Zubov's method for controlled diffusions with state constraints

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Abstract. We consider a controlled stochastic system in presence of state-constraints. Under the assumption of exponential stabilizability of the system near a target set, we aim to characterize the set of points which can be asymptotically driven by an admissible control to the target with positive probability. We show that this set can be characterized as a level set of the optimal value function of a suitable unconstrained optimal control problem which in turn is the unique viscosity solution of a second order PDE which can thus be interpreted as a generalized Zubov equation.

Mathematics Subject Classification (2010). Primary 93B05; Secondary 93E20, 49L25.

Keywords. Controllability for diffusion systems, Hamilton-Jacobi-Bellman equations, viscosity solutions, stochastic optimal control.

1. Introduction

In this paper we aim to study the asymptotic controllability property of controlled stochastic systems in presence of state constraints.

The basic problem in this context is the existence of a control strategy that asymptotically steers the system to a certain target set with positive probability. In the uncontrolled framework, the idea, due to Lyapunov, of linking the stability properties of a system with the existence of a continuous function (in the nowadays literature called a "Lyapunov function") that decreases along the trajectories of the system, represents a fundamental tool for the study of this kind of problems. In his seminal thesis [27], Lyapunov

This work was partially supported by the EU under the 7th Framework Programme Marie Curie Initial Training Network "FP7-PEOPLE-2010-ITN", SADCO project, GA number 264735-SADCO. Parts of the research for this paper were carried out while the second author was visiting the University of Bayreuth as part of her SADCO secondment.

proved that the existence of such a function is a sufficient condition for the asymptotic stability around a point of equilibrium of a dynamical system

$$\dot{x} = b(x), \qquad x(t) \in \mathbb{R}^d, t \geq 0. \tag{1.1}$$

This theory was further developed in later years, see [20, 28, 23], and also the converse property was established. Since the 60s, Lyapunov's method was extended to stochastic diffusion processes. The main contributions in this framework come from [21, 25, 24, 26], where the concepts of stability and asymptotic stability in probability, as well as the stronger concept of almost sure stability, are introduced.

An important domain of research concerns the developments of constructive procedure for the definition of Lyapunov functions. In the deterministic case an important result was obtained by Zubov in [31]. In this work the domain attraction of an equilibrium point $x_{\scriptscriptstyle E} \in \mathbb{R}^d$ for the system (1.1), i.e. the set of initial points that are asymptotically attracted by $x_{\scriptscriptstyle E}$, is characterized by using the solution ϑ of the following first order equation

$$\begin{cases} D\vartheta(x)b(x) = -f(x)(1-\vartheta(x))\sqrt{1+\|b(x)\|^2} & x \in \mathbb{R}^d \setminus \{x_{\scriptscriptstyle E}\} \\ \vartheta(x_{\scriptscriptstyle E}) = 0, \end{cases} \tag{1.2}$$

for a suitable choice of a scalar function f (see [31] and [20]). Equation (1.2) is referred to in the literature as Zubov equation. In particular, what is proved in [31] is that the domain of attraction coincides with the set of points $x \in \mathbb{R}^d$ such that $\vartheta(x) < 1$. Further developments and applications of this method can be found in [3, 20, 1, 18, 22, 11].

More recently, this kind of approach has been applied to more general systems, included control systems, thanks also to the advances of the viscosity solution theory that allows to consider merely continuous solutions of fully nonlinear PDE's. While for systems of ordinary differential equations the property of interest is stability, for systems that involve controls, the interest lies on "controllability", i.e. on the existence of a control such that the associated trajectory asymptotically reaches the target represented by the equilibrium point (see [2, 29]). The case of deterministic control systems was considered in [13]. Here, through the formulation of a suitable optimal control problem, it is proved that the domain of attraction can be characterized by the solution of a nonlinear PDE (that we can consider as a generalized Zubov equation) which turns out to be a particular kind of Hamilton-Jacobi-Bellman (HJB) equation.

In this case the existence of smooth solutions is not guaranteed and therefore the equation is considered in the viscosity sense. The state constrained case, where we aim to steer the system to the target satisfying at the same time some constraints on the state, has been treated in [19].

The Zubov method has been extended to the stochastic setting in [14] and [10] taking into account diffusion processes. The controlled case was later considered in [12] and [9]. In this last paper, under some property of local exponential stabilizability in probability of the target set (that weakens the "almost sure" stabilizability assumption made in [12] and [14]), the set of points $x \in \mathbb{R}^d$ that can be asymptotically steered with positive probability towards the target, is characterized by means of the unique viscosity solution with value zero on the target of the following equation

$$\sup_{u \in U} \Big\{ - f(x,u)(1 - \vartheta(x)) - D\vartheta(x)b(x,u) - \frac{1}{2}Tr[\sigma\sigma^T(x,u)D^2\vartheta(x)] \Big\} = 0.$$

In this paper we aim to add state-constraints in this framework, trying to exploit the ideas proposed in [19]. By the way the results in terms of PDE characterization of the domain of attraction will be very different. In [19] the state constrained controllability is characterized by the solution of an obstacle problem, whereas in our case we will deal with a mixed Dirichlet-Neumann boundary problem in an augmented state space (see Section 5). As in [19], for satisfying the state constrained requirement at any time $t \geq 0$ we use a cost in a maximum form. In the stochastic case this requires the introduction of an additional state variable (that we will denote by y), leading to a generalized Zubov equation which involves oblique derivative boundary conditions. Because of the particular feature of Zubov-type problems comparison results cannot be proved by standard techniques (this is mainly due to the degeneracy of the function f) and the comparison principle stated by Theorem 6.4 is proved providing sub- and super- optimality principles for PDEs of the following form:

$$\begin{cases} H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D_x^2\vartheta) = 0 & \text{in} \quad \mathcal{O} \\ \vartheta = 1 & \text{on} \quad \partial_1\mathcal{O} \\ -\partial_y\vartheta = 0 & \text{on} \quad \partial_2\mathcal{O}. \end{cases}$$

It should be mentioned that — similar to [14] — in this paper we characterize the domain of controllability with arbitrary positive probability without specifying the exact probability of controllability. We conjecture that it will be possible to extend the approach introduced in this paper to obtain such a specific characterization, similar to how [10] extends [14]. However, due to the fact that the treatment of the Zubov problem with mixed boundary conditions covered in this paper already requires a very involved analysis, we decided to postpone this extension to a later publication, see also Remark 2.2.

The paper is organized as follows: in Section 2 we introduce the setting and the main assumptions. Section 3 is devoted to the study of some properties of the domain of attraction. In Section 4 is defined our level set function v as the value function associated with an optimal control problem with a maximum cost and the domain of attraction is characterized as a sub-level set of v. In Section 5 the domain of attraction is characterized by the viscosity solution of second order nonlinear PDE with mixed Dirichlet-Neumann boundary conditions. A comparison principle for bounded viscosity sub- and super-solution of this problem is provided in Section 6.

2. Setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space supporting a p-dimensional Brownian motion $W(\cdot)$, where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ denotes the $\mathbb{P}$-augmentation of filtration generated by W.

We consider the following system of stochastic differential equations (SDE's) in $\mathbb{R}^d$ ($d \geq 1$)

$$\begin{cases} \mathrm{d}X(t) = b(X(t), u(t))\mathrm{d}t + \sigma(X(t), u(t))\mathrm{d}W(t) & t > 0,\ x \in \mathbb{R}^d, \\ X(0) = x \end{cases} \tag{2.1}$$

where $u \in \mathcal{U}$, and $\mathcal{U}$ denotes the set of $\mathbb{F}$-progressively measurable processes taking values in a compact set $U \subset \mathbb{R}^m$. The following classical assumption will be considered for the coefficients b and σ.

(H1) $b : \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times U \to \mathbb{R}^{d \times p}$ are bounded and Lipschitz continuous in their first arguments in the following sense: there exist $L \geq 0$ such that for every $x, y \in \mathbb{R}^d$ and $u \in U$

$$|b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| \leq L|x - y|.$$

It is well-known (see for instance [30, Theorem 3.1]) that, under these assumptions, for any choice of the control $u \in \mathcal{U}$ and any initial position $x \in \mathbb{R}^d$ there exists a unique strong solution of equation (2.1). We will denote this solution by $X^u_x(\cdot)$.

By $\mathcal{T} \subset \mathbb{R}^d$ we denote a target set for the system, i.e., a nonempty and compact set towards which we want to asymptotically drive the trajectories. The open set $\mathcal{C} \subseteq \mathbb{R}^d$ represents the state constraints for system (2.1), i.e., the set where we want to maintain the state $X_x(t)$ with a positive probability for all $t \geq 0$, cf. the definition of the set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ below. For simplicity we assume that $\mathcal{T} \subset \mathcal{C}$. Note that this implies that for r small enough one has $\mathcal{T}_r := \{x \in \mathbb{R}^d : d(x, \mathcal{T}) \leq r\} \subset \mathcal{C}$, where $d(\cdot, \mathcal{T})$ denotes the positive Euclidean distance to $\mathcal{T}$. We impose the following assumptions on the target.

(H2) (*i*) $\mathcal{T}$ is viable for (2.1): for any $x \in \mathcal{T}$ there exists $u \in \mathcal{U}$ such that

$$X^u_x(t) \in \mathcal{T} \quad \forall t \geq 0 \qquad \text{a.s.;}$$

(*ii*) $\mathcal{T}$ is locally exponentially stabilizable in probability for (2.1): there exist positive constants r, λ such that for every $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that for every $x \in \mathcal{T}_r$ there us a control $u \in \mathcal{U}$ for which one has

$$\mathbb{P}\left[\sup_{t \geq 0} d(X^u_x(t), \mathcal{T})e^{\lambda t} \leq C_\varepsilon d(x, \mathcal{T})\ ,\ X^u_x(t) \in \mathcal{C}\ \forall t \geq 0\right] \geq 1 - \varepsilon. \tag{2.2}$$

Remark 2.1. We point out that assumption (H2) implies that for any $x \in \mathcal{T}_r$

$$\sup_{u \in \mathcal{U}} \mathbb{P}\left[\lim_{t \to +\infty} d(X^u_x(t), \mathcal{T}) = 0\ ,\ X^u_x(t) \in \mathcal{C}\ \forall t \geq 0\right] = 1.$$

Indeed, for any $\varepsilon > 0$ and for suitable positive constants λ and C_ε, the local exponentially stabilizability implies the existence of a control $u \in \mathcal{U}$ such that

$$\begin{aligned}(1-\varepsilon) &\le \mathbb{P}\left[\sup_{t\ge 0} d(X_x^u(t),\mathcal{T})e^{\lambda t} \le C_\varepsilon d(x,\mathcal{T})\ ,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\right]\\ &\le \mathbb{P}\left[\lim_{t\to+\infty} d(X_x^u(t),\mathcal{T}) = 0\ ,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\right],\end{aligned}$$

and the result follows by the arbitrariness of ε. We also note that without loss of generality we may assume that $r > 0$ in (H2)-(ii) is so small that $\mathcal{T}_r \subset C$.

Aim of this work is to characterize the set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ of initial states $x \in \mathbb{R}^d$ which can be driven by an admissible control to the target $\mathcal{T}$ with positive probability:

$$\begin{aligned}\mathcal{D}^{\mathcal{T},\mathcal{C}} :=& \Big\{ x\in\mathbb{R}^d : \exists u\in\mathcal{U}\ \text{ s.t.}\\ &\quad \mathbb{P}\left[\lim_{t\to+\infty} d(X_x^u(t),\mathcal{T}) = 0\ ,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\right] > 0\Big\}\\ =& \left\{ x\in\mathbb{R}^d : \sup_{u\in\mathcal{U}} \mathbb{P}\left[\lim_{t\to+\infty} d(X_x^u(t),\mathcal{T}) = 0\ ,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\right] > 0\right\}.\end{aligned}$$

The set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ is called the *domain of asymptotic controllability* (with positive probability) of $\mathcal{T}$.

Remark 2.2. We conjecture that the approach in this paper can be extended to a characterization of the sets

$$\left\{ x\in\mathbb{R}^d : \sup_{u\in\mathcal{U}} \mathbb{P}\left[\lim_{t\to+\infty} d(X_x^u(t),\mathcal{T}) = 0\ ,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\right] = p\right\}$$

for given probabilities $p \in [0, 1]$, similar to how [10] extends [14]. However, in order no to overload this paper we decided to postpone this extension to a future paper.

3. Some results on the set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$

For any $x \in \mathbb{R}^d$ and $u \in \mathcal{U}$ we introduce the random hitting time $\tau(x, u)$ as the first time instant when the trajectory starting at point x and driven by the control u hits the set $\mathcal{T}_r$, that is for any $\omega \in \Omega$

$$\tau(x,u)(\omega) := \inf\left\{t\ge 0 : X_x^u(t)(\omega)\in\mathcal{T}_r\right\}. \tag{3.1}$$

Remark 3.1. We remark that under our assumptions on the set of control processes $\mathcal{U}$, the property of *stability under bifurcation* is satisfied, that is for any $u_1, u_2 \in \mathcal{U}$ and any stopping time $\tau \ge 0$ one has

$$u_1 \mathbb{1}_{[0,\tau]} + (u_1\mathbb{1}_A + u_2\mathbb{1}_{A^C})\mathbb{1}_{(\tau,+\infty)} \in \mathcal{U}.$$

In [8] it is shown how this property automatically follows from stability under concatenation and that stability under bifurcation is important in order to rigorously establish a Dynamic Programming Principle (DPP). In our context, this property also plays another important role in ensuring the controllability of the system. Indeed, for every $y \in \mathcal{T}_r$ the exponential stabilizability property guarantees the existence of a control $u_y \in \mathcal{U}$ such that (2.2) holds. Intuitively, this means that once a controlled path hits the boundary of $\mathcal{T}_r$, we can control it to $\mathcal{T}$ by switching to the process $u_{X_x^u(\tau(x,u))}$. However, this is only possible if the process

$$\bar{u}(t) = u\mathbb{1}_{\{t\le\tau(x,u)\}} + \left(u\mathbb{1}_{\{\tau(x,u)=+\infty\}} + u_{X_x^u(\tau(x,u))}\mathbb{1}_{\{\tau(x,u)<\infty\}}\right)\mathbb{1}_{\{t>\tau(x,u)\}}$$

belongs to $\mathcal{U}$ and this is exactly what the stability under bifurcation property guarantees.

Our goal is now to establish a relation between the set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ and the hitting time $\tau(x,u)$. To this end, we start with the following preliminary result. Therein and in the rest of the paper we use the notation $X_\tau^u := X_x^u(\tau(x,u))$.

Lemma 3.2. *Let assumptions (H1)–(H2) be satisfied. Then for the hitting time $\tau(x,u)$ from* (3.1) *there exist positive constants λ, C such that*

$$\begin{aligned}
&\sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u)<+\infty\,,\ X_x^u(t)\in\mathcal{C}\ \forall t\in[0,\tau(x,u)]\Big]>0\\
&\Rightarrow \sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u)<+\infty\,,\ X_x^u(t)\in\mathcal{C}\ \forall t\ge 0\,,\\
&\qquad\qquad \sup_{t\ge0} d(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t),\mathcal{T})e^{\lambda t}\le C\Big]>0.
\end{aligned}$$

Proof. The statement is proved using the exponential stabilizability assumption. By assumption there exists $\nu\in\mathcal{U}$ such that $\mathbb{P}[\tau(x,\nu)<+\infty$ and $X_x^\nu(t)\in\mathcal{C},\forall t\in[0,\tau(x,\nu)]]>0$. Moreover, thanks to assumption (H2)-(ii), constants $\lambda, C>0$ can be found such that for any $y\in\mathcal{T}_r$, there is $u_y\in\mathcal{U}$ with

$$\mathbb{P}\Big[\sup_{t\ge0} d(X_y^{u_y}(t),\mathcal{T})e^{\lambda t}\le C\,,\ X_y^{u_y}(t)\in\mathcal{C}\ \forall t\ge 0\Big]\ge\frac{1}{2}.$$

Therefore, defining the control

$$\bar{\nu}(t) := \nu\mathbb{1}_{\{t\le\tau(x,\nu)\}} + \left(\nu\mathbb{1}_{\{\tau(x,\nu)=+\infty\}} + u_{X_\tau^\nu}\mathbb{1}_{\{\tau(x,\nu)<\infty\}}\right)\mathbb{1}_{\{t>\tau(x,\nu)\}},$$

see Remark 3.1, and abbreviating $\tau=\tau(x,\nu)=\tau(x,\bar{\nu})$ one obtains

$$\begin{aligned}
&\mathbb{P}\Big[\tau(x,\bar\nu)<+\infty\,,\ X_x^{\bar\nu}(t)\in\mathcal{C}\ \forall t\ge0\,,\ \sup_{t\ge0} d(X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t),\mathcal{T})e^{\lambda t}\le C\Big]\\
&=\mathbb{P}\Big[\tau(x,\bar\nu)<+\infty\,,\ X_x^{\bar\nu}(t)\in\mathcal{C}\ \forall t\in[0,\tau(x,\bar\nu)]\,,\ X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t)\in\mathcal{C}\ \forall t\ge0\,,\\
&\quad\sup_{t\ge0} d(X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t),\mathcal{T})e^{\lambda t}\le C\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^\nu = y \ , \ \tau(x,\nu) = s \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,\nu)]\Big] \\
&\quad \cdot \mathbb{P}\Big[X_y^{u_y}(t) \in \mathcal{C} \ \forall t \geq 0 \ , \ \sup_{t\geq 0} d(X_y^{u_y}(t),\mathcal{T})e^{\lambda t} \leq C \Big| X_s^\nu = y\Big] dyds \\
&\geq \frac{1}{2}\int_0^{+\infty}\int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^\nu = y \ , \ \tau(x,\nu) = s \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,\nu)]\Big] dyds \\
&= \frac{1}{2}\, \mathbb{P}\Big[\tau(x,\nu) < +\infty \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,\nu)]\Big] > 0.
\end{aligned}
$$

□

Thanks to the previous result, the following alternative characterization of $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ is obtained.

Proposition 3.3. *Let assumptions (H1)–(H2) be satisfied. Then*

$$
\mathcal{D}^{\mathcal{T},\mathcal{C}} = \Big\{x \in \mathbb{R}^d : \sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty \ , \ X_x^u(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,u)]\Big] > 0\Big\}.
$$

Proof. The "$\subseteq$" inclusion is immediate since for every $u \in \mathcal{U}$ one has

$$
\begin{aligned}
&\Big\{\omega \in \Omega : \lim_{t\to+\infty} d(X_x^u(t),\mathcal{T}) = 0 \ , \ X_x^u(t) \in \mathcal{C} \ \forall t \geq 0\Big\} \\
&\subseteq \Big\{\omega \in \Omega : \tau(x,u) < +\infty \ , \ X_x^u(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,u)]\Big\}.
\end{aligned}
$$

For the converse inclusion, consider $x \in \mathbb{R}^d$ with

$$
\sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty \ , \ X_x^u(t) \in \mathcal{C} \ \forall t \in [0,\tau(x,u)]\Big] > 0.
$$

Then, Lemma 3.2 yields

$$
\begin{aligned}
\sup_{u\in\mathcal{U}} \mathbb{P}\Big[&\tau(x,u) < +\infty \ , \ \sup_{t\geq 0} d(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t),\mathcal{T})e^{\lambda t} \leq C \ , \\
&X_x^u(t) \in \mathcal{C} \ \forall t \geq 0\Big] > 0
\end{aligned}
$$

which immediately implies

$$
\sup_{u\in\mathcal{U}} \mathbb{P}\Big[X_x^u(t) \in \mathcal{C} \ \forall t \geq 0 \ , \ \lim_{t\to\infty} d(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t),\mathcal{T}) = 0\Big] > 0
$$

and thus $x \in \mathcal{D}^{\mathcal{T},\mathcal{C}}$. □

Proposition 3.4. *Assume assumptions (H1)-(H2) be satisfied. Then $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ is an open set.*

Proof. Let us start observing that for any $x \in \mathcal{D}^{\mathcal{T},\mathcal{C}}$, there is a time $T > 0$ and a control $\nu \in \mathcal{U}$ such that

$$
\mathbb{P}\Big[d(X_x^\nu(T),\mathcal{T}) \leq \frac{r}{2} \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \geq 0\Big] =: \eta > 0.
$$

Thanks to assumptions (H1), one has that for any $\varepsilon > 0$

$$\lim_{|x-y|\to 0} \mathbb{P}\Big[\sup_{s\in[0,T]} \big|X_x^\nu(t) - X_y^\nu(t)\big| > \varepsilon\Big] = 0,$$

therefore we can find $\delta_\eta > 0$ such that for any x, y such that $|x-y| \le \delta_\eta$

$$\mathbb{P}\Big[\sup_{s\in[0,T]} \big|X_x^\nu(t) - X_y^\nu(t)\big| > \varepsilon\Big] \le \frac{\eta}{2}.$$

It follows that for any fixed $\varepsilon > 0$ if $y \in B(x, \delta_\eta)$, the set $\Omega_1 \subset \mathcal{F}$ defined by

$$\Omega_1 := \Big\{\omega \in \Omega : d(X_x^\nu(T)(\omega), \mathcal{T}) \le \frac{r}{2}\ ,\ X_x^\nu(t)(\omega) \in \mathcal{C}\ \forall t \ge 0\ ,$$
$$\sup_{s\in[0,T]} \big|X_x^\nu(t) - X_y^\nu(t)\big|(\omega) \le \varepsilon\Big\}$$

satisfies

$$\begin{aligned}
\mathbb{P}[\Omega_1] &= \mathbb{P}\Big[d(X_x^{\bar u}(T), \mathcal{T}) \le \frac{r}{2}\ ,\ X_x^{\bar u}(t) \in \mathcal{C}\ \forall t \ge 0\ ,\ \sup_{s\in[0,T]} \big|X_x^{\bar u}(t) - X_y^{\bar u}(t)\big| \le \varepsilon\Big] \\
&= 1 - \mathbb{P}\Big[\Big(d(X_x^\nu(T), \mathcal{T}) \le \frac{r}{2}\ ,\ X_x^\nu(t) \in \mathcal{C}\ \forall t \ge 0\Big)^C \\
&\quad \cup\ \Big(\sup_{s\in[0,T]} \big|X_x^\nu(t) - X_y^\nu(t)\big| > \varepsilon\Big)\Big] \\
&\ge 1 - \mathbb{P}\Big[\Big(d(X_x^\nu(T), \mathcal{T}) \le \frac{r}{2}\ ,\ X_x^\nu(t) \in \mathcal{C}\ \forall t \ge 0\Big)^C\Big] \\
&\quad - \mathbb{P}\Big[\sup_{s\in[0,T]} \big|X_x^\nu(t) - X_y^\nu(t)\big| > \varepsilon\Big] \\
&\ge 1 - 1 + \eta - \frac{\eta}{2} = \frac{\eta}{2} > 0.
\end{aligned}$$

For any $\omega \in \Omega_1$, since $X_x^\nu(t) \in \mathcal{C}, \forall t \ge 0$ and $\mathcal{C}$ is an open set one has

$$\delta(x,\nu)(\omega) := \inf_{t\in[0,T]} d(X_x^\nu(t), \mathcal{C}^C)(\omega) > 0.$$

and

$$\sup_{t\in[0,T]} |X_x^\nu(t) - X_y^\nu(t)|(\omega) < \delta(x,\nu)(\omega) \Rightarrow X_y^\nu(t)(\omega) \in \mathcal{C}, \quad \forall t \in [0,T].$$

Furthermore it is also possible to prove that there exist $M > 0$ and $\tilde\Omega_1 \subseteq \Omega_1$ with $\mathbb{P}[\tilde\Omega_1] > 0$ such that

$$\forall \omega \in \tilde\Omega_1 \qquad \delta(x,\nu)(\omega) > M. \tag{3.2}$$

Indeed defined

$$B_n := \Big\{\omega \in \Omega_1 : \delta(x,\nu)(\omega) \in [\frac{1}{n+1}, \frac{1}{n})\Big\}$$

one has

$$0 < \mathbb{P}[\Omega_1] = \mathbb{P}[\bigcup_{n\geq 0} B_n] = \sum_{n\geq 0} \mathbb{P}[B_n].$$

It means that there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{P}[B_{\bar{n}}] > 0$ and defined

$$\tilde{\Omega}_1 := \left\{ \omega \in \Omega_1 : \delta(x,\nu)(\omega) \geq \frac{1}{\bar{n}+1} \right\}$$

we have $\mathbb{P}[\tilde{\Omega}_1] \geq \mathbb{P}[B_{\bar{n}}] > 0$. We have now all the elements necessary for concluding the proof. Taking $\varepsilon \leq \min\{M/2, r/2\}$ we have that for any $\omega \in \tilde{\Omega}_1$

$$X_y^\nu(t)(\omega) \in \mathcal{C}, \forall t \in [0,T]$$

and

$$d(X_y^\nu(T), \mathcal{T})(\omega) \leq d(X_x^\nu(T), \mathcal{T})(\omega) + |X_x^\nu(T) - X_y^\nu(T)|(\omega) \leq \frac{r}{2} + \varepsilon \leq r$$

that is $\tau(y,\nu)(\omega) \leq T$.
In conclusion we have proved that there exists a control $\nu \in \mathcal{U}$ such that for any $y \in B(x, \delta_\eta)$

$$\mathbb{P}\Big[\tau(y,\nu) < +\infty \ , \ X_y^\nu(t) \in \mathcal{C} \ \forall t \in [0, \tau(y,\nu)]\Big] > 0,$$

that means $y \in \mathcal{D}^{\mathcal{T},\mathcal{C}}$. □

4. The "level set" function v

We are now going to define a function v that we will use in order to characterize the domain $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ as a sub-level set. Let us start introducing two functions $g : \mathbb{R}^d \times U \to \mathbb{R}$ and $h : \mathbb{R}^d \to [0, +\infty]$ such that

(H3) there exist constants L_g, M_g and $g_0 > 0$ such that for any $x, x' \in \mathbb{R}^d$, $u \in U$ and $\mathcal{T}$, $\mathcal{T}_r$ from (H2)

$$|g(x,u) - g(x',u)| \leq L_g|x - x'|;$$
$$g(x,u) \leq M_g;$$
$$g \geq 0 \quad \text{and} \quad g(x,u) = 0 \Leftrightarrow x \in \mathcal{T};$$

and

$$\inf_{u\in U} g(x,u) \geq g_0 > 0, \quad \forall x \in \mathbb{R}^d \setminus \mathcal{T}_r; \tag{4.1}$$

(H4) h is a locally Lipschitz continuous function in $\mathcal{C}$ such that

(i) $h(x) = +\infty \Leftrightarrow x \notin \mathcal{C}$;
$h(x_n) \to +\infty, \quad \forall x_n \to x \notin \mathcal{C}$;
$h(x) = 0, \quad \forall x \in \mathcal{T}$;

(ii) there exists a constant $L_h \geq 0$ such that

$$\left| e^{-h(x)} - e^{-h(x')} \right| \leq L_h |x - x'| \tag{4.2}$$

for any $x, x' \in \mathbb{R}^d$.

Let the function $v : \mathbb{R}^d \to [0,1]$ be defined by:

$$v(x) := \inf_{u \in \mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t \geq 0} \left(-e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \right] \right\}. \tag{4.3}$$

We will now show that the function v can be used in order to characterize the domain of controllability $\mathcal{D}^{\mathcal{T},\mathcal{C}}$. In particular, we are going to prove that $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ consists of the set of points x where v is strictly lower than one.

Theorem 4.1. *Let assumptions (H1)–(H4) be satisfied, then*

$$x \in \mathcal{D}^{\mathcal{T},\mathcal{C}} \Leftrightarrow v(x) < 1.$$

Proof. "$\Leftarrow$" We show $v(x) = 1$ for every $x \notin \mathcal{D}^{\mathcal{T},\mathcal{C}}$. If $x \notin \mathcal{D}^{\mathcal{T},\mathcal{C}}$ then Proposition 3.3 implies

$$\sup_{u \in \mathcal{U}} \mathbb{P}\left[\tau(x,u) < +\infty \ , \ X_x^u(t) \in \mathcal{C} \ \forall t \in [0, \tau(x,u)] \right] = 0.$$

This means that for any control $u \in \mathcal{U}$ and almost every realization $\omega \in \Omega$

$$\tau(x,u)(\omega) = +\infty \quad \text{or} \quad \exists \bar{t} \in [0, \tau(x,u)(\omega)] : X_x^u(\bar{t})(\omega) \notin \mathcal{C}.$$

On the one hand, if $\tau(x,u)(\omega) = +\infty$, $\not\exists t$ such that $X_x^u(t)(\omega) \in \mathcal{T}_r$. By assumption (H3) it follows that

$$g(X_x^u(t), u(t))(\omega) > g_0, \qquad \forall t \geq 0$$

with $g_0 > 0$, that is

$$e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}(\omega) \leq e^{-g_0 t - h(X_x^u(t))}(\omega) \qquad \forall t \geq 0.$$

On the other hand, if $X_x^u(\bar{t})(\omega) \notin \mathcal{C}$ for a certain $\bar{t} \in [0, \tau(x,u)(\omega)]$, one has $h(X_x^u(\bar{t}))(\omega) = +\infty$. In both cases, for every $u \in \mathcal{U}$ the argument of the expectation in (4.3) almost surely has the value 0, implying

$$1 + \mathbb{E}\left[\sup_{t \geq 0} \left(-e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \right] = 1$$

for every $u \in \mathcal{U}$ from which $v(x) = 1$ follows by the definition of v.

"$\Rightarrow$" We will prove that $\sup_{u \in \mathcal{U}} \mathbb{E}[\inf_{t \geq 0} e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}] > 0$ for every $x \in \mathcal{D}^{\mathcal{T},\mathcal{C}}$. Let us start observing that, since there exists a control $\nu \in \mathcal{U}$ such that

$$\mathbb{P}\left[\tau(x,\nu) < +\infty \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \in [0, \tau(x,\nu)] \right] > 0,$$

then there exist $T, M > 0$ large enough such that for

$$\Omega_1^u := \left\{ \omega \in \Omega : \tau(x,u) < T \ , \ \max_{t \in [0,\tau(x,u)]} h(X_x^u(t)) \leq M \right\}$$

one has $\delta := \sup_{u \in \mathcal{U}} \mathbb{P}[\Omega_1^u] > 0$. Indeed, defining

$$\begin{aligned} \Omega_\infty := & \left\{ \omega \in \Omega : \tau(x,\nu) < +\infty \ , \ X_x^\nu(t) \in \mathcal{C} \ \forall t \in [0, \tau(x,\nu)] \right\} \\ = & \left\{ \omega \in \Omega : \tau(x,\nu) < +\infty \ , \ h(X_x^u(t)) < \infty \ \forall t \in [0, \tau(x,\nu)] \right\} \end{aligned}$$

and

$$\Omega_n := \Big\{\omega \in \Omega : \tau(x,u) < n \;,\; \max_{t\in[0,\tau(x,u)]} h(X_x^u(t)) \le n\Big\}$$

one has

$$0 < \mathbb{P}[\Omega_\infty] = \mathbb{P}[\bigcup_{n\ge 0} \Omega_n] \le \sum_{n\ge 0} \mathbb{P}[\Omega_n].$$

Hence, there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{P}[\Omega_{\bar{n}}] > 0$ and thus $\sup_{u\in\mathcal{U}} \mathbb{P}[\Omega_1^u] > 0$ for $T = M = \bar{n}$.
Moreover, thanks to the assumption of local exponential stabilizability in probability, there exist constants $\lambda, C > 0$ such that for any $y \in \mathcal{T}_r$

$$\sup_{u\in\mathcal{U}} \mathbb{P}[A_y^u] \ge 1 - \frac{\delta}{2}$$

for $A_y^u := \Big\{\omega \in \Omega : \sup_{t\ge 0} d(X_y^u(t), \mathcal{T}) e^{\lambda t} \le C \;,\; X_y^u(t) \in \mathcal{C}\ \forall t \ge 0\Big\}$. In what follows we will denote by $\tau = \tau(x,u)$ the hitting time (3.1) if no ambiguity arises. For any $u \in \mathcal{U}$ one has (recall that $g \ge 0$):

$$\begin{aligned}
&\mathbb{E}\Big[\inf_{t\ge 0} \exp\Big\{-\int_0^t g(X_x^u(\xi), u(\xi))d\xi - h(X_x^u(t))\Big\}\Big] \\
&\ge \mathbb{E}\Big[\exp\Big\{-\int_0^{+\infty} g(X_x^u(\xi), u(\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_x^u(\xi))\Big\}\Big] \\
&\ge \int_{\Omega_1^u} \exp\Big\{-\int_0^{+\infty} g(X_x^u(\xi), u(\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_x^u(\xi))\Big\} d\mathbb{P} \\
&\ge \int_{\Omega_1^u} \exp\Big\{-\int_0^{\tau} g(X_x^u(\xi), u(\xi))d\xi - \int_\tau^{+\infty} g(X_x^u(\xi), u(\xi))d\xi \\
&\quad - \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \vee \max_{\xi\in[\tau,+\infty)} h(X_x^u(\xi))\Big\} d\mathbb{P} \\
&\ge \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y \;,\; \tau = s \;,\; \tau < T \;,\; \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \le M\Big] e^{-g_0 T - M} \\
&\quad \cdot \mathbb{E}\Big[\exp\Big\{-\int_\tau^{+\infty} g(X_x^u(\xi), u(\xi))d\xi - \max_{\xi\in[\tau,+\infty)} h(X_x^u(\xi))\Big\} \Big| \begin{smallmatrix} X_\tau^u = y,\ \tau = s, \\ \omega\in\Omega_1^u \end{smallmatrix}\Big] dyds \\
&\ge e^{-g_0 T - M} \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau < T, \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \le M\Big] \\
&\quad \cdot \mathbb{E}\Big[e^{-\int_0^{+\infty} g(X_y^{u(s+\cdot)}(\xi), u(s+\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_y^{u(s+\cdot)}(\xi))} \Big| X_s^u = y\Big] dyds.
\end{aligned}$$

Here we are using the notation $a \vee b := \max(a, b)$.
Therefore, applying the Lipschitz continuity of g and h, one has

$$
\begin{aligned}
& e^{-g_0 T - M} \sup_{u \in \mathcal{U}} \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau < T, \max_{\xi \in [0,\tau]} h(X_x^u(\xi)) \leq M\Big] \\
& \quad \cdot \mathbb{E}\Big[e^{-\int_0^{+\infty} g(X_y^{u(s+\cdot)}(\xi), u(s+\xi))d\xi - \max\limits_{\xi \in [0,+\infty)} h(X_y^{u(s+\cdot)}(\xi))} \Big| X_s^u = y\Big] dyds \\
& \geq e^{-g_0 T - M} \sup_{u \in \mathcal{U}} \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau < T, \max_{\xi \in [0,\tau]} h(X_x^u(\xi)) \leq M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A_y^u} e^{-\int_0^{+\infty} g(X_y^{u(s+\cdot)}(\xi), u(s+\xi))d\xi - \max\limits_{\xi \in [0,+\infty)} h(X_y^{u(s+\cdot)}(\xi))} \Big| X_s^u = y\Big] dyds \\
& \geq e^{-g_0 T - M} \sup_{u \in \mathcal{U}} \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau < T, \max_{\xi \in [0,\tau]} h(X_x^u(\xi)) \leq M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A_y^u} e^{-L_g \int_0^{+\infty} d(X_y^{u(s+\cdot)}(\xi), \mathcal{T})d\xi - \max\limits_{\xi \in [0,+\infty)} L d(X_y^{u(s+\cdot)}(\xi), \mathcal{T})} \Big| X_s^u = y\Big] dyds \\
& \geq e^{-g_0 T - M} \sup_{u \in \mathcal{U}} \int_0^T \int_{d(y,\mathcal{T})=r} \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau < T, \max_{\xi \in [0,\tau]} h(X_x^u(\xi)) \leq M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A_y^u} e^{-L_g \int_0^{+\infty} C e^{-\lambda\xi} d\xi - \max\limits_{\xi \in [0,+\infty)} L C e^{-\lambda\xi}} \Big| X_s^u = y\Big] dyds \\
& \geq e^{-g_0 T} e^{-M} e^{-\frac{CL_g}{\lambda}} e^{-LC} \sup_{u \in \mathcal{U}} \int_0^T \int_{y \in \mathcal{T}_r} \mathbb{E}\Big[\chi_{A_y^u} \Big| X_s^u = y\Big] \\
& \quad \cdot \mathbb{P}\Big[X_\tau^u = y, \tau = s, \tau(x,u) < T, \max_{\xi \in [0,\tau]} h(X_x^u(\xi)) \leq M\Big] dyds \\
& = e^{-g_0 T} e^{-M} e^{-\frac{CL_g}{\lambda}} e^{-LC} \sup_{u \in \mathcal{U}} \mathbb{P}\Big[\Omega_1^u \cap A_{X_\tau^u}^u\Big] > 0
\end{aligned}
$$

where for the last inequality we used the fact that (thanks again to the arguments in Remark 3.1) one has $\sup_{u \in \mathcal{U}} \mathbb{P}[\Omega_1^u \cap A_{X_\tau^u}^u] > 0$. $\square$

Remark 4.2. The definition of the function v is based on a similar construction used in [19] for a deterministic controlled setting. That paper shows that in the deterministic setting the domain of controllability can alternatively be characterized by a second function, whose definition, translated to the stochastic framework, would be

$$
V(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\sup_{t \geq 0} \int_0^t g(X_x^u(s), u(s))ds + h(X_x^u(t))\Big]. \tag{4.4}
$$

A little computation using Jensen's inequality shows the relation

$$
\Big\{x \in \mathbb{R}^d : V(x) < +\infty\Big\} \subseteq \Big\{x \in \mathbb{R}^d : v(x) < 1\Big\}.
$$

Since, however, it is not clear whether the opposite inclusion holds in the stochastic setting, we will exclusively work with v in the remainder of this paper.

5. The PDE characterization of $\mathcal{D}^{\mathcal{T},\mathcal{C}}$

After having shown that $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ can be expressed as a sub-level set of v, we now proceed to the second main result of this paper, the PDE characterization of v and thus of $\mathcal{D}^{\mathcal{T},\mathcal{C}}$. In order to derive the PDE which is solves by v, we need to establish a dynamic programming principle (DPP) for v. Unfortunately, however, the presence of the supremum inside the expectation in the definition of v prohibits the direct use of the standard dynamic programming techniques. In particular, it is possible to verify that v does not satisfy a fundamental concatenation property that is usually the main tool necessary for the derivation of the associated partial differential equation. To avoid this difficulty, we follow the classical approach to reformulate the problem by adding a new variable $y \in \mathbb{R}$ that, roughly speaking, keeps track of the running maximum (we refer to [6], [7] for general results regarding this kind of problems). For this reason we introduce the function $\vartheta : \mathbb{R}^d \times [-1, 0] \to [0, 1]$ defined as follows:

$$\vartheta(x, y) := \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \left(-e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \vee y \right] \right\}. \tag{5.1}$$

We point out that

$$\vartheta(x, -1) = v(x) \qquad \forall x \in \mathbb{R}^d,$$

therefore ϑ can still be used for characterizing the set $\mathcal{D}^{\mathcal{T},\mathcal{C}}$ and one has

$$\mathcal{D}^{\mathcal{T},\mathcal{C}} = \left\{ x \in \mathbb{R}^d : \vartheta(x, -1) < 1 \right\}. \tag{5.2}$$

Furthermore, it follows from Theorem 4.1 that

$$\vartheta(x, y) = \begin{cases} 1 + y & \text{on} \quad \mathcal{T} \times [-1, 0] \\ 1 & \text{on} \quad (\mathcal{D}^{\mathcal{T},\mathcal{C}})^C \times [-1, 0]. \end{cases} \tag{5.3}$$

In what follows we will also denote

$$G(t, x, u) := \int_0^t g(X_x^u(s), u(s))ds,$$

so that using this notation the function ϑ reads

$$\vartheta(x, y) = \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \left(-e^{-G(t,x,u) - h(X_x^u(t))} \right) \vee y \right] \right\}.$$

For the new state variable y we can define the following "maximum dynamics":

$$Y_{x,y}^u(\cdot) := e^{G(\cdot,x,u)} \left(y \vee \sup_{t\in[0,\cdot\,]} (-e^{-G(t,x,u)-h(X_x^u(t))}) \right) \tag{5.4}$$

We remark that $Y_{x,y}^u(t) \in [-1, 0]$ for any $u \in \mathcal{U}$, $t \geq 0$ and $(x, y) \in \mathbb{R}^d \times [-1, 0]$.

We are now able to prove a DPP for the function ϑ. Since no information is available at the moment on the regularity of ϑ, we state the weak version of the DPP presented in [8] involving the semi-continuous envelopes of ϑ. Let us denote by ϑ^* and ϑ_* respectively the upper and lower semi-continuous envelope of ϑ. One has:

Lemma 5.1. *Let assumptions (H1),(H3) and (H4) be satisfied. Then for any finite stopping time $\theta \geq 0$ measurable with respect to the filtration, one has the dynamic programming principle (DPP)*

$$\inf_{u\in\mathcal{U}} \mathbb{E}\left[e^{-G(\theta,x,u)}\vartheta_*(X_x^u(\theta), Y_{x,y}^u(\theta))) + \int_0^\theta g(X_x^u(s), u(s))e^{-G(s,x,u)}ds\right]$$
$$\leq \vartheta(x,y) \leq \inf_{u\in\mathcal{U}} \mathbb{E}\left[e^{-G(\theta,x,u)}\vartheta^*(X_x^u(\theta), Y_{x,y}^u(\theta))) + \int_0^\theta g(X_x^u(s), u(s))e^{-G(s,x,u)}ds\right].$$

For a rigorous proof of this result we refer to [8]. Here, we only show the main steps that lead to our formulation of the DPP in the non-controlled and continuous case.

Sketch of the proof of Lemma 5.1*.* For any finite stopping time $\theta \geq 0$ one has

$$\begin{aligned}
&\vartheta(x,y) - 1\\
&= \mathbb{E}\left[\sup_{t\geq 0}\big(- e^{-G(t,x)-h(X_x(t))}\big) \vee y\right]\\
&= \mathbb{E}\left[\sup_{t\geq \theta}\big(- e^{-G(t,x)-h(X_x(t))}\big) \vee \sup_{t\in[0,\theta]}\big(- e^{-G(t,x)-h(X_x(t))}\big) \vee y\right]\\
&= \mathbb{E}\left[e^{-G(\theta,x)}\sup_{t\geq \theta}\big(- e^{-\int_\theta^t g(X_x(s))ds-h(X_x(t))}\big) \vee \sup_{t\in[0,\theta]}\big(- e^{-G(t,x)-h(X_x(t))}\big) \vee y\right]\\
&= \mathbb{E}\left[e^{-G(\theta,x)}\left\{\sup_{t\geq \theta}\big(- e^{-\int_\theta^t g(X_x(s))ds-h(X_x(t))}\big) \vee Y_{x,y}(\theta)\right\}\right]
\end{aligned}$$

where the property of the maximum $(a\cdot b)\vee c = a\cdot(b\vee \frac{c}{a})$, $\forall a,b,c\in\mathbb{R}, a>0$, is used. Applying now the tower property of the expectation one obtains

$$\begin{aligned}
&\vartheta(x,y)\\
&= 1 + \mathbb{E}\left[\mathbb{E}\left[e^{-G(\theta,x)}\left\{\sup_{t\geq 0}\big(- e^{-G(t,X_x(\theta))-h(X_{X_x(\theta)}(t))}\big) \vee Y_{x,y}(\theta)\right\}\middle|\mathcal{F}_\theta\right]\right]\\
&= 1 + \mathbb{E}\left[e^{-G(\theta,x)}\mathbb{E}\left[\sup_{t\geq 0}\big(- e^{-G(t,X_x(\theta))-h(X_{X_x(\theta)}(t))}\big) \vee Y_{x,y}(\theta)\middle|\mathcal{F}_\theta\right]\right]\\
&= 1 + \mathbb{E}\left[e^{-G(\theta,x)}\Big(\vartheta(X_x(\theta), Y_{x,y}(\theta)) - 1\Big)\right]
\end{aligned}$$

and the result just follows observing that $1-e^{-G(\theta,x)} = \int_0^\theta g(X_x(s))e^{-G(s,x)}ds$. $\square$

Using the DPP from Lemma 5.1, we can now show that ϑ is actually continuous.

Proposition 5.2. *Let assumptions (H1)–(H4) be satisfied. Then the function* ϑ *from* (5.1) *is continuous in* $\mathbb{R}^{d+1}$.

Proof. The continuity with respect to y is trivial and one has

$$|\vartheta(x,y)-\vartheta(x,y')|\le |y-y'|.$$

For what concerns the continuity with respect to x, in $(\mathcal{D}^{\mathcal{T},\mathcal{C}})^C$ and $\mathcal{T}$ there is nothing to prove thanks to (5.3).
We start by proving the continuity at the boundary of $\mathcal{T}$. Let $x_0\in\partial\mathcal{T}$. We aim to prove that for any ε there exists $\delta>0$ such that for $x\in B(x_0,\delta)$ one has

$$\vartheta(x,y)-\vartheta(x_0,y)=\vartheta(x,y)-(1+y)\le\varepsilon. \tag{5.5}$$

For $\delta>0$ small enough we can assume that $B(x_0,\delta)\subset\mathcal{T}_r$. Hence, for this choice of δ there exists $\lambda>0$ such that for any $\varepsilon>0$ there exists a constant C_ε and a control ν such that one has

$$\mathbb{P}[A_x^C]\le\frac{\varepsilon}{2}$$

for $A_x:=\left\{\omega\in\Omega:\sup_{t\ge0} d(X_x^\nu(t),\mathcal{T})e^{\lambda t}\le C_\varepsilon d(x,\mathcal{T}) \text{ and } X_x^\nu(t)\in\mathcal{C},\forall t\ge 0\right\}$.
From the definition of ϑ and the monotonicity of the exponential one has

$$\begin{aligned}
&\vartheta(x,y)-(1+y)\\
&=\vartheta(x,y)-\big(1+(-1)\vee y\big)\\
&\le\mathbb{E}\left[\sup_{t\ge0}\,(-e^{-G(t,x,\nu)-h(X_x^\nu(t))})\vee y-\big((-1)\vee y\big)\right]\\
&\le\mathbb{E}\left[1+\sup_{t\ge0}(-e^{-G(t,x,\nu)-h(X_x^\nu(t))})\right]\\
&=\mathbb{E}\left[1-e^{-\sup_{t\ge0}(G(t,x,\nu)+h(X_x^\nu(t)))}\right]\\
&=\int_{A_x}1-e^{-\sup_{t\ge0}(G(t,x,\nu)+h(X_x^\nu(t)))}d\mathbb{P}+\int_{A_x^C}1-e^{-\sup_{t\ge0}(G(t,x,\nu)+h(X_x^\nu(t)))}d\mathbb{P}\\
&\le\int_{A_x}1-e^{-\sup_{t\ge0}(G(t,x,\nu)+h(X_x^\nu(t)))}d\mathbb{P}+\frac{\varepsilon}{2}
\end{aligned}$$

for every $T>0$. Therefore in order to conclude (5.5) it will be sufficient to estimate the integral taking into account the events in A_x.
For sufficiently small $\delta>0$ we obtain $C_\varepsilon d(x,\mathcal{T})<r$ and thus $X_x^\nu(t,\omega)\in\mathcal{T}_r$ for all $\omega\in A_x$, all $t\ge0$ and all $x\in B(x_0,\delta)$. Thus, since $\mathcal{T}_r$ is a compact subset of $\mathcal{C}$, the function h is Lipschitz with constant L along all these trajectories. Since g is Lipschitz, too, and since $g(\xi,u)=h(\xi)=0$ $\forall\xi\in\mathcal{T},u\in U$, for any $t\ge0$ one has

$$g(X_x^\nu(t),\nu(t))\le L_g d(X_x^\nu(t),\mathcal{T})\quad\text{and}\quad h(X_x^\nu(t))\le Ld(X_x^\nu(t),\mathcal{T}).$$

Using these inequalities and the definition of A_x, we obtain

$$\begin{aligned}
&\int_{A_x} 1 - e^{-\sup_{t\geq 0}(G(t,x,\nu)+h(X_x^\nu(t)))} d\mathbb{P} \\
&\leq \int_{A_x} 1 - e^{-\int_0^{+\infty} g(X_x^\nu(t),\nu(t))dt - \sup_{t\geq 0} h(X_x^\nu(t)))} d\mathbb{P} \\
&\leq \int_{A_x} 1 - e^{-\int_0^{+\infty} L_g d(X_x^\nu(t),\mathcal{T})dt - \sup_{t\geq 0} L d(X_x^\nu(t),\mathcal{T})} d\mathbb{P} \\
&\leq \int_{A_x} 1 - e^{-\int_0^{+\infty} L_g C_\varepsilon d(x,\mathcal{T}) e^{-\lambda t} dt - \sup_{t\geq 0} L C_\varepsilon d(x,\mathcal{T}) e^{-\lambda t}} d\mathbb{P} \\
&\leq \int_{A_x} 1 - e^{-(L_g/\lambda + L)C_\varepsilon \delta} d\mathbb{P} \;\leq\; 1 - e^{-(L_g/\lambda+L)C_\varepsilon\delta}.
\end{aligned}$$

Now, choosing $\delta > 0$ such that

$$\left(\frac{L_g}{\lambda} + L\right) C_\varepsilon \delta \leq -\ln(1-\varepsilon/2)$$

we have

$$1 - e^{-(L_g/\lambda+L)C_\varepsilon\delta} \leq \varepsilon/2$$

and thus

$$\vartheta(x,y) - (1+y) \leq \int_{A_x} 1 - e^{-\sup_{t\geq 0}(G(t,x,\nu)+h(X_x^\nu(t)))} d\mathbb{P} + \frac{\varepsilon}{2} \leq \varepsilon,$$

for any x with $d(x,\mathcal{T}) < \delta$, which proves (5.5) and thus continuity at $\partial\mathcal{T}$. The proof of the theorem is concluded proving the continuity in $\mathbb{R}^d \setminus \mathcal{T}$. We point out that we already know that $\vartheta(x,y) = 1+y$ in $(\mathcal{D}^{\mathcal{T},\mathcal{C}})^C$, however the proof that follows is independent of whether $x \in \mathcal{D}^{\mathcal{T},\mathcal{C}}$ or not. Let $x \in \overline{\mathcal{D}^{\mathcal{T},\mathcal{C}}} \setminus \mathcal{T}$ and $\xi \in B(x,\delta)$. From the DPP (Lemma 5.1), for any $y \in [-1,0]$ and any finite stopping time θ, there exists a control $\nu = \nu_\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned}
\vartheta(\xi,y) - \vartheta(x,y) \leq &\mathbb{E}\Big[e^{-G(\theta,\xi,\nu)} \vartheta^*(X_\xi^\nu(\theta), Y_{\xi,y}^\nu(\theta)) - e^{-G(\theta,\xi,\nu)} \\
&- e^{-G(\theta,x,\nu)} \vartheta_*(X_x^\nu(\theta), Y_{x,y}^\nu(\theta)) + e^{-G(\theta,x,\nu)} \Big] + \frac{\varepsilon}{4}.
\end{aligned}$$

In order to prove the result we will use the continuity at $\mathcal{T}$ we proved above. We can in fact state that for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$\vartheta^*(z,y) \leq 1 + y + \frac{\varepsilon}{4} \quad \text{if} \quad d(z,\mathcal{T}) \leq \eta_\varepsilon.$$

Let $T \geq -\frac{\ln(\varepsilon/4)}{g^*}$ and $0 < R \leq \frac{\varepsilon/4}{L_h + L_g T}$ where $g^* := \inf_{\{x : d(x,\mathcal{T}) \geq \eta_\varepsilon/2\}} g(x,\nu) > 0$ and L_h, L_g are, respectively, the Lipschitz constant of $e^{-h(x)}$ and g. Denoting

$$E := \Big\{ \omega \in \Omega : \sup_{t\in[0,T]} |X_x^\nu(t) - X_\xi^\nu(t)| \geq R \Big\},$$

under assumption (H1) we can choose δ sufficiently small such that $\mathbb{P}[E] \leq \frac{\varepsilon}{4}$. Then (recalling that $\vartheta^*, \vartheta_* \in [0,1]$), we have

$$\begin{aligned} &\int_E \Big(e^{-G(\theta,\xi,\nu)} \vartheta^*(X^\nu_\xi(\theta), Y^\nu_{\xi,y}(\theta)) - e^{-G(\theta,\xi,\nu)} \\ &\qquad - e^{-G(\theta,x,\nu)} \vartheta_*(X^\nu_x(\theta), Y^\nu_{x,y}(\theta)) + e^{-G(\theta,x,\nu)} \Big) d\mathbb{P} \\ &\leq \int_E e^{-G(\theta,x,\nu)} d\mathbb{P} \leq \mathbb{P}[E] \leq \frac{\varepsilon}{4}. \end{aligned} \tag{5.6}$$

Let us now define the stopping time

$$\tau := \inf \Big\{ t \geq 0 : d(X^\nu_x(t), \mathcal{T}) \leq \eta_\varepsilon \Big\}$$

with the convention that $\tau(\omega) = T$ if $d(X^\nu_x(t)(\omega), \mathcal{T}) > \eta_\varepsilon, \forall t \in [0,T]$ (this ensures the finiteness of the stopping time needed for the DPP). Thanks to (5.6) (which holds for an arbitrary stopping time), we can write

$$\begin{aligned} &\mathbb{E}\Big[e^{-G(\tau,\xi,\nu)} \vartheta^*(X^\nu_\xi(\tau), Y^\nu_{\xi,y}(\tau)) - e^{-G(\tau,\xi,\nu)} \\ &\qquad - e^{-G(\tau,x,\nu)} \vartheta_*(X^\nu_x(\tau), Y^\nu_{x,y}(\tau)) + e^{-G(\tau,x,\nu)} \Big] \\ &\leq \frac{\varepsilon}{4} + \int_{E^C} \cdots = \frac{\varepsilon}{4} + \int_{E^C \cap \{\tau < T\}} \cdots + \int_{E^C \cap \{\tau = T\}} \cdots \end{aligned}$$

and we will provide estimates separately for the last two integrals.
In $E^C \cap \{\tau = T\}$, using again $\vartheta_*, \vartheta^* \in [0,1]$, we get

$$\begin{aligned} &\int_{E^C \cap \{\tau = T\}} \Big(e^{-G(T,\xi,\nu)} \vartheta^*(X^\nu_\xi(T), Y^\nu_{\xi,y}(T)) - e^{-G(T,\xi,\nu)} \\ &\qquad - e^{-G(T,x,\nu)} \vartheta_*(X^\nu_x(T), Y^\nu_{x,y}(T)) + e^{-G(T,x,\nu)} \Big) d\mathbb{P} \\ &\leq \int_{E^C \cap \{\tau = T\}} e^{-G(T,x,\nu)} d\mathbb{P} \leq e^{-g^* T} \leq \frac{\varepsilon}{4} \end{aligned}$$

thanks to the choice of T. In $E^C \cap \{\tau < T\}$ we have

$$\begin{aligned} &\int_{E^C \cap \{\tau < T\}} \Big(e^{-G(\tau,\xi,\nu)} \vartheta^*(X^\nu_\xi(\tau), Y^\nu_{\xi,y}(\tau)) - e^{-G(\tau,\xi,\nu)} \\ &\qquad - e^{-G(\tau,x,\nu)} \vartheta_*(X^\nu_x(\tau), Y^\nu_{x,y}(\tau)) + e^{-G(\tau,x,\nu)} \Big) d\mathbb{P} \\ &\leq \int_{E^C \cap \{\tau < T\}} \Big\{ e^{-G(\tau,\xi,\nu)} \Big(1 + Y^\nu_{\xi,y}(\tau) + \frac{\varepsilon}{4} \Big) - e^{-G(\tau,\xi,\nu)} \\ &\qquad - e^{-G(\tau,x,\nu)} \Big(1 + Y^\nu_{x,y}(\tau) \Big) + e^{-G(\tau,x,\nu)} \Big\} d\mathbb{P} \\ &= \int_{E^C \cap \{\tau < T\}} \Big(e^{-G(\tau,\xi,\nu)} Y^\nu_{\xi,y}(\tau) - e^{-G(\tau,x,\nu)} Y^\nu_{x,y}(\tau) \Big) d\mathbb{P} + \frac{\varepsilon}{4} \end{aligned}$$

where we used the fact that $\vartheta_*(x,y) \geq 1+y$. Recalling the definition of the variable $Y(\cdot)$ given by (5.4) and because of assumptions (H3)–(H4) we have

$$\begin{aligned}
&\int_{E^C\cap\{\tau<T\}} \Big(e^{-G(\tau,\xi,\nu)}Y^\nu_{\xi,y}(\tau) - e^{-G(\tau,x,\nu)}Y^\nu_{x,y}(\tau)\Big)d\mathbb{P}\\
&= \int_{E^C\cap\{\tau<T\}} \Big\{\sup_{t\in[0,\tau]} (-e^{-G(t,\xi,\nu)-h(X^\nu_\xi(t))}) \vee y\\
&\quad - \sup_{t\in[0,\tau]} (-e^{-G(t,x,\nu)-h(X^\nu_x(t))}) \vee y\Big\}d\mathbb{P}\\
&\leq \int_{E^C\cap\{\tau<T\}} \sup_{t\in[0,\tau]}\Big|e^{-G(t,\xi,\nu)-h(X^\nu_\xi(t))} - e^{-G(t,x,\nu)-h(X^\nu_x(t))}\Big|\, d\mathbb{P}\\
&\leq \int_{E^C\cap\{\tau<T\}} \sup_{t\in[0,\tau]} e^{-G(t,\xi,\nu)}\Big|e^{-h(X^\nu_\xi(t))} - e^{-h(X^\nu_x(t))}\Big|\\
&\quad + \sup_{t\in[0,\tau]} e^{-h(X^\nu_\xi(t))}\Big|e^{-G(t,\xi,\nu)} - e^{-G(t,x,\nu)}\Big|\, d\mathbb{P}\\
&\leq \int_{E^C\cap\{\tau<T\}} (L_h+L_gT)\sup_{t\in[0,T]}|X^\nu_\xi(t) - X^\nu_x(t)|d\mathbb{P} \leq \frac{\varepsilon}{4}
\end{aligned}$$

thanks to the choice of R. □

Thanks to Lemma 5.1 and the continuity of ϑ, we can finally characterize ϑ as a solution, in the viscosity sense, of a second order Hamilton-Jacobi-Bellman equation. To this end, we define the open domain $\mathcal{O} \subset \mathbb{R}^d \times [-1,0]$ by

$$\mathcal{O} = \Big\{(x,y) \in \mathbb{R}^{d+1} : -e^{-h(x)} < y < 0\Big\}$$

and the following two components of its boundary

$$\begin{aligned}
\partial_1\mathcal{O} &:= \Big\{(x,y) \in \overline{\mathcal{O}} : y = 0\Big\}\\
\partial_2\mathcal{O} &:= \Big\{(x,y) \in \overline{\mathcal{O}} : y = -e^{-h(x)},\ y<0\Big\}.
\end{aligned}$$

Remark 5.3. We point out that thanks to the relation

$$\vartheta(x,y) = \vartheta(x, -e^{-h(x)}) \qquad \forall y \leq -e^{-h(x)}$$

it is sufficient to determine the values of ϑ in $\overline{\mathcal{O}}$ in order to characterize ϑ in the whole domain of definition $\mathbb{R}^d \times [-1,0]$. We also remark that $\vartheta(x,0) = 1$ for any $x \in \mathbb{R}^d$.

Let us consider the following Hamiltonian $H : \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathcal{S}^d \to \mathbb{R}$, with $\mathcal{S}^d$ denoting the space of $d\times d$ symmetric matrices

$$\begin{aligned}
&H(x,y,r,p,q,Q) :=\\
&\sup_{u\in U}\Big\{g(x,u)(r-1) - p{\cdot}b(x,u) - \frac{1}{2}Tr[\sigma\sigma^T(x,u)Q] - q\,g(x,u)y\Big\}.
\end{aligned} \tag{5.7}$$

The following theorem holds.

Theorem 5.4. *Let assumptions (H1)-(H4) be satisfied. Then ϑ is a continuous viscosity solution of*

$$\begin{cases} H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D^2_x\vartheta)=0 & in \quad \mathcal{O} \\ \vartheta = 1 & on \quad \partial_1\mathcal{O} \\ -\partial_y\vartheta = 0 & on \quad \partial_2\mathcal{O}. \end{cases} \tag{5.8}$$

We refer to [16, Definition 7.4] for the definition of viscosity solution for equation (5.8). It is in fact well-known that boundary conditions may have to be considered in a weak sense in order to obtain existence of a solution. It means that for the viscosity sub-solution (resp. super-solution) of equation (5.8), we will ask that on the boundary $\partial_2\mathcal{O}$ the inequality

$$\min\big(H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D^2_x\vartheta),-\partial_y\vartheta\big)\le 0$$

$$\Big(\text{resp.}\quad \max\big(H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D^2_x\vartheta),-\partial_y\vartheta\big)\ge 0\Big)$$

holds in the viscosity sense. In contrast, the condition on $\partial_1\mathcal{O}$ is assumed in the strong sense.

Proof of Theorem 5.4. The boundary condition on $\partial_1\mathcal{O}$ follows directly by the definition of ϑ. Let us start proving the sub-solution property.

Let be $\varphi \in C^{2,1}(\overline{\mathcal{O}})$ such that $\vartheta-\varphi$ attains a maximum at point $(\bar x,\bar y)\in \overline{\mathcal{O}}$ and let us assume $\bar y<0$. We need to show

$$H(\bar x,\bar y,\vartheta(\bar x,\bar y),D_x\varphi(\bar x,\bar y),\partial_y\varphi(\bar x,\bar y),D^2_x\varphi(\bar x,\bar y))\le 0 \tag{5.9}$$

if $(\bar x,\bar y)\notin\partial_2\mathcal{O}$ and

$$\min\big(H(\bar x,\bar y,\vartheta(\bar x,\bar y),D_x\varphi(\bar x,\bar y),\partial_y\varphi(\bar x,\bar y),D^2_x\varphi(\bar x,\bar y)),-\partial_y\varphi(\bar x,\bar y)\big)\le 0 \tag{5.10}$$

if $(\bar x,\bar y)\in\partial_2\mathcal{O}$.

Without loss of generality we can always assume that $(\bar x,\bar y)$ is a strict local maximum point (let us say in a ball of radius r) and that $\vartheta(\bar x,\bar y)=\varphi(\bar x,\bar y)$. Using continuity arguments, for any $u\in\mathcal{U}$ and for almost every $\omega\in\Omega$ we can find $\theta:=\theta^u$ small enough such that

$$(X^u_{\bar x}(\theta)\ ,\ Y^u_{\bar x,\bar y}(\theta))(\omega)\in B((\bar x,\bar y),r).$$

Let us in particular consider a constant control $u(t)\equiv u\in U$. Thanks to Lemma 5.1 one has

$$\varphi(\bar x,\bar y)\le\mathbb{E}\Big[e^{-G(\theta,\bar x,u)}\varphi(X^u_{\bar x}(\theta),Y^u_{\bar x,\bar y}(\theta))+\int_0^\theta g(X^u_{\bar x}(s),u)e^{-G(s,\bar x,u)}ds\Big]. \tag{5.11}$$

We now take into account two different cases, depending on whether or not we are in $\partial_2\mathcal{O}$.

Case 1: $\bar y>-e^{-h(\bar x)}$. In this case (since we are inside $\mathcal{O}$) for almost every $\omega\in\Omega$, taking the stopping time $\theta(\omega)$ small enough, we can say

$$e^{G(\theta,\bar x,u)}(\bar y\vee\sup_{t\in[0,\theta]}(-e^{-G(t,\bar x,u)-h(X^u_{\bar x}(t))}))(\omega)=(e^{G(\theta,\bar x,u)}\bar y)(\omega).$$

Therefore from (5.11), for this choice of the stopping time θ, for any $u \in U$ we obtain

$$\mathbb{E}\Big[\varphi(\bar{x},\bar{y}) - e^{-G(\theta,\bar{x},u)}\varphi(X^u_{\bar{x}}(\theta), e^{G(\theta,\bar{x},u)}\bar{y}) + \int_0^\theta g(X^u_{\bar{x}}(s),u)e^{-G(s,\bar{x},u)}ds\Big] \leq 0 \tag{5.12}$$

which yields

$$\mathbb{E}\Big[\int_0^\theta -d\Big(e^{-G(s,\bar{x},u)}\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\Big) + g(X^u_{\bar{x}}(s),u)e^{-G(s,\bar{x},u)}ds\Big] \leq 0.$$

Applying the Ito's formula we have

$$\begin{aligned}
&d\Big(e^{-G(s,\bar{x},u)}\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\Big)\\
&= e^{-G(s,\bar{x},u)}\Big\{ - g(X^u_{\bar{x}}(s),u)\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\\
&\quad + D_x\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\cdot dX^u_{\bar{x}}(s) + \partial_y\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\, g(X^u_{\bar{x}}(s),u)\bar{y}\\
&\quad + \frac{1}{2}Tr[\sigma\sigma^T(X^u_{\bar{x}}(s),u)D^2_x\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})]\Big\}.
\end{aligned}$$

Then, replacing the stopping time θ by $\theta_h := \theta \wedge h$ we get

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{h}\int_0^{\theta_h} e^{-G(s,\bar{x},u)}\Big\{ - g(X^u_{\bar{x}}(s),u)\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\\
&+ D_x\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\cdot b(X^u_{\bar{x}}(s),u) + \partial_y\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})\, g(X^u_{\bar{x}}(s),u)\bar{y}\\
&\quad + \frac{1}{2}Tr[\sigma\sigma^T(X^u_{\bar{x}}(s),u)D^2_x\varphi(X^u_{\bar{x}}(s), e^{G(s,\bar{x},u)}\bar{y})]\Big\}ds\Big] \leq 0.
\end{aligned}$$

Letting $h \to 0$ and observing that for ω fixed $\theta_h{=}h$ holds for $h > 0$ sufficiently small, we can apply the mean value theorem inside the integral for any fixed ω. In this way, applying also the dominated convergence theorem, we finally obtain at $(\bar{x},\bar{y})$

$$g(\bar{x},u)(\varphi-1) - D_x\varphi\cdot b(\bar{x},u) - \frac{1}{2}Tr[\sigma\sigma^T(\bar{x},u)D^2_x\varphi] - \partial_y\varphi g(\bar{x},u)\bar{y} \leq 0 \quad \forall u \in U$$

and then thanks to the arbitrariness of u

$$H(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}), D_x\varphi(\bar{x},\bar{y}), \partial_y\varphi(\bar{x},\bar{y}), D^2_x\varphi(\bar{x},\bar{y})) \leq 0,$$

i.e., (5.9).

Case 2: $\bar{y} = -e^{-h(\bar{x})}$. If $-\partial_y\varphi(\bar{x},\bar{y}) \leq 0$, then (5.10) holds. Hence, let us assume that

$$-\partial_y\varphi(\bar{x},\bar{y}) > 0.$$

This means that in a neighborhood of $(\bar{x},\bar{y})$

$$\varphi(x,y_1) \geq \varphi(x,y_2) \qquad \text{if } y_1 \leq y_2.$$

For almost every $\omega \in \Omega$ and for $\theta(\omega)$ small enough, the point

$$\Big(X^u_{\bar{x}}(\theta)\ ,\ e^{G(\theta,\bar{x},u)}(\bar{y} \vee \sup_{t\in[0,\theta]}(-e^{-G(t,\bar{x},u)-h(X^u_{\bar{x}}(t))}))\Big)$$

is in this neighborhood. Because of

$$Y^u_{\bar x,\bar y}(\theta) = e^{G(\theta,\bar x,u)}\left(\bar y \vee \sup_{t\in[0,\theta]} (-e^{-G(t,\bar x,u)-h(X^u_{\bar x}(t))})\right) \geq e^{G(\theta,\bar x,u)}\bar y$$

we obtain for any $u \in U$

$$\begin{aligned}
&\varphi(\bar x,\bar y)\\
&\leq \mathbb{E}\left[e^{-G(\theta,\bar x,u)}\left\{\varphi(X^u_{\bar x}(\theta), Y^u_{\bar x,\bar y}(\theta)) + \int_0^\theta g(X^u_{\bar x}(s),u)e^{-G(s,\bar x,u)}ds\right\}\right]\\
&\leq \mathbb{E}\left[e^{-G(\theta,\bar x,u)}\left\{\varphi(X^u_{\bar x}(\theta), e^{G(\theta,\bar x,u)}\bar y) + \int_0^\theta g(X^u_{\bar x}(s),u)e^{-G(s,\bar x,u)}ds\right\}\right]
\end{aligned}$$

from which we have again (5.12) and thus

$$H(\bar x,\bar y,\varphi(\bar x,\bar y), D_x\varphi(\bar x,\bar y), \partial_y\varphi(\bar x,\bar y), D^2_x\varphi(\bar x,\bar y)) \leq 0,$$

implying (5.10).

For proving the super-solution property let us assume that $\vartheta-\varphi$ attains a strict maximum in $(\bar x,\bar y)$. Starting again from the DPP and taking the stopping time θ small enough one has

$$\varphi(\bar x,\bar y) \geq \sup_{u\in\mathcal{U}} \mathbb{E}\left[e^{-G(\theta,\bar x,u)}\varphi(X^u_{\bar x}(\theta), Y^u_{\bar x,\bar y}(\theta)) + \int_0^\theta g(X^u_{\bar x}(s),u)e^{-G(s,\bar x,u)}ds\right].$$

If either $\bar y > -e^{-h(\bar x)}$ or $\bar y = -e^{-h(\bar x)}$ and $-\partial_y\varphi(\bar x,\bar y) < 0$ we get, for θ small enough

$$\varphi(\bar x,\bar y) \geq \sup_{u\in\mathcal{U}} \mathbb{E}\left[e^{-G(\theta,\bar x,u)}\varphi(X^u_{\bar x}(\theta), e^{G(\theta,\bar x,u)}\bar y) + \int_0^\theta g(X^u_{\bar x}(s),u)e^{-G(s,\bar x,u)}ds\right]$$

and the desired property can be obtained by standard passages, with the usual modifications required for proving the super-solution inequality. □

6. Comparison principle

After having shown that ϑ solves equation (5.8), we now consider the uniqueness question. As usual in viscosity solution theory, we establish uniqueness in form of a comparison principle between USC sub-solutions and LSC super-solutions. In proving such a comparison principle, some additional difficulties arise because of the degeneracy of g in $\mathcal{T}$. In order to overcome this difficulty we will show that for any super-solution (resp. sub-solution) a super-optimality (resp. sub-optimality) principle holds and then we will use this result for proving the comparison principle by a direct calculation. The proof of the optimality principles given here adapts the techniques in presented in [4, Theorem 2.32] to the particular case of the second order boundary value problem (5.8).

Let us start with a preliminary result. We can in fact prove that, thanks to assumption (H4)-(ii), together with (H1) and (H3), for any control $u \in \mathcal{U}$

and $T \geq 0$, aside from the standard estimation for the process $X_x^u(\cdot)$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_x^u(t) - x\right|^2\right] \leq C_T(1+|x|^2)T \tag{6.1}$$

the following inequality also holds for $Y_{x,y}(\cdot)$: If $(x,y) \in \overline{\mathcal{O}}$, then

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_{x,y}^u(t) - y\right|^2\right] \leq C_T\left(|1-e^{M_g T}|^2 + e^{2M_g T}(1+|x|^2)T\right) \tag{6.2}$$

where $C_T = Ce^{CT}$ and C is a constant depending only on the Lipschitz constants of b and σ, and M_g denotes the bound of the function g. We prove the following result for a later use.

Lemma 6.1. *For any $\varepsilon > 0, T \geq 0$ and $(x,y) \in \overline{\mathcal{O}}$ one has*

$$\sup_{u\in\mathcal{U}} \mathbb{P}\left[\sup_{t\in[0,T]} |(X_x^u(t), Y_{x,y}^u(t)) - (x,y)| > \varepsilon\right] \leq \frac{C_{T,x}}{\varepsilon^2}\left(T + |1-e^{M_g T}|^2\right)$$

for $C_{T,x} := C_T(1+e^{2M_g T})(1+|x|^2)$.

Proof. The result is a consequence of Doob's inequality applied to the sub-martingale $M_t := \sup_{s\in[0,t]}(|X(s)-x| + |Y(s)-y|)$ and of inequalities (6.1) and (6.2). Indeed for any $u \in \mathcal{U}$ one has:

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,T]} |(X_x^u(t), Y_{x,y}^u(t)) - (x,y)| > \varepsilon\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,T]} \left(|X_x^u(t)-x| + |Y_{x,y}^u(t)-y|\right) > \varepsilon\right] \\
&\leq \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left(\sup_{t\in[0,T]} |X_x^u(t)-x| + |Y_{x,y}^u(t))-y|\right)^2\right] \\
&\leq \frac{2}{\varepsilon^2}\,\mathbb{E}\left[\sup_{t\in[0,T]} |X_x^u(t)-x|^2 + \sup_{t\in[0,T]} |Y_{x,y}^u(t))-y|^2\right] \\
&\leq \frac{C_T}{\varepsilon^2}\left((1+e^{2M_g T})T(1+|x|^2) + |1-e^{M_g T}|^2\right)
\end{aligned}$$

where C_T is the constant appearing in (6.1) and (6.2). This shows the claim. □

Let us define the domain

$$\mathcal{O}_\delta := \left\{(x,y) \in \overline{\mathcal{O}} : d(x,\mathcal{T}) > \delta,\ y < -\delta\right\}$$

and the associated exit time for the process $(X_x^u(t), Y_{x,y}^u(t))$

$$\tau_\delta^u := \inf\left\{t \geq 0 : (X_x^u(t), Y_{x,y}^u(t)) \notin \mathcal{O}_\delta\right\}.$$

Theorem 6.2. *Let* $\underline{V} \in USC(\overline{\mathcal{O}})$ *be a bounded viscosity sub-solution to equation* (5.8) *such that*

$$\underline{V}(x,y) = 1 \qquad \text{on } \partial_1\mathcal{O}.$$

Then $\underline{V}$ *satisfies*

$$\begin{aligned}\underline{V}(x,y) \le \inf_{u\in\mathcal{U}} \mathbb{E}\Bigg[&e^{-G(\tau_\delta^u(t),x,u)}\underline{V}(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t))) \\ &+\int_0^{\tau_\delta^u(t)} g(X_x^u(s),u(s))e^{-G(s,x,u)}ds\Bigg]\end{aligned} \tag{6.3}$$

for any $(x,y) \in \mathcal{O}_\delta$*,* $t \ge 0$*, where* $\tau_\delta^u(t) := \min(t, \tau_\delta^u)$ *and* τ_δ^u *denotes the exit time of the process* $(X_x^u(\cdot), Y_{x,y}^u(\cdot))$ *from the domain* $\mathcal{O}_\delta$*.*

Proof. This proof is based on an adaptation of classical arguments (see Theorem 2.32 in [4] for instance) to our context. Let us start observing that since $\underline{V}$ is upper semi-continuous we can write for any $(x,y) \in \overline{\mathcal{O}}$

$$\underline{V}(x,y) = \inf_{k\ge 0} V_k(x,y) \tag{6.4}$$

where $\{V_k\}_{k\ge 0}$ is a decreasing sequence of bounded continuous functions. Let us consider for $k \ge 0$ the following evolutionary obstacle problem

$$\begin{cases} \max\left(\partial_t V + H(x,y,V,D_xV,\partial_yV,D_x^2V)\,, V - V_k\right) = 0 & (0,t]\times\mathcal{O} \\ V(t,x,y) = 1 & (0,t]\times\partial_1\mathcal{O} \\ -\partial_y V(t,x,y) = 0 & (0,t]\times\partial_2\mathcal{O} \\ V(0,x,y) = V_k(x,y) & \overline{\mathcal{O}}. \end{cases} \tag{6.5}$$

It is immediate to verify that $\underline{V}$ is a bounded viscosity sub-solution of this problem for any $k \ge 0$ and $t \ge 0$. For $t \ge 0$, we now define the following function

$$L^k(t,x,y) := \begin{cases} \inf\limits_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-G(\tau_\delta^u(t),x,u)}V_k(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t))) & \overline{\mathcal{O}_\delta} \\ \qquad + \int_0^{\tau_\delta^u(t)} g(X_x^u(s),u(s))e^{-G(s,x,u)}ds\Big] & \\ V_k(x,y) & \overline{\mathcal{O}}\setminus\overline{\mathcal{O}_\delta}. \end{cases}$$

Let us start proving that L^k is continuous in $t = 0$. Of course, we only need to prove the result in $\overline{\mathcal{O}_\delta}$. Noting that $L^k(0,x,y) = V_k(x,y)$, for any $u \in \mathcal{U}$,

$(x,y) \in \overline{\mathcal{O}}_\delta$ one has

$$
\begin{aligned}
&\bigg|\mathbb{E}\bigg[e^{-G(\tau_\delta^u(t),x,u)}V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))\\
&\quad+\int_0^{\tau_\delta^u(t)} g(X_x^u(s),u(s))e^{-G(s,x,u)}ds\bigg]-L^k(0,x,y)\bigg|\\
&\le \mathbb{E}\bigg[\,|e^{-G(\tau_\delta^u(t),x,u)}V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))-V_k(x,y)|\,\bigg]+M_g t\\
&\le \mathbb{E}\bigg[e^{-G(\tau_\delta^u(t),x,u)}|V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))-V_k(x,y)|\bigg]\\
&\quad+\mathbb{E}\bigg[+|V_k(x,y)|\Big(1-e^{-G(\tau_\delta^u(t),x,u)}\Big)\bigg]+M_g t\\
&\le \mathbb{E}\bigg[e^{-G(\tau_\delta^u(t),x,u)}|V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))-V_k(x,y)|\bigg]\\
&\quad+C-Ce^{-M_g t}+M_g t.
\end{aligned}
$$

Thanks to the continuity of V_k, there exists δ_ε such that

$$|V_k(x,y)-V_k(\xi,\eta)|\le\frac{\varepsilon}{2}$$

for any $(\xi,\eta)\in B((x,y),\delta_\varepsilon)$. Therefore if we define the set

$$E:=\Big\{\omega\in\Omega:|\big(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t))\big)-(x,y)|>\delta_\varepsilon\Big\}$$

we obtain

$$\int_{E^C} e^{-G(\tau_\delta^u(t),x,u)}|V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))-V_k(x,y)|d\mathbb{P}\le\frac{\varepsilon}{2}.$$

Moreover, thanks to the boundedness of V_k we get

$$\int_{E} e^{-G(\tau_\delta^u(t),x,u)}|V_k(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t)))-V_k(x,y)|d\mathbb{P}\le 2M\mathbb{P}[E].$$

Using the result in Lemma 6.1 we can state that there exists a constant $C=C_x$ such that

$$\mathbb{P}[E]\le\frac{Ce^{Ct}}{\delta_\varepsilon^2}\Big(t+(1-e^{M_g t})^2\Big).$$

Therefore, there exists $t_\varepsilon>0$ such that for $t<t_\varepsilon$

$$\mathbb{P}[E]\le\frac{\varepsilon}{2}.$$

In conclusion we have proved that for any $\varepsilon>0$, if t is small enough

$$|L^k(t,x,y)-L^k(0,x,y)|\le\varepsilon$$

which proves continuity of L^k in $t=0$.

Denoting by L^k_* the lower semi-continuous envelope of L^k, it is possible to prove that the following DPP holds (see [5, Theorem 4.3]):

$$\inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_0^{\tau^u_\delta(\theta)} g(X^u_x(s),u(s))e^{-G(s,x,u)}ds + \mathbb{1}_{\{\theta\geq\tau^u_\delta\}}V_k(X^u_x(\tau^u_\delta),Y^u_{x,y}(\tau^u_\delta))e^{G(\tau^u_\delta,x,u)}$$
$$+\mathbb{1}_{\{\theta<\tau^u_\delta\}}L^k_*(t-\theta,X^u_x(\theta),Y^u_{x,y}(\theta))e^{G(\theta,x,u)}\bigg] \leq L^k(t,x,y).$$

for any stopping time $0\leq\theta\leq t$.

Thanks to this result, applying the standard dynamic programming arguments, it is possible to prove (see the proof given in [5]) that L^k_* is a viscosity super-solution of (6.5). We point out that the necessity of the obstacle term $V-V_k$ is a consequence of the possible discontinuity of L^k_* on the boundary of $\overline{\mathcal{O}}_\delta$. The initial condition and the boundary condition on $\partial_1\mathcal{O}$ are on the contrary satisfied in the strong sense because of the continuity of L^k_* in $t=0$ and $y=0$.

For equation (6.5) a comparison principle for semi-continuous viscosity sub- and super-solution holds, see Theorem A.1 in the appendix. It can be obtained by the arguments in [17] adapted to the parabolic case. The necessity of using such a result instead of a more classical comparison principle for fully nonlinear second order elliptic equations with oblique derivative boundary conditions, as that one presented for instance in [16] (see also the references therein), comes from the lack of regularity of the domain $\overline{\mathcal{O}}$. Since the key arguments of the proof in [17] easily extend to our context, we only give a sketch of the proof in the appendix.

Applying Theorem A.1, we obtain for any $(t,x,y)\in[0,+\infty)\times\mathcal{O}_\delta$

$$\underline{V}(x,y)\leq L^k_*(t,x,y),$$

which leads to

$$\underline{V}(x,y)\leq\mathbb{E}\bigg[e^{-G(\tau^u_\delta(t),x,u)}V_k(X^u_x(\tau^u_\delta(t)),Y^u_{x,y}(\tau^u_\delta(t)))$$
$$+\int_0^{\tau^u_\delta(t)} g(X^u_x(s),u(s))e^{-G(s,x,u)}ds\bigg] \qquad \forall u\in\mathcal{U}.$$

It remains to pass to the limit for $k\to+\infty$. Recalling expression (6.4) for $\underline{V}$ we have

$$\underline{V}(x,y)=\limsup_{k\to+\infty} V_k(x,y)$$

and then for any $u \in \mathcal{U}$

$$
\begin{aligned}
&\underline{V}(x,y)\\
&\le \limsup_{k\to+\infty} \mathbb{E}\Big[e^{-G(\tau_\delta^u(t),x,u)} V_k(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t)))\\
&\quad + \int_0^{\tau_\delta^u(t)} g(X_x^u(s), u(s)) e^{-G(s,x,u)} ds\Big]\\
&\le \mathbb{E}\Big[\limsup_{k\to+\infty} e^{-G(\tau_\delta^u(t),x,u)} V_k(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t)))\\
&\quad + \int_0^{\tau_\delta^u(t)} g(X_x^u(s), u(s)) e^{-G(s,x,u)} ds\Big]\\
&= \mathbb{E}\Big[e^{-G(\tau_\delta^u(t),x,u)} \underline{V}(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t)))\\
&\quad + \int_0^{\tau_\delta^u(t)} g(X_x^u(s), u(s)) e^{-G(s,x,u)} ds\Big]
\end{aligned}
$$

where for the second inequality we used Fatou's lemma, thanks to the boundedness of the functions V_k. Hence, the desired result is obtained thanks to the arbitrariness of $u \in \mathcal{U}$. □

The same techniques can be applied in order to prove the super-optimality principle for LSC super-solutions. In this case, however, compactness assumption on the dynamics (considering weak solutions of the SDE) are necessary in order to guarantee the last passage to the limit (see [15]). The version of the super-optimality principle we state below avoids this kind of assumption by taking into account only continuous super-solutions.

Theorem 6.3. *Let $\overline{V} \in C(\overline{\mathcal{O}})$ be a bounded viscosity super-solution to equation* (5.8)*. Then for any $(x,y) \in \mathcal{O}_\delta$ and $t \ge 0$*

$$
\begin{aligned}
\overline{V}(x,y) \ge \inf_{u\in\mathcal{U}} \mathbb{E}\Big[&e^{-G(\tau_\delta^u(t),x,u)} \overline{V}(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t)))\\
&+ \int_0^{\tau_\delta^u(t)} g(X_x^u(s), u(s)) e^{-G(s,x,u)} ds\Big].
\end{aligned}
\tag{6.6}
$$

Proof. Let us consider the following evolutionary obstacle problem:

$$
\begin{cases}
\min\left(\partial_t V + H(x,y,V,D_x V, \partial_y V, D_x^2 V)\ , V - \overline{V}\right) = 0 & (0,t] \times \mathcal{O}\\
V(t,x,y) = 1 & (0,t] \times \partial_1 \mathcal{O}\\
-\partial_y V(t,x,y) = 0 & (0,t] \times \partial_2 \mathcal{O}\\
V(0,x,y) = \overline{V}(x,y) & \overline{\mathcal{O}}.
\end{cases}
\tag{6.7}
$$

We can easily observe that $\overline{V}$ is a viscosity super-solution to (6.7). In what follows, we build a viscosity sub-solution for problem (6.7). Let $W : \overline{\mathcal{O}} \to \mathbb{R}$

be defined by

$$W(t,x,y) := \begin{cases} \inf\limits_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-G(\tau_\delta^u(t),x,u)}\overline{V}(X_x(\tau_\delta(t)),Y_{x,y}(\tau_\delta(t))) & \text{in } \overline{\mathcal{O}}_\delta \\ \qquad + \int_0^{\tau_\delta^u(t)} g(X_x^u(s),u(s))e^{-G(s,x,u)}ds\Big] & \\ \overline{V}(x,y) & \text{in } \overline{\mathcal{O}}\setminus\overline{\mathcal{O}}_\delta \end{cases}$$

Let us consider its upper semi-continuous envelope W^*. By similar arguments as in Theorem 6.2 we can prove that W^* is a viscosity sub-solution to (6.7). Indeed, the continuity with respect to time in $t=0$ can be prove as in Theorem 6.2. Moreover, the boundary condition on $\partial_1\mathcal{O}$ is satisfied in the strong sense thanks to the continuity of $\overline{V}$. Therefore, applying the comparison principle Theorem A.1 between sub and super solutions to (6.7) we get

$$\overline{V}(x,y) \geq W^*(t,x,y).$$

This yields

$$\overline{V}(x,y) \geq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-G(\tau_\delta^u(t),x,u)}\overline{V}(X_x^u(\tau_\delta^u(t)),Y_{x,y}^u(\tau_\delta^u(t))) + \int_0^{\tau_\delta^u(t)} g(X_x^u(s))e^{-G(s,x,u)}ds\Big]$$

for any $t\geq 0, (x,y)\in\overline{\mathcal{O}}_\delta$. □

The super-optimality principle from Theorem 6.3 and the sub-optimality principle from Theorem 6.2 are finally used in the next theorem in order to establish the desired comparison result.

Theorem 6.4. *Let* $\underline{V}\in USC(\overline{\mathcal{O}})$ *and* $\overline{V}\in C(\overline{\mathcal{O}})$ *be a bounded viscosity sub- and super-solution to equation* (5.8)*, respectively. Let us also assume that*

$$\underline{V}(x,y) \leq 1+y \leq \overline{V}(x,y) \quad \text{on} \quad \{(x,y)\in\overline{\mathcal{O}} : x\in\mathcal{T}\} \tag{6.8}$$

and

$$\underline{V}(x,0) = \overline{V}(x,0) = 1 \qquad \forall x\in\overline{\mathcal{O}}. \tag{6.9}$$

Then $\underline{V}(x,y)\leq\overline{V}(x,y)$ *for any* $(x,y)\in\overline{\mathcal{O}}$.

Proof. Clearly, if $x \in \mathcal{T}$ there is nothing to prove. Thanks to inequalities (6.6) and (6.3), for any $(x,y) \in \mathcal{O}_\delta$ and $T \geq 0$ we have

$$\begin{aligned}
&\underline{V}(x,y) - \overline{V}(x,y) \\
&\leq \sup_{u\in\mathcal{U}} \mathbb{E}\Bigg[e^{-G(\tau_\delta^u(T),x,u)}\Big(\underline{V}(X_x^u(\tau_\delta^u(T)), Y_{x,y}^u(\tau_\delta^u(T))) \\
&\quad - \overline{V}(X_x^u(\tau_\delta^u(T)), Y_{x,y}^u(\tau_\delta^u(T)))\Big)\Bigg] \\
&= \sup_{u\in\mathcal{U}}\Bigg\{ \int_{\tau_\delta^u \leq T} e^{-G(\tau_\delta^u,x,u)}\Big(\underline{V}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u)) - \overline{V}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u))\Big)d\mathbb{P} \\
&\quad + \int_{\tau_\delta^u > T} e^{-G(T,x,u)}\Big(\underline{V}(X_x^u(T), Y_{x,y}^u(T)) - \overline{V}(X_x^u(T), Y_{x,y}^u(T))\Big)d\mathbb{P}\Bigg\}
\end{aligned}$$

We will study these two integrals separately. Thanks to the (semi-)continuity of $\overline{V}$ and $\underline{V}$ and conditions (6.8) and (6.9), for any $\varepsilon > 0$ it is possible to find δ_ε small enough such that

$$\underline{V}(x,y) \leq 1 + y + \frac{\varepsilon}{2}, \qquad \overline{V}(x,y) \geq 1 + y - \frac{\varepsilon}{2} \qquad \text{if} \quad d(x,\mathcal{T}) \leq \delta$$

and

$$\underline{V}(x,y) \leq 1 + \frac{\varepsilon}{2}, \qquad \overline{V}(x,y) \geq 1 - \frac{\varepsilon}{2} \qquad \text{if} \quad y \geq -\delta.$$

Recalling that τ_δ^u is the exit time from the domain $\mathcal{O}_\delta$, we have that for any $u \in \mathcal{U}$ either $Y_{x,y}^u(\tau_\delta^u) = -\delta$ or $d(X_x^u(\tau_\delta^u), \mathcal{T}) = \delta$. For both these cases, choosing δ small enough, for the first integral we find

$$\begin{aligned}
&\int_{\tau_\delta^u \leq T} e^{-G(\tau_\delta^u,x,u)}\Big(\underline{V}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u)) - \overline{V}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u))\Big)d\mathbb{P} \\
&\leq \varepsilon\, \mathbb{P}[\tau_\delta^u \leq T] \leq \varepsilon.
\end{aligned}$$

For the second integral we can use the boundedness of $\overline{V}$ and $\underline{V}$. Denoting by M a bound for these functions, we obtain for any $u \in \mathcal{U}$

$$\begin{aligned}
&\int_{\tau_\delta^u > T} e^{-G(T,x,u)}\Big(\underline{V}(X_x^u(T), Y_{x,y}^u(T)) - \overline{V}(X_x^u(T), Y_{x,y}^u(T))\Big)d\mathbb{P} \\
&\leq 2M \int_{\tau_\delta^u > T} e^{-G(T,x,u)} d\mathbb{P}.
\end{aligned}$$

If we define

$$g^* := \inf\{g(x,u) \mid x \in \mathbb{R} : d(x,\mathcal{T}) > \delta, u \in U\} > 0$$

we finally obtain for T large enough

$$\underline{V}(x,y) - \overline{V}(x,y) \leq \ \varepsilon + e^{-g^* T} = 2\varepsilon$$

for any $(x,y) \in \mathcal{O}_\delta$ and the result is obtained thanks to the arbitrariness of ε.

Finally. we obtain the desired comparison principle in the whole domain $\overline{\mathcal{O}}$ by sending $\delta \to 0$, thanks to the upper semi-continuity of the function $\underline{V} - \overline{V}$. $\square$

An immediate consequence of this theorem and Theorem 5.4 is the following existence and uniqueness result.

Corollary 6.5. *Let assumptions (H1)–(H4) be satisfied. Then* ϑ *from* (5.1) *is the unique bounded and continuous viscosity solution to equation* (5.8) *such that* $\vartheta(x,y) = 1 + y$ *if* $x \in \mathcal{T}$ *and* $\vartheta(x,0) = 1$ *for any* $x \in \mathbb{R}^d$.

Appendix A. Comparison principle for obstacle problems with Dirichlet-Neumann boundary conditions

In this section we will give a proof of a comparison principle for the obstacle problem (6.7). The result and its proof modify the arguments in [17, Theorem 2.1] in order to take into account the unbounded domain of (6.7). Of course, the same arguments apply to (6.5).

Before starting the proof of the result, we introduce a more compact notation. Let us start defining

$$\tilde{b}(x,y,u) := \begin{pmatrix} b(x,u) \\ yg(x,u) \end{pmatrix} \in \mathbb{R}^{d+1} \text{ and } \tilde{\sigma}(x,y,u) := \begin{pmatrix} \sigma(x,u) \\ 0 \ldots 0 \end{pmatrix} \in \mathbb{R}^{(d+1)\times p}.$$

In what follow we will directly denote with x the variable in the augmented state space $\mathbb{R}^M$ for $M := d+1$, that is $x \equiv (x,y) \in \mathbb{R}^M$. Using this notation we can write the Hamiltonian H in (5.7) in the following compact form

$$H(x,r,q,Q) := \sup_{u\in U}\Big\{ -q\cdot\tilde{b}(x,u) - \frac{1}{2}Tr[\tilde{\sigma}\tilde{\sigma}^T(x,u)Q] + g(x,u)(r-1)\Big\}.$$

The boundary value problem we deal with is the following

$$\begin{cases} \min\Big(\partial_t v + H(x,v,Dv,D^2v)\ ,v-\psi\Big) = 0 & (0,T)\times\mathcal{O} \\ v(t,x) = 1 & (0,T)\times\partial_1\mathcal{O} \\ -\partial_{x_M} v(t,x) = 0 & (0,T)\times\partial_2\mathcal{O} \\ v(0,x) = \psi(x) & \overline{\mathcal{O}} \end{cases} \tag{A.1}$$

(where ∂_{x_M} denotes the partial derivative with respect to the M-th space variable and $\psi(x) = 1$ on $\partial_1\mathcal{O}$). We recall that the boundary conditions in $t = 0$ and $\partial_1\mathcal{O}$ are considered in the strong sense, that is for any viscosity sub-solution $\underline{v}$ (resp. super-solution $\overline{v}$) one has

$$\underline{v}(0,x) \le \psi(x) \qquad \text{(resp. } \overline{v}(0,x) \ge \psi(x)) \qquad \text{on } \overline{\mathcal{O}}$$

and

$$\underline{v}(t,x) \le 1 \qquad \text{(resp. } \overline{v}(t,x) \ge 1) \qquad \text{on } (0,T)\times\partial_1\mathcal{O}.$$

We also recall that on the boundary $\partial_2\mathcal{O}$ the following weak conditions

$$\min\Big(\min\big(\partial_t\underline{v} + H(x,\underline{v},D\underline{v},D^2\underline{v}),\underline{v}-\psi\big)\ ,\ -\partial_{x_M}\underline{v}\Big) \le 0$$

$$\max\Big(\min\big(\partial_t\overline{v} + H(x,\overline{v},D\overline{v},D^2\overline{v}),\overline{v}-\psi\big)\ ,\ -\partial_{x_M}\overline{v}\Big) \ge 0$$

are considered, in the viscosity sense, respectively for sub- and super-solutions. In the sequel we will denote by $|\cdot|_{M-1}$ the norm restricted to the first $M-1$ components of the vector, that is:

$$|x|_{M-1} := |(x_1, \ldots, x_{M-1})|, \qquad \forall x \in \mathbb{R}^M.$$

Theorem A.1. *Let assumptions (H1),(H3) and (H4) be satisfied and* $\psi \in C(\overline{\mathcal{O}})$. *Let* $\underline{v} \in USC([0,T] \times \overline{\mathcal{O}})$ *and* $\overline{v} \in LSC([0,T] \times \overline{\mathcal{O}})$ *be respectively a bounded viscosity sub- and super- solution of* (A.1). *Then for any* $x \in \overline{\mathcal{O}}$ *and* $t \in [0,T)$

$$\underline{v}(t,x) \leq \overline{v}(t,x).$$

Since the main arguments of the proof can be found in [17] we only report below the main lines and the modification necessary for dealing with the possible unboundedness of the domain.

Sketch of the proof. Recalling that the boundary $\partial_2\mathcal{O}$ is defined by the function $-e^{-h(x_1,\ldots,x_{M-1})}$, thanks to the Lipschitz assumption (H4)-(ii), we can easily observe that just taking $\mu := \frac{1}{\sqrt{1+L_h^2}}$, where L_h is the Lipschitz constant appearing in (H4)-(ii), for any $z \in \partial_2\mathcal{O}$ one has

$$\bigcup_{0\leq\xi\leq\mu} B(z-\xi, \xi\mu) \subset \mathcal{O}^C. \tag{A.2}$$

This corresponds to condition (2.9) in [17] and by the same arguments as in [17, Corollary 2.3] the existence of a function $\zeta \in C^2(\overline{\mathcal{O}})$ follows such that $\zeta \geq 0$ on $\overline{\mathcal{O}}$, $-\partial_{x_M}\zeta \geq 1$ on $\partial_2\mathcal{O}$ and $|D\zeta|, \|D^2\zeta\| \leq K_\zeta$ for some constant $K_\zeta \geq 0$. We point out that the in our case the proof of Corollary 2.3 can be strongly simplified because of the constancy of the derivative direction. In particular the local construction presented in [17] will be independent of the choice of the the boundary point allowing us to obtain a uniform bound on $|D\zeta|$ and $\|D^2\zeta\|$ on the whole domain. Let us define for $\delta, \rho, \beta > 0$

$$\underline{v}_{\delta,\rho,\beta}(t,x) := \underline{v}(t,x) - \delta e^{-\rho T}\zeta(x) - \frac{\beta}{T-t}$$

and

$$\overline{v}_{\delta,\rho,\beta}(t,x) := \overline{v}(t,x) + \delta e^{-\rho T}\zeta(x) + \frac{\beta}{T-t}.$$

One has

$$\underline{v}_{\delta,\rho,\beta} \overset{t\to T}{\longrightarrow} -\infty \qquad \text{and} \qquad \overline{v}_{\delta,\rho,\beta} \overset{t\to T}{\longrightarrow} +\infty.$$

It is possible to verify that $\underline{v}_{\delta,\rho,\beta}$ (resp. $\overline{v}_{\delta,\rho,\beta}$) is a sub-solution (resp. super-solution) of an obstacle problem with the following modified boundary condition on $\partial_2\mathcal{O}$:

$$-\partial_{x_M}v + \delta e^{-\rho T} \leq 0 \qquad \left(\text{resp.} \; -\partial_{x_M}v - \delta e^{-\rho T} \geq 0\right). \tag{A.3}$$

Moreover, thanks to the positivity of ζ, one has

$$\underline{v}_{\delta,\rho,\beta} \leq \underline{v} \qquad \text{and} \qquad \overline{v}_{\delta,\rho,\beta} \geq \overline{v},$$

so the boundary conditions for $t=0$ and $y=0$ in (6.7) are trivially satisfied. By using the non negativity of g and the linear growth of $\tilde{b}$ and $\tilde{\sigma}$, in $\overline{\mathcal{O}}$ one has

$$\begin{aligned}
&H(x,\underline{v}_{\delta,\rho,\beta},D\underline{v}_{\delta,\rho,\beta},D^2\underline{v}_{\delta,\rho,\beta})-H(x,\underline{v},D\underline{v},D^2\underline{v})\\
&\le H(x,\underline{v},D\underline{v}_{\delta,\rho,\beta},D^2\underline{v}_{\delta,\rho,\beta})-H(x,\underline{v},D\underline{v},D^2\underline{v})\\
&\le \sup_{u\in U}\left|\tilde{b}(x,u)\cdot\delta e^{-\rho T}D\zeta+\frac{1}{2}Tr[\tilde{\sigma}\tilde{\sigma}^T(x,u)(\delta e^{-\rho T}D^2\zeta)]\right|\\
&\le C_1\delta e^{-\rho T}(1+|x|^2_{M-1}),
\end{aligned}$$

where C_1 only depends on K_ζ and the Lipschitz constants of b and σ. Then if

$$\min\left(\partial_t\underline{v}+H(x,\underline{v},D\underline{v},D^2\underline{v})\ ,\ \underline{v}-\psi(x)\right)\le 0$$

for some $x\in\mathcal{O}\cup\partial_2\mathcal{O}$ one obtains

$$\begin{aligned}
&\min\Bigg(\partial_t\underline{v}_{\delta,\rho,\beta}+H(x,\underline{v}_{\delta,\rho,\beta},D\underline{v}_{\delta,\rho,\beta},D^2\underline{v}_{\delta,\rho,\beta})+\frac{\beta}{T^2}-C_1\delta e^{-\rho T}(1+|x|^2_{M-1})\ ,\\
&\quad\underline{v}_{\delta,\rho,\beta}+\frac{\beta}{T}-\psi(x)\Bigg)\\
&\le\min\left(\partial_t\underline{v}+H(x,\underline{v},D\underline{v},D^2\underline{v})\ ,\ \underline{v}-\psi(x)\right)\le 0.
\end{aligned}$$

The analogous result can be proved for the super-solution $\overline{v}_{\delta,\rho,\beta}$.
Our goal is now to prove the inequality

$$\underline{v}_{\delta,\rho,\beta}(t,x)\le\overline{v}_{\delta,\rho,\beta}(t,x)+2C_1\delta e^{-\rho(T-t)}(1+|x|^2_{M-1}) \tag{A.4}$$

for all $\delta,\rho,\beta>0$. By virtue of the definition of $\underline{v}_{\delta,\rho,\beta}$ and $\overline{v}_{\delta,\rho,\beta}$, this implies the claim of the theorem by letting $\delta,\beta\to 0$.
In order to prove (A.4), we consider the modified obstacle problems given by

$$\begin{cases}
\min\Big(\partial_t v+H(x,v,Dv,D^2v)+\frac{\beta}{T^2}-C_1\delta e^{-\rho T}(1+|x|^2_{M-1})\ , & \\
\qquad v+\frac{\beta}{T}-\psi\Big)\le 0 & (0,T)\times\mathcal{O}\\
v(t,x)\le 1 & (0,T)\times\partial_1\mathcal{O}\\
-\partial_{x_M}v(t,x)+\delta e^{-\rho T}\le 0 & (0,T)\times\partial_2\mathcal{O}\\
v(0,x)\le\psi(x) & \overline{\mathcal{O}}
\end{cases}$$

and

$$\begin{cases}
\min\Big(\partial_t v+H(x,v,Dv,D^2v)-\frac{\beta}{T^2}+C_1\delta e^{-\rho T}(1+|x|^2_{M-1})\ , & \\
\qquad v-\frac{\beta}{T}-\psi\Big)\ge 0 & (0,T)\times\mathcal{O}\\
v(t,x)\ge 1 & (0,T)\times\partial_1\mathcal{O}\\
-\partial_{x_M}v(t,x)-\delta e^{-\rho T}\ge 0 & (0,T)\times\partial_2\mathcal{O}\\
v(0,x)\ge\psi(x) & \overline{\mathcal{O}}
\end{cases}$$

and consider the function

$$\Phi(t,x) := \underline{v}_{\delta,\rho,\beta}(t,x) - \overline{v}_{\delta,\rho,\beta}(t,x) - 2C_1\delta e^{-\rho(T-t)}(1+|x|^2_{M-1}).$$

Thanks to the boundedness and the semi-continuity of $\underline{v}_{\delta,\rho,\beta}$ and $\overline{v}_{\delta,\rho,\beta}$, Φ admits a maximum point $(\hat{t}_{\delta,\rho,\beta}, \hat{x}_{\delta,\rho,\beta}) = (\hat{t}, \hat{x})$. If either $\hat{t} = 0$ or $\hat{x} \in \partial_1\mathcal{O}$, then (A.4) follows from the boundary conditions. Similarly, (A.4) follows immediately in case $\Phi(\hat{t},\hat{x}) \le 0$. If $\hat{x} \in \mathcal{O}$, inequality (A.4) can be proved using classical comparison results for obstacle problems, see [30, Theorem 7.8] (see also the discussion of Case 1 and 2 below).
It remains to consider the case $\hat{x} \in \partial_2\mathcal{O}$, for which we will show that it cannot occur if $\hat{t} > 0$, $\hat{x} \notin \partial_1\mathcal{O}$ and $\Phi(\hat{t},\hat{x}) > 0$ and if $\rho > 0$ is sufficiently large (observe that it is enough to establish (A.4) for all sufficiently large ρ because this will imply (A.4) for all $\rho > 0$). Thanks to the property (A.2) of our domain, the existence of a family of C^2 test functions $\{w_\varepsilon\}_{\varepsilon>0}$ as in [17, Theorem 4.1] can be proved. Among the other properties, $\{w_\varepsilon\}_{\varepsilon>0}$ satisfies:

$$w_\varepsilon(x,x) \le \varepsilon \tag{A.5}$$

$$w_\varepsilon(x,y) \ge C\frac{|x-y|^2}{\varepsilon} \tag{A.6}$$

$$-\partial_{x_M} w_\varepsilon(x,y) \ge -C\frac{|x-y|^2}{\varepsilon} \quad \text{if } x \in \partial_2\mathcal{O}\cap B(\hat{x},\eta), y \in B(\hat{x},\eta) \tag{A.7}$$

$$-\partial_{y_M} w_\varepsilon(x,y) \ge 0 \quad \text{if } y \in \partial_2\mathcal{O}\cap B(\hat{x},\eta), x \in B(\hat{x},\eta) \tag{A.8}$$

for $\varepsilon > 0$ and some $\eta > 0$ small enough.
Applying the doubling variables procedure we define

$$\begin{aligned}
&\Phi_\varepsilon(t,x,y) := \\
&\underline{v}_{\delta,\rho,\beta}(t,x) - \overline{v}_{\delta,\rho,\beta}(t,y) - C_1\delta e^{-\rho(T-t)}(1+|x|^2_{M-1}) - C_1\delta e^{-\rho(T-t)}(1+|y|^2_{M-1}) \\
&- w_\varepsilon(x,y) - |x-\hat{x}|^4 - |t-\hat{t}|^2
\end{aligned}$$

and we denote by $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ its maximum point. By the usual techniques, thanks to the properties (A.5) and (A.6), it is possible to prove that for ε going to 0

$$x_\varepsilon, y_\varepsilon \to \hat{x}, \qquad t_\varepsilon \to \hat{t} \qquad \text{and} \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0.$$

It follows that for ε small enough we can assume that $x_\varepsilon, y_\varepsilon \notin \partial_1\mathcal{O}$ and $t_\varepsilon > 0$. Taking ε small enough we can also say that $x_\varepsilon, y_\varepsilon \in B(\hat{x},\eta)$ and then we can make use of properties (A.7) and (A.8). In particular if $x_\varepsilon \in \partial_2\mathcal{O}$, taking ε small enough, we have

$$\begin{aligned}
&-\partial_{x_M}\Big(w_\varepsilon(x_\varepsilon,y_\varepsilon) + \delta e^{-\rho(T-t_\varepsilon)}(1+|x_\varepsilon|^2_{M-1}) + |x_\varepsilon-\hat{x}|^4\Big) \\
&\ge -C\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon} - 4|x_\varepsilon-\hat{x}|^2|x_{\varepsilon M} - \hat{x}_M| > -\delta e^{-\rho T}.
\end{aligned}$$

On the other hand if $y_\varepsilon \in \partial_2\mathcal{O}$

$$-\partial_{y_M}\Big(-w_\varepsilon(x_\varepsilon,y_\varepsilon) - C_1\delta e^{-\rho(T-t_\varepsilon)}(1+|y_\varepsilon|^2_{M-1})\Big) \le 0 < \delta e^{-\rho T}.$$

This means that for sufficiently small values of ε, we can neglect the derivative boundary conditions in $x_\varepsilon, y_\varepsilon$ and only consider

$$\min\Big(\partial_t \underline{v}_{\delta,\rho,\beta} + H(x_\varepsilon, \underline{v}_{\delta,\rho,\beta}, D\underline{v}_{\delta,\rho,\beta}, D^2\underline{v}_{\delta,\rho,\beta}) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x_\varepsilon|^2_{M-1})\,,$$
$$\underline{v}_{\delta,\rho,\beta} + \frac{\beta}{T} - \psi\Big) \le 0$$
$$\min\Big(\partial_t \overline{v}_{\delta,\rho,\beta} + H(y_\varepsilon, \overline{v}_{\delta,\rho,\beta}, D\overline{v}_{\delta,\rho,\beta}, D^2\overline{v}_{\delta,\rho,\beta}) - \frac{\beta}{T^2} + C_1\delta e^{-\rho T}(1+|y_\varepsilon|^2_{M-1})\,,$$
$$\overline{v}_{\delta,\rho,\beta} - \frac{\beta}{T} - \psi\Big) \ge 0$$

in the viscosity sense.

Case 1: let us assume that

$$\underline{v}_{\delta,\rho,\beta}(t_\varepsilon, x_\varepsilon) + \frac{\beta}{T} - \psi(x_\varepsilon) \le 0.$$

In this case we would get (since $\overline{v}_{\delta,\rho,\beta}(t_\varepsilon, y_\varepsilon) - \frac{\beta}{T} - \psi(y_\varepsilon) \ge 0$ always holds)

$$\underline{v}_{\delta,\rho,\beta}(t_\varepsilon, x_\varepsilon) - \overline{v}_{\delta,\rho,\beta}(t_\varepsilon, y_\varepsilon) + \frac{2\beta}{T} + \psi(y_\varepsilon) - \psi(x_\varepsilon) \le 0.$$

For sufficiently small $\varepsilon > 0$, from $\Phi(\hat{t}, \hat{x}) > 0$ we know that $\Phi_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) > 0$ and this implies $\underline{v}_{\delta,\rho,\beta}(t_\varepsilon, x_\varepsilon) - \overline{v}_{\delta,\rho,\beta}(t_\varepsilon, y_\varepsilon) > 0$, leading to a contradiction for ε going to 0.

Case 2: let us assume that

$$\partial_t \underline{v}_{\delta,\rho,\beta} + H(x, \underline{v}_{\delta,\rho,\beta}, D\underline{v}_{\delta,\rho,\beta}, D^2\underline{v}_{\delta,\rho,\beta}) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x_\varepsilon|^2_{M-1}) \le 0.$$

It follows that

$$\begin{aligned}&\partial_t \underline{v}_{\delta,\rho,\beta}(t_\varepsilon, x_\varepsilon) - \partial_t \overline{v}_{\delta,\rho,\beta}(t_\varepsilon, y_\varepsilon) - C_1\delta e^{-\rho T}(1+|x_\varepsilon|^2_{M-1} + |y_\varepsilon|^2_{M-1})\\ &\quad + H(x_\varepsilon, \underline{v}_{\delta,\rho,\beta}, D\underline{v}_{\delta,\rho,\beta}, D^2\underline{v}_{\delta,\rho,\beta}) - H(y_\varepsilon, \overline{v}_{\delta,\rho,\beta}, D\overline{v}_{\delta,\rho,\beta}, D^2\overline{v}_{\delta,\rho,\beta}) \le -\frac{2\beta}{T^2}.\end{aligned}$$

Using the properties of the Hamiltonian H and of the test function w_ε, we can find a constant C_2 such that at the limit for $\varepsilon \to 0$ one has

$$\begin{aligned}&\delta e^{-\rho(T-\hat{t})}(1+|\hat{x}|^2_{M-1})(\rho - C_2 - C_1)\\ &\le 2\rho\delta e^{-\rho(T-\hat{t})}(1+|\hat{x}|^2_{M-1}) - C_2\delta e^{-\rho(T-\hat{t})}(1+2|\hat{x}|^2_{M-1})\\ &\quad - C_1\delta e^{-\rho T}(1+2|\hat{x}|^2_{M-1})\\ &\le -\frac{2\beta}{T^2}\end{aligned}$$

and a contradiction is obtained as soon as $\rho \ge (C_1 + C_2 + 1)$. □

Acknowledgment

We would like to thank Annalisa Cesaroni for the interesting discussions on optimality principles on occasion of the SADCO NetCo conference (Tours 2014).

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