Typicality in Statistical Mechanics: An Epistemological Approach

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ABSTRACT. The use of typicality has recently enjoyed an increasing popularity among physicists interested in the foundations of statistical mechanics. However, it has been the target of a mounting philosophical critique mainly challenging its explanatory value. After an initial stage of intense dialogue, the debate seems now to have reached a deadlock of mutual incommunicability. Instead of treating typicality as a probabilistic ingredient of an argument, in this paper I unfold the techniques and mathematical practices related with this notion and show that typicality works as a way to combine these techniques in a consistent epistemic story of equilibrium.

Keywords: typicality, statistical mechanics, Boltzmann, celestial mechanics, explanation.

1 A Troubled Notion

In recent years, the debate on the foundations of equilibrium statistical mechanics has increasingly focused upon the notion of typicality (see for example [1], [2], [3], [4], [5]). Briefly said, typicality is a way to explain the central problem of statistical mechanics, that is why systems such as gases tend to evolve toward a state of equilibrium and stay there for indefinitely long periods of time. Intuitively, one says that a property is typical when it holds in the vast majority of cases or, alternatively, the cases in which it does not hold are negligible in number. Let Γ be the set of accessible states of a thermodynamic system and let μ be a measure function. If it is possible to divide Γ into two disjoint subsets, T_1 and T_2, such as (1) only the states in T_1 have the property τ, and (2) $\mu(T_1) \approx 1$, while $\mu(T_2) \approx 0$, then τ is a typical property of the system. The basic argumentative line used by the upholders of the typicality approach can be summarized as follows:

1. Let Γ the accessible region of a thermodynamic system and let M_{eq}, M_{neq} the subsets of the equilibrium and nonequilibrium macrostates, respectively. These subsets form a partition of Γ.

2. Let x be a microstate and $x(t)$ its trajectory under the dynamics of the system. In other words, $x(t) = x_1, x_2, x_3, \ldots$ where $x_i \in \Gamma$.

3. A certain measure function m_L exists, called the Lebesgue measure, such that $m_L(M_{eq}) \approx 1$; the microstates in M_{eq} have the property of "being in equilibrium", hence this property is typical in the thermodynamic system.

4. Also the microstates $x(t) = x_1, x_2, x_3, \ldots$ are typically in equilibrium, hence, the trajectory of an arbitrary state is mainly contained in M_{eq}.

5. Ergo, the system will tend to equilibrium and remain there, because equilibrium is typical.

This straightforward argument has enjoyed a large approval among physicists and aroused an equally large discontent among philosophers. The former like especially its simplicity and its generality. In fact, it has also been extended to interpret Bohmian quantum mechanics ([6], [7], [8]). By contrast, the latter consider the argument above seriously flawed. There are three kinds of criticisms against typicality.

First, the definition of a typical property depends essentially on the size of the macrostate, which in turn depends on the definition of a suitable measure function (step (3) in the argument). In statistical mechanics, the convention is to use the so-called Lebesgue measure. Philosophers object that there is no argument, either philosophical or physical, to claim that Lebesgue measure must enjoy any preference and be considered as the "natural" one. Second, until step (4), the argument only deals with statements concerning measure of macrostates, but the conclusion is a statement about the physical behavior of observable systems. It seems, that (5) concerns the probability that a system will behave in a certain way, so that the argument would require a leap from statements about measures to statements about physical probabilities ([9], [10, 182-191]). Third, no purely measure-theoretical consideration on the macrostates would ever suffice without some dynamical assumption ([1]). In the argument presented above, this assumption is expressed in step (4), where it is supposed that the trajectory contains the same ratio of equilibrium/nonequilibrium states as in the total accessible region.

The effect of these critiques has been to virtually interrupt the dialogue between philosophers and physicists. The eminently logical character of the philosophical analysis has appeared to physicists too detached from their actual foundational problems in statistical mechanics. Thus, many working scientists tend to consider this analysis as hairsplitting and uninformative. On the other side, philosophers have quickly dismissed typicality. From the point of view of traditional philosophical analysis, typicality appears as mere hand-waving at best, or as circular at worst.

In this paper I argue that the problem is partly due to philosophers' conception of explanation. Generally, philosophers working in foundations of statistical mechanics have deployed a Hempelian model according to which an explanation is an argument whose conclusion is equilibrium. Most of the philosophical criticisms against typicality concentrate upon the flaws of arguments containing such notion. I argue, however, that the Hempelian model does not capture what the physicists mean by the explanatory value of typicality. Hence, we have to enlarge our conception of explanation. I submit that typicality provides a *satisfactory causal explanation of the qualitative aspects of equilibrium.* Let me spell out this claim by starting with the final part. By that I mean that typicality only accounts for the general fact that systems exhibit a tendency toward equilibrium, but does not yield any quantitative analysis. Second, by causal explanation is mean that typicality gives us:

1. A set of causal factors for the qualitative aspects of the equilibrium;
2. A formal description of how these factors act.

Here, I adopt Woodward's theory of causal explanation, [11]: the causal factors of an event are those factors that, if properly manipulated, would change the event. Further, condition (2) tells us in which sense we should manipulate the causal factors to obtain a different result. Finally, the satisfactoriness of an explanation does not depend on relations between its parts, but on the resources it uses. I claim that a satisfactory explanation must fulfill the following:

3. Historic-pragmatic value: a sensible use (possibly a reconfiguration) of the traditional practices and techniques deployed in the field.

This element has been totally neglected in philosophical literature on explanation.[1] It is motivated by the almost trivial consideration that explanations do not happen in a vacuum, but are historically situated. Scientists try to construct (and value) explanations that make use of traditional techniques and practices, perhaps providing them with a new meaning and new potentials. Hence, a good explanation must be evaluated relatively to the history of the practices and relatively to the subculture in which it is accepted. In the following sections, I argue that this model illuminates the explanatory value of typicality. I quickly summarize the genealogical lines of the mathematical practices related to the use of typicality in physics (section 2) and I show how these lines converge to the modern approach (section 3).

2 Typicality in Physics: A Genealogy

Current use of typicality is not as clear as many of its supporters would wish. To understand the roots of this notion, it may be useful to begin with examining three definitions of typicality adopted in the literature. The first definition comes from a philosophical paper:

> Intuitively, something is typical if it happens in the 'vast majority' of cases: typical lottery tickets are blanks, typical Olympic athletes are well trained, and in a typical series of 1,000 coin tosses the ratio of the number of heads and the number of tails is approximately one. [2, 997-998]

The second definition comes from a historical paper:

> Generally speaking, a set is typical if it contains an "overwhelming majority" of points in some specified sense. In classical statistical mechanics there is a "natural" sense: namely sets of full phase-space volume. [12, 803]

[1] A note of clarification: the point of requisite (3) is not to provide an explanatory value to dead and buried theories, but to stress that the explanatory value of any theory depends crucially on what a certain community can do with them.

Finally, the third definition comes from one of the most distinguished upholders of the typicality approach, Joel Lebowitz:

> [A] certain behavior is typical if the set of microscopic states [...] for which it occurs comprises a region whose volume fraction goes to one as [the number of molecules] N grows. [13, 7]

Apart from the different levels of technicality and their specific aims, these definitions point out two traits of typicality. First, it relies on *the separation of two families of events*, those which are "almost certain" and those which are "negligible". This evaluation depends on the relative sizes of the corresponding families. Second, Lebowitz's definition stresses the asymptotic character of typical behaviors: they tend to a certain maximal size as the number of degrees of freedom of the problem approaches infinity. The first element is related to the tradition of celestial mechanics that goes back to the notorious three-body problem. The second element is linked to the combinatorial techniques used in statistical mechanics. There are, as we will see, intersections between these traditions, which explain how they can both feature in the definitions of typicality.

2.1 Celestial Mechanics and Topology

Since mid-18th century, mathematicians struggled to show that three bodies interacting according to the gravitational law would never undergo catastrophic collisions or expulsions. The usual strategy to deal with this problem was to solve the equations of motion by means of trigonometric series and to show that these series do not contain diverging (secular) terms. After many failed attempts to provide an explicit solution of the equations of motion, mathematicians grew skeptical that these solutions would ever be discovered. In the second half of the 19th century, it became increasingly clear that there was no way to solve the three-body problem in closed form and other paths were tried.

Instrumental in this change of tack was the adoption of new topological techniques. The undisputed champion of this line of attack was Henri Poincaré [14]. Instead of establishing stability analytically, Poincaré sough for the conditions under which *most of the trajectories* are stable. This method does not require an explicit solution of the equations of motion and do not call for any assumption of randomness. Rather, it aims at classifying trajectories in stable and unstable and then to show under which circumstances the former outnumber the latter [15].

As an example of this procedure, one can consider the famous recurrence theorem [16, III, 847-876]. By a very general topological argument, Poincaré showed that almost all possible mechanical trajectories of a conservative system return, after a very long time, infinitesimally close to their initial state (or, as Poincaré had it, they are Poisson-stable). The set of trajectories that do not behave like that is negligible.

When Poincaré developed his approach, he did not have a precise mathematical notion of "almost-all" or "negligible". This notion became available only in the early 20th century with the development of Henri Lebesgue's theory of measure. The combination of topological and measure-theoretical techniques was successfully put to work on other important problems of celestial mechanics such as the study of singularities and perturbations (for a discussion see [17]). It thus comes as no surprise that contemporary theory of dynamical systems are customarily defined as the study of *typical or generic* properties of systems, that is properties that hold of the vast majority of the possible trajectories. It is important to recognize, though, that these properties are defined asymptotically. Consider, for example, the introduction to one of the most complete and authoritative books on the topic:

> The most characteristic feature of dynamical theories, which distinguishes them from other areas of mathematics dealing with groups of automorphisms of various mathematical structures, is the emphasis on asymptotic behavior [...] that is properties related to the behavior as time goes to infinity. [18, 2]

Typical properties are therefore those properties that come to be recognized as such only in the long run.

2.2 Statistical Mechanics

Although much younger and very different in subject matter, kinetic theory—the predecessor of modern statistical mechanics—faced a similar problem as celestial mechanics. The behavior of a gas composed of many molecules colliding mechanically cannot be predicted by solving the equations of motion. In fact, even the knowledge of the initial conditions is out of reach. Thus, from the beginning, statistical mechanics introduced a set of approximation techniques and assumptions in order to make the problem tractable. For example, the collisions between the molecules and the walls bring in a disturbing effect in the sequence of elastic collisions between molecules. This is the so-called “wall-effect”. To take into account this effect in the equations of the problem leads to innumerable formal complications, therefore it is usually assumed that the container is big enough that the wall effect remains confined to a negligibly small portion of the whole space. Analogously, basically all arguments in kinetic theory are cast by supposing ideal conditions such as the number of molecules grows to infinity, or the duration of a collision tends to zero and so on.

One of Ludwig Boltzmann’s great insights was that the nature of the problem of irreversibility is not affected by the use of these approximation techniques based on asymptotic tendencies. These techniques only cancel out the probabilistic fluctuations and make the results strictly valid. They produce “statistical determinism”. For this reason, Boltzmann made ample use of probabilistic arguments and tools constantly framed within asymptotic assumptions [19].

It was clear to Boltzmann that there are two different, albeit related questions: (1) what is the essence of irreversibility and (2) how to formulate this essence in terms of the specific microscopic arrangements and dynamical laws of the molecules. As for the first question, Boltzmann concluded that irreversibility is due to the extremely large number of molecules in complicate collisions. It is this large number that justifies an assumption of equiprobability for the microstates and thus a probabilistic procedure that leads to the equilibrium distribution as the largest one:

> The great irregularity of the thermal motion and the multiplicity of the forces acting on the body from the outside make it probable that its atoms [...] pass through all the possible positions and velocities consistent with the equation of energy. [20], [21, I, 284]

He illustrates this point most effectively in his famous 1877 combinatorial theory [22], [21, II, 164-223]. Boltzmann assumes very many molecules and calculates the numbers of ways in which energy can be distributed over them. It turns out that the overwhelming majority of these ways are represented by a bell-shaped distribution. This is Maxwell's distribution, which represents the state of equilibrium. It's by far the largest in terms of the number of microscopic molecular allocations of energy compatible with it. The remarkable point is that the dominance of the equilibrium state depends crucially on the huge number of degrees of freedom of the problem: the relative size of the equilibrium state respect to the other increases enormously with the number of degrees of freedom. This behavior is characteristic of asymptotic probability laws such as the law of large numbers or the central limit theorem. For this reason, Boltzmann understood the essence of irreversibility as a probabilistic law valid under suitable asymptotic conditions [19].

The second question was harder. If we assume that molecules obey the laws of mechanics, we run into the reversibility problem. Mechanical motion can be inverted and still remain perfectly mechanical, so how are we to understand irreversibility as a form of mechanical motion? Why a sequence of mechanical collisions leading the system from an arbitrary state to equilibrium should occur more often than its reverse, which is matter-of-factly as mechanical? The most important debate on this question took place on the pages of *Nature* in 1894-95 and involved, besides Boltzmann, as distinguished British physicists as Bryan, Burbury, Watson, and Culverwell. Four possible solutions to this question emerged from the debate.

1. The mechanical reversal of a state violates the formal condition on which Boltzmann's theorem of irreversibility was based (H-theorem). This solution appeared unacceptable to Boltzmann because it emptied the theorem of any physical meaning and downgraded it to a purely mathematical statement.

2. Mechanical reversal is unstable. The situation is analogous to riding a bicycle backwards: it is mechanically possible, but any small perturbation will destroy the equilibrium. Boltzmann liked this option: a reversal presupposes a perfect coordination between the molecules, which is easy destroyed.

3. In its path from the nonequilibrium state to equilibrium, the trajectory branches off in many possible states. It is true that for each whole path the reverse exists, but at each stage there are more ways to go toward equilibrium than in the opposite direction. This is the idea of the H-curve.

4. Microscopic molecular arrangements are molecularly disordered. This is the so-called molecular chaos that Boltzmann introduced in the first volume of his *Gastheorie*.

I will dwell especially upon this last point. Boltzmann's notion of molecular chaos is profound, but not very clear. His basic point is that molecules must be arranged and must behave in a way that leaves all theoretical possibilities open. In other words, any regularity that forces the system out of its typical state of equilibrium must derive from some specially engineered arrangement that made probability considerations invalid:

> If we choose the initial configuration on the basis of a previous calculation of the path of each molecule, so as to violate intentionally the laws of probability, then of course we can construct a persistent regularity. [23, I, 22]

Thus, in making the reversal, we request the molecules to retrace exactly the same sequence of collisions as before. This kind of interventions (or "conspiracy") on the dynamics of the system leads to atypical results. It is important to note that all these solutions of the reversibility objection contain traits characteristic of what is today known as chaos theory. We will see these traits displayed in Lebowitz's paper in the next section. Before concluding this section, however, I want to stress that Boltzmann had clearly in mind also the importance of the notion of negligibility. Poincaré's recurrence theorem is based on the concept of integral invariant, a mathematical technique that Boltzmann had himself introduced and used, albeit imperfectly, since the end of the 1860s [24], [21, I, 49-96]. In the *Gastheorie* he discusses the possibility that a gas, being a conservative and confined mechanical system, passes through its state again and again as prescribed by the recurrence theorem. He finds that this can happen only after an enormous interval of time. He concludes:

> One may recognize that this is practically equivalent to never, if one recalls that in this length of time, according to the laws of probability, there will have been many years in which every inhabitant of a large country committed suicide, purely by accident, one the same day, or every building burned down at the same time—yet the insurance companies get along quite well by ignoring the possibility of such events. If a much smaller probability than this is not practically equivalent to impossibility, then no one can be sure that today will be followed by a night and then a day. [23, II, 254]

Boltzmann was therefore well aware of the topological argument, which aims at distinguishing between typical and negligible events.

3 The Explanatory Value of Typicality

In the 20h century, the theory of dynamical systems and statistical mechanics took up and developed the trends outlined above. Measure theory provided a set of concepts and tools to express typical and negligible events. Furthermore, these tools were used to prove asymptotic statements like in the case of Emil Borel's proof of the law of large numbers (1909). George D. Birkhoff's 1931 ergodic theorem can also be considered a sort of law of large numbers applied to statistical mechanics. Birkhoff showed that dynamical systems have the propriety of ergodicity (from which many statistico-mechanical consequences follow) if and only if the set of trajectories that do not remain in an invariant portion of the phase space is negligible (i.e., it has measure-0). Properties that holds typically or generically are said to hold "almost-everywhere" [25].

Another important development of statistical mechanics in the 20th century is Alexander Khinchin's asymptotic approach [26], [25]. Khinchin claimed that the fundamental proposition of statistical mechanics, the irreversible approach to equilibrium, was just the physical formulation of the central limit theorem. Accordingly, the entire theory could be recast in purely probabilistic terms, leaving aside any physical assumption. Khinchin proved a theorem that systems for which the macroscopic parameters can be expressed by particular functions (sum-functions) reach equilibrium in the long run.

Finally, one of the most successful approach to statistical mechanics focuses on "large systems". The basic tenet is that when we examine the behavior of systems under particular asymptotic circumstances (for example the so-called thermodynamic limit where the number of molecules, the energy, and the volume tend to infinity, but the density and the energy density stay finite), we are able to prove kinetic theorems rigorously [27]. The most impressive result obtained by this approach is Lanford's theorem according to which for a particular gas model and in a particular limit, it is practically certain that the system will reach equilibrium [28], [29].

The upholders of typicality belong to this tradition. Most of them have worked within the framework of the large systems approach. Therefore, it is essential to keep in mind this long-term development to evaluate the meaning of the concept of typicality. The supporters of the typicality approach inscribe themselves in the Boltzmannian line of rigorous mathematical arguments framed within an asymptotic conceptual space where fluctuations become negligible. To illustrate this aspect I briefly discuss a paper by Joel Lebowitz. There are three points that I want to emphasize.

First, the notion of typicality serves the general purpose of understanding the transition from the microscopic to the macroscopic level. Remember the quote given above: typicality is a feature that emerges when the number of molecules approaches infinity. Put in other words, typicality discriminates between behaviors associated with a large number of degrees of freedom and behaviors associated with less complex systems. The former exhibit time-asymmetry, the latter do not:

> The central role in time asymmetric behavior is played by the very large number of degrees of freedom involved in the evolution of the macroscopic systems. It is only this which permits statistical predictions to become "certain" ones for typical individual realizations, where, after all, we actually observe irreversible behavior. This typicality is very robust—the essential features of macroscopic behavior are not dependent on any precise assumptions such as ergodicity, mixing or "equal a

> priori probabilities", being strictly satisfied by the statistical distributions. [13, 3]

This is a point often neglected by philosophers. Typicality is not just shorthand for "very high probability", i.e., another probabilistic notion subject to probabilistic conditions. Typicality is a feature of systems with many degrees of freedom, systems that are handled by certain techniques. More importantly, the high number of degree of freedom plays a real causal role in Woodward's sense. Like in Boltzmann's combinatorics and in Khinchin's probabilistic approach, the equilibrium state dominates over the others *because* there are many particles. Were there just a few of them, the equilibrium would be not so overwhelmingly more probable. Hence, it is by manipulating the number of degrees of freedom that we can make an effect on equilibrium.

The second point is related to the first: Lebowitz introduces a distinction between the qualitative and the quantitative aspects of irreversibility. As said above, the qualitative aspect depends only on the large number of degrees of freedom. From this, the typicality explanation of irreversibility follows. However, this aspect does not yield the hydrodynamical-like equations to predict the concrete behavior of a macroscopic system. For this we need more specific microscopic models, which, however, depend very little on the details of the microscopic dynamics. It is at this level that we find ergodicity, mixing and chaotic dynamics:

> I believe that these models capture the essential features of the transition from microscopic to macroscopic evolution in real physical systems. In all cases, the resulting equations describe the typical behavior of a single macroscopic system chosen from a suitable initial ensemble i.e. there is a vanishing dispersion of the values of the macroscopic variables in the limit of micro/macroscale ratio going to zero. [13, 17]

Again, it is crucial to notice that these models lead to time-asymmetric behavior only because they are applied to a large number of degrees of freedom. As such, chaotic dynamics or ergodicity are time-symmetric:

> This is an important distinction (unfortunately frequently overlooked or misunderstood) between irreversible and chaotic behavior of Hamiltonian systems. The latter, which can be observed in systems consisting of only a few particles, will not have a uni-directional time behavior in any particular realization. [13, 25]

The third point concerns Lebowitz's way of dealing with the reversibility objection. He argues that a reversal of the microscopic motion is conceivable but "effectively impossible to do [...] in practice." To support this claim he uses three arguments, all related to chaos dynamics. The first is that such reversal would be unstable under external perturbations. The second is that mechanical reversal requires a "perfect aiming" and

> [i]t can therefore be expected to be derailed by even smaller imprecisions in the reversal and/or tiny random outside influences. This is somewhat analogous to those pinball machine type puzzle where one is supposed to get a small metal ball into a particular small region. You have to do things just right to get it in but almost anything you do gets it out into larger region. [13, 9]

Lebowitz deploys the example of the pinball, but he might as well mention the example of riding a bicycle backwards: it is the same kind of mechanical situation. Finally, he points out a hidden assumption in the dynamics for typical behavior:

> For the macroscopic systems we are considering the disparity between relative sizes of the comparable regions in the phase space is unimaginably larger. The behavior of such systems will therefore be as observed, in the absence of any "grand conspiracy". [13, 9]

The idea that there must be some artificial intervention for such a system to exhibit an atypical behavior reminds immediately Boltzmann's remark about intentional violations of the laws of probability.

These quotes prove the kinship between the typicality approach and the tradition encompassing celestial mechanics, Boltzmann's statistical mechanics, and the large systems approach. But they also allow us to draw a more general philosophical conclusion. Typicality provides for a plausible *epistemic story* of the qualitative aspects of equilibrium by ascribing it to causal factors i.e., the high number of degrees of freedom, whose action is described by combinatorics and measure-theoretical concepts. It is not a probabilistic ingredient to be added to an argument, although it makes use of a probabilistic argumentative pattern ("given a suitable definition of probability, if the probability of one event is overwhelmingly larger than all alternatives, one can neglect the latter"). More importantly, typicality is a reflective way to classify, organize, and reconfigure a set of theoretical practices as diverse as topological methods, approximations procedures and statistical techniques. It derives from the mathematical practices outlined above and allows to combine them in an explanation of equilibrium. Part of its popularity is due to its historical-pragmatical value. Thus, typicality works as an *epistemic trope*: it is an assemblage of concepts, methods, and argumentative patterns that organize well-established mathematical practices into a specific epistemic story of equilibrium.

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