A New Mathematical Framework for the Balance Sheet Dynamic Modeling

Luca Gentili, Bruno Giacomello

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Abstract

In this paper we intend to introduce a new theoretical framework that will allow us to define a class of balance sheet mathematical models. The elements of this class will enable their user to produce a dynamic representation of an entire balance sheet through finite difference linear systems characterized by the possibility to be presented as a function with a closed form formula.

After a historical introduction about the relationship between mathematics and accounting we will present an in-depth mathematical analysis of the double-entry book-keeping system, to this day the theoretical mathematical framework of the accounting practice and thinking. The results of this analysis will enable us to introduce a new framework based on the concept of a vector specifically built to describe the relationship between a single accounting item and the liquidity. We called it brick-vector since we can build a balance sheet model merging into an algebraic system the brick-vectors of all the accounting items chosen for our modelization. The brick-vector concept and the theoretical framework previously introduced will allow us to reach a dynamic representation of an entire balance sheet constituted by accounting items of our choosing that can also be presented through a closed form formula.

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Introduction

Through the present work we intend to introduce a new theoretical mathematical framework that will enable us to define a new class of balance sheet models. The elements of this class present the following features: they aim to a representation of the entire balance sheet, they model it dynamically through the use of linear finite difference systems and more importantly they can be presented as a function with a closed form formula. In section one we will review the history of the relationship between mathematics and accounting in order to present the many attempts to formalize accounting proposed until now. In section two we will analyse in depth the mathematical framework implied by the double entry book keeping system. In section three, modifying the main features of the double entry framework, we will present our new formalization while section four will propose a simple example of a balance sheet modeled through that formalization. Finally in section five we will briefly discuss possible future developments of the present work.

1 Introduction to a brief history of the relationship between mathematics and accounting

In order to introduce our new theoretical framework it is of pivotal importance to start from the study of the long and close relationship between mathematics and accounting. We think this should be done for several reasons. The first one is to show the strands of research, with their related literature, from which our new theoretical framework stems out. There has been a good deal of research aiming to the mathematical formalization of the accounting discipline, to the use of matrix algebra in order to describe the evolution of the balance sheet as an affine transformation, as well as to the modelization of the accounting practice through information technology, among many other things. The mathematical framework we intend to present moves from those works with the goal of providing a new formalization that could be felt useful, primarily, for being at the same time utilizable via computer as well as via pure mathematics.

The second reason for which we deem important to start to analyze the history of accounting and mathematics, it is of a more general fare. Apart from having a notion of the many brilliant attempts made through the course of history at employing the mathematical "technology" at disposal of the researchers in order to improve the accounting field and the economic knowledge of the firm

in general, it is significant to review this history in order to have the idea of the evolution of the general theoretical framework through which scholars and researchers have looked at the accounting discipline. Moreover we think it's worth noticing that while on the one hand academia seldom complaints about the existence of a gap between the research world and the practitioner one, on the other hand, although there has been a tumultuous change in the way accounting is thought and practiced, especially in the last fifty years, from a mathematical point of view the reference theoretical framework is still mainly the one of the double-entry book-keeping system (introduced five hundred years ago). We do not enter, here, in the discussion concerning the reasons at the core of this condition. At this point we just want to convey the idea of why we think a sound analysis of the history of the relationship between mathematics and accounting represents the foundation of a work like the one that we are developing. Naturally here we give a brief summary of this analysis presenting only the aspects of this review that are more related to the modelization that we intend to pursue. We have divided this history into two parts, the first one from ancient times to the 20th century and the second one from the 20th century to the present day.

1.1 History of accounting and mathematics from ancient times to the 20th century

The relationship between mathematics and bookkeeping or accounting is ancient and more than once, especially in antiquity, proved itself to be very close and very deep indeed.

As a matter of fact the first known formalized accounting system is considered by many to be the precursor of the writing of abstract counting and even of writing itself. We are talking about the Sumerian "token system" (Mattessich 2000) established by the Sumerians even before what is considered as the actual dawn of their civilization.

Another example of accounting giving an essential contribution to mathematics (Colebrook 1973) was the idea and justification of using not only positive but also negative numbers, by Indian mathematicians as Brahamagupta, in the 7th century, or Bhaskara, in the 12th century, in order to devise a mathematical formalization for the accounting practice, conceiving negative numerals as legitimate mathematical notions hudreds of years before the Europeans.

The decimal system itself, one of the greatest, if not the greatest scientific development of the XIII century western world, was introduced and spread through Europe mainly for accounting purposes. The great Fibonacci's "Liber Abaci" (1202) was not in small part a treatise on bookkeeping and accounting that probably relied on some preceding Arab text (Antonini 1996) at a time when the arab scientific culture was conveying much of the Indian tradition on

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the field.

So it is not surprising that the first published treatise on double entry book-keeping ("Particularis de computis et scripturis") appeared in a mathematical text. Naturally we refer to Luca Pacioli's (1494) famous "Summa de Arithmetica, geometria, proportioni et proportionalità" (Part I, Section 9, Treatise 11). There Pacioli presents, not in a theoretical way but in a very practical one, a new method for taking record in the accounting books of every economic and/or financial transaction in which a firm can be involved. We will see an in depth analysis of the mathematical features and implications of the double entry theoretical framework in the next section. Here we would like to underline how successful such method has proved itself to be over the course of the last five hundred years as today it still represents the framework into which practitioner accountants everywhere in the world develop their work, be it through the help of accounting software or be it still through the use of a pen and paper.

For what we have seen so far in classical and medieval times, without forgetting the Indian and Arab world, as well as during the Renaissance and well beyond *it was **arithmetic** that dominated the scene.*

Finally *in the 19th century **algebra** started to play an increasingly important role* in the teaching of bookkeeping and in conveying different classification schemes.

It was mainly in the 19th century that algebra was employed to express more general accounting relations starting a minor revolution that may be difficult to comprehend by modern accountants and scholars. An example of this attitude can be traced back to the many theories of accounts classes that flourished during that period and the harsh controversies that they sparkled. While one author pleaded for the equation: "Assets = Liabilities + Owner's Equity", his opponent would argue in favor of the mathematically equivalent relation: "Assets - Liabilities = Owner's Equity" , intending to emphasize a different kind of classification and introducing an extra-mathematical element that a mathematician may find trivial as for him the two equations are equivalent (Mattessich 2000). Actually in this debate we could see an early example of the issue of the relationship between the mathematical information and the economic information contained in the balance sheet, an issue that we will have to address when we will present differences and similarities between the mathematical framework we are proposing and the one implied by the double entry book keeping system.

Another important innovation, for the scope of the present work, introduced in the 19th century by several accounting scholars on the topic of the modelization of the accounting theory and practice is the use of matrices. Although their use of matrices was substantially different from the use of matrix algebra we are going to do in the mathematical framework we are about to present, we think it's important to mention at least the original contributions to the 19th century theory of accounting by August De Morgan (1846) and Giovanni Rossi (1889). Rossi did even suggest that the accounting matrix ("lo scacchiere a schede" as he called it) could be converted into a sophisticated computing device. This might be regarded as an anticipation of the computerized spreadsheets, though Rossi, like Charles Babbage (1791-1871) had a mechanical device in mind instead of

an electronic one.

Anyway although the classification algebraic schemes of the 19th and early 20th century, the impact of inflation calculations and of present value approaches helped to improve the spread of algebraic thinking in accounting, until the first half of the 20th century systematic mathematical models were rarely used. We have to wait the second half of the 20th century to witness a period through which the relationship between mathematics and accounting flourished as never before.

1.2 History of accounting and mathematics through the 20th century until now

The second half of the 20th century introduced a great variety of mathematical approaches and techniques to the study field of accounting. We will divide this enormous research work into four main areas and we shall briefly address each one of them, starting from the ones less relevant to the scope of our work and ending with those which show the closest relation towards it.

The first area of research that can be identified deals with the statistical methods, particularly sampling techniques, that were introduced to accounting and auditing. The application of statistical sampling methods to accounting and particularly to auditing (audit risk models) has become one of the most successful mathematical tools of the public accounting profession. The pioneer in this field has definitely been Laurence Vance (1950), yet his work has been eclipsed by more sophisticated sampling technics as presented in such texts as Trueblood and Cyert (1954), Trueblood and Cooper (1955), Cyert and Trueblood (1957), Stringer (1961), Arkin (1984) and many others, as well as in research papers like Ijiri and Kaplan (1971). To those efforts might be added the innumerable empirical accounting publications employing hypotheses testing and other statistical techniques that are indispensable to pursue this kind of direction. However it should be stressed that while the foundations of hypotheses testing are mathematical, so still analytical, the application of those tools is on the contrary inductive-empirical. So if those works, in theory, are still dealing with an aspect of the relationship between mathematics and accounting, in essence they fall beyond the very analytical nature of mathematics and more importantly beyond that part of the accounting theory that explores the rigorous formalization of the accounting field, often (but not always) through mathematics, namely analytical accounting (according to its meaning in the Anglo-Saxon academic world).

The second area of research concerns the employment and development of the present value approach and its evolution into the clean surplus theory. During the second half of the century the influence of economics and the emergence of finance as a subject independent of accounting gave a decisive boost to a

further exploration of the present value approach for accounting theory and practice, including statement presentation. The work of Alexander (1948) as well as the one of Corbin (1962) promoted the present value approach and gained wide notoriety. Moreover Gordon (1960) first and Peasnell (1982) later took up Preinreich's ideas and developed them all into the theory of clean surplus. However, the major breakthrough in this area came with a series of publications by Ohlson, Feltham(1995) and others (2003). There are many analytical as well as empirical attempts to develop sufficiently accurate means of predicting future earnings, being such prediction pivotal for any modern value theory of the firm, and the extended clean surplus theory of Feltham and Ohlson has definitely been one of the most popular among the many attempts. Even commercial variations of this model have been widely marketed showing an interest about it that goes beyond academia. As a matter of fact one of the ultimate goals and hopes of our present work is that of building a new mathematical framework for the balance sheet modelization capable of stimulating the flourishing of new ideas about issues like future earnings prediction or firm valuation, with special concern to the link between the firm accounting items dynamics and their most significant macroeconomic drivers.

The third area of research refers to the attempts made to axiomatize accounting, using set-theory and similar mathematical devices in order to attain rigorous formulations of the accounting principles. It all started in the 1950s and early 1960s when the relationship among mathematics and accounting took a new direction, concerning a lot of attempts to rebuild the accounting discipline on more rigorous foundations, given the dissatisfaction with the traditional framework of accounting rules and loosely connected principles. At a time when related disciplines (such as economics, finance, operations research, etc.) reached for more sophisticated mathematical methods and tools, young scholars felt the need for a more analytical and systematic approach in the construction of a conceptual framework for business accounting. This direction was actually pursued by two groups that partly competed, partly cooperated with each other. The first group engaged in an approach that could be qualified as "postulational" since although it aimed at founding the accounting discipline on a limited and sound number of hypotheses and postulates, it tended to avoid rigorous mathematical concepts. The other group instead followed a line of thought that can be defined as axiomatic since it sought a more rigorous methodology with clear assumptions mathematical theorems and corresponding proofs. And it is revealing of an attitude still present in the relationship between the academia and the world of the accounting practice how the approach of this last group was heavily attacked, at the time, for being too mathematically sophisticated, while in comparison to the more recent trend of stochastic-analytical accounting, those earlier applications of mathematical concepts were relatively moderate. The first group, composed by scholars like Moonitz (1961), Sprouse (1962), Givens (1966), got its impetus from Chambers (1955, 1957,1966), while the second one, Winborne (1962) and especially Ijiri (1965, 1967, 1971, 1975), have been stimulated by experiments in Mattessich (1957, 1964).

Although some publications, as Zeff (1982) or Slaymaker (1996) and oth-

ers, indicate that some of those endeavours may ultimately have influenced the Financial Accounting Standards Board, those conceptualizations were occasionally criticized (as Archer (1993)) and experienced quite limited academic success. Axiomatization and related analytical efforts resumed in America and Great Britain in the late 1970s, as in Orbach (1978) and Tippet (1978), and continued in general into the 1980s, 1990s, and even beyond, though by that time this area no longer occupied centre-stage. Anyway it is important to underline that on a purely theoretical level the second (the mathematical) approach influenced a wide range of scholars worldwide in examining in a fairly rigorous way the foundations of several aspects of the accounting field and the present work, apart from borrowing some of the concepts of the axiomatization literature, can be considered to be in line with that trend.

The fourth area of research is very important for the scope of our work, since it concerns the use of accounting matrices, that led to computerized spreadsheets, and more importantly the application of matrix algebra to the accounting theory and practice. Broadly speaking we could divide this area of research into two paths, which anyway in more then one occasion happened to overlap. The first one is represented by the use of matrices in order to formalize and present the practice of the accounting work, coherently with the double-entry bookkeeping system, so that it could be efficiently processed through computerized spreadsheets and a relationship between accounting and the new emerging field of information technology could be established. While the second path consists in the attempt by some scholars to describe the firm dynamics and in particular the balance sheet dynamics through the use of matrix algebra, both theoretically and via computer. As for the first path, as we have previously hinted, accounting matrices were already known in the 19th century in a way that sometimes seemed to precur the use of information technology, but the first suggestion of an electronic spreadsheet applied for accounting purposes is represented by the seminal work of Mattessich (1961) on budgeting models and system simulation. The subsequent elaboration of this idea consisted in presenting as a prototype a mathematical budget model of an entire firm, in Mattessich (1964), as well as a complete computer program (with sub-budgets for cash flows, labour costs, material costs, purchases, sales, overhead expenses with proper allocations, as well as a projected income statement and balance sheet). That is universally considered the first example of an electronic spreadsheet (wrote in Fortran IV) and more importantly the forerunner of such best-selling spreadsheet programmes for personal computer such as VisiCalc, SuperCalc, Lotus 1-2-3 and Excel, from the 1980s onward until today.

It is important to notice that although the use of computers allow to reach the goal of the system simulation, in this first branch of research matrix algebra is not used for the modelization of the firm dynamics in a mathematical way. Through the use of information technology the objective of presenting the firm in a dynamic way is obtained, but there is still no mathematically formalized description of its accounting dynamics. This is one of the most important byproducts, especially with respect to the scope of the present work, of the second path of research.

Matrix algebra seems to have been used first in macro-accounting by Leontieff (1951) and later by Fuerst (1955), then in cost accounting by two germans Pichler (1953) and Wenke (1956, 1959) and in general accounting theory by Mattessich (1957, 1964). Subsequently a flood of pertinent publications in the application of matrix algebra, linear and non-linear programming and other mathematical techniques appeared as in Rosenblatt (1957,1960). Among the books in this area the following must be emphasized: Ijiri's (1965) dissertation on goal-oriented models and the publications by William and Griffin (1964) and Corcoran (1968).

For the scope of the present research it is definitely important also to cite, the aforementioned Mattessich aside, the work by Butterworth (1972, 1974) for his modelization of the firm balance sheet in a dynamic way as well as the work by Melse (2000). Moreover the work by Tippet (2000) that not only describes the firm balance sheet through a finite difference system of the first order but also models its dynamics with the aid of stochastic mathematical tools.

Lastly we think particularly compelled to address the work by Arya et al. (1996, 2000) for its different relations to the bases of the framework that we are about to introduce. In his work, gathering especially from Mattessich and Ijiri, he and his colleagues model the evolution of the balance sheet through a linear finite difference system and then they draw a parallel between the linear algebra modelization and the double-entry bookkeeping framework in order to give a linear algebra interpretation of the error checking capability of the double-entry bookkeeping system.

1.3 History of accounting and mathematics: some remarks and conclusions.

Our mathematical framework moves from a linear algebra interpretation of the balance sheet dynamics very close to that devised by Arya, but instead of looking at the similarities between the linear algebra description and the double entry bookkeeping approach, it starts to build from the differences between the two, analyzing the reasons behind those differences.

In order to do that we have to begin from a thorough scrutiny of the double-entry bookkeeping system from a mathematical point of view. In our summary of the history of mathematics and accounting, we had of course to leave out a lot of research areas, among those we didn't address the attempts to mathematically analyze the double entry bookkeeping system but we did it only because we are going to do it in the next section.

Before that we would like to close this section with a couple of remarks on the results of all this research through history and the general state of analytical accounting, remarks that we think should guide us in the way we conduct and present our research work.

The first thing to notice when we talk about accounting research in general, is the universally perceived gap between the scientific world of academia and the practitioner world. So it is not surprising that even a giant like Mattessich (2005) has an uncertain attitude when he is called to assess the state of analytical accounting as a whole, today.

The case of the research area on axiomatization is emblematic since the postulational approach failed because it wasn't rigorous enough while the second approach failed because it was perceived by many as using too much mathematics.

Moreover the case of the research area about the use of statistical tools and the sampling techniques, which is an example of success, gives us the idea that it is important to present a research which proposes something felt as useful and attempts to formalize it using the most accessible mathematical tools capable of delivering the result.

In conclusion, for now, we can assess that the relationship between mathematics and accounting, though very fruitful, close and deep in ancient times, in the last fifty years, though still considerably fruitful, has produced mixed results. If on the one hand it has proved to be very lively, sparkling ideas in many different directions, on the other hand it has encountered several problems, in part due to shortcomings in the accounting research in general, in part due to the lack, in the accounting world, of a widespread advanced mathematical culture which would be required in order to comply with the mathematical tools that this relationship has often come to imply.

2   The theoretical mathematical framework of the double entry bookkeeping system

In order to try to establish a new theoretical framework for the balance sheet modeling, it is important to start from the analysis of the current theoretical mathematical framework into which the balance sheet is thought and written, namely the double entry bookkeeping system. This mathematical analysis will have to tell us the reasons behind the main features of the double entry approach and the goals that the double entry approach reaches through these characteristics. We think this should be done in order to understand which one of those features could be modified so that our new mathematical modelization could reach its own goals. So let's start from this analysis and more generally from a history of the encounters between mathematics and the double entry book-keeping system.

2.1 Mathematical analysis of the double entry bookkeeping system

A revealing example of the problematic attitude between the accounting academical world and the mathematical academical one, on which we have just touched above, can be represented by the history of the mathematical analysis of the double entry bookkeeping system. To this day it is little known in mathematics and it is even virtually unknown in accounting that the double entry system is based on a well-known mathematical construction of undergraduate algebra, the group of differences, in which the integers are represented as equivalence classes of ordered pairs of natural numbers. The T-accounts of double entry bookkeeping are precisely the ordered pairs of the group of differences construction. With the exception of a paragraph by D.E. Littlewood, until the very important work by Ellerman (1984), unfortunately even to this day not widely known, there is not a single mathematics book which notes that this construction is the theoretical basis of a mathematical technique applied, everyday everywhere, in the mundane world of business for over five centuries.

Through the course of history the encounters between mathematics and double entry have been so sparse that the highlights can be easily specified. A description of double entry bookkeeping was first published by the Italian mathematician Luca Pacioli in 1494. Although Pacioli's system was governed by precise rules, his presentation was in practical and non mathematical form. Let's keep in mind that as an abstract mathematical construction the group of differences seems to have been first published by Sir William Rowan Hamilton in 1837. He made no mention of bookkeeping although accountants, at the time, had been using an intuitive algebra of the ordered pairs, by them called T-accounts, for about four centuries.

Arthur Cayley (1821-1895) was one of the few later mathematicians who wrote about double entry bookkeeping.

In his presidential address to the British Association for Advancement of Science, Cayley hinted that the "notion of a negative magnitude" is "used in a very refined manner in bookkeeping by double entry".

Another brief but insightful observation was made in a semi popular work by D. E. Littlewood in which he noted that the ordered pairs in the group of differences construction function like the debit and credit balances in a bank account.

Some modern accounting theorists believe that the mathematical treatment of the double entry bookkeeping must involve transaction matrices. This is not totally correct, since this transaction matrices represent only a good formal way of representation for the transactions described otherwise through double entry bookkeeping. As a matter of fact the presentation of transactions involving scalars can be facilitated using a square array or table of scalars usually called "transaction matrix". These transaction tables were first used by the En-

glish mathematician August DeMorgan and have been popularized through the history in the ways we presented in the previous section.

Transaction tables have, however in a way, retarded the development of a mathematical formulation of double entry bookkeeping. As we will see onward in this work matrix algebra is the best mathematical tool in order to describe the balance sheet dynamics, not the mathematical essence of double entry bookkeeping. And the seminal work that thoroughly analyzed the double entry bookkeeping system through the instrument of group theory is the 1984 work by Ellerman: "The mathematics of Double Entry Bookkeeping".

2.1.1   The double entry bookkeeping system and the Pacioli group

Ellerman's analysis revolves around two main ideas. The first cornerstone of a mathematical formalization of the double entry bookkeeping system is the acknowledgement that the double entry is based on the construction of the integers (positive and negative) as equivalence classes of ordered pairs of natural numbers (so only positive).

The ordered pairs of this construction correspond to the T-accounts of the double entry bookkeeping, the left hand entry in the ordered pair corresponds to the *debit* side of the T-account and the right hand to the *credit* side.

We can borrow the notation [d // c] from Pacioli himself as the transposition of the following T-account

Debits	Credits
d	c

and we can start to show how the algebraic structure of an additive group can easily be built over the set of those ordered couples, provided of an equivalence relation. Since the label T-account will be used lately in specific accounting contexts, as long as we are dealing with the algebraic structure of this set of ordered couples we will refer to the elements of this set as T-terms.

As last preliminary remark we would like to stress the fact that the numbers in the ordered pairs, in the T-terms, are all positive numbers, in the original idea of the double entry they would have been all natural numbers.

We can define the sum between two T-terms [a // b] and [e // f] as follows:

$$[a//b] + [e//f] = [(a+e)//(b+f)]$$

The result is internal to the original set, since the numbers (a+e) and (b+f) will be two positive numbers, and it carries the algebraic properties of the usual sum.

An equivalence relation between two T-terms is defined by Ellerman in the following way: [a // b] R [e // f] if and only if the cross sums, namely the sums of a couple debit with the other credit and vice-versa, a + f = b + e, are equal.

This definition represents one of the features at the core of the double entry mathematical formalization since basically it states that two T-terms, or two T-accounts in the accounting application, are equivalent if they represent the same amount of wealth flowing into the account, or away from it, depending on which side of the equilibrium equation the account is located

$$[a//b]R[e//f] iff (a-b) = (e-f) or (b-a) = (f-e)$$

Following Ellerman, we call this group of ordered couples the Pacioli group, that will have as zero T-term the class [0//0].

There is a one to one relationship between the classes of ordered pairs in the Pacioli group and the integers (positive and negative), as well as, if we allow the elements of the couples (a, b, e, f, etc. etc.) to be positive real numbers, there would be a one to one relationship, between the classes of ordered pairs and the real numbers (positive and negative). So we can see that the Pacioli group can be interpreted as a way of expressing positive and negative numbers, only through the use of positive quantities.

2.1.2 The double entry bookkeeping system and the equilibrium equation of the balance sheet

The second cornerstone of the in depth analysis performed by Ellerman on the double entry bookkeeping system is represented by the formalization of the relationship between the Pacioli group and the balance sheet equilibrium equation, starting from the realization that the double entry method uses the Pacioli group to perform additive algebraic operations on equations.

We start the analysis of this relationship by describing a method through which we can translate or encode equations into the Pacioli group. We call zero-term a T-term that is equal to the zero T-term [0 // 0], so that basically [b // b], for every b, will be a zero-term. The translation of equations into the Pacioli group is very simple: equations between nonnegative numbers correspond to

zero-terms. As a matter of fact, given any equation where all numbers are nonnegative such as a + ...+ b = e + ... + f, we encode each left-hand-side number as a debit balance T-term, such as [a // 0], we encode every right-hand-side number as a credit balance [0 // e] and we sum them all. With this translation the original equation holds if and only if the result is a zero-term, in the case of our example the equation holds if and only if [a // 0] + ... + [b // 0] + [0 // e] + ... + [0 // f] is a zero-term.

In double entry bookkeeping, transactions must be recorded in such a way as to maintain the truth of the equilibrium equation, or the balance sheet equation:

$$Assets = Liabilities + NetWorth$$

that is transactions must be recorded by valid algebraic operations which transform equations into equations. In the Pacioli group we have just seen that an equation translates into a zero-term, so a valid algebraic operation of that sort would be an operation that transforms zero-terms (equations) into zero-terms (equations). But there is only one such operation: add a zero-term, thus a transaction must be represented by a zero-term to be added to the zero-term representing the balance sheet equation.

In bookkeeping the double entry principle is that each transaction must be recorded with equal debits and credits, the mathematical basis for this principle is that transactions are represented by zero-terms, so the debits must equal the credits (in every transaction) since a zero-term is a translation of an equation. More specifically the zero-terms arising as the representation of equations, as the balance sheet equation or the profit and loss equation, will be called equational zero-terms, while the zero-terms arising from transactions will be called transactional zero-terms. The additive algebraic operations on an equation will work according to the following scheme:

$$(original equational zero-term) + (transactional zero-term) = (final equational zero-term)$$

The last thing to specify is how to reverse the translation process, how to decode zero-terms into equations. An equational zero term is a sum of T-terms, it is not itself an equation with a left and right hand side. Indeed the T-terms can be shuffled around in any order. To decode a zero-term into an equation, one can use any criterion one wishes to divide the T-terms into two sets, left (L) and right (R), and then construct an equation according to the following

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principles: if a T-term is in the set L, for example [a // b], then decode it as the number (a - b) on the left-hand side of the equation, while if [a // b] is in the set R, decode it as (b - a) on the right-hand side of the equation. This procedure will always produce a valid equation given a zero-term.

Naturally in bookkeeping the T-accounts in the final equational zero-term would be put in the sets L and R according to the side of the initial balance sheet equation from which the accounts were originally encoded.

The last thing to underline about this second main feature of the double entry bookkeeping method is that while the first one, seen before, allowed to describe positive and negative quantities in terms only of positive numbers, in the case of the zero-terms or the equivalent equations into or from which they are translated, they provide an instant checking for the validity of the transaction they are referring to, as we are going to see in this next section.

2.2 Conclusions and remarks on the main features of the double entry bookkeeping system mathematical framework

The two main features of the double entry bookkeeping system mathematical framework, that Ellerman has identified, are probably at the basis of the method success. As a matter of fact they give to the user of the system two very useful benefits, especially from the point of view of the historical period in which it was invented as well as, at least, the first four centuries of its use.

The Pacioli group gives the opportunity to express the whole accounting of the firm, positive as well as negative flows of wealth, only through the use of positive numbers, which especially in the 15th century when the system was devised, was definitely a positive thing. Let's see more in detail this characteristic with a very simple exemplification. We have a balance sheet where the simplest of the balance sheet equations is updated:

$$Assets = Liabilities + NetWorth$$

and where we do not have temporary or flow accounts such as revenues or expenses, so basically we do not have a profit and loss statement. In this situation we will have only three accounting items, namely Assets, Liabilities and NetWorth, to which T-terms will refer that, having attached the accounting labels Assets, etc., from now on can properly be called T-accounts. It is the position of the account in the all-positive equation above that identifies the

account as a left-hand side (LHS) or debit-balance account or as a right-hand side (RHS) or credit-balance account. Now in general debiting any x to an account means adding the debit T-term [x // 0] to the T-account, while crediting x to an account means adding the credit T-term [0 // x] to the T-account.

It is a common mistake of non-accountants to think that debit means negative. But it all depends if the account is a LHS account or a RHS account, which, as in our case, can be easily assessed looking at the balance sheet equation. As a matter of fact to debit an account does not necessarily mean to subtract from the balance in the account, that is only true for credit-balance accounts, debiting a debit-balance account, like in our case Assets, means adding to the account's balance.

The second feature of the double entry bookkeeping system, the fact that the zero-term representation of the balance sheet equilibrium equation, gives a quick check of the plausibility of the transaction, for each transaction, is even more important. It remains very useful today in the everyday accounting practice, and it was definitely more so in a period when there were no computers or electronic calculators especially in relation to the recording of a high number of transactions.

Of course this checking opportunity would be present in any kind of transaction recording method that would be based on updating a complete accounting equation, since the double entry system is a system of recording transactions, but its feature relating to the zero-term encoding of equations makes it immediate after some practice. For example if, in relation to the above balance sheet, an event was formulated as the transaction of adding 200 $ to both Liabilities and NetWorth, some thought would be required to see that this formulation of a business event could not possibly be correct and much more would be required for a multiple entry transaction in an accounting system with a lot of accounts. Yet the check is immediate in the double entry system, Liabilities and NetWorth are both credit-balance accounts so the proposed transaction is a double credit transaction in violation of the double entry principle.

3   From the double entry bookkeeping system to a linear algebra modelization of the balance sheet dynamics

So far we have seen the characteristics of the double entry bookkeeping system mathematical framework, now let's approach the linear algebra modelization of the balance sheet dynamics and establish how the features previously discussed can be modified in order to achieve different goals.

We can summarize this approach presenting an example by a work of Arya et al. (2004) in which he states, among other things, that the dynamics of a balance sheet representation can be modeled through an affine transformation.

Let's consider a balance sheet that has all its accounting items at zero at the beginning of the period and then is subjected to the following transactions. Equipment is purchased for $80, let the depreciation expense be $20, the stock is sold for $100, and cash revenue is $30.

The Balance sheet at the end of the period will be the following:

Assets		Liabilities	
$Cash$	50	$Acc.\,Depreciation$	20
$Equipment$	80	$Capital\,Stock$	100
		$Income$	10
$Assets$	130	$Liabilities + Equity$	130

We are in an accounting situation that can be modeled as follows. We can define a Balance sheet vector with the above accounting items, each one having value 0 at the beginning of the period

$$[Cash_0, Equipment_0, Acc.\,Depreciation_0, Capital\,Stock_0] = \;[0, \quad 0, \quad 0, \quad 0\;]$$

and we can define a vector of transactions $\overrightarrow{C_1}$ (expressing the transactions taking place from the time step 0 to the time step 1) with values as above (equipment purchased for $80, depreciation expense of $20, stock sold for $100, cash revenue of $30) namely

$$\overrightarrow{C_1} = \left[\; ca_1, \quad cb_1, \quad cc_1, \quad cd_1 \;\right] = \left[\; 80, \quad 20, \quad 100, \quad 30 \;\right]$$

So at the end of the accounting period the Balance sheet vector will be:

$$\begin{cases} Cash_1 = -(Cash_0 + ca_1) + (Acc.\,Depreciation_0 + cc_1) + \\ \quad + (Capital\,Stock_0 + cd_1) = \\ \quad = -(0 + ca_1) + (0 + cc_1) + 0 + (cd_1) = \\ \quad = -80 + 10 + 30 = 50 \\ Equipment_1 = (Cash_0 + ca_1) = (0 + ca_1) = 80 \\ Acc.\,Depreciation_1 = (Equipment_0 + cb_1) = (0 + cb_1) = 20 \\ Capital\,Stock_1 = (Acc.\,Depreciation_0 + cc_1) = (0 + cc_1) = 100 \end{cases}$$

Naturally a system of first order linear difference equations can be expressed through a matrix representation. So considering the Balance sheet vector at the beginning of the period and the vector of transactions defined above, through the action of the following accounting matrix the Balance sheet vector at the end of the accounting period will be:

$$
\begin{bmatrix} Cash_1 \\ Equipment_1 \\ Acc.\,Depreciation_1 \\ Capital\,Stock_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0+80 \\ 0+20 \\ 0+100 \\ 0+30 \end{bmatrix} = \begin{bmatrix} 50 \\ 80 \\ 20 \\ 100 \end{bmatrix}
$$

Basically we have just established that we can generally formalize the dynamics of the balance sheet mathematically as follows:

$$
S_n = f(S_{n-1}, C_n)
$$

where S_n is the vector of the balance sheet items at time n, C_n represents the vector of the economic and financial transactions occurring (or recorded) between time $(n-1)$ and time n, and f is a linear affine function on both arguments.

Expressing the function above through a matrix representation, applied to the previous example, we will have:

$$
\begin{bmatrix} Cash_n \\ Equipment_n \\ Acc.\,Depreciation_n \\ Capital\,Stock_n \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} Cash_{n-1} + ca_n \\ Equipment_{n-1} + cb_n \\ Acc.\,Depreciation_{n-1} + cc_n \\ Capital\,Stock_{n-1} + cd_n \end{bmatrix}
$$

In the example above we have built the balance sheet vector using only four accounting items, but in general we could theoretically choose as much accounting items as we need for the particular formalization that we have in mind.

Now the first thing that we have to underline, in the formalization that we have just presented, is that we have already modified one of the two main characteristics of the double entry mathematical framework, and rightfully so,

namely the equilibrium balance sheet equation. As we have seen in the previous section the motivations behind the double entry framework can be seen as useful in the everyday accounting practice, especially at the time when it was devised, but, apart from the revolution brought by the computer technology in the last thirty years, in the practitioner field itself, here we are more interested in the mathematical possibility of the formalization we are presenting, as well as its IT applications.

So when we define the balance sheet vector, we do not present in the vector every accounting item that can be found in an equilibrium equation because one of the items, exactly because of the equation, is necessarily linearly dependent on the others.

The item that we choose not to present is the NetWorth, since its value can be devised at every moment summing up the values of all the accounting items that are present in the balance sheet vector, taking into account their sign.

We chose not to present the NetWorth, among all the items that we could have chosen, for a reason. The main purpose of the balance sheet is to record and present an amount of information about the firm, rightfully considered important. And naturally from an economic perspective the information of the net worth is of paramount importance but from a mathematical point of view is the result of all the streams of wealth going in and out of the firm. So if we track the records of the accounting items related to those flows of wealth we will have, not only already the information with respect to the net worth, but hopefully also a better mathematical description of the firm accounting dynamics.

The same will happen for the vectors describing the economic and financial transactions. Anytime we will build a system as a model of a balance sheet we will try to formalize the elements of the transaction vector linking them to accounting items, or aggregations of them, that will mostly be accounting items relating to the profit and loss statement, and again the profit (or loss) item won't be present, since it will be linearly dependent on the other transaction vector elements values.

As for the other characteristic of the double entry bookkeeping framework, meaning the fact of expressing every flow of wealth positive or negative always through the use of positive numbers, actually in the example above we did not modify it completely, since we expressed streams of negative wealth through positive numbers and then we subtracted them from the cash-flow, but nothing constraints us to simply represent every flow of wealth going to the firm through a positive number and every stream of wealth going out of the firm through a negative number.

In this case the accounting item of the equilibrium equation which is not present in our balance sheet modelization will be exactly given by the sum of all the other accounting items, since everyone will appear on the summation with its proper sign.

Now the last aspect of this formalization that we should analyze is related to the fundamental fact that one of our main goals is to express the value of the balance sheet at time n through a closed form formula.

In order to get to a workable closed form formula for the balance sheet vector,

especially in the general case in which it is formed by a reasonably high number of accounting items, the modelization shown above it is not useful. If we will be dealing with a balance sheet vector with k accounting items, we will have to work with a $k \times k$ matrix, and in order to reach the closed form formula we will have to multiply and sum several of this matrices.

Now let's analyze what is the meaning of the matrix elements, using again the example above. If we multiply the matrix and take a look just at the first equation of the resulting system:

$$Cash_n = m_{1,1}(Cash_{n-1} + ca_n) + m_{1,2}(Equipment_{n-1} + cb_n) +$$
$$+m_{1,3}(Acc.\,Depreciation_{n-1} + cc_n) + m_{1,4}(Capital\,Stock_{n-1} + cd_n)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

we can realize that the element $m_{i,j}$ regulates the flow of wealth (positive or negative) from the accounting item of position j at time (n-1), varied by the transaction of position j (happened between time n-1 and time n), to the accounting item of position i.

Basically if we are in a situation where we have modeled the balance sheet using only three accounting items:
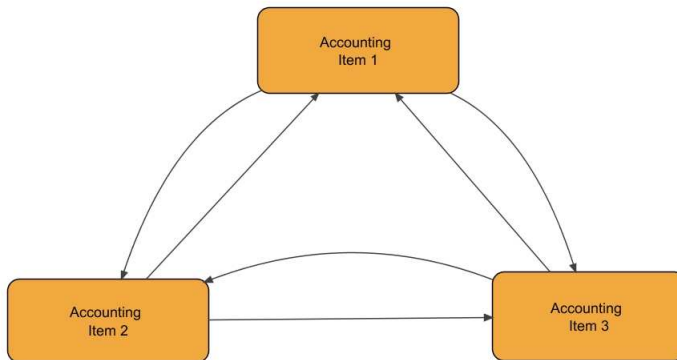
we can see, for example, that the element $m_{2,3}$ will regulate the flow of wealth, happening in the period, from the accounting item 3 to the accounting item 2 (naturally if $m_{2,3}$ assumes a negative value it will regulate a negative flow of wealth from item 3 to item 2):
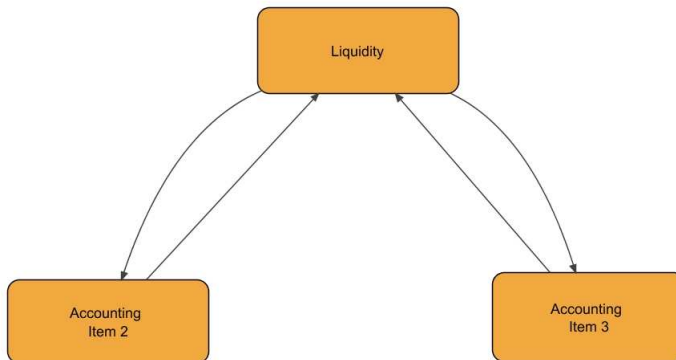
$$AccountingItem3 \longrightarrow AccountingItem2$$

and of course the element $m_{3,2}$ will regulate the flow of wealth in the other direction, namely the stream that during the time period will go from the accounting item 2 to the accounting item 3:

$$AccountingItem3 \longleftarrow AccountingItem2$$

so if we could express all this streams of wealth in a simpler way, from a mathematical standpoint, we could reach a modelization expressed by a system with a mathematical shape that will enable us to present it in a closed form formula as workable as possible.

We start from an idea expressed in the work by Ijiri (1965) on the axiomatization of the accounting discipline, the idea of u-measure. According to this idea if we express, as it is normally done, the value of every accounting item present in a balance sheet through money, specifically through a certain currency, we can consider money, that currency, as the universal unit of measurement of our balance sheet. Consequently we can identify a particular accounting item that can operate as a mediator between any two accounting items: that is the liquidity. Instead of having streams of wealth that directly move from one accounting item to another, we can model the streams of wealth so that in the same time period they all move accordingly to the following scheme, from the starting account to the liquidity and from the liquidity to the ending account:

so that basically all the accounting items will be connected one another only through the liquidity item. If we consider a single flow of wealth starting from the item A and ending in the item B, during the time window that we are facing, the fact that the modelization of this flow of wealth is made so that it passes through the liquidity item, since naturally it doesn't change the end result but only the calculation process through which this end result is reached, can be interpreted just as an evaluation procedure, coherently with the u-measure concept seen above.

Once the modelization proceeds following the above principles it is clear how it becomes of pivotal importance the way in which it is formalized the relationship of each accounting item with the liquidity. We called this formalization brick-vector, since combining the brick-vectors relating to each and every accounting item present in the balance sheet vector, other than the liquidity of course, we can build the system that describes our balance sheet.

3.1 The concept of the brick-vector

For the reasons seen in the previous section the brick-vector will be the atomic element upon which we will build the linear system that will model the dynamics of the balance sheet we are interested in. Since every account is connected to the others only through the liquidity item we can think the liquidity as the sum of a number of sub-liquidities each one related to its specific accounting item:
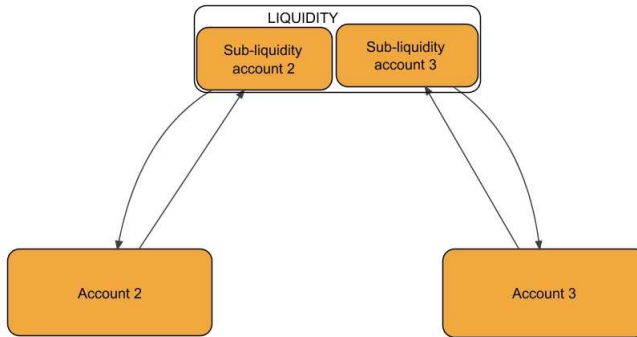
so that we can consider the brick-vector as the modelization of the relationship between each account and its specific sub-liquidity.

In the most general terms the brick-vector will assume the following shape, where, for reasons seen before, every accounting item and vector variable will be expressed in money (so everything except the elements of the array) :

$$\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} (1-\alpha_n) & \beta_n \\ \alpha_n & (1-\beta_n) \end{bmatrix} \begin{bmatrix} LI_{n-1} + A_n^{LI} \\ I_{n-1} + A_n^I \end{bmatrix} + \begin{bmatrix} P_n^{LI} \\ P_n^I \end{bmatrix} \qquad (1)$$

where I_n will be the value of a specific accounting item at time n, and LI_n will be the value of the sub-liquidity related to the account I at time n. As we can see they depend on their values at time $n-1$, and on a couple of vectors describing the transactions happened in the time window between $n-1$ and n, the vector $\overrightarrow{A}$ and the vector $\overrightarrow{P}$. The difference between the two vectors is that on $\overrightarrow{A}$ the accounting matrix acts immediately, while the vector $\overrightarrow{P}$ represents transactions whose values we want to put on certain accounts on which the accounting matrix will operate its redistribution effect the next time step. Naturally the variable A_n^I will represent the value of the transactions in $\overrightarrow{A}$ that it is due to the account I, while A_n^{LI} will represent the one due to the liquidity (via the sub-liquidity), and the same will happen with respect to the vector $\overrightarrow{P}$. Finally the parameters in the accounting matrix α_n and β_n regulate the flows of wealth between the item and the sub-liquidity and vice-versa. The first parameter α_n represents the percentage of wealth that (in the time window between $n-1$ and n) goes from the sub-liquidity, augmented by the transaction A_n^{LI}, to the account I, while β_n carries out the same task for the flow of wealth going from the item I to the liquidity.

The shape seen above for the brick vector is the most general one, a lot of times several of the above variables or parameters won't be needed and will simply carry the value zero. Let's see as example the shape of a brick-vector describing the relationship between the Receivables from clients and its relative sub-liquidity, for a small firm:

$$\begin{bmatrix} LT_n \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & \eta_n \\ 0 & (1-\eta_n) \end{bmatrix} \begin{bmatrix} LT_{n-1} + 0 \\ T_{n-1} + Z_n \end{bmatrix} \qquad (2)$$

where T will be the accounting item modeling the Receivables from clients, LT will be its sub-liquidity, Z_n will be the variable representing the sales and the parameter η_n will be the percentage of the past Receivables plus the sales happened in the time window, that are liquidated during the time window.

In this example we can see an important characteristic of the brick-vector the fact that it allows to represent the flows of wealth between the two items in two different ways, since in theory we could simply avoid the use of the accounting matrix and express the event only through the transaction vector:

$$
\left[\begin{array}{c} LT_n \\ T_n \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{c} LT_{n-1} + Z_n^{LT} \\ T_{n-1} + Z_n^{T} \end{array} \right] \tag{3}
$$

where in this example Z_n^{T} will be the part of the sales happened in the accounting period which are not liquidated, while Z_n^{LT} will be the value of the part of the sales in the accounting period that becomes cash-flow. The choice between the use of one of the two different ways, or a mix of the two, for each brick-vector (so for each accounting item), will naturally depend on what kind of information we are interested in (or is at our disposal), combined with the mathematical tools that the modelization will have to interact with, hence the mathematical goals it is set to achieve.

An example of this kind of attitude, could be even seen in the way we are about to reach the aim of getting to a closed form formula for our balance sheet vector at time n.

As a matter of fact the major shortcoming of the general formalization of the brick-vector we presented so far is that in order to combine the vectors into a single balance sheet system we have to deal with a number of extra-parameters (as much as the number of the brick-vectors), at every time step. These extra parameters should weight the contribution of each sub-liquidity on the total of the liquidity in a situation where one of the most concerning problems in order to find a workable closed form solution is represented by the sheer number of variables and parameters that inevitably a balance sheet modelization brings with itself.

A solution would be once again represented by an attempt to limit the ways through which the streams of wealth can travel between the accounting item and its sub-liquidity. The previous problem arises from the parameter α_n regulating the flow of wealth from the sub-liquidity to the accounting item. Then we should try to find ways so that the only path through which the stream can happen, in both directions, would be the one from the item to the sub-liquidity:

Sub-liquidity I
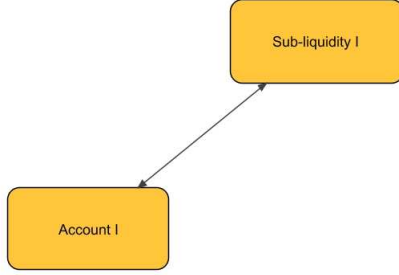
Account I

The stream in the different direction (from the sub-liquidity to the account) then could be modeled in two different ways. The first is to consider it always as a flow of wealth going from the account to the sub-liquidity but of negative value, namely allowing β_n to be negative. The second is through the use of the transaction vectors as seen in the example above. In a situation of that kind the brick vector would get the following general shape:

$$
\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & \beta_n \\ 0 & (1 - \beta_n) \end{bmatrix} \begin{bmatrix} LI_{n-1} + A_n^{LI} \\ I_{n-1} + A_n^{I} \end{bmatrix} + \begin{bmatrix} P_n^{LI} \\ P_n^{I} \end{bmatrix} \tag{4}
$$

where β_n would be in the condition of assuming negative values. And where the modelization of the vectors $\overrightarrow{A}$ and (or) $\overrightarrow{P}$ would be devised not only to model accounting items, or aggregations of accounting items coming from the profit and loss statement, but also a possible stream of wealth occurring from the sub-liquidity to the account I.

4   An example of a balance sheet modelization through a linear finite difference system

Now following the theoretical mathematical framework that we have devised so far we would like to build an example of a balance sheet modelization with its closed form formula representation. Naturally as we have already hinted before,

one of the greatest problems of the closed form setting is the high number of variables that any balance sheet imply. In the last section of this work we will discuss briefly some of the paths that we think this research should follow in order to alleviate the impact of that issue. So every time someone should approach a modelization according to the present framework, the first step should always be a careful assessment of the accounting items to use, possibly through a work of aggregation of different items. In every mathematical modelization there is a heavy trade off between the need for detail and the need for simplicity but here more than ever.

So, in order to provide a first example, we prefer to develop one of the simplest formalizations allowed by our framework. One in which the balance sheet vector will be represented only by three items: naturally the liquidity L, one accounting item for the assets except for the liquidity, we will name it S, and one for the liabilities I. In this situation we will have to develop only two brick vectors. The first will be the one expressing the relationship between our assets account and the liquidity, and it will be defined as follows:

$$
\begin{bmatrix} LS_n \\ S_n \end{bmatrix} = \begin{bmatrix} 1 & \pi_n \\ 0 & (1-\pi_n) \end{bmatrix} \begin{bmatrix} LS_{n-1}+0 \\ S_{n-1}+P_n \end{bmatrix}
\tag{5}
$$

where the variable P_n represents the aggregation of all the accounting items in the profit and loss statement bringing a positive stream of wealth from the outside world to the firm and the parameter π_n represents the percentage of that wealth that is turned into cash flow in the time window considered.

The last will be the brick vector representing the relationship between the liabilities and the liquidity and it will take this form:

$$
\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & \nu_n \\ 0 & (1-\nu_n) \end{bmatrix} \begin{bmatrix} LI_{n-1}+0 \\ I_{n-1}+N_n \end{bmatrix}
\tag{6}
$$

where in this case the variable N_n will represent the aggregation of all the accounting items in the profit and loss statement bringing a negative stream of wealth from the outside to the firm, and the parameter ν_n will be the percentage of that negative wealth that is turned into negative cash-flow in the time window considered.

As we can see, the flow of wealth goes always from the accounting item to the liquidity. Simply in the case of the liabilities I and of the variable N the values will always be negative. This choice has also the positive side that it will enable us to obtain the net worth just by summing up all the accounting items in the balance sheet vector.

The closed form formula expressing the first brick-vector at time n will be the following one:

$$\begin{cases} LS_n = \sum_{l=1}^n \pi_l(\prod_{i=1}^{l-1}(1 - \pi_i)S_0) + \sum_{l=1}^n \pi_l(\sum_{h=1}^l P_h(\prod_{j=h}^{l-1}(1 - \pi_j))) \\ \\ S_n = \prod_{i=1}^n(1 - \pi_i)S_0 + \sum_{h=1}^n(\prod_{j=h}^n(1 - \pi_j))P_h \end{cases} \tag{7}$$

while the second brick-vector will have at time n a closed form formula as follows

$$\begin{cases} LI_n = \sum_{l=1}^n \nu_l(\prod_{i=1}^{l-1}(1 - \nu_i)I_0) + \sum_{l=1}^n \nu_l(\sum_{h=1}^l N_h(\prod_{j=h}^{l-1}(1 - \nu_j))) \\ \\ I_n = \prod_{i=1}^n(1 - \nu_i)I_0 + \sum_{h=1}^n(\prod_{j=h}^n(1 - \nu_j))N_h \end{cases} \tag{8}$$

with the convention that any time $a > b$ we will have $\prod_a^b = 1$ and $\sum_a^b = 0$.

It is worth noting that the shape of the two formulas is exactly the same, as it would have been expected since the systems from which they stem have the same mathematical shape.

Now we combine the two brick-vectors into one single system expressing the whole balance sheet vector at time n. The close form formula will be: for the items the same formulas seen above for them, while for the liquidity it will be the sum of the formulas of the subl-iquidities:

$$\begin{cases} L_n = LS_n + LI_n \\ S_n = S_n \\ I_n = I_n \end{cases} \tag{9}$$

26

As we have already stated before, one of the inevitable main problems of a mathematical modelization of a balance sheet is the huge number of variables and parameters that even the simplest of modelizations necessarily involves. The framework related to the idea of the brick-vector makes no exception, and it couldn't be otherwise. The brick-vector attempts to alleviate the consequences of this issue by breaking down the final formula into smaller ones. So that even the most complex expression, the one of the liquidity, presents itself as a repetition of the same general formula combined by the sum operation.

5   Conclusions and remarks

In the present work we have introduced a new theoretical framework for the mathematical interpretation of the balance sheet that allows to build a class of models describing its evolution through the tool of the linear first order finite difference system. Moreover this framework is devised so that the modelizations would possess the feature of not only being able to be implemented through the use of information technology but also being described by a closed form formula allowing the use of tools typical of pure mathematics.

We started from a summary of the history of the relationship between accounting and mathematics, giving a particular attention to what has happened in the last fifty years. Then we moved to an in depth analysis of the mathematical characteristics of the double-entry bookkeeping system, since it represents the current theoretical framework under which the mathematical interpretation of the balance sheet is done. It could be described as the "looking glasses" through which accountants everywhere in the world, speak of the balance sheet, deals with the balance sheet and more importantly think about it.

Following the ideas by Ellerman (1984) we identified two main features of the double-entry mathematical framework, which are linked to the two main reasons behind its overwhelming success and diffusion over the last five centuries. The use only of positive numbers to describe streams of wealth both positive and negative, and the automatic checking of the correctness of each transactions record, in coherence with the fundamental balance sheet equilibrium equation. Both characteristics appear to be of pivotal importance in a situation where the accounting work needs to be done by the largest number of agents, sometimes with the smallest training possible, without the use of any sort of calculator. Like the world in which the double entry system was devised, that as far as those traits are concerned, remained nearly unaltered until fifty years ago.

Of course everything changes in a situation like ours where the accounting procedures need to be modeled in order to achieve mathematical purposes so that the model obtained can be used, via computer or pure mathematics, in the most different areas of the accounting and economic research such as corporate finance, budgeting simulation, risk management, to name a few.

So starting from the idea of formalizing the dynamics of the balance sheet through linear affine transformations and matrix algebra we decided to change the features discussed above, in order to model the balance sheet as a system. Consequently linear dependence cannot be allowed, and in order to follow as closely as possible the streams of wealth in and out of the firm (as well as among different accounting items), the use of negative values may become important.

The goal of expressing the value of the system at time n through a closed form formula, the consequent attempt to model the flow of wealth among the accounting items through the introduction of the concept of the brick-vector and the need to model the flow of wealth within the brick-vector itself through different ways, completed the introduction of our theoretical mathematical framework. Finally we gave the example of one simple model built according to our approach, one of the infinite number of models belonging to the set coherent with this work.

As we have stated more than once during this paper, one of the main goals if not the main goal of this work, is to reach a class of models that not only can be expressed through a closed form formula but that can be expressed through a workable one. Of course this would be the main problem of any balance sheet formalization, since the evolution of the balance sheet depends on a huge number of variables especially when the number of time steps n increases. Naturally this is a trait that can't be avoided by any formalization. What our formalization tries to do about that issue is to break down the problem into smaller ones, through the brick-vector concept, and then combine them in a way so that the most complex formula (the one of the liquidity) will be a summation of sub-formulas (the ones of the sub-liquidities) all having the same shape.

The present paper is part of an ongoing research about the limits and the potentials of this kind of balance sheet modelization approach, with applications mainly in the risk management field. As far as these limits are concerned we think one of the most important path of development should be about dealing with a way of reducing the impact of the number of variables present in the formalization. For instance through averaging procedures or a specific modeling of the variables and parameters time series. In the second case these variables and parameters should be seen more as a template on which to develop further formalizations.

Actually we hope that this mathematical framework and its class of models could prove to be useful in several areas of research, starting from the risk management area as well as the corporate finance field. With proper time series modelizations and proper averaging procedures, we hope the model could produce a good proxy of the dynamics of the main accounting items, linked through the time series modelizations to the main macroeconomic drivers of the most important accounts. This would prove particularly useful in a situation

where, as for the clean surplus theory, the major problem of the mathematical accounting research, as well as its applications, seems to be the linking of future projections of the values of the accounting items to the past values of the same accounts, which are the accounting data in our possession.

Another hope we want to express about the present work is instead related to the relatively simple shape that we have chosen for the brick vector. We think that a relatively simple template (on which to operate in a second moment for more details or constructions) could result more appropriate in view of the future making of a step towards the bridging of the gap between the world of the accounting academia and the world of the accounting practice. Finally we hope that the broad spectrum of models and formalizations that this theoretical mathematical framework allows could give space to researchers from different academic paths, especially from accounting and finance, to come together for, on the one hand, having a more fruitful use of the incredible amount of data which constitutes the essence of the accounting discipline, and on the other reaching a more integrated approach with the deep mathematical methods that fields like the financial one can offer.

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