Noname manuscript No.
(will be inserted by the editor)

Weakening additivity in adjoining closures

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Received: date / Accepted: date

Abstract In this paper, we weaken the conditions for the existence of adjoint closure operators, going beyond the standard requirement of additivity/co-additivity. We consider the notion of join-uniform (lower) closure operators, introduced in computer science, in order to model perfect lossless compression in transformations acting on complete lattices. Starting from Janowitz's characterization of residuated closure operators, we show that join-uniformity perfectly weakens additivity in the construction of adjoint closures, and this is indeed the weakest property for this to hold. We conclude by characterizing the set of all join-uniform lower closure operators as fixpoints of a function defined on the set of all lower closures of a complete lattice.

Keywords Residuated closures, uniformity, adjoint functions

1 Introduction

In this paper, we weaken the notion of residuated closure operator. We consider closure operators on complete lattices [15] and Janowitz's notion of residuated closures as introduced in [1,11]. Residuated closures are defined on lattices and they become complete join-morphisms on complete lattices. We aim at studying what is preserved of the adjoint relation when residuation, namely additivity, fails and it is replaced by the weaker notion of join-uniformity.

Uniformity has been introduced in [10] in the context of static program analysis for providing a lattice-theoretic characterization of abstract domain compressors in abstract interpretation [3,5]. In abstract interpretation a domain is uniquely determined by an upper closure operator. This models precisely the loss of precision in

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analyzing undecidable program properties, such as the extensional semantic properties Π of programs [14]. If the objects of a complete lattice denote properties (viz., sets) of computed program states in $\mathbb{S}$, then, by extensivity, it means that approximation produces larger sets (i.e., less tight properties), idempotence means that approximation is made all at once, and monotonicity specifies that approximation keeps the relative order of precision of properties. In this context, we are able to associate with each program P and (possibly undecidable) property to verify Π, a (decidable) over-approximation $\rho(P, \Pi)$ of Π (i.e., $\Pi \subseteq \rho(P, \Pi)$) [3]. $\rho(P, \Pi)$ can specify, for instance, the absence of specific run-time errors, such as run-time overflows of integer and floating point variables (see [6]). Of course, when $P \in \rho(P, \Pi)$, nothing can be concluded about P and Π, but when $P \notin \rho(P, \Pi)$ surely we have $P \notin \Pi$. The function $\lambda X.\rho(P, X)$ is, in this context, an upper closure operator [5] on the lattice of all program properties, where the most precise approximation is the identity function, which is the bottom element in the lattice of all upper closures, and the less precise approximation is the function mapping any property to the whole set of all possible values $\mathbb{S}$. Refining approximations, i.e., upper closures, corresponds therefore precisely to act as a lower closure on abstract domains, i.e., any lower closure operator on the lattice of all upper closure operators is a refinement.

The notion of abstraction refinement is extensively studied in computer science, notably in automatic program analysis, system verification, and abstract model checking for removing false alarms and spurious behaviors when approximating undecidable properties of computer programs and systems [5, 13, 2]. In this context, an abstraction refinement is *compressible* if it is join-uniform, namely if the least upper bound of all abstract domains having the same refinement, still has the same refinement. This unique domain is the least one that, by refinement, gives back a given domain. A function f defined on a complete lattice C is join-uniform whenever for any $Y \subseteq C$ such that f is constant on Y with value $f(y)$, $f(\bigvee Y) = f(y)$. When f is join-uniform its inverse $f^{-}(x) = \bigvee \{\, y \mid f(y) = f(x) \,\}$ is such that $f(f^{-}(x)) = f(x)$ and $f^{-}(x)$ is the maximal element for which this happens. Examples include f as the function mapping any subset S of a complete boolean algebra C into the corresponding sub-boolean algebra $f(S)$ of C and f^{-} as the reduction of S into the sets of its join-irreducible elements [9]. This notion of reduction along a given partial order, e.g., file size, is shared by a number of algorithms for *lossless compression*, e.g., see the `ZIP` file format and the GNU tool `gzip`, as well as the `PNG` and `GIF` file formats. The lossless compression is perfectly captured by uniformity.

We show that uniformity plays a central role in weakening additivity and co-additivity in the existence of adjoint closures. In particular, we prove that residuation, which is additivity on complete lattices, can be weakened by join-uniformity yet keeping the possibility of having an adjoint on a modified partial order. We introduce the notion of pseudo-adjoint of a function which corresponds to the adjoint when the function is additive. Then, by weakening additivity, pseudo-adjoint still keeps some of the properties of the adjoint case. In particular, in the case of lower closures, as specified in domain refinement, join-uniformity is the only possible weakening of additivity, yet keeping the adjoint relation on a modified partially ordered set. Finally, we characterize the set of all join-uniform lower closure operators as the set of fix-points of a function defined on the set of all lower closures of a complete lattice.

2 Closures and residuated closures

Closures and notations. In the following, $\langle C, \leq, \top, \bot, \vee, \wedge \rangle$ is a generic complete lattice, denoted $C_{\leq}$ for underlining the considered order on C. A monotone function f on a complete lattice C is denoted $f : C \xrightarrow{\text{m}} C$, the set of all monotone functions on C is denoted

$$\mathcal{F}^{\text{mono}}_C \stackrel{\text{def}}{=} \left\{ f : C \longrightarrow C \,\middle|\, f \text{ monotone} \right\}$$ [1]

A reductive [extensive] function f on C is such that $\forall x \in C.\ f(x) \leq x$ $[f(x) \geq x]$ and it is denoted $f : C \xrightarrow{\text{red}} C$ $[f : C \xrightarrow{\text{ext}} C]$. The set of all reductive [extensive] functions is

$$\mathcal{F}^{\text{red}}_C \stackrel{\text{def}}{=} \left\{ f : C \longrightarrow C \,\middle|\, f \text{ reductive} \right\} [\mathcal{F}^{\text{ext}}_C \stackrel{\text{def}}{=} \left\{ f : C \longrightarrow C \,\middle|\, f \text{ extensive} \right\}].$$

Finally, an idempotent function f on C is such that $\forall x \in C.\ f(f(x)) = f(x)$ and it is denoted $f : C \xrightarrow{\text{idem}} C$. The set of all idempotent functions is

$$\mathcal{F}^{\text{idem}}_C \stackrel{\text{def}}{=} \left\{ f : C \longrightarrow C \,\middle|\, f \text{ idempotent} \right\}.$$

Let f, g be two functions on a complete lattice C, we define the point-wise order as $f \dot{\leq} g$ if and only if $\forall x \in C.\ f(x) \leq_C g(x)$. Let $f : C \to D$ be a monotone function, for each $X \in \wp(D)$ we define its *inverse image set* as the image of the function $f^{-1} : \wp(D) \to \wp(C)$ defined as $f^{-1}(X) = \left\{ y \,\middle|\, f(y) \in X \right\}$.
Two monotone functions between the complete lattices $C_{\leq_C}$ and $A_{\leq_A}$ $\alpha : C \xrightarrow{\text{m}} A$ and $\gamma : A \xrightarrow{\text{m}} C$ form an *adjunction* if for any $x \in C$ and $y \in A$: $\alpha(x) \leq_A y \Leftrightarrow x \leq_C \gamma(y)$. In this case, α [resp. γ] is the *left* [*right*]*adjoint* of γ $[\alpha]$ and it is additive [co-additive], i.e., it preserves *lub*'s [*glb*] of all subsets of the domain (empty set included). Let us define the following function transformations:

$$f^+(x) \stackrel{\text{def}}{=} \bigvee \left\{ y \,\middle|\, f(y) \leq x \right\} = \bigvee f^{-1}(\downarrow x) \qquad \text{and}$$
$$f^-(x) \stackrel{\text{def}}{=} \bigwedge \left\{ y \,\middle|\, x \leq f(y) \right\} = \bigwedge f^{-1}(\uparrow x)$$

In the following, we call f^+ right *pseudo*-adjoint of f, and f^- left *pseudo*-adjoint of f. When f is an additive [co-additive] map then the right [left] adjoint exists and it is precisely f^+ $[f^-]$ (see [1] for notation). In this case, we say that $\langle f, f^+ \rangle$ and $\langle f^-, f \rangle$ are adjoint operators.

An *upper [lower] closure operator* $\rho : C \longrightarrow C$ is monotone, idempotent, and extensive [reductive], i.e., $f \in \mathcal{F}^{\text{mono}}_C \cap \mathcal{F}^{\text{idem}}_C \cap \mathcal{F}^{\text{ext}}_C$ $[f \in \mathcal{F}^{\text{mono}}_C \cap \mathcal{F}^{\text{idem}}_C \cap \mathcal{F}^{\text{red}}_C]$. The set of all upper [lower] closure operators on C is denoted by *uco*(C) [*lco*(C)]. Recall that if C is a complete lattice, then $\langle uco(C), \sqsubseteq, \sqcup, \sqcap, \lambda x.\top, id \rangle$ is a complete lattice [15, 7, 12], where for every $\rho, \eta \in$ *uco*(C), $\{\rho_i\}_{i \in I} \subseteq$ *uco*(C) and $x \in C$:

- $\rho \sqsubseteq \eta$ if and only if $\forall y \in C.\ \rho(y) \leq \eta(y)$ if and only if $\eta(C) \subseteq \rho(C)$.
- $(\sqcap_{i \in I} \rho_i)(x) = \wedge_{i \in I} \rho_i(x)$;
- $(\sqcup_{i \in I} \rho_i)(x) = x \Leftrightarrow \forall i \in I.\ \rho_i(x) = x$;
- $\lambda x.\top$ is the top element, whereas $id \stackrel{\text{def}}{=} \lambda x.x$ is the bottom element.

[1] In the following, we omit the pedex C when clear from the context.

Upper closures are uniquely determined by their fix-points $\rho(C)$. $X \subseteq C$ is the set of fix-points of an upper closure if it is a *Moore family*, i.e., $X = \mathcal{M}(X) \stackrel{\text{def}}{=} \{\bigwedge S \mid S \subseteq X\}$. It is known that upper closures can be made additive. This transformation is called the *disjunctive completion* of δ and it is defined as:

$$\curlyvee(\delta) \stackrel{\text{def}}{=} \bigsqcup \left\{ \rho \in \mathit{uco}(C) \,\middle|\, \rho \sqsubseteq \delta,\ \rho \text{ additive} \right\},$$

which is an upper closure closed by concrete least upper bound, i.e., it is such that $\rho \circ \vee \circ \rho = \vee \circ \rho$ [8][2].

Residuated closures. Janowitz, characterized the order theoretic structure and properties of residuated closure operators [1,11]. Let us first recall the notion of quasi-residuated and residuated map.

Definition 1 [11] Let $f : C \longrightarrow D$ be a monotone map. f is *quasi-residuated* if $\exists h : D \longrightarrow C.\ f \circ h \dot{\leq} \mathit{id}$ if and only if $\forall x \in D.\ \left\{ y \in C \,\middle|\, f(y) \leq x \right\} \neq \varnothing$. f is *residuated* if $\exists h : D \longrightarrow C.\ f \circ h \dot{\leq} \mathit{id} \ \wedge\ h \circ f \dot{\geq} \mathit{id}$.

In the following, this function h is the right adjoint of f (called *residual* in [11]). The following theorem provides the Janovitz characterization of adjunction between residuated closure operators.

Theorem 1 *[11, Theorem 2.10]*
Let $f : C \longrightarrow C$ be residuated *map, i.e., $\langle f, f^+ \rangle$[3] is a pair of adjoint operators on C, then*

$$(1)\ \ f \in \mathit{uco}(C) \ \Leftrightarrow\ f^+ \in \mathit{lco}(C) \ \Leftrightarrow\ f \circ f^+ = f^+ \ \Leftrightarrow\ f^+ \circ f = f$$

and

$$(2)\ \ f \in \mathit{lco}(C) \ \Leftrightarrow\ f^+ \in \mathit{uco}(C) \ \Leftrightarrow\ f \circ f^+ = f \ \Leftrightarrow\ f^+ \circ f = f^+$$

Dually, we have that if $f : C \longrightarrow C$ is a *dual-residuated*[4] map and f^- is its left-adjoint (defined in the previous section), then

$$(3)\ \ f \in \mathit{uco}(C) \ \Leftrightarrow\ f^- \in \mathit{lco}(C) \ \Leftrightarrow\ f \circ f^- = f \ \Leftrightarrow\ f^- \circ f = f^-$$

and

$$(4)\ \ f \in \mathit{lco}(C) \ \Leftrightarrow\ f^- \in \mathit{uco}(C) \ \Leftrightarrow\ f^- \circ f = f \ \Leftrightarrow\ f \circ f^- = f^-$$

In the hypotheses of the theorem, f^+ is trivially dual-residuated, and f^- is residuated.
We observe that, on a complete lattice, a *residuated* function is a monotone function admitting right-adjoint [1,11], namely a residuated function on a complete lattice is an additive function and viceversa. By this observation, when dealing with complete lattices, we can use the notion of additivity and of residuated as synonymous. Dually, co-additivity on complete lattices corresponds to dual-residuation.

[2] Note that this fact always holds for $\eta \in \mathit{lco}(C)$, hence for lower closures it does not corresponds to additivity.
[3] f^+ is precisely the function defined in the previous section.
[4] Defined by duality.

Pseudo-adjoining closures. Let $\eta \in lco(C)$ and $\delta \in uco(C)$, then we can define the following functions:

$$\eta_-(x) \stackrel{\text{def}}{=} \bigwedge\{\eta(y)|\eta(y) \geq x\} \qquad \eta_+(x) \stackrel{\text{def}}{=} \bigvee\{y|\eta(y) = \eta(x)\} = \bigvee \eta^{-1}(\{\eta(x)\})$$

$$\delta_+(x) \stackrel{\text{def}}{=} \bigvee\{\delta(y)|\delta(y) \leq x\} \qquad \delta_-(x) \stackrel{\text{def}}{=} \bigwedge\{y|\delta(x) = \delta(y)\} = \bigwedge \delta^{-1}(\{\delta(x)\})$$

The following results show the relation between the functions introduced above and the pseudo-adjoints of an upper or lower closure operator.

Proposition 1 *Let $\eta \in lco(C)$ and $\delta \in uco(C)$. Then*

$$(1)\ \eta^- = \eta_- \qquad \delta^+ = \delta_+ \tag{2.1}$$

$$(2)\ \eta^+ = \eta_+ \qquad \delta^- = \delta_- \tag{2.2}$$
if η is additive *and δ is* co-additive.

Proof 1. Let us first prove Equation 2.1 on $\eta \in lco(C)$ (the proof for δ is dual). First of all, note that $\eta^-(x) \leq \eta_-(x)$ since by idempotence of η $\{\ \eta(y) \mid \eta(y) \geq x\ \} \subseteq \{\ y \mid \eta(y) \geq x\ \}$. On the other hand, by reductivity of η we have

$$\eta_-(x) = \bigwedge \{\ \eta(y) \mid \eta(y) \geq x\ \} \leq \bigwedge \{\ y \mid \eta(y) \geq x\ \} = \eta^-(x)$$

Hence we have the equality.

2. Consider $\eta \in lco(C)$ additive, let us prove Equation 2.2, for η (for δ is dual). First of all $\eta_+(x) \leq \eta^+(x)$ since, by reductivity of η we have $\{\ y \mid \eta(y) = \eta(x)\ \} \subseteq \{\ y \mid \eta(y) \leq x\ \}$. On the other hand, by Theorem 1 we have that $\eta(x) = \eta(\eta^+(x))$, hence $\eta^+(x) \in \{\ y \mid \eta(y) = \eta(x)\ \}$, and therefore $\eta^+(x) \leq \eta_+(x)$, implying the equality.

These results tell us that the right pseudo-adjoint of an upper closure is always equal to δ_+, while the right pseudo-adjoint of a lower closure is equal to η^+ if the closure is additive, namely when we are computing precisely its right adjoint.

3 Weakening additivity by join-uniformity

A monotone function on a complete lattice C, $f : C \xrightarrow{\text{m}} C$, is *join-uniform* if for all $Y \subseteq C$, $(\exists \bar{x} \in Y.\ \forall y \in Y.\ f(y) = f(\bar{x})) \Rightarrow (\exists \bar{x} \in Y.\ f(\bigvee Y) = f(\bar{x}))$ [10]. We can rewrite this definition as follows: The function f is join-uniform if and only if, for each $x \in C$

$$f(\bigvee \{\ y \mid f(y) = f(x)\ \}) = f(x), \text{ i.e.,} \qquad f(f^{-1}(\{f(x)\})) = f(x).$$

It is worth noting that, if $f \in lco(C)$ then f is join-uniform if and only if $\forall x \in C.\ f(f_+(x)) = f(x)$. We denote join-uniform functions as $f : C \xrightarrow{\text{ju}} C$. Meet-uniformity is dually defined, in particular we denote a meet-uniform function as $f : C \xrightarrow{\text{mu}} C$. It is obvious that join-uniformity is strictly weaker than additivity. It is known (cf. [10]) that a *lifted* partial order can be induced by join-uniform closures, and defined as follows for a join-uniform closure $\eta \in lco(C)$:

$$x \leq_\eta y \Leftrightarrow \eta(x) \leq \eta(y) \wedge (\eta(y) = \eta(x) \Rightarrow x \leq y) \tag{3.1}$$

In general $\leq \ \subseteq \ \leq_\eta$ and, when $C_\leq$ is a complete lattice, then also $C_{\leq_\eta}$ is a complete lattice. Moreover η is always additive on $C_{\leq_\eta}$ [10].

When adjoining an additive/co-additive closure we obtain two different results: (1) the (right/left) adjoint is still a closure operator and (2) they satisfy the relations in Theorem 1. In the following we will weaken additivity by join-uniformity and we study the preserved relation between f and f^+ both when $f \in lco(C)$ and when $f \in uco(C)$.

3.1 Join uniformity of lower closures

Let us consider $\eta \in lco(C)$. Next theorems analyze the results described in Theorem 1 showing when residuation is sufficient, sufficient and necessary or not necessary. In particular, without any hypothesis on η we show that the pseudo-adjoint fails to be an uco but one equation between η and η^+ in Theorem 1 hold anyway, while the other corresponds to additivity.

Theorem 2 *Let C be a complete lattice. Let $\eta \in lco(C)$, the following facts hold*

1. *$\eta^+ \in \mathcal{F}^{mono} \cap \mathcal{F}^{ext}$ (it may lose idempotence);*
2. *$\eta^+ \circ \eta = \eta^+$ always holds.*
3. *$\eta \circ \eta^+ = \eta$ if and only if η additive;*

Proof

1. Let us show that η^+ is monotone and extensive. Let $x \leq z$, then we have that $\{\, y \mid x \geq \eta(y) \,\} \subseteq \{\, y \mid z \geq \eta(y) \,\}$. But this implies that $\bigvee \{\, y \mid x \geq \eta(y) \,\} \leq \bigvee \{\, y \mid z \geq \eta(y) \,\}$. Extensivity holds by reductivity of η. In fact $x \geq \eta(x)$, hence $x \in \{\, y \mid x \geq \eta(y) \,\}$, which trivially implies that $x \leq \bigvee \{\, y \mid x \geq \eta(y) \,\} = \eta^+(x)$. In Fig. 1 we show an example where the lack of additivity implies a lack of idempotence of η^+. On the left η, whose fix-points are represented by circled points, is not additive, while, on the right, η^+ (represented again by circled points) is not idempotent on x.
2. If $\eta(z) \leq \eta(x)$ (for some $z \in C$), being η reductive we have $\eta(z) \leq x$. On the other hand, if $\eta(z) \leq x$ then by idempotence of η, $\eta(z) \leq \eta(x)$, hence we have that $\{\, z \mid \eta(z) \leq x \,\} = \{\, z \mid \eta(z) \leq \eta(x) \,\}$, namely $\eta^+(x) = \eta^+(\eta(x))$.
3. If η is additive then the thesis holds by Theorem 1(2). Suppose that $\eta \circ \eta^+ = \eta$. We observe also that, being $\eta \in lco(C)$, it is closed by concrete least upper bound, hence, for any $X \subseteq C$ we have $\bigvee_{x \in X} \eta(x) = \eta(\bigvee_{x \in X} \eta(x))$. Let us prove that, for any X we have $\eta(\bigvee X) = \bigvee_{x \in X} \eta(x)$. Note that

$$\eta(\eta^+(\bigvee_{x \in X} \eta(x))) = \eta(\bigvee_{x \in X} \eta(x)) = \bigvee_{x \in X} \eta(x)$$

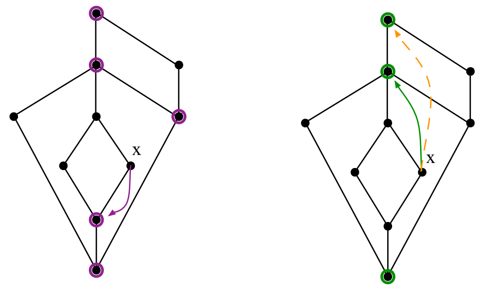


Fig. 1 (left) η not additive, (right) η^+, the dashed line is $\eta^+ \circ \eta^+(x)$.

by hypothesis and because $\eta \in lco(C)$. Observe that $\forall x \in X$ we have $\eta(x) \leq \bigvee_{x \in X} \eta(x)$, hence $X \subseteq \{\, y \,|\, \eta(y) \leq \bigvee_{x \in X} \eta(x) \,\}$ namely

$$\bigvee X \leq \bigvee \{\, y \,|\, \eta(y) \leq \textstyle\bigvee_{x \in X} \eta(x) \,\} = \eta^+(\bigvee_{x \in X} \eta(x)).$$

By reductivity of η this implies that $\eta(\bigvee X) \leq \eta^+(\bigvee_{x \in X} \eta(x))$, then by idempotence of η, this implies that

$$\eta(\bigvee X) \leq \eta(\eta^+(\bigvee_{x \in X} \eta(x))) = \bigvee_{x \in X} \eta(x)).$$

On the other hand, $x \leq \bigvee X$ implies by monotonicity that $\eta(x) \leq \eta(\bigvee X)$, hence $\bigvee_{x \in X} \eta(x) \leq \eta(\bigvee X)$, implying the equality and therefore additivity of η.

Consider $\eta \in lco(C_{\leq})$ is join-uniform, in [10], the authors observed that η^+ is not the right adjoint of η on $C_{\leq}$, while it is its right adjoint on $\leq_\eta$, denoting the order lifted by η as defined in Equation 3.1. The following result shows that this happens if and only if η is join-uniform [10] and, in this case, η_+ is precisely its right adjoint.

Theorem 3 *Let $\eta \in lco(C)$. Then the following facts are equivalent*

1. *η is join-uniform;*
2. *η is additive on $\leq_\eta$;*
3. *$\langle \eta, \eta_+ \rangle$ are adjoints on $\leq_\eta$.*

Proof Let's prove first that η is join-uniform on $\leq$ if and only if η is additive on $\leq_\eta$. When η is join-uniform on the standard order then η is additive on the lifted order by duality from [10, Theorem 5.10]. Let us prove the other implication, and consider the definition of lifted least upper bound [10] rewritten as:

$$\bigvee_\eta Y = \begin{cases} \bigvee Y & \text{if } \exists x \in Y \,.\, \forall y \in Y.\, \eta(y) = \eta(x) \\ \bigvee \{\, \eta(y) \,|\, y \in Y \,\} & \text{otherwise} \end{cases}$$

Let $Y = \{\, y \,|\, \eta(y) = \eta(x) \,\}$: then

$$\textstyle\bigvee_\eta \{\, y \,|\, \eta(y) = \eta(x) \,\} = \bigvee \{\, y \,|\, \eta(y) = \eta(x) \,\} \qquad (*)$$

Hence we have

$$\eta(\bigvee \{\, y \,|\, \eta(y) = \eta(x) \,\}) = \eta(\bigvee_\eta \{\, y \,|\, \eta(y) = \eta(x) \,\}) \\ = \bigvee_\eta \{\, \eta(y) \,|\, \eta(y) = \eta(x) \,\} = \eta(x)$$

This is join-uniformity of η on the order $\leq$.
Finally, η is join-uniform if and only if $\langle \eta, \eta_+ \rangle$ are adjoints on $\leq_\eta$ is consequence of the previous result $(*)$ and of Proposition 1.

We have just shown that η_+ is the right adjoint of η on the lifted order $\leq_\eta$ if and only if η is join-uniform. Next theorem shows how the relations in Theorem 1 change for η_+ on $\leq$ when η is only join-uniform.

Theorem 4 *Let C be a complete lattice. $\eta \in \mathrm{lco}(C)$*

1. *If η is join-uniform then $\eta_+ \in \mathcal{F}^{idem} \cap \mathcal{F}^{ext}$ (it may lose monotonicity);*
2. *η is join-uniform if and only if $\eta \circ \eta_+ = \eta$;*
3. *$\eta_+ \circ \eta = \eta_+$ always holds.*

Proof 1. Extensivity is trivial being $x \in \{\, y \,|\, \eta(y) = \eta(x) \,\}$. Idempotence comes directly from join-uniformity.

$$\eta_+(\eta_+(x)) = \bigvee \{\, y \,|\, \eta(y) = \eta(\eta_+(x)) \,\} \\ = \bigvee \{\, y \,|\, \eta(y) = \eta(\bigvee \{\, y \,|\, \eta(y) = \eta(x) \,\}) \,\} \\ = \bigvee \{\, y \,|\, \eta(y) = \eta(x) \,\} \quad \text{(By join-uniformity)} \\ = \eta_+(x)$$

In Figure 2 we show that in general η_+ is not monotone [10].

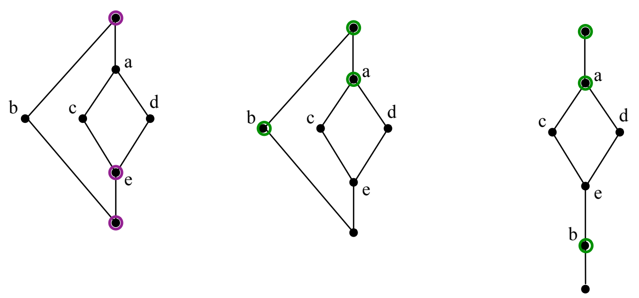


Fig. 2 (left) η not additive, (center) η_+ on $\leq$, (right) η_+ on $\leq_\eta$.

2. η is join-uniform if and only if $\eta \circ \eta_+ = \eta$, by definition of join-uniformity.
3. $\eta_+ \circ \eta = \eta_+$ trivially holds by idempotence of η.

3.2 Join-uniformity of upper closures

In the previous section, we proved that join-uniformity of lower closures, when it holds, weakens additivity in the adjunction relation. In this section, we show that join-uniformity of upper closures does not allow in general to weaken additivity of the adjoint relation. First of all, let us recall that an uco is always join-uniform [8]. At this point, while it is trivial to show that $\delta^+ \circ \delta = \delta$ always holds for any $\delta \in uco(C)$, we prove that the key property which characterizes the right-adjoint of an upper closure, $\delta \circ \delta^+ = \delta^+$, cannot be weakened.

Theorem 5 *Let C a complete lattice. Let $\delta \in \mathrm{uco}(C)$, the following facts hold*

1. $\delta^+ \in \mathrm{lco}(C)$;
2. $\delta^+ \circ \delta = \delta$;
3. $\delta \circ \delta^+ = \delta^+$ *if and only if* δ *additive.*

Proof 1. Let us prove that δ^+ is monotone. Let $x, y \in C$ such that $x \leq y$: then $\{\, z \mid \delta(z) \leq x \,\} \subseteq \{\, z \mid \delta(z) \leq y \,\}$, which implies that $\bigvee \{\, z \mid \delta(z) \leq x \,\} \leq \bigvee \{\, z \mid \delta(z) \leq y \,\}$, namely $\delta^+(x) \leq \delta^+(y)$.
Note that $\{\, y \mid \delta(y) \leq x \,\} \subseteq \{\, y \mid y \leq x \,\}$ hence

$$\delta^+(x) = \bigvee \{\, y \mid \delta(y) \leq x \,\} \leq \bigvee \{\, y \mid y \leq x \,\} = x.$$

Therefore δ^+ is a lower operator. Finally, we have just proved that $\delta^+\delta^+(x) \leq \delta^+(x)$. Let $w \in C$ such that $\delta(w) \leq x$: then $\delta(w) \in \{\, z \mid \delta(z) \leq x \,\}$. Hence, $\delta^+(x) = \bigvee \{\, z \mid \delta(z) \leq x \,\} \geq \delta(w)$, i.e., $w \in \{\, z \mid \delta(z) \leq \delta^+(x) \,\}$. We, therefore, proved that $\{\, z \mid \delta(z) \leq x \,\} \subseteq \{\, y \mid \delta(y) \leq \delta^+(x) \,\}$. But then

$$\delta^+(x) = \bigvee \{\, z \mid \delta(z) \leq x \,\} \leq \bigvee \{\, y \mid \delta(y) \leq \delta^+(x) \,\} = \delta^+\delta^+(x).$$

2. $\delta^+(\delta(x)) = \delta_+(\delta(x)) = \bigvee \{\, \delta(z) \mid \delta(z) \leq \delta(x) \,\} = \delta(x)$.
3. First of all, by Equation 2.1 (Theorem 1) $\delta^+(x) = \delta_+(x) = \bigvee \{\, \delta(y) \mid \delta(y) \leq x \,\}$.
If δ is additive then

$$\begin{aligned} \delta(\bigvee \{\, \delta(y) \mid \delta(y) \leq x\}) &= \bigvee \{\, \delta\delta(y) \mid \delta(y) \leq x\} \quad \text{(by idempotence of } \delta\text{)} \\ &= \bigvee \{\, \delta(y) \mid \delta(y) \leq x\} \end{aligned}$$

Suppose that the equality holds. Recall that, $\delta \in uco$ is additive if and only if for all $Z \subseteq \delta(C)$ we have $\delta(\bigvee Z) = \bigvee Z$. Note that $\bigvee Z \geq \bigvee \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\}$ and that

$$\delta(y) \in Z \;\Rightarrow\; \delta(y) \leq \bigvee Z \;\Rightarrow\; \delta(y) \in \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\}$$

Namely, $Z \subseteq \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\}$ and $\bigvee Z \leq \bigvee \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\}$. So we have the equality. Therefore we can conclude that

$$\delta(\bigvee Z) = \delta(\bigvee \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\}) = \bigvee \{\, \delta(y) \mid \delta(y) \leq \bigvee Z \,\} = \bigvee Z$$

where the second equality holds by hypothesis and because $\delta^+ = \delta_+$.

Let $Fix(\delta) \stackrel{\text{def}}{=} \{ x \in C \mid \delta(x) = x \}$, we observe that $Fix(\delta) \subseteq Fix(\delta^+)$ if and only if $\delta^+ \circ \delta = \delta$. Analogously, $Fix(\delta) \supseteq Fix(\delta^+)$ if and only if $\delta \circ \delta^+ = \delta^+$. This implies that, if δ^+ is the right adjoint of δ, when δ is additive, then $Fix(\delta) = Fix(\delta^+)$. We proved in Theorem 5 that, if δ is not additive, namely only join-uniform by construction, $\delta \circ \delta^+ = \delta^+$ does not hold, namely $Fix(\delta) \subsetneq Fix(\delta^+)$. We are interested in characterizing the points that are in $Fix(\delta^+) \setminus Fix(\delta)$, namely the points added by δ^+. In particular, we observe that these points are precisely those elements making δ not additive. This provides a further characterization of disjunctive completion as pseudo-adjoint of an upper closure operator.

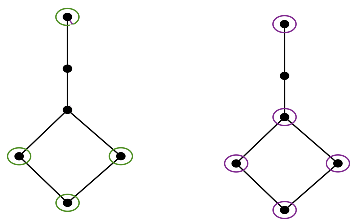
Lemma 1 *Let $\delta \in uco(C)$: then $\delta^+ = (\Upsilon(\delta))^+$.*

Proof We have to prove that $x \in Fix(\delta^+)$ if and only if $x \in Fix((\Upsilon(\delta))^+)$. Since Υ-closed upper closures are trivially additive, then $\Upsilon(\delta)$ admits right adjoint and by Theorem 1-(1) we have that $Fix(\Upsilon(\delta)) = Fix((\Upsilon(\delta))^+)$. Moreover, we observed above that $Fix(\delta) \subseteq Fix(\delta^+)$, then

$$\begin{aligned} x \in Fix(\delta^+) &\text{ iff } x \in Fix(\delta) \ \vee \ \exists y.\, x = \bigvee \{ \delta(z) \mid \delta(z) \leq y \} \\ &\text{ iff } x \in Fix(\delta) \ \vee \ \exists Y \subseteq Fix(\delta).\, x = \bigvee Y \\ &\text{ iff } x \in Fix(\Upsilon(\delta)) = Fix((\Upsilon(\delta))^+) \end{aligned}$$

Proposition 2 *Let $\delta \in uco(C)$, let us define $\tilde{\delta} \in uco(C)$ by the set of fix-point* $\mathrm{Fix}(\tilde{\delta}) \stackrel{\text{def}}{=} \mathrm{Fix}(\delta^+)$*, then $\tilde{\delta} = \Upsilon(\delta)$.*

Example 1 Consider the small lattice in the picture on the left. The circled points are the fix-points of an upper closure δ. On the right, the circled points are the fix-points of $\delta^+ = \delta_+$ which, as we can observe, adds the points of disjunctive completion, making the closure additive.



Let's collect all the results (and their duals) concerning pseudo-adjoints of closure operators in Table 1 and Table 2.

4 Making (lower) closures uniform

We aim at characterizing the set of all uniform lower closures. We will follow [4] and we provide a characterization of uniform closures as images of suitable transformers on lower closure operators.

		JOIN-UNIFORM	ADDITIVE
$\delta \in uco(C)$		$\delta^+ \in lco(C)$ $\delta^+ \circ \delta = \delta$ $\delta^+ = (\curlyvee(\delta))^+$	$\delta \circ \delta^+ = \delta^+$ $(\langle \delta, \delta^+ \rangle$ GC)
$\eta \in lco(C)$	$\eta^+ \circ \eta = \eta^+$ $\eta_+ \circ \eta = \eta_+$	$\eta \circ \eta_+ = \eta$ $\eta_+ \in uco(C, \leq_\eta)$ $\langle \eta, \eta_+ \rangle$ GC on $\leq_\eta$	$\eta \circ \eta^+ = \eta$ $\eta^+ \in uco(C)$ $(\langle \eta, \eta^+ \rangle$ GC)

Table 1 Properties of right pseudo-adjoints of closures

		MEET-UNIFORM	CO-ADDITTIVE
$\delta \in uco(C)$	$\delta_- \circ \delta = \delta_-$ $\delta^- \circ \delta = \delta^-$	$\delta \circ \delta_- = \delta$ $\delta_- \in lco(C, \leq_\delta)$ $\langle \delta_-, \delta \rangle$ GC on $\leq_\delta$	$\delta \circ \delta^- = \delta$ $\delta^- \in lco(C)$ $(\langle \delta^-, \delta \rangle$ GC)
$\eta \in lco(C)$		$\eta^- \in uco(C)$ $\eta^- \circ \eta = \eta$ $\eta^- = \mathcal{M}(\eta)^-$	$\eta \circ \eta^- = \eta^-$ $(\langle \eta^-, \eta \rangle$ GC)

Table 2 Dual properties of left pseudo-adjoints of closures

In [10, Theorem 4.2] the authors proved that the subdomain of $uco(C)$ of all the meet-uniform closures is a Moore family of $uco(C)$. This fact says that this domain forms itself a closure operator on $uco(C)$. Dually, on lower closures, this means that, for any $\eta \in lco(C)$, the best join-uniform lower approximation

$$\bigsqcup \left\{ \delta \in lco(C) \,\middle|\, \delta \sqsubseteq \eta, \delta \text{ is join-uniform} \right\}$$

exists. At this point, we precisely focus on this transformer mapping any lower closure operator η into the *greatest* join-uniform lower closure operator *smaller* than η. In the following, if $\eta \in lco(C)$, we denote as $\eta(Z) = \mathit{const}$ the fact $\exists w \in \eta \,.\, \forall z \in Z \,.\, \eta(z) = w$ and as $Z \not\geq x$ the fact $\forall z \in Z \,.\, z \not\geq x$.

Theorem 6 *Let $\eta \in lco(C)$: then η is join-uniform if and only if*

$$\forall x \in \eta \,.\, \forall Z \subseteq C \,.\, \Big((\eta(Z) = \text{const} \;\wedge\; Z \not\geq x) \Rightarrow \bigvee Z \not\geq x\Big),$$

Proof

($\Rightarrow$) We prove that if $\exists x \in \eta \,.\, \exists Z \subseteq C \,.\, (\eta(Z) = \mathit{const} \;\wedge\; Z \not\geq x \;\wedge\; \bigvee Z \geq x)$ then η is not join-uniform. We know that Z is such that $\exists w \in \eta \,.\, \forall z \in Z \,.\, \eta(z) = w$, this means that $\forall z \in Z \,.\, z \geq w$. Moreover the hypothesis $\bigvee Z \geq x$ implies, by monotonicity, that $\eta(\bigvee Z) \geq x$ and if $\eta(\bigvee Z) = w$ then we would have $w \geq x$ and this is absurd because otherwise we would have $\forall z \in Z \,.\, z \geq x$, which is avoided by the hypotheses on x and Z. Therefore $\eta(\bigvee Z) \neq w$, namely the closure η is not join-uniform.

($\Leftarrow$) We prove that if η is not join-uniform then $\exists x \in \eta \,.\, \exists Z \subseteq C \,.\, (\eta(Z) = \textit{const} \;\wedge\; Z \not\geq x \;\wedge\; \bigvee Z \geq x)$. Consider $w \in \eta$ and $Z = \{\, z \in C \,|\, \eta(z) = w \,\}$, then $\forall z \in Z \,.\, z \geq w$ and this implies that $\bigvee Z \geq w$. By monotonicity this implies $\eta(\bigvee Z) \geq w$. We supposed that η was not join-uniform, this means that $\eta(\bigvee Z) > w$, i.e. there isn't the equality. Let $x = \eta(\bigvee Z)$, therefore $\bigvee Z \geq \eta(\bigvee Z) = x$. Moreover we have $\forall z \in Z \,.\, z \not\geq x$, otherwise if it exists $z \in Z$ such that $z \geq x$ we would have also $\eta(z) = w < x = \eta(x)$ by definition of x, which is absurd for the monotonicity of η. All these facts imply that Z is such that $\eta(Z) = \textit{const}$, by construction, and $Z \not\geq x$ and $\bigvee Z \geq x$ for what we have just proved.

This theorem implies that we can isolate a set of fix-points in $\eta(C)$ which represent the closest join-uniform closure with respect to η.

$$(\eta)^{\vartriangle} \stackrel{\text{def}}{=} \left\{\, x \in \eta \,\middle|\, \forall Z \subseteq C \,.\, ((\eta(Z) = \textit{const}, Z \not\geq x) \Rightarrow \bigvee Z \not\geq x) \,\right\}$$

Lemma 2 *If $\eta \in lco(C)$ then $(\eta)^{\vartriangle} \in lco(C)$ and*

$$(\eta)^{\vartriangle} = \bigsqcup \left\{\, \delta \in lco(C) \,\middle|\, \delta \sqsubseteq \eta, \delta \text{ is join-uniform} \,\right\}.$$

Proof Consider a set Y of elements y such that $y \in (\eta)^{\vartriangle}(C)$, we have to prove that $\bigvee Y \in (\eta)^{\vartriangle}(C)$. The hypotheses imply that for each $y \in Y$ we have that $\forall Z \subseteq C \,.\, (\eta(Z) = w \;\wedge\; Z \not\geq y) \;\Rightarrow\; \bigvee Z \not\geq y$. Consider $Z \subseteq C$ such that $\eta(Z) = \textit{const}$ then we prove that $Z \not\geq \bigvee Y$ implies $\bigvee Z \not\geq \bigvee Y$. Therefore suppose $Z \not\geq \bigvee Y$ and suppose $\bigvee Z \geq \bigvee Y$, this condition implies that $\forall y \in Y \,.\, \bigvee Z \geq y$. Consider now the condition $Z \not\geq \bigvee Y$. This means that $\forall z \in Z \,.\, z \not\geq \bigvee Y$, i.e. $\forall z \in Z \,.\, \exists y \in Y \,.\, z \not\geq y$. If we prove that, with these hypotheses, $\exists y \in Y \,.\, \forall z \in Z \,.\, z \not\geq y$, i.e $Z \not\geq y$, than we would have an absurd because we have $\bigvee Z \geq y$ and that $Z \not\geq y$ when the hypothesis was that $y \in (\eta)^{\vartriangle}(C)$. Suppose that $\forall y \in Y \,.\, \exists z \,.\, z \geq y$. Then we know that $\forall z \in Z \,.\, z \geq w$ and by monotonicity this implies that $w = \eta(z) \geq \eta(y) = y$. Therefore $\forall y \in Y \,.\, w \geq y$ and this implies that $\forall z \in Z \,.\, z \geq w \geq \bigvee Y$, by definition of $\bigvee$, that is absurd by the hypothesis made. This means that $\exists y \in Y \,.\, \forall z \in Z \,.\, z \not\geq y$, that for the absurd described above implies that $\bigvee Z \not\geq \bigvee Y$. Indeed if $Z \not\geq \bigvee Y$ then $\bigvee Z \not\geq \bigvee Y$, so $\bigvee Y \in (\eta)^{\vartriangle}(C)$.

Finally, we can prove that $(\eta)^{\vartriangle}$ is the greatest closure join-uniform smaller than η. Clearly, by construction and by Theorem 6, $(\eta)^{\vartriangle}$ is join-uniform. Suppose that there exists $\eta' \sqsubseteq \eta$ join-uniform and $(\eta)^{\vartriangle} \sqsubseteq \eta'$. This means that η' has more fix-points than $(\eta)^{\vartriangle}$, i.e., $\exists x \in \eta'(C) \subseteq \eta(C)$ such that $x' \notin (\eta)^{\vartriangle}$. Now, since η' is join-uniform, by Theorem 6 x is such that $\forall Z \subseteq C \,.\, ((\eta(Z) = \textit{const}, Z \not\geq x) \Rightarrow \bigvee Z \not\geq x)$, but by construction all the elements in $\eta(C)$ satisfying this property are in $(\eta)^{\vartriangle}$, hence $x \in (\eta)^{\vartriangle}$, namely $\eta' = (\eta)^{\vartriangle}$.

Theorem 7 *$\eta \in lco(C)$ is join-uniform if and only if $(\eta)^{\vartriangle} = \eta$.*

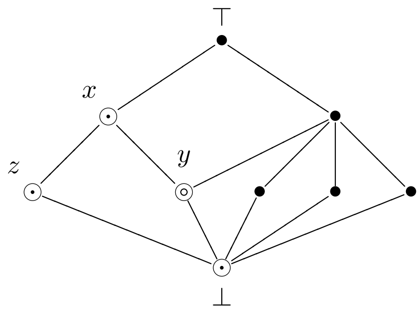
Proof Trivially, we have that if η is join-uniform then it is image of itself. Indeed, by Theorem 6 all the elements of η satisfy the condition imposed by $(\eta)^{\vartriangle}$. Analogously for each $\eta \in Ico(C)$ we have that $(\eta)^{\vartriangle}$ is join-uniform, again by Theorem 6. Moreover, $(\eta)^{\vartriangle}$ is the most concrete join-uniform closure contained in η. Indeed, if there exists another join-uniform closure operator η' contained in η such that $\eta' \sqsubseteq (\eta)^{\vartriangle}$, then there exists at least one element $x \in \eta'$ such that $x \notin (\eta)^{\vartriangle}$. By Theorem 6, this means that $x \in (\eta)^{\vartriangle}$ and consequently η' cannot be join-uniform.

Join-uniformity can be characterized on join-irreducible elements. Recall that x is join-irreducible if $x \neq \bot$ and $x = y \vee z$ implies that either $x = y$ or $x = z$.

Theorem 8 $(\eta \smallsetminus (\eta)^{\vartriangle}) \cap Jirr(\eta) = \varnothing \;\Leftrightarrow\; \eta = (\eta)^{\vartriangle}$.

Proof Consider $\eta \neq (\eta)^{\vartriangle}$, namely $\exists x \in \eta$ such that $Z \not\geq x$ and $\bigvee Z \geq x$. Consider $Y \subseteq Jirr(\eta)$ such that $x = \bigvee Y$. We have to prove that there exists $y \in Y$ such that $Z \not\geq y$. Suppose that $\forall y \in Y . Z \geq y$, namely $\forall z \in Z . \forall y \in Y . z \geq y$. Let $w = \eta(Z)$, we can note that $z \geq y$ implies, by monotonicity of η, that $w = \eta(z) \geq \eta(y) = y$, and this holds for each $y \in Y$. We supposed that $x = \bigvee Y$, so by definition of $\bigvee$ we have that $w \geq x$, but we know that for each z we have $z \geq w$, this would mean that $\forall z \in Z . z \geq x$, which is avoided by the hypotheses on Z and x. Therefore $\exists y \in Y . Z \not\geq y$. Finally, if we consider this y then we have that $\bigvee Z \geq x \geq y$, i.e. we have the thesis.
On the other hand, if $\eta = (\eta)^{\vartriangle}$ then $\eta \smallsetminus (\eta)^{\vartriangle} = \varnothing$ and therefore the intersection is $\varnothing$.

Example 2 Consider the following lattice where the circled points, $\odot$ and $\circledcirc$, are the points in the lower closure η. η is clearly not join-uniform due to y. y is precisely the element to be removed to make η a join-uniform closure.



Acknowledgement

We would like to thank the anonymous reviewers for the useful suggestions and comments helping us in improving the presentation of our work.

Compliance with Ethical Standards

- Conflict of Interest: The authors declare that they have no conflict of interest.
- The authors declare that the research presented in this work does not involve Human participants and/or Animals.

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