$

Therefore, using the displayed equation (and separating out the leading 1 term), we get that $s(c) + \sum_{i=1}^n k_i c^i$ is equal to

$$1+\sum_{i=1}^n k_i+\sum_{i=1}^n \binom{i}{1} k_i(c-1)+\cdots+\sum_{i=j}^n \binom{i}{j} k_i(c-1)^j+\cdots+k_n(c-1)^n+(-1)^{n-1}(1-s(c))(c-1)^n.$$

Since $\big(1 - s(c)\big)(c - 1) = -\big(1 - s(c)\big)$, the final summand $(-1)^{n-1}(1 - s(c))(c - 1)^n$ coincides with $(-1)^{m+n-1}(1 - s(c))(c - 1)^{m+n}$ for each $m \geq 0$, and so it belongs to $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. Therefore, the element $s(c) + \sum_{i=1}^n k_i c^i$ belongs to $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ if and only if

$$1 + \sum_{i=1}^n k_i + \sum_{i=1}^n \binom{i}{1} k_i(c - 1) + \cdots + \sum_{i=j}^n \binom{i}{j} k_i(c - 1)^j + \cdots + k_n(c - 1)^n$$

belongs to $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. In such a situation, the displayed element must annihilate in particular the elements α_1, ..., α_n. By successively multiplying this equation in turn by $\alpha_1, \alpha_2, \ldots, \alpha_n$, and using the displayed observation made prior to Remark 5.5, we get that

$$0 = 1 + \sum_{i=1}^n k_i = \cdots = \sum_{i=j}^n \binom{i}{j} k_i = \cdots = k_n,$$

which contradicts that $k_n \neq 0$.

(4) The first claim follows immediately by point (2). Moreover,

$$\begin{aligned} r\alpha_i &= r\psi_{E,i}(1 + L_K(E)(c-1)^i) = \psi_{E,i}(r + L_K(E)(c-1)^i) \\ &= \psi_{E,i}(1 + L_K(E)(c-1)^i)r = \alpha_i r \end{aligned}$$

for each $r \in L_K(E)$ and $i \geq 1$.

(5) Any $u \in U_{E,c-1}$ can be written as

$$u = k_1\alpha_1 + \cdots + k_n\alpha_n$$

for a suitable $n \geq 1$. Since $M_{E,j,c-1} \cong L_K(E)\alpha_j$, we have

$$(c-1)^i u = 0 \quad \forall i \geq n$$

and, if $i < n$, then

$$(c-1)^i u = k_{i+1}(c-1)^i\alpha_{i+1} + \cdots + k_n(c-1)^i\alpha_n = k_{i+1}\alpha_1 + \cdots + k_n\alpha_{n-i}.$$

Therefore $(c-1)^i u = 0$ if and only if $k_{i+1} = \cdots = k_n = 0$ if and only if $u \in L_K(E)\alpha_i$. □

Remark 8.2. *We note that although each two-sided ideal $L_K(E)(c-1)^n$ is not graded (because it contains neither c nor 1), the intersection $J = \bigcap_{n\in\mathbb{N}} L_K(E)(c-1)^n$ is graded (because it has been shown to be generated as a two-sided ideal by a set of vertices).*

Proposition 8.3. *Let c be a source loop. For any $j \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ there exists $n \in \mathbb{N}$ such that $c^{*n} j = 0$.*

Proof. By Proposition 8.1(3), any nonzero $j \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ is a K-linear combination of elements of the form $\alpha\beta^* w\gamma\delta^* \neq 0$, with α, β, γ and δ real paths and $w \neq s(c)$ a vertex in E. Let us concentrate on one of these elements. If $\alpha\beta^* w = w$ then $c^*\alpha\beta^* w\gamma\delta^* = c^* w\gamma\delta^* = 0$. If $\alpha\beta^* w = \beta^* w \neq w$ then $s(\beta^*) = r(\beta) \neq s(c)$, otherwise β would be a path which starts in $w \neq s(c)$ and ends at $s(c)$, contrary to c being a source loop; then $c^*\alpha\beta^* w\gamma\delta^* = c^*\beta^* w\gamma\delta^* = 0$.

In all the other cases $\alpha = c^t\eta_1\cdots\eta_s$ with $c \neq \eta_1 \in E^1$, $t \geq 0$ and $s \geq 1$. Then

$$(c^{t+1})^*\alpha\beta^* w\gamma\delta^* = (c^{t+1})^* c^t\eta_1\cdots\eta_s\beta^* w\gamma\delta^* = c^*\eta_1\cdots\eta_s\beta^* w\gamma\delta^* = 0.$$

Since j is a finite sum of terms of the form $\alpha\beta^* w\gamma\delta^*$, we achieve the desired conclusion. □

Proposition 8.4. *For any $\ell \in L_K(E)\setminus \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ and for any $u \in U_{E,c-1}$, there exists $X \in U_{E,c-1}$ such that $\ell X = u$. That is, u is divisible by any element in $L_K(E) \setminus \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$.*

Proof. Let us consider $u \in U_{E,c-1}$. Then, as observed at the beginning of this section, we have

$$u = k_1\alpha_n + k_2\alpha_{n-1} + \cdots + k_n\alpha_1$$

where $k_i \in K$. Since $\ell \notin \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$, by Proposition 8.1 there exists $m \in \mathbb{N}$ such that ℓ is not right-divisible by $(c-1)^m$. Therefore

$$\ell = h_1 + h_2(c-1) + \cdots + h_m(c-1)^{m-1} + q_m(c-1)^m$$

with $h_i \in K$ for $i = 1, ..., m$, $q_m \in L_K(E)$ and

$$(h_1, ..., h_m) \neq (0, 0, ..., 0).$$

Let s be the minimum index such that $h_{s+1} \neq 0$. It is not restrictive to assume $m \geq n + s$: otherwise we apply the division algorithm to q_m, q_{m+1}, ... until we get

$$\ell = h_1 + h_2(c-1) + \cdots + h_m(c-1)^{m-1} + \cdots + h_{n+s}(c-1)^{n+s-1} + q_{n+s}(c-1)^{n+s}.$$

We claim that the equation $\ell X = u$ has solutions in $L_K(E)\alpha_{n+s}$, as follows. Set $X = X_1\alpha_{n+s} + \cdots + X_{n+s-1}\alpha_2 + X_{n+s}\alpha_1$. We solve

$$\ell\big(X_1\alpha_{n+s} + \cdots + X_{n+s-1}\alpha_2 + X_{n+s}\alpha_1\big) = u,$$

that is

$$\big(h_1 + \cdots + h_m(c-1)^{m-1} + q_m(c-1)^m\big)\big(X_1\alpha_{n+s} + \cdots + X_{n+s}\alpha_1\big) = k_1\alpha_n + \cdots + k_n\alpha_1.$$

This yields the following equations in the field K:

$$h_1X_1 = 0, \ ..., \ \sum_{i=1}^{s} h_iX_{s+1-i} = 0, \ \sum_{i=1}^{s+1} h_iX_{s+2-i} = k_1,$$

$$\sum_{i=1}^{s+2} h_iX_{s+3-i} = k_2, ..., \ \sum_{i=1}^{s+n} h_iX_{s+n+1-i} = k_n.$$

Since $0 = h_1 = \cdots = h_s$ we get

$$h_{s+1}X_1 = k_1, \ h_{s+1}X_2 + h_{s+2}X_1 = k_2, ..., \sum_{i=s+1}^{s+n} h_iX_{s+n+1-i} = k_n$$

from which we obtain the values of X_1, ..., X_n. The values of X_{n+1}, ..., X_{n+s} can be chosen arbitrarily. □

Corollary 8.5. *If $0 \neq u \in U_{E,c-1}$ then $(c^*)^m u \neq 0$ for all $m \in \mathbb{N}$.*

Proof. Since $c \notin L_K(E)(c-1) \supseteq \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$, by Proposition 8.4 there exists $0 \neq x \in U_{E,c-1}$ with $cx = u$. Since $cs(c) = c$ we may assume that $s(c)x = x$. Then $0 \neq x = s(c)x = c^*cx = c^*u$. Repeating the same argument for $0 \neq c^*u \in U_{E,c-1}$, we get $(c^*)^2u \neq 0$; iterating, we get the result. □

Proposition 8.6. *Let c be a source loop in E. Let I_f be a finitely generated left ideal of $L_K(E)$, and let $\varphi : I_f \to U_{E,c-1}$ be a $L_K(E)$-homomorphism. Then there exists $\psi : L_K(E) \to U_{E,c-1}$ such that $\psi|_{I_f} = \varphi$. Consequently,* $\mathrm{Ext}^1(L_K(E)/I_f, U_{E,c-1}) = 0$.

Proof. It has been established in [3] that $L_K(E)$ is a Bézout ring, i.e., that every finitely generated left ideal of $L_K(E)$ is principal. So $I_f = L_K(E)\ell$ for some $\ell \in I_f$. Assume on one hand that $\ell \in \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$, and hence $I_f \leq \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. By Proposition 8.3, any element of $\mathrm{Ann}_{L_K(E)}(U_{E,c-1})$ is annihilated by a suitable ${c^*}^n$. Further, ${c^*}^n u \neq 0$ for any $0 \neq u \in U_{E,c-1}$ by Corollary 8.5. Thus in this case we must have $\mathrm{Hom}_{L_K(E)}(I_f, U_{E,c-1}) = 0$, so that $\varphi = 0$ and the conclusion follows trivially.
Assume on the other hand that $\ell \notin \mathrm{Ann}_{L_K(E)}(U_{E,c-1})$. By Proposition 8.4, there

exists $x \in U_{E,c-1}$ for which $\ell x = \varphi(\ell)$. Let $\psi : L_K(E) \to U_{E,c-1}$ be the extension of the map defined by setting $\psi(1) = x$. Then, for each $i = r_i\ell \in I_f$, we have

$$\psi(i) = \psi(r_i\ell) = r_i\ell\psi(1) = r_i\ell x = r_i\varphi(\ell) = \varphi(r_i\ell) = \varphi(i),$$

which establishes the desired conclusion in this case as well.
The final statement is then immediate. □

A submodule N of a module M is *pure* if for each finitely presented module F, the functor $\mathrm{Hom}(F, -)$ preserves exactness of the short exact sequence $0 \to N \to M \to M/N \to 0$. Modules that are injective with respect to pure embeddings are called *pure-injective.*

By [11, Lemma 4.2.8], a module which is linearly compact over its endomorphisms ring is pure-injective. We will prove that $U_{E,c-1}$ is artinian over the ring $\mathrm{End}(U_{E,c-1})$ and therefore linearly compact.

Proposition 8.7. *Let E be a finite graph, and c a source loop in E. Then the endomorphism ring of the left $L_K(E)$-module $U_{E,c-1}$ is isomorphic to the ring of formal power series $K[[x]]$.*

Proof. Let $\varphi \in \mathrm{End}(U_{E,c-1})$. Since

$$(c-1)^i\varphi(\alpha_i) = \varphi\big((c-1)^i\alpha_i\big) = \varphi(0) = 0,$$

by Proposition 8.1 $\varphi(\alpha_i)$ belongs to $L_K(E)\alpha_i$. Therefore

$$\varphi(\alpha_i) = h_{1,i}\alpha_i + \cdots + h_{i,i}\alpha_1.$$

Since

$$\begin{aligned} h_{1,i}\alpha_i + \cdots + h_{i,i}\alpha_1 &= \varphi(\alpha_i) = \varphi((c-1)\alpha_{i+1}) = (c-1)\varphi(\alpha_{i+1}) \\ &= (c-1)(h_{1,i+1}\alpha_{i+1} + \cdots + h_{i,i+1}\alpha_2 + h_{i+1,i+1}\alpha_1) \\ &= h_{1,i+1}\alpha_i + \cdots + h_{i,i+1}\alpha_1, \end{aligned}$$

we get $h_{j,i+1} = h_{j,i} =: h_j$ for each $1 \le j \le i$. Denote by $H_\varphi(x)$ the formal power series

$$\sum_{j=1}^{\infty} h_j x^{j-1}.$$

It is easy to check that the map $\varphi \mapsto H_\varphi(x)$ defines a ring monomorphism Φ between $\mathrm{End}(U_{E,c-1})$ and $K[[x]]$. Given any formal power series $H(x) = \sum_{j=1}^{\infty} h_j x^{j-1}$, setting

$$\varphi_H(\alpha_i) = h_1\alpha_i + \cdots + h_i\alpha_1$$

one defines an endomorphism of $U_{E,c-1}$. Indeed

$$\begin{aligned} (c-1)\varphi_H(\alpha_{i+1}) &= (c-1)(h_1\alpha_{i+1} + \cdots + h_i\alpha_2 + h_{i+1}\alpha_1) \\ &= \ h_1\alpha_i + \cdots + h_i\alpha_1 + h_{i+1}0 \\ &= \varphi_H(\alpha_i) \ = \ \varphi_H((c-1)\alpha_{i+1}). \end{aligned}$$

□

Corollary 8.8. *Any endomorphism of $U_{E,c-1}$ is the right product by a formal power series $\sum_{j=1}^{\infty} h_j(c-1)^{j-1}$ with coefficients $h_j \in K$.*

Proof. Following the notation of Proposition 8.7, the endomorphism φ_H associated to the formal power series $H = \sum_{j=1}^{\infty} h_j x^{j-1}$ sends α_i to

$$\begin{aligned} h_1\alpha_i + \cdots + h_i\alpha_1 &= (h_1 + h_2(c-1) + \cdots + h_i(c-1)^{i-1})\alpha_i \\ &= \alpha_i(h_1 + h_2(c-1) + \cdots + h_i(c-1)^{i-1}), \end{aligned}$$

where the latter equality follows by point (4) of Proposition 8.1. Since $\alpha_i(c-1)^j = (c-1)^j\alpha_i = 0$ for each $j \geq i$, we can define

$$\alpha_i \sum_{j=1}^{\infty} h_j(c-1)^{j-1} := \alpha_i(h_1 + h_2(c-1) + \cdots + h_i(c-1)^{i-1}) = \varphi_H(\alpha_i). \qquad \square$$

Proposition 8.9. *Let E be a finite graph, and c a source loop in E. Then $U_{E,c-1}$ is pure-injective. Consequently, if $(N_\alpha, f_{\alpha,\beta})$ is a direct system of left $L_K(E)$-modules and $L_K(E)$-homomorphisms, then* $\mathrm{Ext}^1(\varinjlim N_\alpha, U_{E,c-1}) = \varprojlim \mathrm{Ext}^1(N_\alpha, U_{E,c-1})$.

Proof. The left $L_K(E)$-module $U_{E,c-1}$ is the union of its submodules $\{L_K(E)\alpha_i \mid i \geq 1\}$. Let T denote the endomorphism ring $\mathrm{End}(U_{E,c-1})$. By Corollary 8.8, $L_K(E)\alpha_i$ is a right T-submodule of $U_{E,c-1}$ for each $i \geq 1$. We show that these are the unique right T-submodules of $U_{E,c-1}$.

If N is a finitely generated T-submodule of $U_{E,c-1}$, let i_N be the smallest natural number i such that $N \leq L_K(E)\alpha_i$. If $i_N = 1$, $L_K(E)\alpha_{i_N} = L_K(E)\alpha_1$ is a one dimensional K-vector space and hence a simple T-module. If $i_N \geq 2$, consider $n \in N \setminus L_K(E)\alpha_{i_N-1}$. Then

$$n = k_1\alpha_{i_N} + \cdots + k_{i_N}\alpha_1 = \alpha_{i_N}(k_1 + \cdots + k_{i_N}(c-1)^{i_N-1})$$

with $k_1 \neq 0$. Again invoking Corollary 8.8, let $\sum_{j=1}^{\infty} h_j(c-1)^{j-1}$ be the inverse of $k_1 + \cdots + k_{i_N}(c-1)^{i_N-1}$ in $K[[c-1]]$, which exists as $k_1 \neq 0$. Then

$$n\sum_{j=1}^{\infty} h_j(c-1)^{j-1} = \alpha_{i_N}$$

and hence $N = L_K(E)\alpha_{i_N}$.

If on the other hand N is not finitely generated, write $N = \varinjlim N_\lambda$, where the N_λ are the finitely generated right T-submodules of N. For any λ, by the previous paragraph, there exists j_λ such that $N_\lambda = L_K(E)\alpha_{j_\lambda}$. Since $N \neq N_\lambda$ for any λ, the sequence $(j_\lambda)_\lambda$ is unbounded and so $N = U_{E,c-1}$.

Thus, since $\{L_K(E)\alpha_i : i \geq 1\}$ has been shown to be the lattice of the proper right T-submodules of $U_{E,c-1}$, we conclude that $U_{E,c-1}$ is an artinian right T-module and hence linearly compact. By [11, Lemma 4.2.8] we get that the left $L_K(E)$-module $U_{E,c-1}$ is pure-injective. (The quoted result says: If a module is linearly compact over its endomorphism ring, then it is algebraically compact and hence pure-injective.) Therefore we may invoke [9, Lemma 3.3.4] to conclude that the functor $\mathrm{Ext}^1(-, U_{E,c-1})$ sends direct limits to inverse limits. $\square$

Proposition 8.10. *Let E be a finite graph with source loop c. Then the Prüfer module $U_{E,c-1}$ is injective. Indeed, $U_{E,c-1}$ is the injective envelope of $L_K(E)\alpha_1 \cong V_{[c^\infty]}$.*

Proof. In order to check the injectivity of $U_{E,c-1}$, we apply Baer's Lemma. We need only check that $U_{E,c-1}$ is injective relative to any short exact sequence of the form $0 \to I \to L_K(E) \to L_K(E)/I \to 0$. This is equivalent to showing that $\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) = 0$ for any left ideal I of $L_K(E)$. Write $I = \varinjlim I_\lambda$, where the I_λ are the finitely generated submodules of I. It is standard that $L_K(E)/I = \varinjlim L_K(E)/I_\lambda$. So now applying the functor $\mathrm{Ext}^1_{L_K(E)}(-, U_{E,c-1})$, we get

$$\begin{aligned}\mathrm{Ext}^1_{L_K(E)}(L_K(E)/I, U_{E,c-1}) &= \mathrm{Ext}^1_{L_K(E)}(\varinjlim L_K(E)/I_\lambda, U_{E,c-1})\\ &= \varprojlim \mathrm{Ext}^1(L_K(E)/I_\lambda, U_{E,c-1}) \quad \text{(by Proposition 8.9)}\\ &= 0 \quad \text{(by Proposition 8.6).}\end{aligned}$$

Since $L_K(E)\alpha_1$ is an essential submodule of $U_{E,c-1}$, the last statement follows. □

Proof of Theorem 6.4(1). This now follows immediately from Propositions 7.1, 8.7, and 8.10.

Acknowledgements

The first author is partially supported by a Simons Foundation Collaboration Grants for Mathematicians Award #208941. The second and third authors are supported by Progetto di Eccellenza Fondazione Cariparo "Algebraic structures and their applications: Abelian and derived categories, algebraic entropy and representation of algebras". Part of this work was carried out during a visit by the second and third authors to the University of Colorado Colorado Springs, and during a visit by the first author to the Università degli Studi di Padova. The authors are grateful for the support of these institutions.

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