Stock Financial Bubbles: a trinomial trees based analysis

Luca Di Persio [*]   Francesco Guida [†]

Abstract— In the present work we consider a novel approach to model the dynamic of financial bubbles. In particular, we exploit a technique based on trinomial trees, which is mainly inspired by the typical *Market Order Book* (MOB) structure. We use a bottom-up approach, according to the typical MOB rules, to derive the relevant generator process for the financial quantities characterizing the market we are considering. Our proposal pays specific attention to consider the real world changes in probability levels characterizing the bid-ask preferences driven by market movements. We show that financial bubbles are, indeed, originated by these movements which, in turn, amplify their growth. Numerical experiments are provided to show the effectiveness of our results within real volatility wars scenarios. Namely, we study realistic economic frameworks characterized by volatility levels showing great fluctuations, in relatively small times.

Keywords— Bubbles, Constraints, Stochastic Dynamics, Trinomial Models.

I. Introduction

The rise of financial bubbles is a well-known phenomenon in real world markets. It substantially consists in episodes in which abnormal and unpredictable increasing of prices are followed by sudden collapses of the same.

Famous examples could be easily recognized in the past as well as during recent times. The general dynamic of such financial phenomena present rather similar characteristics, even if each bubble has its own history. In fact, the financial assets involved are part of very different economic scenarios, spanning from contracts based on natural products to exclusively speculative structured derivatives. So that, during history, we can recognize both old bubble bursts, as in the case of *Dutch Tulipomania* (1634/6-1637), see, e.g., [7], the *Kipper- und Wipperzeit* bubble, see, e.g., [27], which was triggered by the lack of an effective taxation and proper methods to identify the real value of currencies coined by different countries which aimed at raise revenue for the *Thirty Years' War*, and more recent rises and sudden falls of strictly financial products, as happened during the 2007-2008 US crisis of of sub-prime mortgages market, see, e.g., [10], whose effects have clearly shown the effectiveness of the so called *glocalisation phenomena*, see, e.g., [23], namely the intimate interconnections between economic markets around the world, see also,e,g [12] for more nested commodities related bubbles, and, e.g., [3, 13, 26], and references therein, with respect to (also potential) bubbles based on the new digital economy.

Therefore, researchers have been focused on these phenomena, rigorously studying them from the macroeconomic perspective as well as from the mathematical point of views, see, e.g., the recent papers [17, 18, 33], where the local martingale approach is used to study stock financial bubbles within both complete and incomplete markets, also providing a methodology for bubble detecting. In what follows, we focus our attention on the relation between bubbles formation and how stocks are exchanged through market order books. Inspired by the MOB structure and rules, we built up a model for stock prices. In particular we exploit a methodology based on trinomial tree. Thus, in this framework the market price is given endogenously. It is worth to mention that within the MOB scenario, characterized by a rather strict structure, a trader who wants to buy or sell a certain security can put orders in two different way: limit orders and market orders. The trader could place a limit order to buy or sell a security, or, broadly speaking, a general type of *a financial instrument*, at a specified price.

That kind of orders may not be executed if the price set by the trader cannot be met during the period of time in which the order is left open. Alternatively, the trader could place a market order and instantly transact the desired quantity at the best bid or at the best ask price. Additionally, a market order will cause a shock to the order book removing liquidity. At each time the MOB consists of a queue of buyers and a queue of sellers. Prices and orders arrival times, both determine the place where a new order is ordered in the MOB. The ranking procedure is of the *First In First Out* (FIFO) type, with respect to any specific price. When there is an overlap between the two queues, the intersection of all orders will be transacted, and a new price is formed. We will analyse the dynamic of possible bubbles in such a MOB scheme.

Another really important characteristic of stocks that we consider here is the birth of a bubble. We underline that the model we are considering are based on the macroeconomic theory that sees the great investors as triggering factors for the birth of the financial bubbles. In order to model this phenomenon we considered the *financial movements*, characterizing the bid-ask preferences, once some great investor put its order, shifting real world probabilities for subsequent orders.

The volatility clustering property of financial returns comes in a naturally way from the model's dynamic. We give much more insights also on the relation between the boundedness of the MOB and the volatility clustering property. The paper is structured as follows: in Section II we provide the model setup, also by showing some examples illustrating the model's dynamic; in Section III, we shall describe the way we model the birth and the death of financial bubbles within the endogenously determined stock price; while in Section IV, resp. in Section V, we will provide, using different probability distributions, numeri-

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*luca.dipersio@univr.it, Dept. of Computer Science, University of Verona, Strada le Grazie 15 37134 Verona
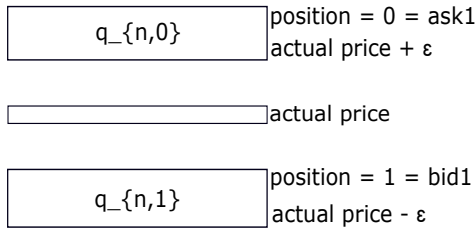
†francesco.guida@unitn.it, Dept. of Mathematics, University of Trento, Via Sommarive, 14, 38123 Povo TN

cal examples and simulations, resp. we will discuss possible developments and caveats.

II. THE SETUP

We consider a market consisting of a single stock which is traded through a *Limit Order Book (LOB)*. In general LOBs have different bid-ask levels so that the resulting structure could easily become rather nested. In order to simplify this structure, we will consider just two LOB levels, namely the first bid, resp. the first ask, levels. The latter implies that traders can only exercise their orders using such two levels. We underline that this hypothesis is not unrealistic, since any more general order, can be subdivided as a sum of the aforementioned twos. Fig. 1 represents the typical LOB queue at time $n \geq 0$.

In particular, the first ask level has position 0 with quantity $q_{n,0}$ that is the quantity that can be traded at time n, while the first bid level has position 1, with quantity $q_{n,1}$. The bid-ask spread is fixed as 2ε, where $\varepsilon > 0$ represents the distance between the actual price and the bid/ask price offers.

| q_{n,0} | position = 0 = ask1 |
| | actual price + ε |

| | actual price |

| q_{n,1} | position = 1 = bid1 |
| | actual price - ε |

Order book considered at time n.

It follows that the LOB representation here provided is of static type, while the dynamic of the order book will be determined by orders arrival. In particular, traders put their orders at each trading day, composed by N trading instants. As an example, N could be the number of seconds or the number of milliseconds between the opening of the market and its closure. Then, in each trading day we have $0 \leq n \leq N$ trading moments, with orders arrivals that are randomly distributed over time. Moreover their magnitude is random, too. Last but not least, each order could be of bid, or ask type. In what follows, we model orders arrivals supposing that at each time $0 \leq n \leq N$, an order can be placed or not with certain probabilities. Sources of randomness characterizing the LOB are

- Order's arrival;
- Bid or ask level of the order;
- Quantity of the order.

It is worth to mention that the first two sources of randomness are binary. In fact an order can arrive or not and the order can be on bid level or ask level, While we model the last component by a probability distribution for the orders quantities. Furthermore traders can make limit orders or market orders. Market orders can only remove a specific quantity from the level chosen, because a market order is traded at the best bid or ask level. Instead, limit orders can also add quantity to the level chosen.

We will model this property of orders by assuming a symmetric probability distribution and considering the sign of the quantity as a discriminant factor. If the quantity is positive we will add quantity to the level chosen, otherwise we will subtract it.

Let us consider an example of the just introduced LOB dynamic:

- $n = 0$, $q_{n,0} = 50$, $q_{n,1} = 38$. We have an order of bid level (position 1), with quantity -20.
- $n = 1$, $q_{n,0} = 50$, $q_{n,1} = 18$. We have no order.
- $n = 2$, $q_{n,0} = 50$, $q_{n,1} = 18$. We have an order of ask level (position 0), with quantity $+50$.
- $n = 3$, $q_{n,0} = 100$, $q_{n,1} = 18$. We have an order of bid level (position 1), with quantity -30.
- $n = 4$, $q_{n,0} = 100$, $q_{n,1} = 12$. We have no order.

The last two steps of the above example produce a price change. Indeed, since the last order fills completely the bid quantity, then a new price is formed. The remaining quantity $12 = 30 - 18$, becomes the new quantity for the new bid level. Remind that a price is formed whenever the ask or the bid level is completely filled by an order, namely when there is an overlap between the bid queue and the ask queue.

In particular, at time $n+1$, considering a price variation $\varepsilon > 0$, the price X_{n+1} can be $X_n \pm \varepsilon$ or still X_n if there is no overlap. For example if a trader buy $q_{n,0}$ price shares at $X_n + \varepsilon$ then a new price $X_{n+1} = X_n + \varepsilon$ is formed. Equivalently, if a trader sell $q_{n,1}$ price shares at $X_n - \varepsilon$ then the new price $X_{n+1} = X_n - \varepsilon$ is formed. In other cases a price is not formed and the resulting price will be still $X_{n+1} = X_n$. We will now make the whole structure more precise and formal. Let $(\Omega, F, \mathbb{P})$ be a probability space, and, discretizing time, let us define $o_n : \Omega \to \mathbb{R}$ as the binary random variable describing if an order is arrived at time n, being $0 \leq n \leq N$, for a given horizon $N \in \mathbb{N}^+$. We assume that $\mathbb{P}(o_n = 0) = \lambda = 1 - \mathbb{P}(o_n = 1)$, where $\lambda \in (0,1)$. Moreover, we define $P_n : \Omega \to \mathbb{R}$, as the random variable describing the bid-ask level of the market order, once this is arrived. Since also P_n is a Bernoulli random variable, taking $p \in (0,1)$, we set $\mathbb{P}(P_n = 0) = p = 1 - \mathbb{P}(P_n = 1)$. Concerning the magnitude q_n^* of the $n-$th order, we model it as follows: $q_n^* \sim N(\mu, \sigma^2)$. It is worth to mention that latter Gaussian-type assumption, is coherent with our aim to have a benchmark, or starting point, to calibration purposes. Let us remind that once an order has arrived, and the trader has chosen its *position*, a negative, resp. positive, quantity corresponds to a subtraction, resp. an addition. Since the formation of a new price, when dealing with a *bid order*, depends on the fact that $q_n^* \leq q_{n,0}$, or $q_{n,1}$, then the price at time $n+1$ is defined as follows:

$$
\begin{aligned}
X_{n+1} = {} & (X_n + \varepsilon) \mathbb{1}_{\{o_n=1\}} \mathbb{1}_{\{q_n^* < -q_{n,0}\}} \mathbb{1}_{\{P_n=0\}} \\
& + (X_n - \varepsilon) \mathbb{1}_{\{o_n=1\}} \mathbb{1}_{\{q_n^* < -q_{n,1}\}} \mathbb{1}_{\{P_n=1\}} \\
& + X_n \mathbb{1}_{\{o_n=1\}} \mathbb{1}_{\{q_n^* > -q_{n,0}\}} \mathbb{1}_{\{P_n=0\}} \\
& + X_n \mathbb{1}_{\{o_n=1\}} \mathbb{1}_{\{q_n^* > -q_{n,1}\}} \mathbb{1}_{\{P_n=1\}} + X_n \mathbb{1}_{\{o_n=0\}}
\end{aligned}
\tag{1}
$$

$$= X_n + \varepsilon \left[\mathbb{1}_{\{o_n = 1\}} \left(\mathbb{1}_{\{q_n^* < -q_{n,0}\}} \mathbb{1}_{\{P_n = 0\}} - \mathbb{1}_{\{q_n^* < -q_{n,1}\}} \mathbb{1}_{\{P_n = 1\}} \right) \right] \tag{2}$$

To simplify notation we define:

$$J_n \doteq \mathbb{1}_{\{o_n = 1\}} \left(\mathbb{1}_{\{q_n^* < -q_{n,0}\}} \mathbb{1}_{\{P_n = 0\}} - \mathbb{1}_{\{q_n^* < -q_{n,1}\}} \mathbb{1}_{\{P_n = 1\}} \right),$$

so that

$$X_{n+1} = X_n + \varepsilon J_n.$$

Moreover, we have that

$$X_{n+1}(\omega) = \begin{cases} X_n + \varepsilon & \mathbb{P}(X_{n+1} = X_n + \varepsilon) = \lambda \, p N(-q_{n,0}) \\ X_n - \varepsilon & \mathbb{P}(X_{n+1} = X_n - \varepsilon) = \lambda (1-p) N(-q_{n,1}) \\ \\ X_n & \mathbb{P}(X_{n+1} = X_n) = \begin{aligned} & 1 - \lambda [pN(-q_{n,0}) \\ & - (1-p)N(-q_{n,1})] \end{aligned} \end{cases}$$

and

$$\begin{aligned} \mathbb{E}[X_{n+1}] &= \mathbb{E}[X_n] + \varepsilon \mathbb{E}[J_n] \\ &= X_n + \varepsilon [\lambda (p \cdot N(-q_{n,0}) - (1-p) \cdot N(-q_{n,1}))] \end{aligned} \tag{3}$$

Hence to be a martingale, the price process has to fulfill the following condition

$$\varepsilon \lambda (p \cdot N(-q_{n,0}) - (1-p) \cdot N(-q_{n,1})) = 0. \tag{4}$$

Therefore, we completely define the order book process $\{q_{n,0}, q_{n,1}\}_{0 \leq n \leq N}$, exploiting the triple $\{o_n, P_n, q_n^*\}_{0 \leq n \leq N}$, as well as the intra-day price process $\{X_n\}_{0 \leq n \leq N}$, supposing that o_n, P_n, q_n^* are mutually independent. We note that by using the quadruple $\{n, o_n, P_n, q_n^*\}_{0 \leq n \geq N}$, we have indeed defined the whole supply curve associated to the price process.

III.   THE BIRTH OF A BUBBLE

In this section we will focus our attention on the financial bubbles bursting mechanism, within the framework defined along the previous section. A ever present feature of a bubble is that it is characterized by changes that strongly deviate a market price from its intrinsic value, see, e.g, [33]. We will characterize the birth of a bubble by the arrival of a great order in the order book.

It is worth to mention that such *great orders* could be triggered by different micro as well as macro economic factors, as, e.g., market uncertainty, speculative politics acted by big financial players, insider trading and mass media attacks, etc.

As a matter of fact, empirical evidences based on the analysis of a huge quantity of time series, show that small to medium investors are likely to suddenly change their trading intentions when great magnitude orders hit the markets. Such type of dynamics for crowds, or collective aggregation, which are mostly ethero directed have been widely analysed, also because of similarities across etherogeneous frameworks, and then applied in population biology, particle physics, social sciences, etc., see, e.g., [2, 9, 11, 20, 22, 35], and references therein. In our setting, the latter means that preferences on bid/ask level change as well as probabilities. In particular, we have a change of

the probability level p. We identify the arrival of huge orders with q_n^* being under a certain threshold. Having assumed that $q_n^* \sim N(\mu, \sigma^2)$, we choose $q_n^* \sim N(-1, 1)$, so that to have a threshold $q_n^* = -5.6$, if $N = 10000$ and the number of days of the simulation $n^* = 252 * 10$. The reason of the latter choice lies in the fact that the intra-day price process simulation brings N realization of the process $\{q_n^*\}_{0 \leq n \leq N}$ each day and for $n^* = 252 * 10$ days, then taking $q_n^* = -5.6$ we have $N(-q_n^*) < 2.11 \cdot 10^{-6}$. Is such an order arrive, then many traders start to replicate it, memorizing its arrival position, in time, and then changing accordingly the probabilities in the bid/ask level choice. We underline that this approach has been inspired by to the one suggested in [5], see Remark 3.3.

A further peculiarity of bubbles concerns their duration. Even limiting our attention to the financial scenario, bubbles could last days or months. Moreover, due to the increasing relevance of high frequency trading, smaller bubbles have been studied with very short durations, see, e.g., [8, 19, 32]. We will focus our study on bubbles with fixed duration t_{vol}, analysing their decay dynamic. Indeed, whenever we are in a bubble regime, we can exploit empirical data to predict the time region where the bubble will end up. It should be underlined that, in most of the cases the end of a financial bubble does not consist in a instantaneous renormalization of prices, but rather it follows a renormalization regime that may happens at slow speed. In what follows we will consider the changes in probability until we arrive at $\frac{t_{vol}}{2}$ and after that we use reversed probabilities. As to make an example, supposing $p = 0.5$, with a huge order arrival at time n, namely $(o_n = 1, P_n = k, q_n^* < -5.6), k \in \{0, 1\}$ being fixed, then for $n \in [n+1, N]$, in the next $t_{vol}/2$ days, the probabilities of the P_n process change, so that

$$P_n^{\leq t_{vol}/2}(\omega) = \begin{cases} k & p_1 = 0.6 \\ 1-k & 1-p_1 = 0.4 \end{cases}$$

From $t_{vol}/2$ to t_{vol} we invert probabilities to simulate the bubble's death, therefore

$$P_n^{> t_{vol}/2}(\omega) = \begin{cases} k & p_1 = 0.4 \\ 1-k & 1-p_1 = 0.6 \end{cases},$$

and we will refer to this scheme as to the *first scheme*. Even if we have considered instantaneous jumps in probability, to make the example more clear, probabilities changes could happen in less sudden ways, e.g. following some smooth dynamic of exponential type. Indeed, in real world settings, bubbles are characterized by three important phases: birth, death and a period, separating the first twos, of more quiescent volatility levels. Such time interval results is tipically longer than the bubble's burst and death periods. Aiming at improving our approach, we consider the bubble life as subdivided into four parts, as follows

$$P_n^{< t_{vol}/4}(\omega) = \begin{cases} k & p_1 = 0.6 \\ 1-k & 1-p_1 = 0.4 \end{cases},$$

hence recovering a *middle part* where we restore the initial probabilities. Therefore, we are now exploiting a regime switching approach

$$P_n^{\left[\frac{t_{vol}}{4}, \frac{3t_{vol}}{4}\right)}(\omega) = \begin{cases} k & p_1 = 0.5 \\ 1-k & 1-p_1 = 0.5 \end{cases},$$

where, from $\frac{3t_{vol}}{4}$ to t_{vol}, we invert probabilities, to model the bubble's death. In particular, we will have

$$P_n^{\geq \frac{3t_{vol}}{4}}(\omega) = \begin{cases} k & p_1 = 0.4 \\ 1-k & 1-p_1 = 0.6 \end{cases},$$

and we define such scheme as the "*second scheme*".

IV.   SIMULATION RESULTS

i.   Gaussian case

In this section we provide numerical results related to the methods introduced during previous sections. We consider $n^* = 252 \cdot 10$, ten financial years, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments.

We fix $\varepsilon = 0.2$, and the initial price $X_0 = 100$. We suppose that the initial order book is normalized and characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n \geq 0}$, and that quantities of orders are distributed as $q_n^* \sim N(\mu = -1, \sigma^2 = 1)$, the choice of $\mu = -1$ being forced to obtain non trivial dynamics. Indeed, choosing $\mu = 0$, implies 50% of orders adding/subtracting some quantity to the bid/ask level chosen.

To optimize our approach, we give a greater probability of having subtracting operation on bid/ask level, instead of addition ones. n. Regarding probabilities levels, we set $\lambda = \frac{1}{2}$, $p = \frac{1}{2}$.

Great orders arrive within the threshold $q_n^* < -5.6$ and we fixed $t_{vol} = 200$ and $p_1 = 0.515$. Figures from 2a to 3b, show the results obtained following the *first scheme*. Fig.2a shows the price process during days namely we report the last price of each

In figure 2b we consider the volatility process fitted with the *Auto Regressive Conditional Heteroskedasticity* (GARCH) method
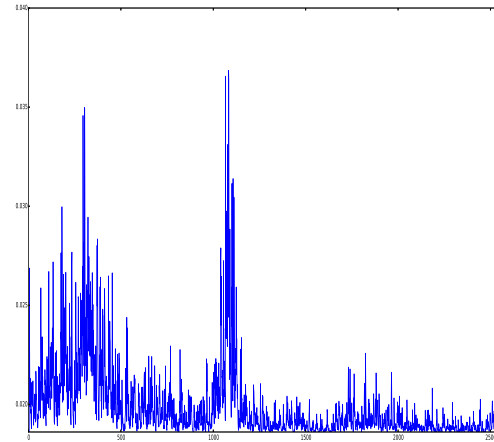

Fig.2b:*Volatility fitted using GARCH methodologies. First Scheme.*

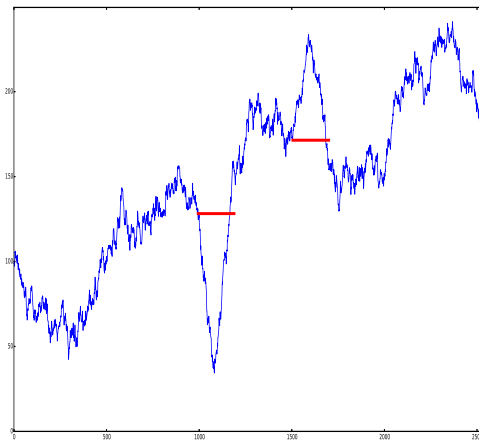The logarithmic returns of prices are represented in figure 3a below

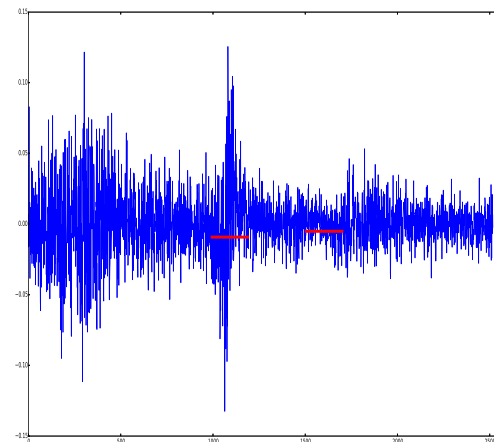**Fig.2a:***Price process simulated. First Scheme.*

Fig.(3a):*Logarithmic returns.*

simulation of the intra-day price process $\{X_n\}_{0 \leq n \leq N}$, plotting the new sequence $(X_{N_1}, X_{N_2}, X_{N_3}, ..., X_{N_{n^*}})$, X_{N_i} which represents the last price of the *i*-th day. Furthermore, we made a simple analysis of our price as if they are given as exogenous quantities.

while in figure 3b we report their empirical distribution, again using the *first scheme* approach.
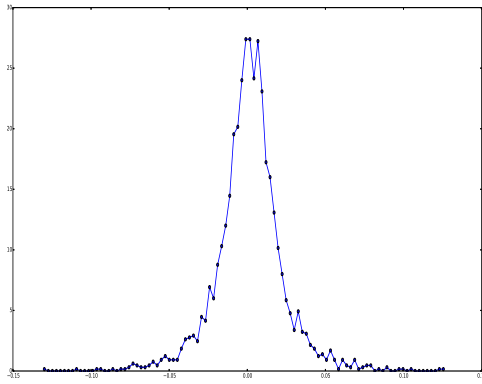
Fig.(3b):*Distribution of logarithmic returns. First Scheme.*

We would like to underline how the obtained results clearly show that the proposed model well represent realistic market dynamics. In particular, we outline how the computations highlight that logarithmic returns share the volatility clustering property. It is also worth to mention that the variables here considered are independent between each other. Hence, it seems that the volatility clustering property is intrinsic, being mainly realized by the structure of the order book, instead of being triggered by the structure of the involved random variables. Furthermore this may suggest that, in a realistic framework where order book have various bid/ask levels, as well as different characteristics due to social, economic and financial factors, it is reasonable that the volatility clustering property comes from the order book structure. It is worth to mention that the red lines which compare in figures 2a and 3a, represent the bubble's regime. As we can see, initially the bubble burst and after $\frac{t_{vol}}{2} = 100$ days, the preferences in bid/ask level change. The graphs presented in figures from 4a to 5b, are related to simulation results obtained using the same constant aforementioned specified, but now the method used is the *second scheme* one.
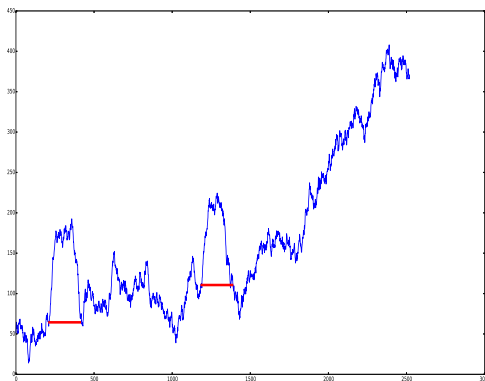
Looking at figure 4a

an average volatility behavior. Then, the bubble tends to its end. As before we can fit the volatility levels obtained by computations, using the GARCH method, in order to have the following graph
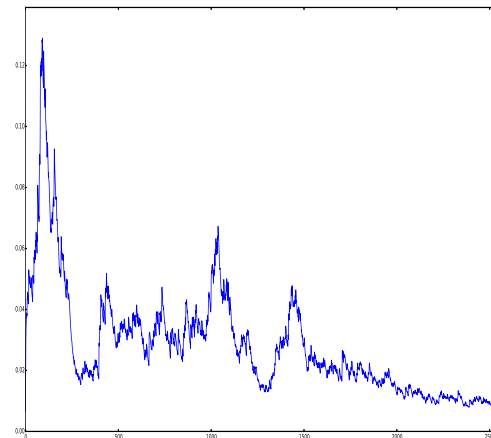

Fig.(4b): *Volatility fitted using GARCH methodologies. Second Scheme.*

While figures 5a and 5b respectively show the logarithmic returns behavior and the associated empirical distribution.
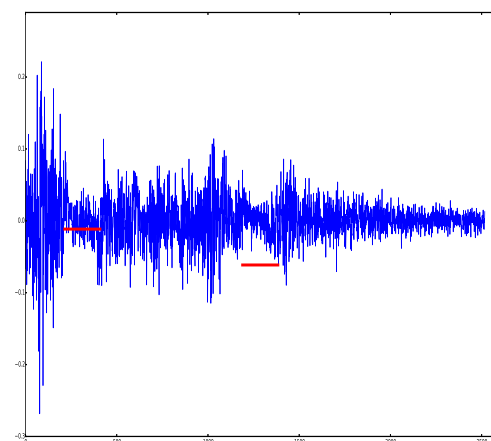

Fig.(4a):*Price process simulated. Second Scheme.*

Fig.(5a):*Logarithmic returns. Second Scheme.*

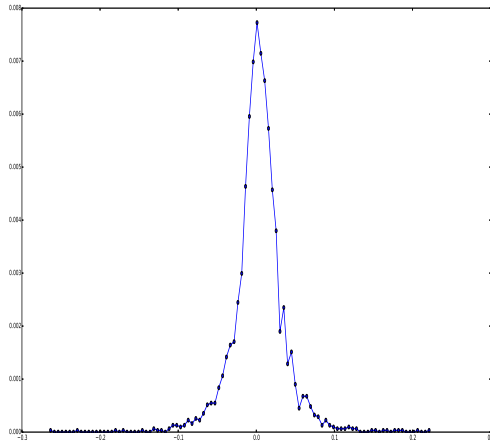we can see that, after the bubble burst, price fluctuate showing

Fig.(5b):_Distribution of logarithmic returns. Second Scheme._

Remark 1. _Aiming at deriving a more complete analysis of our approach, in what follows we consider different type of probability distributions, hence moving away from the Gaussian perspective. In particular, we will show how the flexibility of the proposed model allows to consider the dynamic of analyzed financial quantities by using different types of probability distributions. The latter permits to see how the involved priced behaviors change passing from a given random framework to another. By the very definition of our model, possible distribution choices have to be done with respect to random variable distributed over all the real line. This is motivated by the fact that quantities related to our trinomial model must have a sign. Therefore, we will discard all distribution defined just in the real positive line and we consider the Cauchy distribution and the t-Student distribution. Of course, previous options have to be handled with care, namely a great attention has to be put concerning the parameters selection. This is to have dynamics similar to the Gaussian case in regarding the presence of bubbles, their lifetime as well as concerning the probability assigned to the presence of orders of great magnitude._

ii.   Cauchy case

As in the Gaussian case, we again consider $n^* = 252 \cdot 10$, namely ten financial years, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments. We fix $\varepsilon = 0.2$, and the initial price $X_0 = 100$. We suppose that the initial order book is normalized and characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n \geq 0}$, and that quantities of orders are distributed as $q_n^* \sim C(-q_{n-1}^*, 1)$, where we considered a Cauchy distribution defined as:

$$C(x_0, \gamma) = \frac{1}{\pi \gamma (1 + (\frac{x - x_0}{\gamma})^2)} \,, \tag{5}$$

so that we choose the location parameter to be the quantity of the previous financial moment and the scale interpreted by the gamma factor as 1. Regarding probabilities levels, we set $\lambda = \frac{1}{2}$, $p = \frac{1}{2}$. We also fix $t_{vol} = 200$ and $p_1 = 0.515$. To model great

orders arrival, we then must choose a threshold in such a way that the probability of such an event results as a very rare case. In order to do this we consider $q_n^* < -150000$. Therefoere, the resulting cumulative distribution function of the Cauchy distribution give us a very little probability to have a great order. Namely, we have $C(-q_n^*) = 2.12 \cdot 10^{-6}$. Figures from 6a to 7b show the computational results for the present *Cauchy approach*. In Fig. 6a we can see the simulation for the price dynamic, according with the *second scheme*
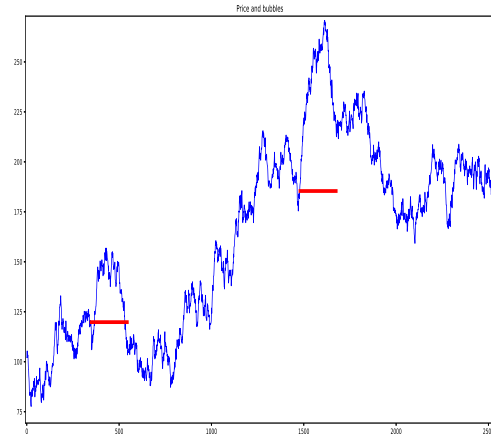

Fig.(6a):_Price process simulated in the Cauchy case. Second Scheme._

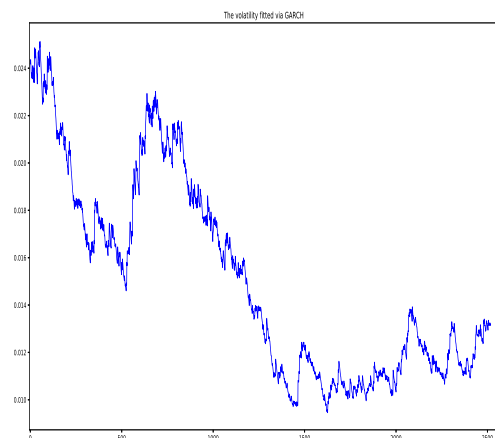while the following figure 6b is about the volatility levels

Fig.(6b):_Volatility in the Cauchy case. Second Scheme._

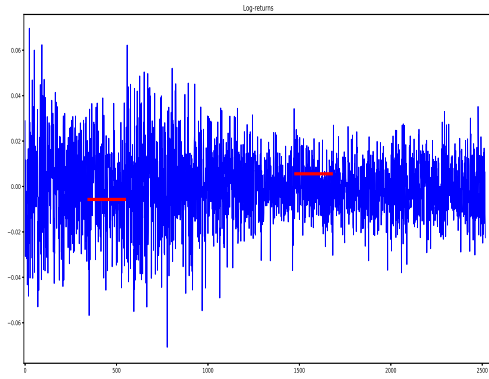The logarithmic returns as well as their empirical distribution, are shown in figures 7a and 7b below

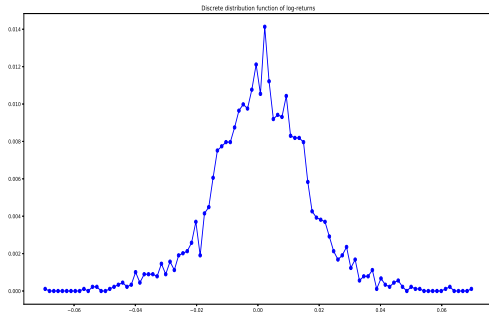**Fig.(7a):***Logarithmic returns in the Cauchy case. Second Scheme.*

Fig.(7b):*Distribution of logarithmic returns in the Cauchy case. Second Scheme.*

iii.   t-Student case

In what follows we adopt the t-Student probability distribution to model the random quantities we are interesting in. Therefore, as before, we set $n^* = 252 \cdot 10$, namely ten financial years, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments. We also fix $\varepsilon = 0.2$, while the initial price is taken as $X_0 = 100$. We suppose that the initial order book is normalized, being also characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n \geq 0}$. Moreover, the quantities of orders are distributed as follow $q_n^* \sim t(-q_{n-1}^*, 1, 1)$, where we have considered the t-Student probability distribution function defined by:

$$t(x, df, \gamma) = \frac{\gamma((df+1)/2)}{\sqrt{(\pi * df)} \cdot \gamma(df/2) \cdot (1 + x^{\frac{2}{df}})^{((df+1)/2)}} , \quad (6)$$

therefore we choose the location parameter to be the quantity of the previous financial moment, the parameter df as 1, while is taken as $\gamma = 1$. Regarding probabilities levels, we set $\lambda = \frac{1}{2}$, $p = \frac{1}{2}$. We also fix $t_{vol} = 200$ and $p_1 = 0.515$. Great orders arrivals are modeled by considering a a threshold in such a way

that the probability of the correspondent random event results as a very rare case.

The latter implies large negative values for q_n^*. In particular, we consider $q_n^* < -150000$, so that the resulting cumulative distribution function results in a tiny probability og having a great order, namely we obtain $t(-q_n^*) = 2.12 \cdot 10^{-6}$.

In Fig. 8a we show the price process in days. This is the last price of each simulation of the intra-day price process $\{X_n\}_{0 \leq n \leq N}$. In particular we plotted the new sequence $(X_{N_1}, X_{N_2}, X_{N_3}, ..., X_{N_{n^*}})$, X_{N_i} representing the last price of the i-th day.
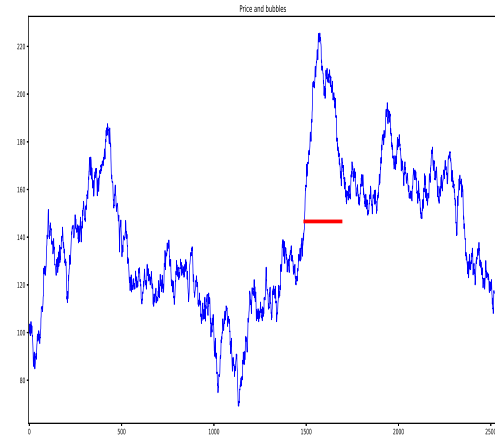

Fig.(8a):*Price process simulated in the t-Student case. Second Scheme.*

while Fig.8b below represents the associated volatility levels fitted with GARCH method.
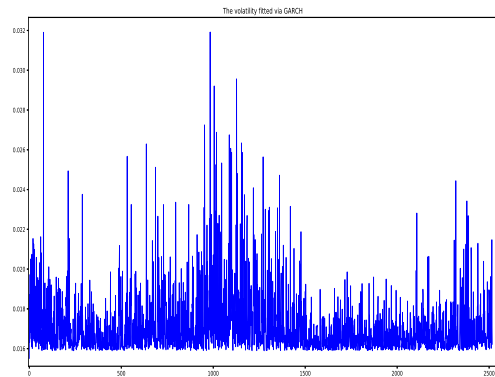

Fig.(8b):*Volatility in the t-Student case. Second Scheme.*

Moreover, in figures 9a, 9b we show the related logarithmic returns of the price, with their empirical distribution.
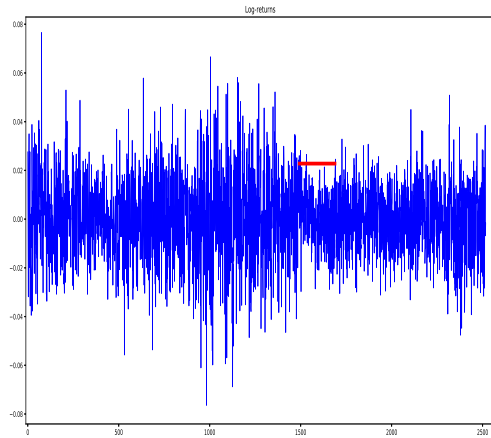

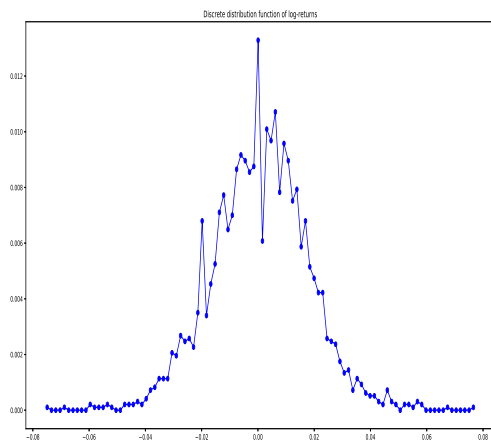**Fig.(9a):***Logarithmic returns in the t-Student case. Second Scheme.*

Fig.(9b):*Distribution of logarithmic returns in the t-Student case. Second Scheme.*

V. CONCLUSION

In this paper we provided a mathematical method, discrete in time, to model stocks' bubbles dynamics, focusing the attention on the market micro-structure in terms of *Market Order Book* (MOB). In particular we described how micro-economic trading interactions determine a shift in probabilistic traders' perspectives. We proposed a constructive methodology that determine endogenously price processes. We also provided a model that, starting from the changes in bid-ask preferences, determines the formation of a bubble.

Numerical simulations have been also reported to show the effectiveness of the proposed approach. Our setting is very flexible, indeed it can be adapted to a large class of financial scenario, also taking into account different types of market and socio-economics factors. The latter is also witnessed by the use of different probability distributions setting, namely spanning from the Gaussian to the t-Student one, we have considered. It is worth to mention that obtained results seem to be very stable under such kind of random changes, hence maintaining the possibility to sharp model bubbles' dynamics. Future improvements are related to the continuous counterpart of the proposed schemes, as well as concerning its mathematical relationships with the general local martingale approach.

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