# Optimal execution strategy in liquidity framework under exponential temporary market impact

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**Abstract**

In the present work we compute the optimal liquidation strategy for an investor who intends to entirely extinguish his position in an illiquid asset so as to minimize a criterion involving mean and variance of the strategies implementation shortfall. The market impact due to illiquidity is modeled by splitting it into two different component, namely the permanent market impact, which is assumed to be linear in the rate of trading, and the temporary market impact, which follows an exponential-type function.

**Keywords**— Stochastic mean-variance optimization, non-liquid markets, non linear market impact factors, Lambert function.

## 1 Introduction

We consider an optimal execution problem in a non-liquid market for a risky asset, hence allowing for an agent to influence the asset price process by participating in the market. The price variation due to the agent's actions is called market impact, and, usually, when large trades are executed, price moves in the trader's unfavorable direction, proportionally to sales volume. Therefore a common practice is to divide a large trade into many smaller ones. The main aim of this work is indeed to find the best strategy for a big sale, that is how to split it into smaller orders so as minimize

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the corresponding implementation cost or a cost criterion stated a priori, which may also involve risk parameters.

We solve latter problem for a model characterized by a market impact composed by two factors: the *permanent market impact* and the *temporary market impact*. Let us recall that the permanent impact refers to the long lasting modifications of prices under the action of a given sell order, otherwise such effects are considered as temporary market impacts.

Taking into account both analytical and empirical research, see, e.g., [4], we shall consider a model given by a linear permanent market impact plus an exponential-type temporary market impact which is characterized by properly chosen parameters whose meaning will be later clarified, see Section 3.

The quest for optimal selling strategies in liquidity frameworks has become a central topic in financial mathematics during recent years. In [2] and [3], Almgren and Chriss consider an asset price process following an arithmetic random walk, with constant volatility over the strategy's lifetime, in the discrete-time, and an arithmetic Brownian motion (ABM), in continuous-time. In both cases, the optimal trading strategy is, by definition, the one which minimizes a linear combination of the expected cost and the variance of the cost of each strategy. Moreover in [2] and in [3], the market impact is assumed to be linear in block trades, while in [1], it is modeled by a nonlinear function of power-law type. It is worth to mention that the latter results are based on the assumptions that the drift, the volatility of the price process and the liquidity parameters are constant over all the liquidation interval. Improvements can be achieved by modeling such parameters by stochastic processes, see, e.g., [11], where the drift of the price process is forced equal to zero, or [5], where the drift is a stochastic process. In particular in [6], although the price process is still assumed to be an ABM, the authors replace the variance approach by considering both Value at Risk (VaR) and Expected Shortfall (ES) as risk parameters for which they exhibit related optimal execution strategies, hence obtaining more realistic results.

In [13], resp. in [7], the same problem outlined before is studied under the assumption that the unaffected price process is modeled as a geometric Brownian motion, resp. as a displaced diffusion process, when both components of the market impact are still assumed to be linear the in trading speed. Under latter hypothesis in [17] the authors provide a robustness property for the optimal strategies. Indeed, under a specified cost criterion, the form of the solution is independent of the unaffected price process as long as it is a square integrable martingale.

It is worth to mention that the market impact has emerged as a fundamental topic in modern electronic market. Indeed, the use of computer algorithms, and related high-frequency trading strategies, have changed a lot how transactions are currently executed. In particular the execution's speed has been modified with several implications concerning the volume size of trades. Latter scenario has been studied, e.g., in [8] and in [9], from the *limit order book market* point of view.

The remainder of this paper is organized as follows. In Section 2, following [3] and [1], we state the mathematical setting of the optimal trading problem we want to study. In Section 3, the originality of our approach is outlined and the optimal execution strategy is computed in terms of the Lambert W function, when the temporary market impact is modeled by an exponential-type function. Lastly, in Appendix A, the main characteristics and properties of the Lambert W function are summarized.

## 2 The model framework

The model is built following the framework given in [3]. A trader holds $X \in \mathbb{R}^+$ shares of a non-liquid asset and he aims at completely liquidate his position within a fixed deadline (*fixed horizon*), $T > 0$. We divide the time interval $[0, T]$ into a finite number $N \in \mathbb{N}^+$ of subintervals of equal length $\tau := \frac{T}{N}$. Then at every discrete time $t_{n-1} = (n-1)\tau$ the trader chooses how many shares $y_n$ to sell in the subsequent subinterval $(t_{n-1}, t_n]$. The $N-$tuple $(y_1, \ldots, y_N)$, which is called *trading list*, takes into account all the sold quantities. Notice that, since the trader sells his entire position over the whole time interval $[0, T]$, a trading list has to satisfy the liquidation constraint $\sum_{n=1}^{N} y_n = X$. By knowing the trading list, we can compute the execution strategy $x$, defined as a $(N+1)$-tuple $x = (X, x_1, \ldots, x_N)$ where $x_n$ stands for the volume of shares held by the trader at time $t_n$, $n = 0, \ldots, N$. Since the sold quantity in the time interval $(t_{n-1}, t_n]$ matches the difference between the quantities held at the endpoints, i.e. $y_n = x_{n-1} - x_n$, then, exploiting the liquidation constraint, we have that

$$x_n = X - \sum_{k=1}^{n} y_k = \sum_{k=n+1}^{N} y_k \,,$$

which implies $x_N = 0$.

In order to model the illiquid features of the asset we split the impact due to the acting of the trader in the market, that is called market impact, into two parts, namely the permanent, resp. the temporary, component. In particular, while the temporary market impact refers to the asset price modification in the $k$-th time interval due to the sale occurred in the immediately preceding time interval, the permanent market impact takes into account the price variation that persists throughout the remaining trading time.

According to well established literature, see, e.g., [2], we assume that the unaffected price process $S$, that is the price per share of the asset which occurs in a *market impact-free world* or, similarly, the one we have if the trader does not participate in the market, follows an arithmetic random walk. It follows that, when the initial asset price is a known value $S_0$, the price per share at time $t_n$ is given by

$$S_n = S_{n-1} + \sigma\sqrt{\tau}\xi_n \, ,$$

where $\xi_1, \dots, \xi_N$ are independent and identically distributed random variables, having zero mean and unitary variance, $\sigma$ being the volatility of the asset process, which is assumed to be constant over the whole time interval $[0, T]$, as in the Black-Scholes model.

Nevertheless, the presence of liquidity effects implies that the trader does not receive the price $S_t$ per share. Actually, the value at which the asset will be sold may be rather different. Let us underline that the price that the trader actually receives on each trade per share is called actual price (process, since it depends on time), and it will be denoted by $\tilde{S}_t$. This latter *price process*, also depending on the unaffected price as well as on the behaviour of the trader in the market, can be defined in two steps:

*step 1*: First we consider the permanent market impact, $S_t$ being redefined as follows

$$S_n = S_{n-1} + \sigma\sqrt{\tau}\xi_n - \tau g(v_n) = S_0 + \sigma\sqrt{\tau}\sum_{k=1}^{n}\xi_k - \tau\sum_{k=1}^{n} g(v_k) \, , \tag{1}$$

where $v_n$ is the speed of selling, i.e. it indicates the rate $\frac{y_n}{\tau}$, while the function $g(v)$ models the permanent market impact.

*step 2*: Then we consider the temporary component of the market impact, which is modeled by the function $h(v)$, $\tilde{S}_n$ being defined by

$$\tilde{S}_n := S_{n-1} - h(v_n) \, , \tag{2}$$

for $n = 1, \dots, N$.

Exploiting (1) and (2), we can explicitly provide the difference between the two components of the market impact. In particular, at a fixed time $t_n > 0$, the actual price $\tilde{S}_n$ depends, through the temporary market impact $h$, only on the sale executed at this time, i.e. $y_n$, while, *vice versa*, it depends through the permanent market impact $g$ by all the previous sold quantities $y_1, \ldots, y_n$.

The *total capture*, indicated by $G(x)$ with respect to the chosen strategy $x$, is nothing but the total cash received over the strategy lifetime $[0, T]$, namely

$$
\begin{aligned}
G(x) &:= \sum_{n=1}^{N} y_n \tilde{S}_n \\
&= \sum_{n=1}^{N} y_n S_0 + \sum_{n=1}^{N} \sum_{k=1}^{n-1} \left[ (\sigma\sqrt{\tau}\xi_k - \tau g(v_k)) y_n \right] - \sum_{n=1}^{N} y_n h(v_n) \\
&= X S_0 + \sigma\sqrt{\tau} \sum_{n=1}^{N} x_n \xi_n - \tau \sum_{n=1}^{N} x_n g(v_n) - \tau \sum_{n=1}^{N} v_n h(v_n) \,,
\end{aligned}
$$

where the last equality follows from the relation between the trading strategy $x$ and the related sold quantities $y$, as stated above. The quantity $XS_0$ is the *market-to-market* value of the trader's initial position, hence the difference $C(x) := XS_0 - G(x)$ is the cost due to the illiquidity. Latter quantity is often called *implementation shortfall* and it represents the ex-post measure of transaction cost. By previous assumptions on $\xi_n$, the cost related to a trading strategy $x$, i.e. $C(x)$, becomes a random variable with mean

$$
\mathbb{E}[C(x)] = \tau \sum_{n=1}^{N} x_n g(v_n) + \tau \sum_{n=1}^{N} v_n h(v_n) \,, \tag{3}
$$

and variance

$$
\mathrm{Var}[C(x)] = \tau \sum_{n=1}^{N} \sigma^2 x_n^2 \,. \tag{4}
$$

In order to work in a continuous-time framework, we let the time step $\tau$ go to zero. Then a strategy is represented by a continuous function $x \colon [0, T] \to \mathbb{R}_0^+$ which satisfies the initial condition and the liquidation constraint if its boundary conditions are $x(0) = X$ and $x(T) = 0$. Moreover we assume that the sold size $y_n$ are such that $v_n \to v(k\tau)$ when $\tau \to 0$, with $v(t) = -\dot{x}(t)$. All such strategies are called admissible and the set of all the admissible strategies will be denoted by $\mathcal{A}$. In this case, the expected value and the variance of the implementation shortfall $C(x)$, i.e. equations (3) and (4), have the following finite limits:

$$
\mathbb{E}[C(x)] = \int_0^T x(t) g(v(t)) + v(t) h(v(t)) \, dt \,,
$$

and

$$\mathrm{Var}[C(x)] = \int_0^T \sigma^2 x(t)^2 \, dt \, .$$

In order to decide the optimal strategy within the set $\mathcal{A}$, we assume that the trader's goal is to find the strategy which minimizes the mean-variance of the cost functional $U$, defined as follows

$$U(x) := \mathbb{E}[C(x)] + \lambda \mathrm{Var}[C(x)] \, ,$$

where $\lambda$ is a positive constant. We would like to underline that the mean-variance cost criterion is one of the most popular tool used to compare different trading strategies. Indeed, it is equivalent at fixing the highest values of risk, equivalently of variance, the trader is willing to tolerate, say $V^*$, and then looking for the strategy that minimizes the expected cost, within all the admissible strategies with $variance \leq V^*$. It follows that the risk aversion of the trader can be efficiently modeled by the parameter $\lambda$.

In our case study, the trader's problem reads as follows.

**Problem 2.1** (Minimization Problem)**.** The objective of the trader is to find, among all the admissible strategies $\mathcal{A}$, which one minimizes the cost functional $U$, i.e.

$$x^* = \arg\min_{x \in \mathcal{A}} U(x) \tag{5}$$

where

$$U(x) = \int_0^T x(t) g(v(t)) + v(t) h(v(t)) + \lambda \sigma^2 x(t)^2 \, dt \, . \tag{6}$$

In order to find the optimal trading strategy $x^*$ we argue as in [1]. First of all, the integrand function in (6) reads as follows

$$F(x, v) = x g(v) + v h(v) + \lambda \sigma^2 x^2 \, .$$

Then, the Euler-Lagrange equation guarantees that the strategy $x^*$ in (5) has to satisfy

$$F_x(x, -\dot{x}) + \frac{d}{dt} F_v(x, -\dot{x}) = 0 \, ,$$

that is

$$0 = F_x(x, -\dot{x}) + \dot{x} F_{vx}(x, -\dot{x}) - \ddot{x} F_{vv}(x, -\dot{x}) \, .$$

Furthermore, this implies that

$$\frac{d}{dt}\left(F(x, -\dot{x}) + \dot{x} F_v(x, -\dot{x})\right) = \dot{x} \left[F_x(x, -\dot{x}) + \dot{x} F_{vx}(x, -\dot{x}) - \ddot{x} F_{vv}(x, -\dot{x})\right] = 0 \tag{7}$$

and hence, integrating both sides of eq. (7) from 0 to $T$, it follows that the optimal strategy makes the functional $F(x, -\dot{x}) + \dot{x} F_v(x, -\dot{x})$ constant. Straightforward computations give that

$$F(x, -\dot{x}) + \dot{x} F_v(x, -\dot{x}) = x(g(-\dot{x}) + \dot{x} g'(-\dot{x})) - P(-\dot{x}) \tag{8}$$

where the function $P$ in (8) is defined as

$$P(v) := v^2 h'(v) . \tag{9}$$

Then, if we denote by $v_0$ the speed at which $x(t)$ hits $x = 0$, it holds that

$$F(0, v_0) - v_0 F_v(0, v_0) = -P(v_0) ,$$

and therefore

$$P(-\dot{x}) - P(v_0) = x(g(-\dot{x}) + \dot{x} g'(-\dot{x})) + \lambda \sigma^2 x^2 . \tag{10}$$

*Remark* 2.2. In order to obtain explicit solutions, we assume that the permanent impact is linear in the trading rate $v$, that is $g(v) = \beta v$ with $\beta$ positive constant. Then, no matter the strategy the trader follows, we have

$$x(g(-\dot{x}) + \dot{x} g'(-\dot{x})) = x(-\beta \dot{x} + \beta \dot{x}) = 0 ,$$

and then the condition stated in eq. (10) reduces to

$$P(-\dot{x}) = \lambda \sigma^2 x^2 + P(v_0) .$$

By separation of variables, and assuming that $P^{-1}$ is well defined and $P^{-1}(\lambda \sigma^2 x^2 + P(v_0)) \neq 0$ for all $t \in [0, T]$, we obtain that the optimal strategy solves the following equation

$$\int_{x(t)}^{X} \frac{1}{P^{-1}(\lambda \sigma^2 x^2 + P(v_0))} \, dx = t . \tag{11}$$

# 3 Exponential market impact function

In what follows we still consider a linear permanent market impact $g(v) = \beta v$ with $\beta > 0$, and we specify the temporary market impact as an exponential function, namely we assume

$$h(v) := \begin{cases} \gamma e^{-\frac{\theta}{v}} & \text{for } v > 0, \\ 0 & \text{for } v = 0, \end{cases} \tag{12}$$

where $\gamma$ and $\theta$ are strictly positive constants. The reason why $h$ is defined only for positive value of the trading rate will be clarified later. Through the parameters $\gamma$ and $\theta$ we can control the shape of the temporary market impact. Notice that the function $h(v)$ is strictly increasing in its domain, it is convex on the set $\left[0, \frac{\theta}{2}\right]$ and concave for $v \geq \frac{\theta}{2}$. Experimental analysis confirms the concavity of the temporary impact function, see [4] and references therein. Nevertheless we choose to allow it to be convex for small values of $v$. In fact such values are difficult to estimate, since they correspond to small change in the price, moreover the system may be extremely fragile around a critical point.

Under previous assumptions, and without fixing a deadline $T$, we can explicitly compute the optimal trading strategy in the case when the set of admissible strategies is narrowed by considering only those of pure selling type. We recall that a trading strategy is called *pure selling strategy*, if its rate process is strictly positive, namely if the strategy itself is strictly decreasing. From now on, a strategy $x$ will be admissible if, besides satisfying the conditions mentioned above, it is of pure selling type.

**Theorem 3.1.** *Let us assume that no deadline is exogenously imposed on the sale. If the permanent market impact is linear in the trading rate and the temporary market impact is given as in* (12)*, the optimal solution among all the admissible pure selling strategies of the minimization Problem 2.1 is given by*

$$x^*(t) = \begin{cases} \dfrac{\exp\left\{W_{-1}\left(\dfrac{\kappa\left(\frac{\theta}{2}t + X(\ln[\kappa X] - 1)\right)}{e}\right) + 1\right\}}{\kappa} & \text{for } t < \frac{2}{\theta}X\ln\left[\frac{e}{\kappa X}\right] \\ 0 & \text{for } t \geq \frac{2}{\theta}X\ln\left[\frac{e}{\kappa X}\right] \end{cases} \tag{13}$$

*where $W$ is the Lambert $W$ function and $\kappa = \sqrt{\frac{\lambda\sigma^2}{\gamma\theta}}$, provided that*

$$\kappa X < 1\,. \tag{14}$$

*Proof.* According to the considerations outlined in Section 2, the optimal trading strategy we are looking for satisfies equation (11). Under the assumptions of the theorem, the function $P$, as defined in (9), becomes

$$P(v) = v^2 \frac{d}{dv} e^{-\gamma\frac{\theta}{v}} = \gamma\theta e^{-\frac{\theta}{v}}\,,$$

which has as inverse function $P^{-1}(v) = -\frac{\theta}{\ln[\frac{v}{\gamma\theta}]}$, which is only defined for $v > 0$. Therefore a necessary condition for the well-posedness of problem (11) is indeed to consider strategies with strictly positive rate process, which implies to consider only sell programs. Therefore problem (11) turns out to be

$$\frac{1}{\theta}\int_{x(t)}^{X} -\ln\left[\frac{\lambda\sigma^2 x^2+\gamma e^{-\frac{\theta}{v_0}}}{\gamma\theta}\right]dx = t\ . \tag{15}$$

If, as in this case, no time horizon is exogenously imposed then we obtain the longest possible liquidation time, denoted in the following by $T$, by setting $v_0 = 0$, and therefore the problem stated in (15) reduces to

$$\int_{x(t)}^{X} -\ln\left[\frac{\lambda\sigma^2 x^2}{\gamma\theta}\right]dx = \theta t\ . \tag{16}$$

In order to the problem be well-posed, the candidate solution has to satisfy the constraint $P^{-1}(\kappa^2 x(t)^2) \neq 0$ for all $t \in [0,T]$, i.e. $-\frac{1}{\ln[\kappa^2 x(t)^2]} \neq 0$ for all $t \in [0,T]$, then the optimal execution strategy must satisfy one, and only one, of the following conditions

$$\kappa x(t) < 1 \quad \forall t \in [0,T] \quad \text{or} \quad \kappa x(t) > 1 \quad \forall t \in [0,T]\ .$$

Since at the final time $T$ the strategy's value is $x(T) = 0$, i.e. $\kappa x(T) < 1$, the optimal solution $x^*$ can only meet the first constraint. Latter condition is verified since each admissible trading strategy $x \in \mathcal{A}$ is decreasing with $\kappa X < 1$ at the initial time as required by the theorem. It can be seen that this condition is a constraint on the model's parameters, indeed it reads as $\frac{\lambda\sigma^2 X^2}{\gamma\theta} < 1$. Equation (16) implies that the quantity

$$\int_{x(t)}^{X} -\ln\left[\frac{\lambda\sigma^2 x^2}{\gamma\theta}\right]dx = 2x(t)\Bigg(\ln\left[\frac{\sqrt{\lambda}\sigma x(t)}{\sqrt{\gamma\theta}}\right]-1\Bigg) - 2X\Bigg(\ln\left[\frac{\sqrt{\lambda}\sigma X}{\sqrt{\gamma\theta}}\right]-1\Bigg)$$

is equal to $\theta t$, and therefore the optimal strategy fulfills

$$x(t)(\ln[\kappa x(t)]-1) = \frac{\theta}{2}t + X(\ln[\kappa x(t)]-1)\ ,$$

that can be rewritten as

$$x(t)\ln\left[\frac{\kappa x(t)}{e}\right] = \frac{\theta}{2}t + X(\ln[\kappa X]-1)\ . \tag{17}$$

Equation (17) has two solutions for each $t < \frac{2}{\theta}X\ln\left[\frac{e}{\kappa X}\right]$. Nevertheless since we have assumed to perform a pure selling strategy and then $x$ is a continuous and decreasing function, there exists a unique trading strategy which satisfies equation (17), namely (13). Notice that the optimal strategy reaches $x^* = 0$ in a finite time $T = \frac{2}{\theta}X\ln\left[\frac{\sqrt{\gamma\theta}}{\sqrt{\lambda}\sigma X}e\right]$. $\square$

See Appendix A for further details on the Lambert W function.

*Remark* 3.2. We want to directly verify that the optimal execution strategy stated in Theorem 3.1 satisfies the initial condition $x(0) = X$. By definition the initial value of the optimal strategy is

$$x(0) = \frac{e^{W_{-1}\left(\frac{\kappa X \sigma}{e} \ln\left[\frac{\kappa X}{e}\right]\right)+1}}{\kappa} ,$$

and since $W_{-1}$ is upper bounded by $-1$, then the initial value $x(0)$ belongs to the interval $\left[0, \frac{1}{\kappa}\right]$. Moreover, by manipulating the previous equation, we have that $x(0)$ solves

$$\frac{\kappa x(0)}{e} = e^{W_{-1}\left(\frac{\kappa X}{e} \ln\left[\frac{\kappa X}{e}\right]\right)} .$$

Hence, taking the logarithm of both sides and using the definition of the Lambert W function, we obtain the equality

$$\frac{\kappa x(0)}{e} \ln\left[\frac{\kappa x(0)}{e}\right] = \frac{\kappa X}{e} \ln\left[\frac{\kappa X}{e}\right] ,$$

which is verified by $x(0) = X$. In fact this is the unique solution of the latter equation, since the function $\frac{\kappa y}{e} \ln\left[\frac{\kappa y}{e}\right]$ is strictly decreasing in the interval $\left[0, \frac{1}{\kappa}\right]$.

## 3.1 Evaluation of $W_{-1}$

Even if the Lambert W function can not be expressed in terms of elementary functions, we want to describe its behaviour in order to sketch the optimal trading strategy (13). Since the Lambert W function is defined by mean of an inverse relation, arbitrary-precision evaluations can be obtained by iterative root-finding methods. Given a value $z$, its corresponding value $w = W(z)$ satisfies $we^w = z$, that is the root of the function $f(w) = we^w - z$. Notice that since the Lamber W function $W(z)$ is bi-valued in $\left(-\frac{1}{e}, 0\right)$, we have to take into account that we will find two solutions: the one that is greater than $-1$ is the value of the so called principal branch $W_0(z)$, while the second real branch, the lower branch, is indeed $W_{-1}(z)$.

Several numerical methods for the root finding problem have been developed, which differ each other for complexity of implementation, conditions and rate of convergence. A natural choice in our setting, is to use the third-order Halley's method which starts with an initial guess $w_0$ for the root, and then performs the following iteration scheme

$$w_{n+1} = w_n - \frac{2f(w_n)f'(w_n)}{2(f'(w_n))^2 - f(w_n)f''(w_n)} = w_n - \frac{(w_n e^{w_n} - z)}{e^{w_n}(w_n + 1) - \frac{(w_n e^{w_n} - z)(w_n+2)}{2(w+1)}} ,$$

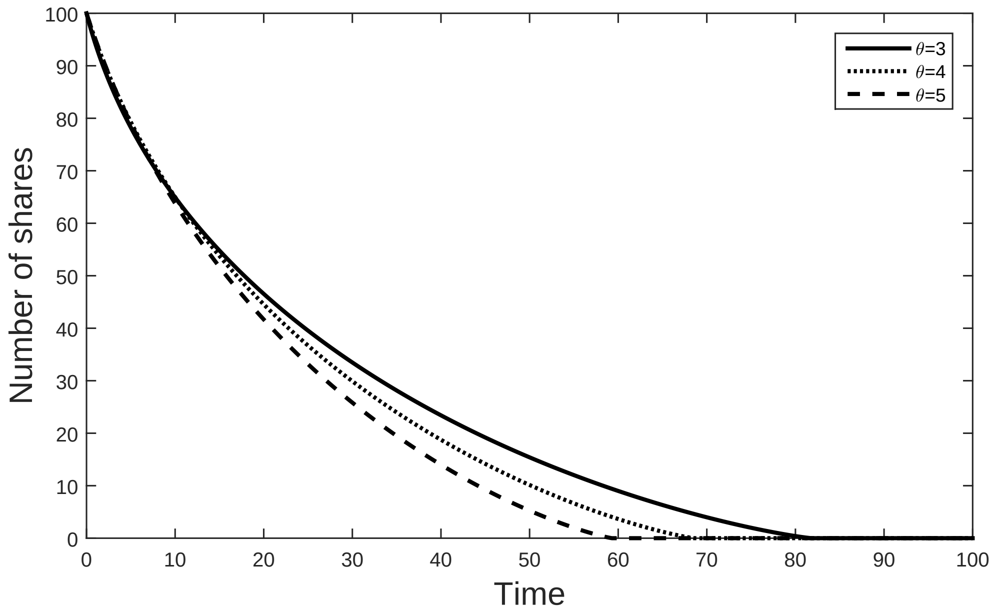


Figure 1: Optimal solution strategy

which converges to the desired value.

Figure 1 shows the behaviour of the optimal solution $x^*$, see (13), for different values of the parameter $\theta$, the other parameters being fixed as in Table 1. Notice that Theorem 3.1 applies, in fact the parameters always satisfy the condition (14), i.e. $\frac{\lambda\sigma^2}{\gamma\theta}X < 1$.

By the definition of function $h$ in (12), it can be seen that, fixed a sales volume, the effect on the price, due to the temporary market impact, is lower for a higher value of $\theta$. This means in particular that, when $\theta$ is higher the decrease in price is smaller and therefore the trader sells the illiquid asset faster.

# 4 Conclusion

We have found the explicit optimal execution strategy that minimize a criterion containing the expected cost and the variance of the implementation shortfall for trading a non-liquid asset when the market impact is model by an exponential-type function. In doing so, we extend the case studies by considering different nonlinear impact functions from the ones introduced in [1].

We have also provided the optimal execution strategy under reasonable assump-

Table 1: Parameters' value

| Parameter | Value |
|---|---|
| $X$ | 100 |
| $\lambda$ | 5 |
| $\sigma$ | 0.02 |
| $\gamma$ | 10.5 |

tions, in particular exploiting the Lambert W function. In particular, it is possible to numerical treat latter result in order to obtain the solution behavior as well as its intrinsic properities.

# A Lambert W function

In this Section, we recall the main characteristics and properties of the Lambert W function. The Lambert W function, also called the omega function, is defined to be the function satisfying $z = W(z)e^{W(z)}$, that is the inverse function of $f(w) = we^w$, which is not injective, hence the relation $W$ is multivariate. In particular, if $x$ is real $W(x)$ is double-valued on $\left(-\frac{1}{e}, 0\right)$. Then the Lambert W function has two real branches with a branching point located at $(-e^{-1}, -1)$. Indeed if we consider $W$ under the constraint $W \geq -1$ or $W \leq -1$, they are two well defined real valued functions. The branch satisfying $W \geq 1$ is called *principal branch* and it is denoted by $W_0$, or just $W$, if no ambiguity exists, while the branch satisfying $W \leq -1$, the *lower branch*, is denoted by $W_{-1}$. The Lambert W function has the special values $W(-e^{-1}) = -1$, $W(0) = 0$, and $W(1) \simeq 0.567143$, called the *omega constant*, that satisfies $\exp(-W(1)) = W(1)$, that is $\ln\left[\frac{1}{W(1)}\right] = W(1)$.

## A.1 Taylor series for $-\frac{1}{e} < z < 0$

The Lambert W function $W_{-1}(z)$ is (upper)-bounded and infinitely differentiable in $(-\frac{1}{e}, 0) \in \mathbb{R}$. By differentiating the defining expressions $z = W(z)e^{W(z)}$, it follows $1 = W'(z)e^{W(z)} + W(z)W'(z)e^{W(z)}$, and then the first derivative of $W$ turns out to be

$$W'(z) = \frac{1}{e^{W(z)}(1 + W(z))}$$

provided that $W(z) \neq -1$, i.e. $z \neq -\frac{1}{e}$ or equivalently $W'(z) = \frac{W(z)}{z(1+W(z))}$, with the additional condition $W(z) \neq 0$, i.e. $z \neq 0$. The $n$th derivative of $W$ is

$$\frac{d^n W(x)}{dx^n} = \frac{e^{-nW(x)} P_n(W(x))}{(1+W(x))^{2n-1}} \text{ , for } n \geq 1 \text{ ,} \tag{18}$$

where the polynomials $P_n(w)$ are defined by the recurrence relation

$$P_{n+1}(w) = (1 - nw - 3n)P_n(w) + (1+w)P'(w) \quad \text{for } n \geq 2$$

and the initial polynomial $P_1(w) = 1$. Indeed

$$\begin{aligned}
\frac{d^{n+1}W(z)}{dx^{n+1}} &= \frac{d}{dz}\frac{e^{-nW(z)}P_n(W(z))}{(1+W(z))^{2n-1}} \\
&= \frac{[-nW'(z)e^{-nW(z)}P_n(W(z)) + e^{-nW(z)}P_n'(W(z))W'(z)](1+W(z))^{2n-1}}{(1+W(z))^{4n-2}} \\
&\quad - \frac{(2n-1)(1+W(z))^{2n-2}W'(z)e^{-nW(z)}P_n(W(z))}{(1+W(z))^{4n-2}} \\
&= \frac{e^{-(n+1)W(z)}[(1+W(z))P_n'(W(z)) + (-3m - nW(z) + 1)P_n(W(z))]}{(1+W(z))^{2n+1}} \, .
\end{aligned}$$

Then for any $z_0$ and $z$ in the domain $(-\frac{1}{e}, 0)$ we can write the Taylor series for the function $W_{-1}$ as

$$W_{-1}(z) = W_{-1}(z_0) + \sum_{n=1}^{\infty} \frac{1}{n!} W_{-1}^{(n)}(z_0)(z - z_0)^n \, .$$

Notice that since the $n$th derivative of $W_1$ in $z_0$, i.e. $W_1^{(n)}(z_0)$, can be computed just by knowing $W_1(z_0)$, see (18), it is enough to estimate the function $W_{-1}$ in $z_0$ to know also the derivatives values.

## A.2 Series expansions about the branch point $z = -\frac{1}{e}$

For a fixed value $z \in \big[-\frac{1}{e}, 0\big)$, let us consider the point $p = -\sqrt{2(ez+1)}$, which is such that $ez = \frac{p^2}{2} - 1$. Then

$$W(z)e^{W(z)} = z \implies W(z)e^{z+1} = \frac{p^2}{2} - 1 \, .$$

By expanding the exponential function in power of $W(z)+1$ we have that

$$\frac{p^2}{2}-1 = W\sum_{k=0}^{\infty}\frac{(W(z)+1)^k}{k!} = -1+\sum_{k=1}^{\infty}\left(\frac{1}{(k-1)!}-\frac{1}{k!}\right)(1+W(z))^k\,,$$

then we have $W_{-1}(z) = \sum_{k=0}^{\infty}\mu_k p^k$ , where $\mu_k = \frac{k-1}{k+1}\left(\frac{\mu_{k-2}}{2}+\frac{\alpha_{k-2}}{4}\right)-\frac{\alpha_k}{2}-\frac{\mu_{k-1}}{k+1}$, $\alpha_k = \sum_{j=2}^{k-1}\mu_j\mu_{k+1-j}$, with $\mu_0=-1,\quad \mu_1=1,\quad \alpha_0=2,\quad \alpha_1=-1$, and the series converges in the whole domain of existence of $W_{-1}$. For details, see e.g. [10].

### A.3 Asymptotic series for $z<0$

A real-valued asymptotic series can be found when $z\to 0^-$. Indeed, by using the Lagrange inversion theorem, it can be found

$$W_{-1}(z) = \ln[-z]-\ln[-\ln(z)]+\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}C_{km}(\ln[-z])^{-(k+m)}(\ln[-\ln[-z]])^m\,,$$

with

$$C_{km} = \frac{(-1)^k S(k+m,k+1)}{m!}$$

where $S(k+m,k+1)$ is a non-negative Stirling number of the first kind. They count the number of permutations of n elements with k disjoint cycles and also arise as coefficients of the rising factorial

$$(x)^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n}S(n,m)x^m\,.$$

Moreover they are computable via the recursive formula

$$S(n,m) = S(n-1,m-1)+(n-1)S(n-1,m)\quad n>1\,.$$

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