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OPTIMAL CONTROL FOR THE STOCHASTIC FITZHUGH-NAGUMO MODEL WITH RECOVERY VARIABLE

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Abstract. In the present paper we derive the existence and uniqueness of the solution for the optimal control problem governed by the stochastic FitzHugh-Nagumo equation with recovery variable. Since the drift coefficient is characterized by a cubic non-linearity, standard techniques cannot be applied, instead we exploit the Ekeland's variational principle.

1. Introduction.

The mathematical formulation of the signal propagation in a neural cell has been firstly introduced by A. L. Hodgkin, and A. F. Huxley in [27], where the authors proposed a mathematical model based on a system of four non-linear, coupled differential equations describing how action potentials in neurons are initiated and propagated. In particular, the above mentioned system describes the evolution in time of four state variables. Due to the high complexity of the above model, several attempts have been tried in order to simplifies the *Hodgkin–Huxley model*. The most succesfull one is perhaps the celebrated FitzHugh-Nagumo model (FHN), see [26, 30], where the system is reduced to two equations describing the evolution in time of the (neuronal) voltage variable and of the so called *recovery variable*. It is worth to mention that the previous description, as noted by the authors in their seminal papers, is an example of relaxation oscillator. In fact, FitzHugh referred to his model as the BonhoefferVan der Pol oscillator.

During recent years, the mathematical study of the FHN model has gained great attention, particularly to consider the influence of random perturbations of the original deterministic description, see, e.g., [1, 10, 29] In fact, from the experimental point of view, many neuronal activities can be better understood allowing for random components which affect the transmission of signals, as well as the inaccuracy of laboratory measures and the lack of a complete knowledge of the particular cerebral activity under study.

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Aiming at considering such a generalized, random framework, we will analyze the following stochastic system

$$\begin{cases} \partial_t v(t,\xi) & = (\Delta_\xi - I_{ion})\, v(t,\xi) - w(t,\xi) - f(\xi)v(t,\xi) + \partial_t \beta_1(t)\,, \text{ in } [0,T]\times\mathcal{O}\,, \\ \partial_t w(t,\xi) & = \gamma v(t,\xi) - \delta w(t,\xi) + \partial_t\beta_2(t)\,, \text{ in } [0,T]\times\mathcal{O}\,, \\ \partial_\nu v(t,\xi) & = 0\,, \quad \text{ on } [0,T]\times\partial\mathcal{O}\,, \\ v(0,\xi) & = v_0(\xi), \quad w(0,\xi) = w_0(\xi)\,, \text{ in } [0,T]\times\mathcal{O}\,. \end{cases}, \tag{1}$$

where, as mentioned above, the variable v represents the voltage quantity, w denotes the recovery variable and β_1 and β_2 are two independent Brownian motions; all components appearing in equation (1) will be specified in a while. For the moment, let us note that the function I_{ion} is a polynomial of degree 3, then standard existence and uniqueness results do not hold for eq. (1), since the non-linear term I_{ion} fails to be Lipschitz continuous. Latter problem is often overcome taking into account some additional regularity properties of the infinitesimal generator, namely the Laplacian Δ appearing in eq. (1), such as the so-called $m-$dissipativity assumption, see, e.g., [2, 3, 21] and references therein, for details.

In the present paper we will consider a controlled version of equation (1) where the control variable u appear in the drift of the stochastic PDE (SPED) (1). In particular we focus our attention on the existence and uniqueness of the optimal control for above stochastic system. We would like to underline that in [6], the existence and uniqueness of an optimal control has been proven for a similar equation, but without the recovery variable w. It is worth to mentiont that deriving the existence of an optimal control in the stochastic case is a rather delicate point, mainly because technical problems arise when one tries to pass to the limit in the weak topology, fact that implies the use of non trivial results. In particular the main result of the present work is based, following [6], on the Ekelands's variational principle.

The present work is so structured, in section 2 we introduce the main notation and assumptions used throughout the work, also stating the existence and uniqueness result for the main equation of interest. Then, in section 3, we derive the main result, namely we prove the existence and uniqueness solution of the optimal control problem associated to the FH-N model with recover variable, exploiting the Ekelands's variational principle

2. **The abstract setting.**

Let us consider the following controlled stochastic FitzHugh-Nagumo system of equations

$$\begin{cases} \partial_t v(t,\xi) & = (\Delta_\xi - I_{ion})\, v(t,\xi) - w(t,\xi) - f(\xi)v(t,\xi) + B_{\mathcal{O}}u(t,\xi) + \partial_t \beta_1(t)\,, \text{ in } [0,T]\times\mathcal{O}\,, \\ \partial_t w(t,\xi) & = \gamma v(t,\xi) - \delta w(t,\xi) + \partial_t\beta_2(t)\,, \text{ in } [0,T]\times\mathcal{O}\,, \\ \partial_\nu v(t,\xi) & = 0\,, \quad \text{ on } [0,T]\times\partial\mathcal{O}\,, \\ v(0,\xi) & = v_0(\xi), \quad w(0,\xi) = w_0(\xi)\,, \text{ in } [0,T]\times\mathcal{O}\,. \end{cases}, \tag{2}$$

where $v = v(t,\xi)$ represents the transmembrane electrical potential, $w = w(t,\xi)$ is a recovery variable, also known as gating variable and which can be used to describe the potassium conductance, $\mathcal{O}\subset\mathbb{R}^d$, $d = 2,3$, is a bounded and open set with smooth boundary $\partial\mathcal{O}$. Furthermore Δ_ξ is the Laplacian operator with respect to the spatial variable ξ, while γ and δ are positive constants representing phenomenological coefficients, ν is the outer unit normal direction to the boundary $\partial\mathcal{O}$ and ∂_ν denotes the derivative in the direction ν, $f(\xi)$ is a given external forcing term, I_{ion} represents the *Ionic current* assumed to be as in the FitzHugh-Nagumo model, namely it is taken as a cubic non-linearity of the following form $I_{ion}(v) = v(v-a)(v-1)$, v_0, $w_0 \in L^2(\mathcal{O})$. and β_1 and β_2 two independent Q_i-Brownian motions, $i = 1, 2$, Q_i being positive trace class commuting operators. In particular we assume that

$$\beta_i \in C\left([0,T]; L^2\left(\Omega, L^2\left(\mathcal{O}\right)\right)\right)\,, \quad i = 1,\, 2\,,$$

with

$$\beta_i(t,\cdot) \sim \mathcal{N}\left(0, t\sqrt{Q_i}\right)\,, \quad i = 1,\, 2\,.$$

Eventually we assume that the two operators Q_1 and Q_2 diagonalize on the same basis $\{e_k\}_{k\geq 1}$, namely we assume that there exists a sequence of positive real numbers $\{\lambda_k^i\}_{k\geq 1}$, $i = 1,\, 2$ such that

$$Q_i\, e_k = \lambda_k^i\, e_k\,, \quad i = 1,\, 2\,, \quad k \geq 1\,.$$

In order to rewrite (2) in a more compact form as an infinite dimensional stochastic evolution equation, let us define the Hilbert space $H := L^2(\mathcal{O}) \times L^2(\mathcal{O})$ endowed with the inner product

$$\langle (v_1, w_1), (v_2, w_2) \rangle_H = \gamma \langle v_1, v_2 \rangle_2 + \langle w_1, w_2 \rangle_2\,, \tag{3}$$

where $\langle \cdot, \cdot \rangle_2$ denotes the usual scalar product in $L^2(\mathcal{O})$, and the corresponding norm will be indicated by $|\cdot|_2$; also $\langle \cdot, \cdot \rangle_H$, resp. $|\cdot|_H$ will indicate the scalar product, resp. the norm, in H. Let us further introduce the space $V := H^1(\mathcal{O}) \times L^2(\mathcal{O})$ with the norm

$$|X|_V^2 = \gamma |v|_{H^1}^2 + |w|_2^2\,, \quad X = (v, w) \in V\,.$$

We then define the operator $A : D(A) \subset H \to H$ as follows

$$A = \begin{pmatrix} -A_0 + f & I \\ -\gamma & \delta \end{pmatrix}\,, \quad A_0 = \Delta_\xi\,,$$

with domain given by

$$\begin{aligned} &D(A) := D(A_0) \times L^2(\mathcal{O})\,,\\ &D(A_0) := \{u \in H^2(\mathcal{O}) : \partial_\nu u(\xi) = 0 \text{ on } \partial\mathcal{O}\}. \end{aligned}$$

In particular, we have that A generates a C_0-semigroup satisfying

$$\|e^{-tA}\| \leq e^{-\omega t}\,, \quad \omega > 0\,,$$

see, e.g. [11].

We further define the non-linear operator

$$F : D(F) := L^6(\mathcal{O}) \times L^2(\mathcal{O}) \to H\,,$$

as

$$F\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} fv + I_{ion}(v) \\ 0 \end{pmatrix} = \begin{pmatrix} fv + v(v-a)(v-1) \\ 0 \end{pmatrix}.$$

In what follows we will assume that it holds

$$3\bar{f} - (a^2 - a + 1) \geq 0\,, \tag{4}$$

where we have denoted by $\bar{f} := \min_{\xi \in \mathcal{O}} f(\xi) > 0$.

Notice that the above condition implies that choosing $\bar{a} := \frac{1}{3}\left(a^2 - a + 1\right)$ we have that $A + F$ is *accreative* in $H \times H$, for $\bar{f} \geq \bar{a}$, that is

$$\langle (A+F)X - (A+F)\bar{X}, X - \bar{X} \rangle_H \geq 0\,, \quad \forall\, X\,,\, \bar{X} \in D(A) \cap D(F)\,.$$

Moreover we have that

$$\langle AX, FX\rangle \geq 0\,, \quad \forall\, X \in D(A)\cap D(F)\,,$$

and this implies (see, e.g., [4, Pag. 44]) that $A+F$ is $m-$*accreative*.

Let us thus consider the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, such that the two independent Wiener processes β_1 and β_2 are adapted to the filtration $\mathcal{F}_t$, $\forall\, t \geq 0$, and we define $W(t) = (\beta_1(t), \beta_2(t))$ a cylindrical Wiener process on H and by Q the operator

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\,,$$

being clearly Q a nuclear operator from H to itself. Exploiting previously introduced notation, equation (2), in the uncontrolled case, can be rewritten as follows

$$\begin{cases} dX(t) + [AX(t) + F(X(t))]dt = \sqrt{Q}dW(t), \\ X(0) = x_0 \in H\,, \quad t\in[0,T]\,, \end{cases} . \tag{5}$$

In what follows we will employ the subsequent notation. We denote by $C_W([0,T];H)$ the space of all $H-$valued $(\mathcal{F}_t)-$adapted processes such that $X \in C\left([0,T]; L^2(\Omega;H)\right)$. Similarly we will denote by $L^2_W([0,T];V)$ the space of all $V-$valued $(\mathcal{F}_t)-$adapted processes such that $X \in L^2\left([0,T]; L^2(\Omega;V)\right)$; here $V = H^1(\mathcal{O}) \times L^2(\mathcal{O})$.

Definition 2.1. We say that the function $X \in C_W([0,T];H) \cap L^2_W([0,T];V)$ is called a *strong solution* to (5) if $X(t) : [0,T] \to H$ is continuous $\mathbb{P}-$a.s., $\forall\, t \in [0,T]$ and it satisfies the stochastic integral equation

$$X(t) = x - \int_0^t (AX(s) + F(s))\,ds + \int_0^t \sqrt{Q}dW(s), \quad \forall\, t\in[0,T]\,.$$

The we have the following existence and uniqueness result concerning equation (5).

Theorem 2.2. *For any $x \in V$, there exists a unique solution X to (5) which satisfies*

$$X \in L^2_W\left(\Omega; C([0.T];H)\right) \cap L^2_W\left(\Omega; L^2([0.T];V)\right)\,.$$

Proof. Consider the approximating equation

$$\begin{cases} dX_\lambda(t) + [AX_\lambda(t) + F_\lambda(X_\lambda(t))]dt = \sqrt{Q}dW(t), \\ X_\lambda(0) = x_0 \in H\,, \quad t\in[0,T]\,, \end{cases} , \tag{6}$$

where

$$F_\lambda := \frac{1}{\lambda}\left(Id - (Id + \lambda F)^{-1}\right)\,,$$

is the *Yosida approximation* of F (see, e.g. [2]), being Id is the identity operator on H.

Since F_λ is Lipschitz, equation (6) has a unique solution

$$X_\lambda \in L^2_W\left(\Omega; C([0.T];H)\right) \cap L^2_W\left(\Omega \times [0.T];V\right)\,.$$

Let j_λ be defined by

$$\nabla j_\lambda(x) = F_\lambda(x)\,, \quad \forall\, x\in H\,.$$

By Itô's formula it follows that

$$\int_{\mathcal{O}} j_\lambda\left(X_\lambda(t)\right) d\xi + \int_0^t |X_\lambda(s)|_V^2 ds \leq \int_{\mathcal{O}} j_\lambda\left(x\right) d\xi + \\ + C \int_0^t \int_{\mathcal{O}} |F_\lambda\left(X_\lambda(s)\right)|_H^2 d\xi ds + \int_0^t \int_{\mathcal{O}} \langle \sqrt{Q} dW(s), j_\lambda'(X_\lambda(s)\rangle d\xi \,.$$

This yield

$$\mathbb{E} \sup_{t\in[0,T]} \int_{\mathcal{O}} j_\lambda\left(X_\lambda(t)\right) d\xi + \mathbb{E} \int_0^t |X_\lambda(s)|_V^2 ds \leq \\ \leq \mathbb{E} C \int_0^t \int_{\mathcal{O}} |f\left(X_\lambda(s)\right)|_H^2 ds d\xi \leq C \,.$$

Using the Burkholder-Davis-Gundy inequality we have that

$$\mathbb{E} \sup_{t\in[0,T]} \{|X_\lambda(t) - X_\epsilon(t)|_H^2\} \leq C(\epsilon + \lambda)\,,$$

which implies that letting $\lambda \to 0$, we have that

$$X = \lim_{\lambda \to 0} X_\lambda\,,$$

with $X \in L^2(\Omega, C([0,T]; H))$ from which the continuity of X follows. □

3. **The optimal control problem.** Let us now consider a controlled version of equation (5). Let U be a Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle_U$, we have that $u : [0,T] \to U$ denotes the control and $B_{\mathcal{O}} \in L(U, L^2(\mathcal{O}))$. Let $B \in L\left(U; H\right)$ defined as

$$Bu = \begin{pmatrix} B_{\mathcal{O}} u \\ 0 \end{pmatrix}, \quad B_{\mathcal{O}} \in L(U; L^2(\mathcal{O}))\,.$$

We shall denote by $\mathcal{U}$ the space of all $\left(\mathcal{F}_t\right)_{t\geq 0}$ −adapted processes $u : [0,T] \to U$ s.t. $\mathbb{E}\left[\int_0^T |u(t)|_U^2 dt\right] < \infty$. The space $\mathcal{U}$ is a Hilbert space with the norm $|u|_{\mathcal{U}} = \left(\mathbb{E}\left[\int_0^T |u(t)|_U^2 dt\right]\right)^{\frac{1}{2}}$ and scalar product

$$\langle u, v \rangle_{\mathcal{U}} = \mathbb{E}\left[\int_0^T \langle u(t), v(t)\rangle_U dt\right], \quad \forall\, u,\, v \in \mathcal{U}\,,$$

where $\langle \cdot, \cdot \rangle_U$ is the scalar product of U.

Consider the functions $g,\, g_0 : \mathbb{R} \to \mathbb{R}$ and $h : U \to \bar{\mathbb{R}} :=] - \infty, \infty]$, which satisfy the following conditions

(i): $g,\, g_0 \in C^1\left(H\right)$ and $Dg,\, Dg_0 \in Lip\left(H; H\right)$, where D stands for the Fréchet differential

(ii): h is convex, lower semi–continuous and $\left(\partial h\right)^{-1} \in Lip(U)$ where $\partial h : U \to U$ is the subdifferential of h, see, e.g., [8, p. 82], and $Lip(U)$ is the space of Lipschitz function from U to itself equipped with the standard norm. Moreover we assume that $\exists\ \alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$ s.t. $h(u) \geq \alpha_1 |u|_U^2 + \alpha_2$, $\forall\ u \in U$, and we set $L = \|(\partial h)^{-1}\|_{Lip(U)}$.

We consider the following optimal control problem

$$\text{Minimize}\,\mathbb{E}\left[\int_0^T (g(X(t)) + h(u(t)))\,dt\right] + \mathbb{E}\left[g_0(X(T))\right]\,, \tag{P}$$

subject to $u \in \mathcal{U}$ and to state equation

$$\begin{cases} dX(t) + [AX(t) + F(X(t))]dt = Bu(t)dt + \sqrt{Q}dW(t)\,, \\ X(0) = x_0 \in H\,, \quad t \in [0,T]\,, \end{cases} . \tag{7}$$

Existence and uniqueness of a solution, in the sense of Definition 2.1, follows with similar argument mentioned above. As regard existence in P we have.

Theorem 3.1. *Let $x \in D(A)$. Then there exists $C^* > 0$ independent of x such that for $LT + \|Dg_0\|_{Lip} < C^*$ there is a unique solution (u^*, X^*) to problem (P).*

Proof. The argument is similar to the one used in [6], (see also [7]).

Let us consider the function $\Psi : \mathcal{U} \to \bar{\mathbb{R}}$ defined by

$$\Psi(u) = \mathbb{E}\left[\int_0^T (g(X^u(t)) + h(u(t)))\,dt\right] + \mathbb{E}\left[g_0(X^u(T))\right]\,,$$

where X^u is the solution to (7). Recall that Ψ is lower semi–continuous and convex, see, e.g. [8].

By Ekeland's variational principle, see [23], there is a sequence $(u_\epsilon) \subset \mathcal{U}$ such that

$$\begin{aligned} &\Psi(u_\epsilon) \le \inf\{\Psi(u)\,;u \in \mathcal{U}\} + \epsilon\,, \\ &\Psi(u_\epsilon) \le \Psi(u) + \sqrt{\epsilon}\,|u_\epsilon - u|_{\mathcal{U}}\,, \quad \forall\, u \in \mathcal{U}\,. \end{aligned} \tag{8}$$

In other words,

$$u_\epsilon = \arg\min_{u\in\mathcal{U}}\{\Psi(u) + \sqrt{\epsilon}\,|u_\epsilon - u|_{\mathcal{U}}\}\,.$$

Hence $(X^{u_\epsilon}, u_\epsilon)$ is a solution to the optimal control problem

$$\begin{aligned} \min\Bigg\{&\mathbb{E}\left[\int_0^T (g(X^u(t) + h(u(t)))\,dt\right] + \mathbb{E}\left[g_0\left(X^u(T)\right)\right] + \\ &+ \sqrt{\epsilon}\left(\mathbb{E}\left[\int_0^T |u(t) - u_\epsilon(t)|_U^2\,dt\right]\right)^{\frac{1}{2}}\,; u \in \mathcal{U}\Bigg\}\,. \end{aligned} \tag{9}$$

From the optimality of u_ϵ, it follows by equation (9) that for any $v \in \mathcal{U}$ and any $\lambda > 0$ we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left(g(X^{u_\epsilon+\lambda v}(t) + h((u_\epsilon + \lambda v)(t))\right)dt\right] + \mathbb{E}\left[g_0(X^{u_\epsilon+\lambda v}(T))\right] + \\ &+ \lambda\sqrt{\epsilon}\left(\mathbb{E}\left[\int_0^T |v(t)|_U^2\,dt\right]\right)^{\frac{1}{2}} \ge \\ &\ge \mathbb{E}\left[\int_0^T (g(X_\epsilon(t)) + h(u_\epsilon(t)))\,dt\right] + \mathbb{E}\left[g_0(X_\epsilon(T))\right]\,. \end{aligned} \tag{10}$$

Dividing equation (10) by λ and taking the limit as $\lambda \to 0$ we get

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \langle Dg(X_\epsilon(t)), Z^v(t)\rangle_2 \, dt\right] + \mathbb{E}\left[\int_0^T h'(u_\epsilon(t), v(t))dt\right] + \\ &+ \mathbb{E}\left[\langle Dg_0(X_\epsilon(T)), Z^v(T)\rangle_2\right] + \sqrt{\epsilon}\left(\mathbb{E}\left[\int_0^T |v(t)|_U^2 dt\right]\right)^{\frac{1}{2}} \geq 0\,, \quad \forall\, v \in \mathcal{U}\,, \end{aligned} \tag{11}$$

where Z^v solves the system in variations associated with (7), that is

$$\begin{cases} \frac{\partial}{\partial t} Z^v(t) + AZ^v(t) + DF(X_\epsilon(t))Z^v(t) = Bv(t)\,, t \in [0,T]\,, \\ Z^v(0) = 0\,, \end{cases} \tag{12}$$

and $h' : U \times U \to \mathbb{R}$ is the directional derivatives of h, see, e.g., [8, p.81], namely

$$h'(u_\epsilon, v) = \lim_{\lambda \downarrow 0} \frac{h(u_\epsilon + \lambda v) - h(u_\epsilon)}{\lambda}\,, \quad \forall\, v \in U\,.$$

We thus associate with (9) the dual stochastic backward equation, see, e.g. [7],

$$\begin{cases} dp_\epsilon(t) = \left[Ap_\epsilon(t)dt + DF(X_\epsilon)p_\epsilon(t) + Dg(X_\epsilon(t))\right] dt + \kappa_\epsilon(t)\sqrt{Q}dW(t)\,, t \in [0,T]\,, \\ p_\epsilon(T) = -Dg_0(X_\epsilon(T))\,, \end{cases} . \tag{13}$$

It is well-known that equation (13) has a unique solution $(p_\epsilon, \kappa_\epsilon)$ satisfying

$$\begin{aligned} &p_\epsilon \in L^\infty_W([0,T];H) \cap L^2_W([0,T];V)\,, \\ &k_\epsilon \in L^2_W([0,T];H)\,, \end{aligned}$$

(See, e.g., [25, Prop. 4.2] or [32]). By Itô's formula we have from (12) and (13) that

$$d\langle p_\epsilon, Z^v\rangle_H = \langle dp_\epsilon, Z^v\rangle_H + \langle p_\epsilon, dZ^v\rangle_H\,,$$

and this yields

$$\mathbb{E}\left[\int_0^T \langle Dg(X_\epsilon(t)), Z^v(t)\rangle_H \, dt\right] + \mathbb{E}\left[\langle Dg_0(X_\epsilon(T)), Z^v(T)\rangle_H\right] = \mathbb{E}\left[\int_0^T \langle Bv(t), p_\epsilon(t)\rangle_H \, dt\right]\,,$$

which substituted in (11) yields that $\forall\, v \in \mathcal{U}$, the following inequality holds

$$\begin{aligned} &\mathbb{E}\left[\int_0^T h'(u_\epsilon(t), v(t))dt\right] + \sqrt{\epsilon}\left(\mathbb{E}\left[\int_0^T |v(t)|_U^2 dt\right]\right)^{\frac{1}{2}} + \\ &+ \mathbb{E}\left[\int_0^T \langle B^* p_\epsilon(t), v(t)\rangle_U \, dt\right] \geq 0\,. \end{aligned} \tag{14}$$

Let $G(u) := \mathbb{E}\left[\int_0^T h(u(t))dt\right]$, the sub-differential $\partial G : \mathcal{U} \to \mathcal{U}$ in u_ϵ is given by

$$\partial G(u_\epsilon) = \left\{ v^* \in \mathcal{U} \,:\, \langle v, v^*\rangle_{\mathcal{U}} \leq \mathbb{E}\left[\int_0^T h'(u_\epsilon(t), v(t))dt\right]\,, \forall\, v \in \mathcal{U} \right\}.$$

(See, e.g., [8, p.81]). Then we infer from equation (14), where $v(t) = -u_\epsilon(t) + \bar{v}$, that it holds

$$u_\epsilon(t) = (\partial h)^{-1}\left(B^* p_\epsilon(t) + \sqrt{\epsilon}\tilde{\theta}_\epsilon\right)\,,\ t \in [0,T]\,, \quad \mathbb{P} - a.s.\,,$$

where $\tilde{\theta}_\epsilon \in \mathcal{U}$ and $|\tilde{\theta}_\epsilon|_U \leq 1$, $\forall\, \epsilon > 0$.

Therefore, we have shown that

$$
\begin{aligned}
&u_\epsilon = (\partial h)^{-1}\left(B^* p_\epsilon + \theta_\epsilon\right), \|\theta_\epsilon\|_{L^2([0,T]\times\Omega;U)} \leq \sqrt{\epsilon}\,,\\
&dp_\epsilon(t) = \left[Ap_\epsilon(t)dt + DF(X_\epsilon)p_\epsilon(t) + Dg(X_\epsilon(t))\right]dt + \kappa_\epsilon(t)\sqrt{Q}dW(t)\,, t\in[0,T]\,,.\\
&p_\epsilon(T) = -Dg_0(X_\epsilon(T))\,,
\end{aligned}
\tag{15}
$$

Using the Itô formula applied to $|X|_2^2$, we have that $\forall\ \epsilon > 0$ it holds

$$
\begin{aligned}
|X_\epsilon(t)|_H^2 = |x|_H^2 - 2\int_0^t \langle AX_\epsilon(s) + F(X_\epsilon(s)) - Bu_\epsilon(s), X_\epsilon(s)\rangle_H\, ds +\\
+ TrQt + 2\int_0^t \left\langle X_\epsilon(s), \sqrt{Q}dW(s)\right\rangle_H\,;
\end{aligned}
\tag{16}
$$

notice that above the application of Itô formula is only formal, nevertheless the following bounds can be made rigorous by a truncation argument, see, e.g. [10, 11].

(Here and everywhere in the following we shall denote by C several positive constants independent of ϵ.)

From the fact that $\int_0^t \left\langle X_\epsilon(s), \sqrt{Q}dW(s)\right\rangle_H$ is a square integrable martingale, [18, Th. 3.14, Th. 4.12], we have that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \left\langle X_\epsilon(s), \sqrt{Q}dW(s)\right\rangle_H\right|\right] \leq CTr(Q)\mathbb{E}\left[\int_0^T |X_\epsilon(t)|_H^2\, dt\right].
$$

We have

$$
\int_0^t \langle AX_\epsilon(s), X_\epsilon(s)\rangle_H\, ds \geq C_1\int_0^t |X_\epsilon(s)|_V^2 ds\,.
$$

We also have that it holds,

$$
\int_0^t \langle F(X_\epsilon(s)), X_\epsilon(s)\rangle_H\, ds \geq C|X_\epsilon(t)|_H^2\,,
$$

see, e.g. [2, 11] for details. Eventually from assumption (ii) we have

$$
\int_0^t \langle Bu(s), X_\epsilon(s)\rangle_H\, ds \leq L^{-1}\int_0^T |u_\epsilon(t)|_U^2 dt\,.
$$

Taking then the expectation on both side of (16) yields, via Burkholder-Davis-Gundy inequality

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|X_\epsilon(t)|_H^2\right] + \mathbb{E}\left[\int_0^T |X_\epsilon(t)|_V^2 dt\right] \leq C_1 + C_2\int_0^T \mathbb{E}\left[\sup_{s\in[0,t]}|X_\epsilon(s)|_H^2\, dt\right]
$$

and applying Gronwall's lemma it follows eventually that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|X_\epsilon(t)|_H^2\right] + \mathbb{E}\left[\int_0^T |X_\epsilon(t)|_V^2\, dt\right] \leq C(1+|x|_H^2)\,. \tag{17}
$$

In an analogous manner, applying Itô formula to $|p_\epsilon|_H^2$ by (15) we obtain that

$$
\begin{aligned}
&\frac{1}{2}d|p_\epsilon(t)|_H^2 = \langle Ap_\epsilon(t) + DF(X_\epsilon(t))p_\epsilon(t) + Dg(X_\epsilon(t)), p_\epsilon(t)\rangle_H +\\
&= \frac{1}{2}\langle \kappa_\epsilon(t), \kappa_\epsilon(t)\rangle_H\, dt + \left\langle p_\epsilon(t), \kappa_\epsilon(t)\sqrt{Q}dW(t)\right\rangle_H\,.
\end{aligned}
$$

which yields after applying arguments similar to the ones above

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}|p_\epsilon(t)|_H^2\right]+\mathbb{E}\left[\int_0^T|p_\epsilon(t)|_V^2\,dt\right]+\mathbb{E}\left[\int_0^T|\kappa_\epsilon(t)|_H^2dt\right]\leq\\&\leq C+\mathbb{E}\left[|X_\epsilon(T)|_H^2\right]\leq C\,,\quad\forall\,\epsilon>0\,.\end{aligned}\tag{18}$$

Denoting by X_λ the solution with control u_λ, we have that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(X_\epsilon(t)-X_\lambda(t)\right)+A\left(X_\epsilon(t)-X_\lambda(t)\right)+\left(F\left(X_\epsilon(t)\right)-F\left(X_\lambda(t)\right)\right)=\\&=BB^*(p_\epsilon(t)-p_\lambda(t))+B(\theta_\epsilon(t)-\theta_\lambda(t))\,.\end{aligned}\tag{19}$$

In virtue of (18) this yields

$$\begin{aligned}&\frac{1}{2}\left|X_\epsilon(t)-X_\lambda(t)\right|_H^2+\int_0^t\left|X_\epsilon(s)-X_\lambda(s)\right|_V^2ds+\\&+\int_0^t\left\langle F\left(X_\epsilon(s)\right)-F\left(X_\lambda(s)\right),X_\epsilon(s)-X_\lambda(s)\right\rangle_H\,ds\leq\\&\leq L\int_0^t|p_\epsilon(s)-p_\lambda(s)|_H|X_\epsilon(s)-X_\lambda(s)|_Hds\\&+C\int_0^t|\theta_\epsilon(s)-\theta_\lambda(s)|_U|X_\epsilon(s)-X_\lambda(s)|_Hds\,,\quad\forall\,t\in[0,T]\,,\end{aligned}$$

where $L=\|(\partial h)^{-1}\|_{Lip}$.

From the definition of F, we further have that,

$$\left\langle F(X_\epsilon)-F(X_\lambda),X_\epsilon-X_\lambda\right\rangle_H\geq-C\left|X_\epsilon-X_\lambda\right|_H^2\,,$$

which yields, for $t\in[0,T]$, applying Young inequality,

$$\begin{aligned}&|X_\epsilon(t)-X_\lambda(t)|_2^2+\int_0^t|X_\epsilon(s)-X_\lambda(s)|_V^2\,ds\leq\\&\leq C\left(L\int_0^t|p_\epsilon(s)-p_\lambda(s)|_2^2ds+\int_0^t|X_\epsilon(s)-X_\lambda(s)|_H^2ds+\epsilon+\lambda\right)\,.\end{aligned}\tag{20}$$

Applying Gronwall's lemma in (20), we have

$$\begin{aligned}&|X_\epsilon(t)-X_\lambda(t)|_2^2+\int_0^t|X_\epsilon(s)-X_\lambda(s)|_V^2\,ds\leq\\&\leq C\left(L\int_0^T|p_\epsilon(s)-p_\lambda(s)|_2^2ds+\epsilon+\lambda\right)\,,\quad\forall\,\epsilon\,,\,\lambda>0\,,\,t\in[0,T]\,.\end{aligned}\tag{21}$$

Similarly we get by the Itô formula

$$
\begin{aligned}
&|p_\epsilon(t)-p_\lambda(t)|_H^2+\int_t^T |p_\epsilon(s)-p_\lambda(s)|_V^2\,ds+\frac{1}{2}\int_t^T|\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2ds=\\
&=|Dg_0(X_\epsilon(T))-Dg_0(X_\lambda(T))|_H^2+\\
&\quad+\int_t^T\langle DF(X_\epsilon(s))p_\epsilon(s)-DF(X_\lambda(s))p_\lambda(s),p_\epsilon(s)-p_\lambda(s)\rangle_H\,ds+\\
&\quad-\int_t^T\left\langle\kappa_\epsilon(s)-\kappa_\lambda(s))\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_H\leq\\
&=\int_t^T\langle DF(X_\epsilon(s))(p_\epsilon(s)-p_\lambda(s)),p_\epsilon(s)-p_\lambda(s)\rangle\,ds+\\
&\quad+\int_t^T\langle p_\lambda(s)(DF(X_\epsilon(s))-DF(X_\lambda(s))),p_\epsilon(s)-p_\lambda(s)\rangle_H\,ds+\\
&\quad+\int_t^T\left\langle\kappa_\epsilon(s)-\kappa_\lambda(s))\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_H+\\
&\quad+|Dg_0\left(X_\epsilon(T)\right)-Dg_0\left(X_\lambda(T)\right)|_H^2\leq\\
&\leq C\left(\int_t^T(|X_\epsilon(s)|_H^2+1)|p_\epsilon(s)-p_\lambda(s)|_H^2\,ds\right)+\\
&\quad+C\left(\int_t^T\left(1+|X_\epsilon(s)|^2+|X_\lambda(s)|^2\right)|X_\epsilon(s)-X_\lambda(s)|_H|p_\epsilon(s)-p_\lambda(s)|_H|p_\epsilon(s)|_H\,ds\right)+\\
&\quad+\int_t^T\left\langle\kappa_\epsilon(s)-\kappa_\lambda(s))\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_H+\\
&\quad+\|Dg_0\|_{Lip}|X_\epsilon(T)-X_\lambda(T)|_H^2\,,\quad t\in[0,T]\,,\mathbb{P}-a.s.\,.
\end{aligned}
\tag{22}
$$

Exploiting again Young's inequality, and denoting for short

$$
T_{\epsilon,\lambda}:=(1+|X_\epsilon|_H^2+|X_\lambda|_H^2)|p_\epsilon|_H\,,
$$

we get,

$$
\begin{aligned}
&(|X_\epsilon(s)-X_\lambda(s)|_H|p_\epsilon(s)-p_\lambda(s)|_H)\,T_{\epsilon,\lambda}\leq\\
&\leq C\left(|X_\epsilon-X_\lambda|_H^2+|p_\epsilon-p_\lambda|_H^2\right)T_{\epsilon,\lambda}\,.
\end{aligned}
\tag{23}
$$

Substituting now (23) into (20), (22), we obtain $\mathbb{P}-$a.s.

$$
\begin{aligned}
&|X_\epsilon(t)-X_\lambda(t)|_H^2+|p_\epsilon(t)-p_\lambda(t)|_H^2+\int_0^t|X_\epsilon(s)-X_\lambda(s)|_V^2ds+\\
&\quad+\int_t^T|p_\epsilon(s)-p_\lambda(s)|_V^2\,ds+\int_t^T|\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2ds\leq\\
&\leq C\left(L\int_0^t|p_\epsilon(s)-p_\lambda(s)|_H^2\,ds+\epsilon+\lambda\right)+C\int_t^T|p_\epsilon(s)-p_\lambda(s)|_2^2\,|X_\epsilon(s)|_H^2ds+\\
&\quad+\|Dg_0\|_{Lip}|X_\epsilon(T)-X_\lambda(T)|_2^2+\\
&\quad+C\int_t^T\left(|X_\epsilon(s)-X_\lambda(s)|_H^2+|p_\epsilon(s)-p_\lambda(s)|_H^2\right)T_{\epsilon,\lambda}(s)ds+\\
&\quad-\int_t^T\left\langle\kappa_\epsilon(s)-\kappa_\lambda(s))\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_H\,,\quad\forall\,t\in[0,T]\,.
\end{aligned}
\tag{24}
$$

Exploiting thus the fact that the process

$$
r\mapsto\int_t^r\left\langle(\kappa_\epsilon-\kappa_\lambda)\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_2\,,
$$

is a local martingale on $[t,T]$, hence again by the Burkholder-Davis-Gundy inequality, see, e.g., [20, p.58], we have for all $r\in[t,T]$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[t,T]}\left|\int_t^r\left\langle(\kappa_\epsilon(s)-\kappa_\lambda(s))\sqrt{Q}dW(s),X_\epsilon(s)-X_\lambda(s)\right\rangle_H\right|\right]\leq\\
&\leq C\left(\mathbb{E}\left[\int_0^r|\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2|X_\epsilon(s)-X_\lambda(s)|_H^2ds\right]\right)^{\frac{1}{2}}\leq\\
&\leq C\mathbb{E}\left[\sup_{s\in[t,r]}|X_\epsilon(s)-X_\lambda(s)|_H^2\right]+C\mathbb{E}\left[\int_t^r|\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2ds\right]\,.
\end{aligned}
\tag{25}
$$

Taking then the expectation in and by (24), and using (25) we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}\left(|X_\epsilon(s)-X_\lambda(s)|_H^2+|p_\epsilon(s)-p_\lambda(s)|_H^2\right)\right]\\
&\quad+\mathbb{E}\left[\int_0^T|X_\epsilon(s)-X_\lambda(s)|_V^2ds+\int_t^T|p_\epsilon(s)-p_\lambda(s)|_H^2ds\right]\\
&\quad+\mathbb{E}\left[\int_t^T|\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2ds\right]\leq\\
&\leq\|Dg_0\|\mathbb{E}\left[|X_\epsilon(T)-X_\lambda(T)|_H^2\right]+C\left(L\mathbb{E}\left[\int_0^T|p_\epsilon(s)-p_\lambda(s)|_H^2\,ds\right]+\epsilon+\lambda\right)\\
&\quad+C\mathbb{E}\left[\sup_{s\in[t,T]}|X_\epsilon(s)-X_\lambda(s)|_H^2\right]\\
&\quad+C\mathbb{E}\left[\int_t^T\left(|p_\epsilon(s)-p_\lambda(s)|_H^2+|X_\epsilon(s)-X_\lambda(s)|_H^2\right)\left(|X_\epsilon(s)|_H^2+T_{\epsilon,\lambda}(s)\right)ds\right]\,.
\end{aligned}
\tag{26}
$$

Taking into account estimates (21) and (22), from (26) we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}\left(|X_\epsilon(s)-X_\lambda(s)|_H^2+|p_\epsilon(s)-p_\lambda(s)|_H^2\right)\right]\\
&\quad+\mathbb{E}\left[\int_0^T |X_\epsilon(s)-X_\lambda(s)|_V^2 ds+\int_t^T |p_\epsilon(s)-p_\lambda(s)|_H^2 ds\right]\\
&\quad+\mathbb{E}\left[\int_t^T |\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2 ds\right]\leq\\
&\leq \tilde{C}\left(L\mathbb{E}\left[\int_0^T |p_\epsilon(s)-p_\lambda(s)|_H^2\, ds\right]\right)\\
&\quad+\tilde{C}\left(\mathbb{E}\left[\int_t^T |p_\epsilon(s)-p_\lambda(s)|_H^2\left(|X_\epsilon(s)|_H^3+T_{\epsilon,\lambda}(s)\right)ds\right]\right)\\
&\quad+\tilde{C}\|Dg_0\|_{Lip}\mathbb{E}\left[|X_\epsilon(T)-X_\lambda(T)|_H^2\right]+\tilde{C}(\epsilon+\lambda)\,.
\end{aligned}\tag{27}$$

where $\tilde{C}$ is a positive constant independent of ϵ and λ. It follows that if $\tilde{C}(LT+\|Dg_0\|_{Lip})<1$, then, for any $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}\left(|X_\epsilon(s)-X_\lambda(s)|_H^2+|p_\epsilon(s)-p_\lambda(s)|_H^2\right)\right]\\
&\quad+\mathbb{E}\left[\int_0^T |X_\epsilon(s)-X_\lambda(s)|_V^2 ds+\int_t^T |p_\epsilon(s)-p_\lambda(s)|_H^2 ds\right]\\
&\quad+\mathbb{E}\left[\int_t^T |\kappa_\epsilon(s)-\kappa_\lambda(s)|_H^2 ds\right]\leq\\
&\leq C\mathbb{E}\left[\int_t^T |p_\epsilon(s)-p_\lambda(s)|_H^2\left(|X_\epsilon(s)|_H^2+T_{\epsilon,\lambda}(s)\right)ds\right]+C(\epsilon+\lambda)\,.
\end{aligned}\tag{28}$$

Let us define for $j\in\mathbb{N}$

$$\Omega_j:=\left\{\omega\in\Omega\,:\sup_{\epsilon}\sup_{t\in[0,T]}\left(|X_\epsilon(t)|_H^2+|X_\epsilon(t)|_V^2+|p_\epsilon(t)|_H^2\right)dt\leq j\right\}\,,$$

then estimates (17)–(18) implies that

$$\mathbb{P}\left(\Omega_j\right)\geq 1-\frac{C}{j}\,,\quad \forall\, j\in\mathbb{N}\,,$$

for some constant C independent of ϵ.

If we set $X_\epsilon^j:=\mathbb{1}_{\Omega_j}X_\epsilon$, $p_\epsilon^j:=\mathbb{1}_{\Omega_j}p_\epsilon$ and $\kappa_\epsilon^j:=\mathbb{1}_{\Omega_j}\kappa_\epsilon$, then such quantities satisfy the system (15), with $\mathbb{1}_{\Omega_j}\sqrt{Q}dW$. The latter means that estimate (28) still holds in this context, so that we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{s\in[t,T]}|X^j_\epsilon(s)-X^j_\lambda(s)|^2_H+\sup_{s\in[t,T]}|p^j_\epsilon(t)-p^j_\lambda(t)|^2_H\right]\\ &\quad+\mathbb{E}\left[\int_t^T|p^j_\epsilon(s)-p^j_\lambda(s)|^2_V ds\right]+\mathbb{E}\left[\int_t^T|(\kappa_\epsilon(s)-\kappa_\lambda(s))\chi_j|^2_H ds\right]\leq \\ &\leq C_j\int_t^T\mathbb{E}\left[|p^j_\epsilon(s)-p^j_\lambda(s)|^2_H\right]ds+C\left(\epsilon+\lambda\right)\,,\quad j\in\mathbb{N}\,. \end{aligned} \tag{29}$$

By Gronwall's lemma we get, for any $t\in[0,T]$

$$\mathbb{E}\left[\sup_{s\in[t,T]}|X^j_\epsilon(s)-X^j_\lambda(s)|^2_H+\sup_{s\in[t,T]}|p^j_\epsilon(s)-p^j_\lambda(s)|^2_H\right]\leq C(\epsilon+\lambda)e^{C_jT}\,, \tag{30}$$

hence, for $\epsilon\to 0$ and all $j\in\mathbb{N}$ and all $t\in[0,T]$, we obtain

$$\begin{aligned} &X^j_\epsilon\to X^j \quad \text{in} \quad L^2\left(\Omega_j;L^2([0,T]\times\mathcal{O})\times L^2([0,T]\times\mathcal{O})\right)\,,\\ &p^j_\epsilon\to p^j \quad \text{in} \quad L^2\left(\Omega_j;L^2([0,T]\times\mathcal{O})\times L^2([0,T]\times\mathcal{O})\right)\,. \end{aligned} \tag{31}$$

Therefore for each $\omega\in\Omega$, we have that $\{X_\epsilon(t,\omega),p_\epsilon(t,\omega)\}$ are Cauchy sequences in $L^2([0,T]\times\mathcal{O})$, with respect to ϵ and by estimates (17) and (18) it follows that taking related subsequences, still denoted by ϵ, we have

$$\begin{aligned} &X_\epsilon\rightharpoonup X^* && \text{in } L^2\left([0,T]\times\Omega;V\right)\,,\\ &p_\epsilon\rightharpoonup p^* && \text{in } L^2\left([0,T]\times\Omega\times\mathcal{O}\times\mathcal{O}\right)\,,\\ &p_\epsilon\rightharpoonup p^* && \text{in } L^2\left([0,T]\times\Omega;V\right)\,,\\ &u_\epsilon\rightharpoonup u^* && \text{in } L^\infty\left([0,T];L^2\left(\Omega\times U\right)\right)\,, \end{aligned} \tag{32}$$

where $\rightharpoonup$ means weak (respectively, weak-star) convergence, so we have for $\epsilon\to 0$

$$X_\epsilon\to X^*\,,\quad p_\epsilon\to p^*\,, a.e. \text{ in } [0,T]\times\Omega\times\mathcal{O}\times\mathcal{O}\,. \tag{33}$$

We also have, since $\{I_{ion}(v_\epsilon)\}$ is bounded in $L^{\frac{4}{3}}([0,T]\times\Omega\times\mathcal{O})$, then it is weakly compact in $L^1([0,T]\times\Omega\times\mathcal{O})$ and by (33) we have that for a subsequence $\{\epsilon\}\to 0$,

$$I_{ion}(v_\epsilon)\to I_{ion}(v^*)\,,\quad a.e. \text{ in } [0,T]\times\Omega\times\mathcal{O}\,,$$

which implies that

$$I_{ion}(v_\epsilon)\to I_{ion}(v^*) \qquad \text{in } L^1\left([0,T]\times\Omega\times\mathcal{O}\right)\,. \tag{34}$$

Then, letting $\epsilon\to 0$ we obtain

$$\begin{cases} dX^*(t)+AX^*(t)dt+F(X^*(t))dt+\sqrt{Q}dW(t)=Bu^*(t)dt\,, t\in[0,T]\,,\\ X^*(0)=x\,, \end{cases}\,.$$

Taking into account that Ψ is weakly lower semi–continuous in $\mathcal{U}$ we infer by (8) that

$$\Psi(u^*)=\inf\left\{\Psi(u);u\in\mathcal{U}\right\}\,,$$

therefore (X^*,u^*) is optimal for the problem (P) and the proof of existence is therefore complete. □

Concerning the uniqueness for the optimal pair (X^*,u^*) given by Th. 3.1, we have that it follows by the same argument via the maximum principle result for problem (P), namely one has the following result.

Theorem 3.2. *Let* (X^*, u^*) *be optimal in problem* (P)*, then*

$$u^* = (\partial h)^{-1}(B^* p)\,, a.e.\, t \in [0,T]\,, \tag{35}$$

where p *is the solution to the backward stochastic equation* (13)*.*

Proof. If (X^*, u^*) is optimal for the problem (P), then by the same argument used to prove Th. 3.1, see (11), we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \langle Dg(X^*(t)), Z^v(t)\rangle_2\, dt\right] + \mathbb{E}\left[\int_0^T h'(u^*(t), v(t))dt\right] \\ &+ \mathbb{E}\left[\langle Dg_0(X^*(T)), Z^v(T)\rangle_2\right] \le 0\,, \quad \forall\, v \in \mathcal{U}\,, \end{aligned} \tag{36}$$

where Z^v is solution to equation (12) with X_ϵ replaced by X^*. This implies as above that (35) holds. □

The uniqueness in (P)*.* If (X^*, u^*) is optimal in (P) then it satisfies systems (5), (35) and (36), so that arguing as in the proof of Th. 3.1, the same set of estimates implies that the previous system has at most one solution if $LT + \|Dg_0\|_{Lip} < C^*$, where C^* is sufficiently small. □

Remark 1. Let us apply the change of variable $Y := X - \sqrt{Q}W$ to equation (5), so that we obtain the following random PDE

$$\begin{cases} Y(t) + AY + F\left(Y + \sqrt{Q}W\right) = -A\sqrt{Q}W\,, \\ Y(0) = x_0\,; \end{cases} \tag{37}$$

Setting

$$F_H Y := AY + F\left(Y + \sqrt{Q}W\right)\,,$$

which is a continuous and monotone function from $V \to V'$; then equation (37) has a unique solution (for each $\omega \in \Omega$) with $y \in C([0,T]; H) \cap L^2([0,T]; V)$, $\frac{d}{dt}Y(t) \in L^2([0,T]; V')$, see, e.g. [4, Th. 4.17]. We also have that

$$\mathbb{E}\int_0^T \|Y(t)\|_V^2 dt < \infty\,.$$

Moreover the optimal control problem (P) can be treated in terms of the random PDE (37).

4. **Conclusions.** In the present work we have derived the existence and uniqueness of the solution to the control problem associated to a FH-N system of equations perturbed by a Gaussian noise and in presence of a recovery variable. We would like to underline that the aforementioned result has potential applications in medicine, particularly from the point of view of neuronal diseases care. Indeed, the scheme of equations we have studied is linked to the Bonhoeffervan der Pol oscillator, namely a nonlinear damping governed by a second-order differential equation we are able to treat in presence of random (Gaussian) noise. The latter aspect is of great relevance in desincronized abnormal electrical activities that happen under the influence of pathology as the Parkinson's one, or during epileptic attacks. Possible generalizations of the proposed analysis will concern the study of the full Hodgkin-Huxley model, when a random source of noise has to be taken into consideration, as well as the study of the aforementioned models over networks of interconnected neurons, mainly following the approach derived in [15, 16]. Such topics are the subjects of our ongoing research.

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