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Optimal Solution of Kinodynamic Motion Planning for the Cart-Pole System

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Abstract: The aim of this work is motion planning for a class of underactuated mechanical systems. To illustrate the theory, we introduce and investigate, from a geometric and numerical point of view, the solution of kinodynamic planning for the cart–pole. More precisely, given an initial condition for the configuration of the cart–pole, we want to plan an optimal trajectory making the inverted pendulum on the cart to avoid an obstacle during its motion, and to attain a prescribed final configuration.

Keywords: Optimal trajectory, trajectory planning, geometric control theory, autonomous mobile robots, robotics.

1. INTRODUCTION

The study of optimal control and motion planning of underactuated mechanical systems starting from given initial and final conditions is a challenging problem in robotics and control theory (see for example Lynch (2000), Arai et all (1998), Spong (2008)).

The aim of this work is to propose a geometric method, based on the theory exposed in Colombo (2010), to plan a trajectory and to find the corresponding optimal controls for an underactuated mechanical system, by applying suitable external forces and avoiding fixed obstacles. In the light of global aspects of the problem (from the mathematical point of view), we adopted the geometric approach point of view outlined in Bloch and Crouch (1994), Colombo (2010), Blach et all (2015) in which, starting from a constrained variational problem for a mechanical control system, the authors present a geometrical approach that allows to compute the dynamics of the system and, in principle, to solve the related optimal control problem. Roughly speaking, we will consider an optimization problem with second order constraints (i.e. on the acceleration) and we will reformulate the problem as a truly Hamiltonian problem on a suitable symplectic manifold W_1. Then, after the integration of Hamilton equations, we will be able to reconstruct control forces and solving the original problem. In practice the integration of Hamilton equations as well as the optimization are performed numerically. More precisely, we integrate numerically the equations of motion, and then, by using a shooting method, we optimize trajectories and find the control forces. We stress that the proposed method is coordinate independent and, moreover, being Hamiltonian, one can use energy preserving or symplectic algorithms to integrate the equation of motions.

As an application of the theory, we consider the control of the classical cart–pole system, to which we add an external obstacle. We want to study the optimal control and motion planning of this system, in such a way that the cart–pole, starting from a given initial configuration arrives to a given final configuration, avoiding the obstacle. While the problem of the stabilization of the pendulum around the unstable equilibrium is well studied and understood, finding an optimal solution considering the kinodynamic constraints of the cart-pole system is apparently new in the control theory community (see e.g. Boubaker (2013) and references therein).

The paper is organized as follows. In Section 2 we introduce the problem from a general point of view. In Section 3 we recall the basic mathematical aspects of the approach introduced in Colombo (2010) and references therein. In Section 4 we describe our motion planning algorithm, whereas Section 5 is devoted to the solution of the kinodynamic motion planning problem. Conclusions together with future perspectives are drawn in the last section.

Throughout the paper, Einstein's convenction over repeated indices is used, where with lower indices we denote covariant quantities and with upper indices contravariant quantities. Moreover, all manifolds, distributions and maps are assumed to be smooth and regular. Most of the mathematical background to understand the method proposed can be found in Bloch (2015) and Bullo and Lewis (2004).

2. PROBLEM STATEMENT

We consider the class of underactuated mechanical systems such that the n–dimensional configuration space $Q = Q_1 \times Q_2$ is the Cartesian product of two differentiable manifolds, Q_1 on which forces are applied, and Q_2 on which dynamics evolves freely. Let $\dim Q_1 = r$ and (q^a), $a = 1, \ldots, r$ be local coordinates on Q_1, and (q^μ), $\mu = r+1, \ldots, n$ be local coordinates on Q_2. We denote by $q^A := (q^a, q^\mu)$, with $a = 1, \ldots, r$ and $\mu = r+1, \ldots, n$,

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the corresponding local coordinates on Q, with, obviously, $A = 1, \ldots, n$. [1]

The mechanical system is described by a Lagrangian function $L : TQ := TQ_1 \times TQ_2 \to \mathbb{R}$. Since we supposed to apply external (control) forces only to Q_1, Euler–Lagrange equations reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = u^a \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) - \frac{\partial L}{\partial q^\mu} = 0 \tag{2}$$

with $a = 1, \ldots, r$ and $\mu = r+1, \ldots, n$, and where u^a, $a = 1, \ldots, r$, are the external forces or control inputs.

Given initial and final conditions $(q^A(t_0), \dot{q}^A(t_0))$ and $(q^A(t_f), \dot{q}^A(t_f))$, our goal is to provide a trajectory $(q^A(t), u^a(t))$ of the configuration variables and control inputs which satisfies (1) and (2) by minimizing the cost functional

$$\mathcal{A}(q(\cdot), u(\cdot)) = \int_0^{t_f} C\left(q^a(t), q^\mu(t), \dot{q}^a(t), \dot{q}^\mu(t), u^a(t)\right) \, dt \, . \tag{3}$$

where $C(\cdot)$ is the cost function.

3. VARIATIONAL CONSTRAINED SYSTEMS PROBLEM, OPTIMAL CONTROL AND MOTION PLANNING

According to Bloch and Crouch (1994), there are two equivalent methods to solve an optimal problem for a constrained mechanical system. The first one is the *Lagrangian multipliers method* , and the second one, which we will focus on, is the so–called *variational constrained system problem*.

We adopt the latter geometric approach since it allows to intrinsically consider constraints in the problem. More precisely, on the one hand it allows treating intrinsically constraints on accelerations, which are otherwise difficult to investigate with the standard methods, and, on the other hand, detecting the preservation of fundamental geometric objects (such as a symplectic two form and a suitable "energy", see below for details).

As outlined in the previous Section, the solution of the optimal control problem of finding a pair $\left(q^A(t), u^a(t)\right)$ $t \in [t_0, t_f]$ satisfying equations (1) and (2), with initial $(q^A(t_0), \dot{q}^A(t_0))$ and final $(q^A(t_f), \dot{q}^A(t_f))$ conditions is given by the minimization of the cost functional (3). The minimization of (3) is equivalent to minimize the cost function:

$$\tilde{\mathcal{A}}(q(\cdot)) = \int_0^{t_f} \tilde{L}(q^a(t), q^\mu(t), \dot{q}^a(t), \dot{q}^\mu(t), \ddot{q}^a(t), \ddot{q}^\mu(t))dt \tag{4}$$

subject to the constraints

$$\Phi^\mu(q^a(t), q^\mu(t), \dot{q}^a(t), \dot{q}^\mu(t), \ddot{q}^a(t)) := \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) - \frac{\partial L}{\partial q^\mu} = 0 \tag{5}$$

and to the boundary conditions. The function $\tilde{L} : T^2Q \to \mathbb{R}$ is defined on the second tangent space T^2Q by

$$\begin{aligned}&\tilde{L}(q^a(t), q^\mu(t), \dot{q}^a(t), \dot{q}^\mu(t), \ddot{q}^a(t), \ddot{q}^\mu(t)) := \\ &C\left(q^a(t), q^\mu(t), \dot{q}^a(t), \dot{q}^\mu(t), \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a}\right) .\end{aligned} \tag{6}$$

Observe that the cost functional $\tilde{\mathcal{A}}$ is independent of the controls $u(\cdot)$, then the minimization will not give the controls, but the optimal trajectories. The evaluation of equations (1) along the optimal trajectories will provide the corresponding controls.

According to the theory developed by Colombo (2010) the dynamics of the higher–order constrained variational problem is determined by a pre–symplectic Hamiltonian system on a suitable fiber bundle W_0 over TQ. In the following we recall the basic constructions of the fundamental geometric tools and of the equations of motion.

Let $\mathcal{M} \subset T^2Q$ be the submanifold given by the regular values of the constrained function Φ^μ defined by equations (5). If equations (2) can be written in normal form, that is if the matrix $(W_{\mu\nu})$, $r+1 \le \mu, \nu \le n$, with coefficients given by

$$W_{\mu\nu} := \frac{\partial^2 L}{\partial \dot{q}^\mu \partial \dot{q}^\nu}$$

is not singular, [2] then

$$\ddot{q}^\mu = W^{\mu\nu} F_\nu(q^A, \dot{q}^A, \ddot{q}^a) =: G^\mu(q^A, \dot{q}^A, \ddot{q}^a) \, , \tag{7}$$

where $(W^{\mu\nu})$ denotes the inverse of the matrix $(W_{\mu\nu})$ and

$$F_\nu(q^A, \dot{q}^A, \ddot{q}^a) = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^\nu}\ddot{q}^a + \frac{\partial^2 L}{\partial q^A \partial \dot{q}^\nu}\dot{q}^A - \frac{\partial L}{\partial q^\nu} \, .$$

Therefore $(q^A, \dot{q}^A, \ddot{q}^a)$, $A = 1, \ldots, n$ and $a = 1, \ldots, r$, defines local coordinates on $\mathcal{M}$. We observe the non singularity of the matrix $W_{\mu\nu}$ is guaranteed, for example, if the Lagrangian is of the mechanical type, i.e. kinetic minus potential energy.

The behaviour of an underactuated system is thus described on a submanifold $\mathcal{M}$ of the second tangent space T^2Q. If $\iota_{\mathcal{M}} : \mathcal{M} \to T^2Q$ denotes the canonical inclusion, we can define the restricted Lagrangian $\tilde{L}_{\mathcal{M}} := \tilde{L}|_{\mathcal{M}}$. Generalizing the classical Skinner–Rusk formalism (Skinner (1983)) to higher-order equations (see Colombo (2010)), as described in Figure 1, allows us to define the suitable spaces where studying our problem. Let $W_0 = T^*(TQ) \times_{TQ} \mathcal{M}$ be a fiber product over TQ, locally described by coordinates $(q^A, \dot{q}^A, p_A^0, p_A^1, \ddot{q}^a)$. The coordinates p_A^0 and p_A^1 are the conjugate momenta of q^A and $\dot{q}^A$, respectively. Precisely p_A^0 are the classical conjugate momenta, p_A^1 are conjugate momenta of the generalized velocities $\dot{q}^A$ and thus have the physical dimensions of a force.

Let $\Omega_{W_0} = \pi_1^*(\omega_{TQ})$ be the pull-back on W_0 of the standard 2–form ω_{TQ} of TQ and $H_{W_0}(\alpha_x, v_x) := \langle \alpha_x, \iota_{\mathcal{M}}(v_x) \rangle - \tilde{L}_{\mathcal{M}}(v_x)$ the Hamiltonian on W_0, where $x \in TQ$, $v_x \in$

[1] In general we will denote by lowercase latin letters apexes local coordinates on Q_1, by lowercase greek letters apexes local coordinates on Q_2, and by uppercase latin letter apexes local coordinates on Q.

[2] Observe that a similar condition, which gives the trasversality of the constraint, is needed also in the Lagrangian multiplier method.

$$\begin{tikzcd}
& W_0 = T^*(TQ)\times_{TQ}\mathcal{M} \arrow[dl, "\pi_{\mathcal{M}}"'] \arrow[dd, "\pi"] \arrow[dr, "\pi_{T^*(TQ)}"] & \\
\mathcal{M} \arrow[dr, "\tau_{\mathcal{M}}"'] & & T^*(TQ) \arrow[dl, "\pi_{T^*Q}"] \\
& TQ &
\end{tikzcd}$$

Fig. 1. Skinner-Rusk formalism

$\mathcal{M}_x = \tau_{\mathcal{M}}^{-1}(x)$, $\alpha_x \in T_x^*TQ$ and $\langle\cdot,\cdot\rangle$ denotes the standard pairing of forms with vectors.

We can better understand the previous constructions using local coordinates, the 2–form Ω_{W_0} reads

$$\Omega_{W_0} = dq^A \wedge dp_A^0 + d\dot{q}^A \wedge dp_A^1 \tag{8}$$

and the Hamiltonian is

$$H_{W_0} = p_A^0\dot{q}^A + p_a^1\ddot{q}^a + p_\mu^1 G^\mu(q^A,\dot{q}^A,\ddot{q}^a) - \tilde{L}_{\mathcal{M}}(q^A,\dot{q}^A,\ddot{q}^a) \tag{9}$$

The equations of motion of our constrained variational problem are Hamilton equations for H_{W_0}:

$$i_{X_{H_{W_0}}}\Omega_{W_0} = dH_{W_0}\,, \tag{10}$$

where $i_X\Omega$ denotes the contraction of the vector field X with the differential form Ω.

By construction, the 2–form Ω_{W_0} is a pre–symplectic 2–form, that is it is a closed, possibly degenerate, 2–form. This is easy to be verified in local coordinates, since the coordinates $\ddot{q}^a$ do not appear in the local representation (8) of Ω_{W_0}, thus its kernel is locally represented by

$$\ker\Omega_{W_0} = \mathrm{span}_{\mathbb{R}}\left(\frac{\partial}{\partial\ddot{q}^a}\right) \tag{11}$$

Following Gotay–Nester–Hinds's algorithm (see Gotay and Nester (1979)), we allow a primary constraint:

$$dH_{W_0}\left(\frac{\partial}{\partial\ddot{q}^a}\right) = 0 \tag{12}$$

that in local coordinates reads

$$\varphi_a^1 := \frac{\partial H_{W_0}}{\partial\ddot{q}^a} = p_a^1 + p_\mu^1\frac{\partial G^\mu}{\partial\ddot{q}^a} - \frac{\partial\tilde{L}_M}{\partial\ddot{q}^a} = 0\,. \tag{13}$$

The zero level set of the constraint φ_a^1 defines a $4n$–dimensional manifold W_1 equipped with local coordinates $(q^A,\dot{q}^A,\ddot{q}^a,p_A^0,p_\mu^1)$, $A=1,\ldots,n$, $a=1,\ldots,r$ and $\mu = r+1,\ldots,n$. Denoting by $\iota_{W_1} : W_1 \longrightarrow W_0$ the canonical inclusion of W_1 in W_0, under some mild condition, namely the matrix $(\mathcal{R}_{ab})$, with coefficients given by

$$\mathcal{R}_{ab} = \frac{\partial^2\tilde{L}_{\mathcal{M}}}{\partial\ddot{q}^b\partial\ddot{q}^a} - p_\mu^1\frac{\partial^2 G^\mu}{\partial\ddot{q}^b\partial\ddot{q}^a} \tag{14}$$

being not singular, the manifold (W_1,Ω_{W_1}) is a symplectic manifold, that is a manifold endowed with a closed and non–degenerate 2–form, where $\Omega_{W_1} := \iota_{W_1}^*\Omega_{W_0}$ is the pullback of the 2–form Ω_{W_0} to W_1.

We now compute the *Hamilton equation* (10) in local coordinates. Let

$$X = X^{q^A}\frac{\partial}{\partial q^A} + X^{\dot{q}^A}\frac{\partial}{\partial\dot{q}^A} + X^{\ddot{q}^a}\frac{\partial}{\partial\ddot{q}^a} + X^{p_A^0}\frac{\partial}{\partial p_A^0} + X^{p_A^1}\frac{\partial}{\partial p_A^1} \tag{15}$$

be the generic vector field on W_0. We contract X with the pre–symplectic form Ω_{W_0}

$$i_X\Omega_{W_0} = X^{q^A}dp_A^0 + X^{\dot{q}^A}dp_A^1 - X^{p_A^0}dq^A - X^{p_A^1}d\dot{q}^A \tag{16}$$

and equating term by term the righthand side of (16) with the differential of H_{W_0}, we obtain the coefficients of the Hamiltonian vector field $X_{H_{W_0}}$:

$$X^{q^A} = \dot{q}^A, \quad X^{\dot{q}^A} = G^\mu + \ddot{q}^a,$$
$$X^{p_A^0} = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial q^A} - p_\mu^1\frac{\partial G^\mu}{\partial q^A}, \quad X^{p_A^1} = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^A} - p_A^0 - p_\mu^1\frac{\partial G^\mu}{\partial\dot{q}^A}$$

Hamilton equation (10) in local coordinates reads:

$$\frac{dq^A}{dt} = \dot{q}^A, \qquad \frac{d^2q^a}{dt^2} = \ddot{q}^a \tag{17}$$

$$\frac{d^2q^\mu}{dt^2} = G^\mu\left(q^A,\frac{dq^A}{dt},\frac{d^2q^a}{dt^2}\right) \tag{18}$$

$$\frac{dp_A^0}{dt} = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial q^A} - p_\mu^1\frac{\partial G^\mu}{\partial q^A} \tag{19}$$

$$\frac{dp_A^1}{dt} = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^A} - p_A^0 - p_\mu^1\frac{\partial G^\mu}{\partial\dot{q}^A} \tag{20}$$

$$p_a^1 = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\ddot{q}^a} - p_\mu^1\frac{\partial G^\mu}{\partial\ddot{q}^a} \tag{21}$$

Equation (21) is a condition on the vanishing of the coefficient of the differential of $\ddot{q}^a$, it defines the primary constraint φ_a^1 and then the symplectic manifold W_1. Combining equations (20) and (21) we obtain an evolution equation for p_a^1:

$$\frac{d}{dt}p_a^1 = \frac{d}{dt}\left(\frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\ddot{q}^a} - p_\mu^1\frac{\partial G^\mu}{\partial\ddot{q}^a}\right) = -p_a^0 - p_\mu^1\frac{\partial G^\mu}{\partial\dot{q}^a} + \frac{\partial\tilde{L}_{\mathcal{M}}}{d\dot{q}^a}\,.$$

Differentiating with respect to time and substituting the evolution equation (19) of p_a^0 we obtain

$$\frac{d^2}{dt^2}\left(\frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\ddot{q}^a} - p_\mu^1\frac{\partial G^\mu}{\partial\ddot{q}^a}\right) + \frac{d}{dt}\left(p_\mu^1\frac{\partial G^\mu}{\partial\dot{q}^a} - \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^a}\right) + \\ + \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial q^a} - p_\mu^1\frac{\partial G^\mu}{\partial q^a} = 0\,. \tag{22}$$

The same procedure for p_μ^1 gives

$$\frac{d^2p_\mu^1}{dt^2} = \frac{d}{dt}\left(\frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^\mu} - p_\nu^1\frac{\partial G^\nu}{\partial\dot{q}^\mu}\right) + p_\nu^1\frac{\partial G^\nu}{\partial q^\mu} - \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial q^\mu}\,. \tag{23}$$

Remark 1. We observe that solving equations (22) and (23) allows to find p_μ^0 and p_a^0 by equation (19). More precisely by (19) one gets

$$p_\mu^0 = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^\mu} - p_\nu^1\frac{\partial G^\nu}{\partial\dot{q}^\mu} - \frac{dp_\mu^1}{dt}\,,$$

and by (19) and using the primary constraint φ_a^1 we end up with:

$$p_a^0 = \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\dot{q}^a} - p_\nu^1\frac{\partial G^\nu}{\partial\dot{q}^a} - \frac{d}{dt}\left(p_\nu^1\frac{\partial G^\nu}{\partial\ddot{q}^a} - \frac{\partial\tilde{L}_{\mathcal{M}}}{\partial\ddot{q}^q}\right)\,.$$

Therefore the solutions of (the second of) equations (17), (22) and (23) are sufficient to determine $q^A(t)$ without explicitly computing $p^0_A(t)$.

Under the same assumption that guarantees the symplecticity of the manifold W_1, equation (23) can be posed in normal form, then the interesting equations of motion read

$$\begin{aligned} &\frac{d^4 q^a}{dt^4} = \Gamma^a\left(q^A, \dot{q}^A, \ddot{q}^a, \dddot{q}^a, p^1_\mu, \dot{p}^1_\mu\right) \\ &\frac{d^2 q^\mu}{dt^2} = G^\mu\left(q^A, \dot{q}^A \ddot{q}^a\right), \quad \frac{d^2 p^1_\mu}{dt^2} = \frac{d}{dt}\left(\frac{\partial \tilde{L}_{\mathcal{M}}}{\partial \dot{q}^\mu} - p^1_\nu \frac{\partial G^\nu}{\partial \dot{q}^\mu}\right) \end{aligned} \tag{24}$$

where the function Γ^a is [3]

$$\begin{aligned} \Gamma^a\left(q^A, \dot{q}^A, \ddot{q}^a, \dddot{q}^a, p^1_\mu, \dot{p}^1_\mu\right) := \mathcal{R}^{ab}\Big[&\mathcal{H}_b + \frac{d}{dt}\mathcal{F}_b - \frac{d}{dt}\mathcal{L}_b \\ &- \dddot{q}^c \frac{d}{dt}\mathcal{R}_{bc}\Big] \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_a &= \frac{\partial \tilde{L}_{\mathcal{M}}}{\partial \dot{q}^a} - p^1_\mu \frac{\partial G^\mu}{\partial \dot{q}^a}, \quad \mathcal{H}_a = p^1_\mu \frac{\partial G^\mu}{\partial q^a} - \frac{\partial \tilde{L}_{\mathcal{M}}}{\partial q^a} \\ \mathcal{L}_a &= \frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial q^A \partial \ddot{q}^a}\dot{q}^A + \frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \dot{q}^b \partial \ddot{q}^a}\ddot{q}^b + \frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \dot{q}^\beta \partial \ddot{q}^a} G^\beta - \dot{p}^1_\mu \frac{\partial G^\mu}{\partial \ddot{q}^a} + \\ &- p^1_\mu\left(\frac{\partial^2 G^\mu}{\partial q^A \partial \ddot{q}^a}\dot{q}^A + \frac{\partial^2 G^\mu}{\partial \dot{q}^b \partial \ddot{q}^a}\ddot{q}^a + \frac{\partial^2 G^\mu}{\partial q^\beta \partial \ddot{q}^a}\ddot{q}^\beta\right) \end{aligned} \tag{25}$$

Fact 2. As previously mentioned, the flow of equations (24) allows to reconstruct the momenta p^0_A, thus, together with the constraint equation (21), the flow of the Hamiltonian vector field $X_{H_{W_1}}$. This geometric fact is worth to be stressed, since it yields two conserved quantities along the flow of $X_{H_{W_1}}$: the Hamiltonian H_{W_1} and the symplectic form Ω_{W_1}. This aspect is extremely important from the numerical analysis viewpoint and will be deeply investigated in a future work.

Remark 3. We included and developed the explicit expressions of the formulae involved, since in practical examples all the computations can be implemented with a symbolic computational tools, such as Mathematica© or Python-SymPy.

4. IMPLEMENTATION

Our goal to find an optimal trajectory from an initial to a final configuration is accomplished by solving the initial values problem given by equations (24), and once we compute an optimal trajectory, we can calculate the related controls u^a, $a = 1, \ldots, r$, needed to drive the robot from a starting position to the target, by substituting the optimal curve in (1) and solving it with respect to the controls.

The method described in Section 3 solves an initial values problem, while we aim to solve a boundary values problem in which the initial and final configurations of the system are given. Indeed, to solve equations (24) we have to assign all the initial values, in particular, we have to assign the initial values of the p^1's and of their derivatives, which is a practical absurd. In practice, to solve the problem numerical methods are implemented to find the "optimal" values of p^1_a and $\dot{p}^1_a$ to solve our two-point problem. To do this, we implement the following 3–steps procedure:

- **Numerical integration.** We numerically integrate equations (24). possibly exploiting the geometric properties of the method.
- **Minimization.** After the integration of the equations of motion, we minimize the difference between the computed final configuration and the prescribed one. We will look for such a minimum leaving as free parameters the initial values of the p^1_μ's and of the $\dot{p}^1_\mu$'s of the initial value problem. To optimize our search, we will search for a minimum along a grid of p^1_μ's and $\dot{p}^1_\mu$'s.
- **Optimal control.** Once a trajectory $(q^A(t), \dot{q}^A(t))$ is computed, we will compute the corresponding controls $u^a(t)$ by equation (1).

To better understand the procedure and to present a way to implement it, we now provide the pseudo-code:

Function errorFun(par, $q^a_0, q^\mu_0, q^a_f, q^\mu_f$, time)
begin
    $q_0 = [q^a_0, 0, 0, 0, q^\mu_0, 0, par[0], par[1]]$
    $q_{curr}(t) =$ odeSolv(dynEvo, q_0, time)
    $q^a_{curr} = q_{curr}[0](t_f)$
    $q^\mu_{curr} = q_{curr}[4](t_f)$
    return[$q^a_0 - q^a_{curr}, q^\mu_0 - q^\mu_{curr}$]
end

Program cartPole
begin
    tMax= 1
    t= [0, 0.1, ..., t_f]
    optPar= optim(errorFun(par, $q^a_0, q^\mu_0, q^a_f, q^\mu_f$, t)
    $qInit_{opt} = [q^a_0, 0, 0, 0, q^\mu_0, 0, par[0], par[1]]$
    $q_{opt} =$ odeInt(dynEvo, $qInit_{opt}$, time)
    $u^a = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a_{opt}}\right) - \frac{\partial L}{\partial q^a}$
end

where function `dynEvo` $(q^a, \dot{q^a}, \ddot{q^a}, \dddot{q^a}, q^\mu, \dot{q^\mu}, p^1_\mu, \dot{p}^1_\mu)$ implements the system dynamic evolution as solutions of equations (24).

5. THE KINODYNAMIC MOTION PLANNING AND RELATED RESULTS

5.1 The cart-pole example

As an applicative–example illustrating our approach, we study the classical system of the cart–pole: a cart with an inverted pendulum on it, on which we force an external constraint: while the cart moves, the pendulum has to avoid an obstacle (a fixed point at a certain height). As a first approach we want planning an optimal trajectory (from the point of view of the cost $\mathcal{A}(x(\cdot), \theta(\cdot), u(\cdot)) = \frac{1}{2}\int_0^{t_f} u^2 dt$) $(x(t), \theta(t), u(t))$ of the configuration variables and of the controls that starting from a given initial con-

[3] Recall that $(\mathcal{R}^{ab})$ is the inverse matrix of the matrix $(\mathcal{R}_{ab})$ defined in (14).

figuration $(x(0), \theta(0), \dot{x}(0), \dot{\theta}(0))$, avoids the obstacle and stops at a prefixed final position $(x(t_f), \theta(t_f), \dot{x}(t_f), \dot{\theta}(t_f))$.

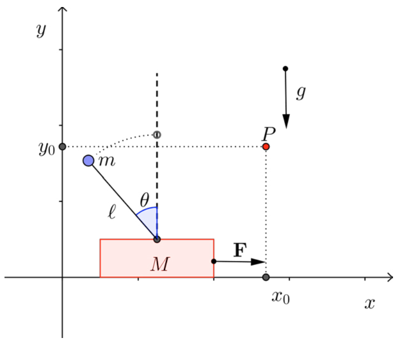

Fig. 2. The cart-pole example

The configuration space of the systems is $Q = \mathbb{R} \times \mathbb{S}^1$ equipped with local coordinates (x, θ), where x identifies the position of the center of mass of the cart and θ is the pole angle with respect to the vertical direction. The phase space is TQ with local coordinates $(x, \theta, \dot{x}, \dot{\theta})$. The Lagrangian of the system is :

$$L(x, \theta, \dot{x}, \dot{\theta}) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2) - mg\ell\cos\theta \tag{26}$$

where M is the mass of the cart, m and ℓ are the mass and the length of the pendulum, respectively, and g is the gravity acceleration constant.

The system is subjected to a control force $\boldsymbol{F} = (u, 0)$ along the x–axis and the degree of freedom defined by θ is not actuated. The equations of motion of the controlled system are then

$$(M + m)\ddot{x} - m\ell\dot{\theta}^2\sin\theta + m\ell\ddot{\theta}\cos\theta = u,$$
$$\ddot{x}\cos\theta + \ell\ddot{\theta} - g\sin\theta = 0\,.$$

From the second equation we obtain

$$G^\theta(x, \theta, \dot{x}, \dot{\theta}, \ddot{x}) = \frac{g\sin\theta - \ddot{x}\cos\theta}{\ell} \tag{27}$$

and then the constrained Lagrangian $\tilde{L}\,|_{\mathcal{M}}$ is

$$\tilde{L}\,|_{\mathcal{M}}\,(x, \theta, \dot{x}, \dot{\theta}, \ddot{x}) = \frac{1}{2}[(M + m)\ddot{x} - m\ell\dot{\theta}^2\sin\theta + mg\cos\theta\sin\theta - m\ddot{x}\cos^2\theta]^2 \tag{28}$$

where $\mathcal{M} = \left\{(x, \theta, \dot{x}, \dot{\theta}, \ddot{x}, \ddot{\theta}) \in T^2Q \mid \ddot{\theta} = G^\theta(x, \theta, \dot{x}, \dot{\theta}, \ddot{x})\right\}$ is the constraint manifold.

The pre–symplectic 2–form Ω_{W_0} and the Hamiltonian H_{W_0} are, respectively

$$\Omega_{W_0} = dx \wedge dp^0_x + d\theta \wedge dp^0_\theta + d\dot{x} \wedge dp^1_x + \mathrm{d}\dot{\theta} \wedge dp^1_\theta$$

$$H_{W_0} = p^0_x\dot{x} + p^0_\theta\dot{\theta} + p^1_x\ddot{x} + p^1_\theta G^\theta - \frac{1}{2}\left[(M + m)\ddot{x} - m\ell\dot{\theta}^2\sin\theta + m\ell\ddot{\theta}\cos\theta\right]^2$$

The primary constraint is

$$\varphi^1_x = p^1_\theta + p^1_\theta\frac{\partial G^\theta}{\partial \ddot{x}} - \frac{\partial \tilde{L}_{\mathcal{M}}}{\partial \ddot{x}} = 0\,.$$

Symbol	Description	Value
M	mass of the car	$1\,Kg$
m	mass of the pole	$0.01\,Kg$
ℓ	length of the pole	$1\,m$
g	gravity acceleration	$9.81\,ms^{-2}$
q_0	initial configuration	$(0; 0)$
q_f	final configuration	$(1; -0.5)$
P	position of the obstacle	$(1; 0.8)$
t_0	initial time	$0\,s$
t_f	final time	$1\,s$

Table 1. Mechanical properties of the cart-pole example using the International system of unit

The submanifold W_1 of W_0, locally defined by φ^1_x, equipped with the restriction Ω_{W_1} of Ω_{W_0} is a symplectic manifold, indeed

$$\mathcal{R} = M + m\sin^2\theta \neq 0\,. \tag{29}$$

Thus Gotay–Nester–Hinds's algorithm stabilizes at the first step, and there exists a unique vector field X_{W_1} on W_1 that satisfies $i_{X_{W_1}}\Omega_{W_1} = dH_{W_1}$, where H_{W_1} denotes the restriction to W_1 of the Hamiltonian H_{W_0}. As a consequence there exists a unique control, given by equation (1), which minimizes the cost functional $\mathcal{A}$.[4]

We recall that the developed theory guarantees the conservation along the flow of the Hamiltonian vector field X_{W_1} of the symplectic form Ω_{W_1} and of the Hamiltonian H_{W_1}. These two geometrical invariants will play a crucial role in the numerical simulations.

The equations of motions (24) for the controlled cart–pole are

$$\ddot{\theta}(t) = G^\theta(x, \theta, \dot{x}, \dot{\theta}, \ddot{x})$$
$$\frac{d^2p^1_\theta}{dt^2} = \frac{d}{dt}\frac{\partial \tilde{L}_{\mathcal{M}}}{\partial \dot{\theta}} - \frac{\partial \tilde{L}_{\mathcal{M}}}{\partial \theta} - p^1_\theta\frac{\partial G^\theta}{\partial \theta}$$
$$\frac{d^4x}{dt^4} = -\ddot{x}\frac{d}{dt}\mathcal{R} - \frac{d}{dt}\left(\frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \theta\,\partial \ddot{x}}\dot{\theta} + \frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \dot{x}\,\partial \ddot{x}}\ddot{x} + \frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \dot{\theta}\,\partial \ddot{x}}G^\theta - \dot{p}^1_\theta\frac{\partial G^\theta}{\partial \ddot{x}}\right)$$

with $\mathcal{R}$ defined in (29).

5.2 *Optimal solution for kinodynamic motion planning*

As a first partial answer to the kinodynamic problem, we generate an optimal trajectory for a two–point values problem in which the cart–pole stops just under the obstacle. More precisely, the cart–pole starts from an initial configuration $q_0 = (x_0, \theta_0)$ and stops at a final configuration $q_f = (x_f, \theta_f)$ avoiding the obstacle placed at point P of coordinates (x_f, y_P), with $y_P < \ell$ and with θ_f chosen so that $\ell\cos\theta_f < y_P$.

The numerical results of the implementation in Python of the method exposed in Section 4, using the values of the parameters outlined in Table 1, are shown in Figures 3

[4] We observe that the strict convexity of the cost functional $\mathcal{A}$ ensures by itself the uniqueness of the solutions of the optimal control problem. Nevertheless the uniqueness is not guaranteed passing to the Hamiltonian side, unless one con fully apply Gotay–Nester–Hinds's algorithm.

and 4.[5] The first plot in Figure 3 shows the evolution of the center of mass of the cart: we can observe that the cart goes ahead until a maximum near $0.9\,s$ outlined by the dashed vertical line, and then it goes back to the prescribed final position. The second plot describes the evolution of the pole's angle θ. The pendulum rotates anticlockwise (over $1\,rad$) and then, with a (small) delay with respect to the x maximum, stops increasing and rotates clockwise to the final position, without touching the obstacle (see Figure 4, that illustrates the time evolution of the height inverted pendulum-blue line-compared with the height of the (fixed) obstacle-horizontal red line). The second plot in Figure 4 shows the evolution of the optimal control: at the beginning the applied control force is positive to move the cart toward the positive direction of the x axe, then, after $0.6\,s$ (and before $0.8\,s$) it changes sign and first slows down the cart–pole, then reverses the direction of the motion to reach the final position.

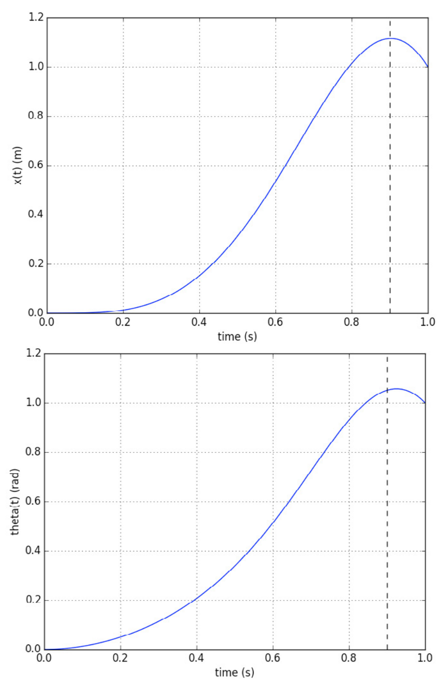

Fig. 3. Time evolution of $x(t)$ and the control $u(t)$.

6. CONCLUSIONS AND FUTURE PERSPECTIVES

In this work we discuss the problem of planning a trajectory for underactuated mechanical systems, and propose an algorithm to solve it. We apply the method to solve the kinodynamic motion planning for the cart-pole system. In a future work, we plan to investigate the numerical aspects of the problem exploiting the geometric invariants to explore other approaches to avoid external obstacles, and to study the controllability of the system.

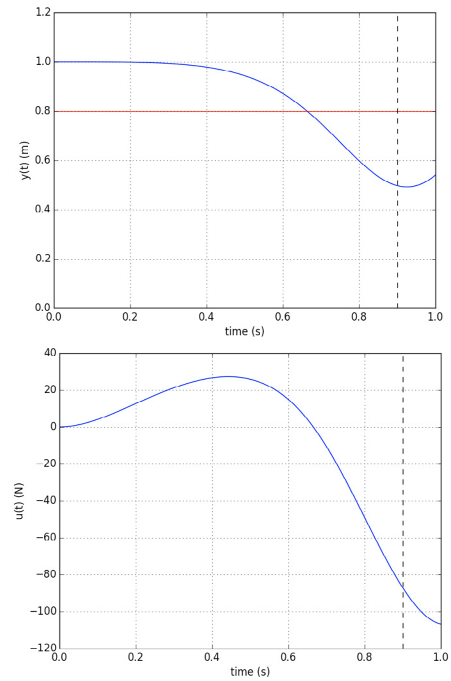

Fig. 4. Time evolution of $y(t)$ and $\theta(t)$

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[5] The vertical lines identify the maximum of the evolution of $x(t)$.