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A Stochastic Model of Optimal Debt Management and Bankruptcy*

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Abstract. A problem of optimal debt management is modeled as a noncooperative interaction between a borrower and a pool of lenders, in an infinite time horizon with exponential discount. The yearly income of the borrower is governed by a stochastic process. When the debt-to-income ratio $x(t)$ reaches a given size x^*, bankruptcy instantly occurs. The interest rate charged by the risk-neutral lenders is precisely determined in order to compensate for this possible loss of their investment. For a given bankruptcy threshold x^*, existence and properties of optimal feedback strategies for the borrower are studied, in a stochastic framework as well as in a limit deterministic setting. The paper also analyzes how the expected total cost to the borrower changes, depending on different values of x^*.

Key words. debt management and bankruptcy, infinite time horizon, optimal feedback solution

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1. Introduction.

We consider a problem of optimal debt management in an infinite time horizon, modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders. Since the debtor may go bankrupt, lenders charge a higher interest rate to offset the possible loss of part of their investment.

In the models studied in [8, 9], the borrower has a fixed income, but large values of the debt determine a bankruptcy risk. Namely, if at a given time t the debt-to-income ratio $x(t)$ is too big, there is a positive probability that panic spreads among investors and bankruptcy occurs within a short time interval $[t, t+\varepsilon]$. This event is similar to a bank run. Calling T_b the random bankruptcy time, this means

$$\text{Prob}\Big\{T_b \in [t,\, t+\varepsilon] \,\Big|\, T_b > t\Big\} \;=\; \rho(x(t))\cdot\varepsilon + o(\varepsilon).$$

Here the "instantaneous bankruptcy risk" $\rho(\cdot)$ is a given, nondecreasing function.

At all times t, the borrower must allocate a portion $u(t) \in [0,1]$ of his income to service the debt, i.e., paying back the principal together with the running interest. Our analysis will be mainly focused on the existence and properties of an optimal repayment strategy $u = u^*(x)$ in feedback form.

In the alternative model proposed by Nuño and Thomas in [16], the yearly income $Y(t)$ is modeled as a stochastic process:

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$$dY(t) \;=\; \mu Y(t)\,dt + \sigma Y(t)\,dW. \tag{1}$$

Here $\mu \geq 0$ is an exponential growth rate, while W denotes Brownian motion on a filtered probability space. Differently from [8, 9], in [16] it is the borrower himself who chooses when to declare bankruptcy. This decision will be taken when the debt-to-income ratio reaches a certain threshold x^*, beyond which the burden of servicing the debt becomes worse than the cost of bankruptcy.

At the time T_b when bankruptcy occurs, we assume that the borrower pays a fixed price B, while lenders recover a fraction $\theta(x(T_b)) \in [0,1]$ of their outstanding capital. Here $x \mapsto \theta(x)$ is a nonincreasing function of the debt size. For example, the borrower may hold an amount R_0 of collateral (gold reserves, real estate, ...) which will be proportionally divided among creditors if bankruptcy occurs. In this case, when bankruptcy occurs each investor will receive a fraction

$$\theta(x(T_b)) \;=\; \max\left\{\frac{R_0}{x(T_b)}\,,\; 1\right\} \tag{2}$$

of his outstanding capital.

The aim of the present paper is to provide a detailed mathematical analysis of some models closely related to [16]. We stress that these problems are very different from a standard problem of optimal control. Indeed, the interest rate charged by lenders is not given a priori. Rather, it is determined by the expected evolution of the debt at all future times. Hence it depends globally on the entire feedback control $u(\cdot)$. An "optimal solution" for the borrower must be understood as the best trade-off between the sustainability of his debt, related to the interest rate charged by the lenders, and the need to keep the repayment rate as small as possible.

A precise description of our model is given in section 2. Here the strategy of the borrower comprises a feedback control $u = u(x)$, determining the fraction of income allocated to servicing the debt, and a stopping set $S \subset \mathbb{R}_+$, where bankruptcy is declared. In a way, this resembles the classical problem of stochastic control with stopping time, as in [6]. We remark that, in a naive formulation, the optimization problem always admits the trivial solution

$$u(x) \;\equiv\; 0, \qquad\qquad S \;=\; \emptyset. \tag{3}$$

This corresponds to a Ponzi scheme: no payment is ever made, bankruptcy is never declared, and the interest on old loans is paid by initiating more and more new loans. This strategy guarantees zero cost and is clearly optimal.

To rule out the trivial solution and achieve a more realistic model, we assume that some upper bound x^* for the debt is given, beyond which bankruptcy must instantly occur. For example, one can think of x^* as the maximum amount of cash that all financial markets can provide. It can be very large, but certainly finite. In this modified setting, the optimization problem is formulated for $x \in [0, x^*]$, and the stopping set $S \subset [0, x^*]$ must contain the point x^*.

The main results of the paper can be summarized as follows:

- Given an upper bound x^* for the debt, we show that the optimal choice of the stopping set is $S = \{x^*\}$. In other words, it is never convenient for the borrower to declare bankruptcy, unless he is forced to do it.

We then seek an optimal feedback control $u = u^*(x)$, $x \in [0, x^*]$ which minimizes the expected cost to the borrower. For any value $\sigma \geq 0$ of the diffusion coefficient in (1), we prove that the problem admits at least one solution, in feedback form. In the deterministic case where $\sigma = 0$, the solution can be constructed by concatenating solutions of a system of two ODEs, with terminal data given at $x = x^*$.

- We then study how the expected total cost of servicing the debt together with the bankruptcy cost (exponentially discounted in time) depends on the upper bound x^*. Let $\theta(x^*) \in [0, 1]$ be the salvage rate, i.e., the fraction of outstanding capital that will be paid back to lenders if bankruptcy occurs when the debt-to-income ratio is x^*. If

$$\lim_{s \to +\infty} \theta(s)\, s \;=\; +\infty, \tag{4}$$

then, letting $x^* \to +\infty$, the total expected cost to the borrower goes to zero. On the other hand, if

$$\lim_{s \to +\infty} \theta(s)\, s \;<\; +\infty, \tag{5}$$

then the total expected cost to the borrower remains uniformly positive as $x^* \to +\infty$.

Our analysis shows that if the debtor can access a large amount x^* of credit, when (4) holds he can postpone the bankruptcy time far into the future. Due to the exponential discount, as $x^* \to +\infty$ his expected cost will thus approach zero. On the other hand, when (5) holds, after the debt has reached a certain threshold, bankruptcy must occur within a fixed time regardless of the size of x^*. We remark that the assumption (5) is more realistic. For example, if (2) holds, then $\theta(x^*)\, x^* = R_0$ for all x^* large enough.

The remainder of the paper is organized as follows. In section 2 we describe more carefully the model, deriving the equations satisfied by the value function V and the discounted bond price p. In sections 3 and 4 we construct equilibrium solutions in feedback form, in the stochastic case ($\sigma > 0$) and in the deterministic case ($\sigma = 0$), respectively. Finally, sections 5 and 6 contain an analysis of how the expected cost to the borrower changes, depending on the bankruptcy threshold x^*.

In the economics literature, some related models of debt and bankruptcy can be found in [1, 3, 8, 12, 13]. For the basic theory of optimal control and viscosity solutions of Hamilton–Jacobi equations we refer to [4, 10].

2. A model with stochastic growth. We consider a slight variant of the model in [16]. We denote by $X(t)$ the total debt of a borrower (a government, or a private company) at time t. The annual income $Y(t)$ of the borrower is assumed to be a random process, governed by the stochastic evolution equation (1).

The debt is financed by issuing bonds. When an investor buys a bond of unit nominal value, he receives a continuous stream of payments with intensity $(r + \lambda)e^{-\lambda t}$. Here

- r is the interest rate paid on bonds, which we assume coincides with the discount rate,
- λ is the rate at which the borrower pays back the principal.

If no bankruptcy occurs, the payoff for an investor will thus be

$$\int_0^\infty e^{-rt}(r + \lambda)e^{-\lambda t}\, dt \;=\; 1.$$

In case of bankruptcy, a lender recovers only a fraction $\theta \in [0,1]$ of his outstanding capital. Here θ can depend on the total amount of debt at the time on bankruptcy. To offset this possible loss, the investor buys a bond with unit nominal value at a discounted price $p \in [0,1]$. As in [9, 16], at any time t the value $p(t)$ is uniquely determined by the competition of a pool of risk-neutral lenders.

We call $U(t)$ the rate of payments that the borrower chooses to make to his creditors, at time t. If this amount is not enough to cover the running interest and pay back part of the principal, new bonds are issued, at the discounted price $p(t)$. As in [16], the nominal value of the outstanding debt thus evolves according to

$$\dot{X}(t) \;=\; -\lambda X(t) + \frac{(\lambda + r)X(t) - U(t)}{p(t)}. \tag{6}$$

For a more detailed derivation of (6) from the economic primitives, we refer the reader to [16].

The debt-to-income ratio is defined as $x = X/Y$. In view of (1) and (6), Ito's formula [17, 18] yields the stochastic evolution equation

$$dx(t) \;=\; \left[\left(\frac{\lambda + r}{p(t)} - \lambda + \sigma^2 - \mu\right) x(t) - \frac{u(t)}{p(t)}\right] dt - \sigma\, x(t)\, dW. \tag{7}$$

Here $u = U/Y$ is the fraction of the total income allocated to pay for the debt. Throughout the following we assume $r > \mu$.

In this model, the borrower has two controls. At each time t he can decide the portion $u(t)$ of the total income which he allocates to repay the debt. Moreover, he can decide at what time T_b bankruptcy is declared.

Throughout the following, we assume that an upper bound x^* for the debt is a priori given (as an external constraint, imposed by the size of the markets) and consider strategies in feedback form. These comprise

(i) a closed set $S \subset [0, x^*]$, with $x^* \in S$, where bankruptcy is declared, and

(ii) a feedback control determining the repayment rate

$$u \;=\; u^*(x) \;\in\; [0,1] \qquad\qquad \text{for} \quad x \in [0, x^*] \setminus S. \tag{8}$$

For a given choice of the stopping set S, the bankruptcy time is thus the random variable

$$T_b \;\doteq\; \inf\big\{t \geq 0\,;\;\; x(t) \in S\big\}\,. \tag{9}$$

Given an initial size x_0 of the debt, the goal of the borrower is to minimize the total expected cost, exponentially discounted in time. Namely,

$$\text{minimize:}\;\; J\left(x_0, u^*, S\right) \;\doteq\; E\left[\int_0^{T_b} e^{-rt} L\left(u^*(x(t))\right)\, dt + e^{-rT_b} B\right]_{x(0)=x_0}. \tag{10}$$

Here B is a large constant, accounting for the bankruptcy cost, while $L(u)$ is the instantaneous cost to the borrower for implementing the control u.

To complete the model, we need an equation determining the discounted bond price p in the evolution equation (7). For every $x > 0$, let $\theta(x)$ be the salvage rate, i.e., the fraction of

the outstanding capital that can be recovered by lenders, if bankruptcy occurs when the debt has size x. Given an initial debt size x_0, the expected payoff to a lender purchasing a coupon with unit nominal value is computed by the right-hand side of

$$p(x_0) \;=\; E\left[\int_0^{T_b} (r+\lambda)e^{-(r+\lambda)t}dt + e^{-(r+\lambda)T_b}\theta\left(x(T_b)\right)\right]_{x(0)=x_0}. \tag{11}$$

Assuming that the bond price is determined by the competition of a large pool of risk-neutral lenders, this expected payoff should coincide with the discounted bond price $p(x_0)$. This motivates (11).

Notice that the stopping time T_b in (9), and hence $p(x_0)$, depends on the initial state x_0, on the stopping set S, and on all values of the feedback control $u^*(\cdot)$. Since the salvage rate $\theta(\cdot)$ is nonincreasing, we have

$$p(x) = \theta(x) \quad \text{if} \quad x \in S, \qquad\qquad p(x) \;\in\; [\theta(x^*), 1] \quad \text{for all} \quad x \in [0, x^*]. \tag{12}$$

Having described the model, we can introduce the definition of optimal solution, in feedback form.

Definition 2.1 (stochastic optimal feedback solution). *In connection with the above model, we say that a set $S \subset [0, x^*]$ and a pair of functions $u = u^*(x)$, $p = p(x)$ provides an optimal solution to the problem of optimal debt management (7)–(11) if*

(i) *given the function $p(\cdot)$, for every initial value $x_0 \in [0, x^*]$ the feedback control $u^*(\cdot)$ with stopping time T_b as in (9) provides an optimal solution to the stochastic control problem (10), with dynamics (7);*

(ii) *given the feedback control $u^*(\cdot)$ and the set S, for every initial value x_0 the discounted price $p(x_0)$ satisfies (11), where T_b is the stopping time (9) determined by the dynamics (7).*

We emphasize that in our model, if $x(t) = x^*$, then bankruptcy must instantly occur. The following simple observation shows that, for the borrower, it is never convenient to voluntarily declare bankruptcy at any earlier time.

Lemma 2.2. *Let S, $u^*(\cdot)$, $p(\cdot)$ be an optimal solution to the debt management problem (7)–(11). Then $S = \{x^*\}$.*

Proof. Assume that, on the contrary, there is a value $x_0 < x^*$ such that $x_0 \in S$. We show that condition (i) in the above definition cannot hold. Indeed, consider the optimization problem with initial datum $x(0) = x_0$. If $x_0 \in S$, then bankruptcy instantly occurs at time $T_b = 0$, and the expected cost in (10) is $J[x_0, u^*, S] = B$. However, the alternative strategy $u(t) \equiv 0$, with bankruptcy occurring at the first time where $x(t) = x^*$, provides the strictly smaller expected cost

$$J\left(x_0, 0, \{x^*\}\right) \;=\; E\left[e^{-rT_b}B\right] \;<\; B. \qquad \blacksquare$$

Motivated by Lemma 2.2, from now on we shall always take $S = \{x^*\}$ as the stopping set. The random stopping time is thus

$$T_b \;\doteq\; \inf\left\{t \geq 0\,;\;\; x(t) = x^*\right\}. \tag{13}$$

Concerning the cost function L in (10), we shall assume the following:

(A) The function L is twice continuously differentiable for $u \in [0,1[$ and satisfies

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad \lim_{u \to 1-} L(u) \;=\; +\infty. \tag{14}$$

For example, for some $c, \alpha > 0$, one may take

$$L(u) = c\,\ln \frac{1}{1-u} \qquad \text{or} \qquad L(u) = \frac{cu}{(1-u)^\alpha}\,.$$

For a given function $p = p(x)$, we denote by $V(\cdot)$ the value function for the stochastic optimal control problem (10) with dynamics (7). Namely,

$$V(x_0) \;\doteq\; \inf_{u(\cdot)} \; J\big(x_0, u, \{x^*\}\big)\,. \tag{15}$$

Denote by

$$H(x,\xi,p) \;\doteq\; \min_{\omega \in [0,1]} \left\{ L(\omega) - \frac{\xi}{p}\,\omega \right\} + \left(\frac{\lambda + r}{p} - \lambda + \sigma^2 - \mu \right) x\,\xi \tag{16}$$

the Hamiltonian associated to the dynamics (7) and the cost function L in (10). Notice that, as long as $p > 0$, the function H is differentiable with Lipschitz continuous derivatives w.r.t. all arguments.

By standard arguments, the value function V provides a solution to the second order ODE

$$rV(x) \;=\; H\big(x, V'(x), p(x)\big) + \frac{(\sigma x)^2}{2} V''(x) \tag{17}$$

with boundary conditions

$$V(0) \;=\; 0\,, \qquad\qquad V(x^*) \;=\; B\,. \tag{18}$$

As soon as the function V is determined, the optimal feedback control is recovered by

$$u^*(x) = \operatorname*{argmin}_{\omega \in [0,1]} \left\{ L(\omega) - \frac{V'(x)}{p(x)}\,\omega \right\}.$$

By (A) this yields

$$u^*(x) \;=\; \begin{cases} 0 & \text{if} \quad \dfrac{V'(x)}{p(x)} \;\leq\; L'(0)\,, \\[2ex] (L')^{-1}\left(\dfrac{V'(x)}{p(x)} \right) & \text{if} \quad \dfrac{V'(x)}{p(x)} \;>\; L'(0)\,. \end{cases} \tag{19}$$

On the other hand, if the feedback control $u = u^*(x)$ is known, then by the Feynman–Kac formula $p(\cdot)$ satisfies the equation

$$(r+\lambda)(p(x)-1) \;=\; \left[\left(\frac{\lambda + r}{p(x)} - \lambda + \sigma^2 - \mu \right) x - \frac{u^*(x)}{p(x)} \right] \cdot p'(x) + \frac{(\sigma x)^2}{2} p''(x) \tag{20}$$

with boundary values

$$p(0) = 1, \qquad p(x^*) = \theta(x^*)\,. \tag{21}$$

Combining (17) and (20), we are thus led to the system of second order ODEs

$$\begin{cases} rV(x) &= H\big(x, V'(x), p(x)\big) + \dfrac{(\sigma x)^2}{2} \cdot V''(x)\,, \\ (r+\lambda)(p(x)-1) &= H_\xi\big(x, V'(x), p(x)\big) \cdot p'(x) + \dfrac{(\sigma x)^2}{2} \cdot p''(x) \end{cases} \tag{22}$$

with the boundary conditions

$$\begin{cases} V(0) = 0, \\ V(x^*) = B, \end{cases} \qquad \begin{cases} p(0) = 1, \\ p(x^*) = \theta(x^*). \end{cases} \tag{23}$$

In the next section, an optimal feedback solution to the problem (7)–(11) will be obtained by solving the above system of ODEs for the value function $V(\cdot)$ and for the discounted bond price $p(\cdot)$.

We close this section by collecting some useful properties of the Hamiltonian function.

Lemma 2.3. *Let the assumptions* (A) *hold. Then, for all* $\xi \geq 0$ *and* $0 < p \leq 1$*, the function* H *in* (16) *satisfies*

$$\left(\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu)x\right)\xi \;\leq\; H(x,\xi,p) \;\leq\; \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\xi, \tag{24}$$

$$\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu)x \;\leq\; H_\xi(x,\xi,p) \;\leq\; \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x. \tag{25}$$

Moreover, for every $x, p > 0$ *the map* $\xi \mapsto H(x,\xi,p)$ *is concave down and satisfies*

$$H(x,0,p) = 0, \tag{26}$$

$$H_\xi(x,0,p) = \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\,, \tag{27}$$

$$\lim_{\xi\to+\infty} H(x,\xi,p) = \begin{cases} -\infty & \text{if} \quad \dfrac{1}{p} > \left(\dfrac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\,, \\ +\infty & \text{if} \quad \dfrac{1}{p} \leq \left(\dfrac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\,. \end{cases} \tag{28}$$

Proof.

1. Since $H(x,\cdot,p)$ is defined as the infimum of a family of affine functions, it is concave down. We observe that (16) implies

$$H(x,\xi,p) = \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu\right) x\xi \qquad \text{if} \quad 0 \leq \xi \leq pL'(0). \tag{29}$$

This yields the identities (26)–(27).

2. Taking $\omega = 0$ in (16) we obtain the upper bound in (24). By the concavity property, the map $\xi \mapsto H_\xi(x,\xi,p)$ is nonincreasing. Hence (27) yields the upper bound in (25).
3. Since $L(w) \geq 0$ for all $w \in [0,1]$, we have

$$H(x,\xi,p) \;\geq\; \min_{w\in[0,1]} \left\{ -\frac{\xi}{p} w \right\} + \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu \right) x\,\xi$$

and obtain the lower bound in (24). On the other hand, using the optimality condition, one computes from (16) that

$$H_\xi(x,\xi,p) \;=\; \frac{(\lambda+r)x - u^*(\xi,p)}{p} + (\sigma^2 - \lambda - \mu)x, \tag{30}$$

where

$$u^*(\xi,p) \;=\; \operatorname*{argmin}_{\omega\in[0,1]} \left\{ L(\omega) - \frac{\xi}{p}\,\omega \right\} \;=\; (L')^{-1}\left(\frac{\xi}{p}\right) \;<\; 1\,.$$

Observe that, as $\xi \to +\infty$, one has $u^*(\xi,p) \to 1$ in (30). The nonincreasing property of the map $\xi \to H_\xi(x,\xi,p)$ yields the lower bound in (25).
4. To prove (28) we observe that, in the first case, there exists $\omega_0 < 1$ such that

$$\frac{\omega_0}{p} \;>\; \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu \right) x.$$

Hence, letting $\xi \to +\infty$ we obtain

$$\lim_{\xi\to+\infty} H(x,\xi,p) \;\leq\; \lim_{\xi\to+\infty} \left[L(\omega_0) - \frac{\omega_0}{p}\xi + \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu \right) x\,\xi \right] \;=\; -\infty\,.$$

To handle the second case, we observe that, for $\xi > 0$ large, the minimum in (16) is attained at the unique point $\omega(\xi)$ where $L'(\omega(\xi)) = \xi/p$. Hence $\lim_{\xi\to+\infty} \omega(\xi) = 1$ and

$$\begin{aligned} \lim_{\xi\to+\infty} H(x,\xi,p) = \; & \lim_{\xi\to+\infty} \left[L(\omega(\xi)) - \frac{\omega(\xi)}{p}\xi + \left(\frac{\lambda+r}{p} - \lambda + \sigma^2 - \mu \right) x\,\xi \right] \\ \geq \; & \lim_{\xi\to+\infty} L(\omega(\xi)) \;=\; +\infty. \end{aligned}$$

■

Remark 2.4. We summarize here the main differences between the proposed model and the model presented in [16]. In [16], the borrower is a government that can control the primary surplus ratio, the inflation rate, and the time of declaring bankruptcy. The control on the inflation rate can be used by the government as a monetary policy to temporarily deflate the actual debt value, by paying a price in terms of welfare cost. While controlling the primary surplus ratio is actually equivalent in our model to the choice of $u(\cdot)$, and in both models the borrower can choose the bankruptcy time, in our model the borrower cannot choose the inflation rate r. This simplification can be motivated assuming either that the borrower is not a government or that the monetary policy of the government has been delegated to an independent central banker which acts in order to keep it constant (e.g., 2% in Eurozone), no matter of the consequences on the borrower's debt sustainability.

In [16], the instantaneous preferences of the borrower are expressed by a (discounted) utility function of logarithmic type, while our analysis deals with more general cost functions $L(\cdot)$.

3. Existence of solutions. Let $x^* > 0$ be given. If a solution (V, p) to the boundary value problem (22)–(23) is found, then the feedback control $u = u^*(x)$ defined at (19) and the function $p = p(x)$ will provide an equilibrium solution to the debt management problem, as in Definition 2.1.

To construct a solution to the system (22)–(23), we consider the auxiliary parabolic system

$$\begin{cases} V_t(t,x) &= -rV(t,x) + H\big(x, V_x(t,x), p(t,x)\big) + \dfrac{(\sigma x)^2}{2} \cdot V_{xx}(t,x)\,, \\ \\ p_t(t,x) &= (r+\lambda)(1-p(t,x)) + H_\xi\big(x, V_x(t,x), p(t,x)\big) \cdot p_x(t,x) + \dfrac{(\sigma x)^2}{2} \cdot p_{xx}(t,x) \end{cases} \tag{31}$$

with boundary conditions

$$\begin{cases} V(t,0) &= 0, \\ \\ V(t,x^*) &= B, \end{cases} \qquad \begin{cases} p(t,0) &= 1 \\ \\ p(t,x^*) &= \theta(x^*) \end{cases} \qquad \text{for all } t \geq 0\,.$$

Following [2], the main idea is to construct a compact, convex set of functions $(V, p) : [0, x^*] \mapsto [0, B] \times [\theta(x^*), 1]$ which is positively invariant for the parabolic evolution problem. A topological technique will then yield the existence of a steady state, i.e., a solution to (22)–(23).

Theorem 3.1. *In addition to* (A), *assume that* $\sigma > 0$ *and* $\theta(x^*) > 0$. *Then the system of second order ODEs* (22) *with boundary conditions* (23) *admits a* $\mathcal{C}^2$ *solution* $(\overline{V}, \bar{p})$, *such that* $\overline{V} : [0, x^*] \to [0, B]$ *is increasing and* $\bar{p} : [0, x^*] \to [\theta(x^*), 1]$ *is decreasing.*

Proof.

1. For any $\varepsilon > 0$, consider the parabolic system

$$V_t = -rV + H(x, V_x, p) + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xx}\,, \qquad \begin{cases} V(0) = 0, \\ V(x^*) = B, \end{cases} \tag{32}$$

$$p_t = (r+\lambda)(1-p) + H_\xi(x, V_x, p) p_x + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) p_{xx}\,, \qquad \begin{cases} p(0) = 1, \\ p(x^*) = \theta(x^*), \end{cases} \tag{33}$$

   obtained from (31) by adding the terms εV_{xx}, εp_{xx} on the right-hand sides. For any $\varepsilon > 0$, this renders the system uniformly parabolic, also in a neighborhood of $x = 0$.

2. Recalling [2, Theorem 1], for every initial data $V_0, p_0 \in \mathcal{C}^2([0, x^*])$, the system (32)–(33) with initial data

$$V(0,x) = V_0(x), \qquad p(0,x) = p_0(x) \tag{34}$$

   admits a unique solution $V(t,x), p(t,x)$ in $\mathcal{C}^2([0,T] \times [0, x^*])$ for all $T > 0$. Adopting a semigroup notation, let $t \mapsto (V(t,\cdot), p(t,\cdot)) = S_t(V_0, p_0)$ be the solution of the system (32)–(33) with initial data (34).

Consider the closed, convex set of functions in $\mathcal{C}^2([0,x^*])$

$$\mathcal{D} \;=\; \Big\{(V,p):[0,x^*]\mapsto[0,B]\times[\theta(x^*),1]\;;\quad V,p\in\mathcal{C}^2,\,V_x\geq 0,\,p_x\leq 0,\;\text{ and (23) holds}\Big\}. \tag{35}$$

We claim that the above domain is positively invariant under the semigroup S, namely,

$$S_t(\mathcal{D}) \;\subseteq\; \mathcal{D} \qquad\qquad \text{for all } t\geq 0\,. \tag{36}$$

Indeed, consider the constant functions

$$\begin{cases} V^+(t,x) \;=\; B,\\ \\ V^-(t,x) \;=\; 0, \end{cases} \qquad\qquad \begin{cases} p^+(t,x) \;=\; 1,\\ \\ p^-(t,x) \;=\; \theta(x^*)\,. \end{cases}$$

Recalling (26), one easily checks that V^+ is a supersolution and V^- is a subsolution of the scalar parabolic problem (32). Indeed

$$-rV^+ + H(x,V_x^+,p) + \left(\varepsilon+\frac{(\sigma x)^2}{2}\right)V^+_{xx} \;\leq\; 0, \qquad V^+(t,0)\geq 0, \qquad V^+(t,x^*)\geq B,$$

$$-rV^- + H(x,V_x^-,p) + \left(\varepsilon+\frac{(\sigma x)^2}{2}\right)V^-_{xx} \;\geq\; 0, \qquad V^-(t,0)\leq 0, \qquad V^-(t,x^*)\leq B.$$

A standard comparison principle (see, for example, Theorem 9.1 in [15]) yields

$$0 \;=\; V^-(t,x) \;\leq\; V(t,x) \;\leq\; V^+(t,x) \;=\; B \qquad \text{for all } (t,x)\in[0,T]\times[0,x^*]\,.$$

Similarly, since p^+ is a supersolution and p^- is a subsolution of the scalar parabolic problem (33), one has that

$$\theta(x^*) \;=\; p^-(t,x) \;\leq\; p(t,x) \;\leq\; p^+(t,x) \;=\; 1 \qquad \text{for all } (t,x)\in[0,T]\times[0,x^*]\,.$$

This proves that if the initial data V_0, p_0 in (34) take values in the box $[0,B]\times[\theta(x^*),\,1]$, then for every $t\geq 0$ the solution of the system (32)–(33) will satisfy

$$0 \;\leq\; V(t,x) \;\leq\; B, \qquad\qquad \theta(x^*) \;\leq\; p(t,x) \;\leq\; 1 \tag{37}$$

for all $x\in[0,x^*]$. In turn, this implies

$$\begin{cases} V_x(t,0) \;\geq\; 0,\\ \\ V_x(t,x^*) \;\geq\; 0, \end{cases} \qquad\qquad \begin{cases} p_x(t,0) \;\leq\; 0,\\ \\ p_x(t,x^*) \;\leq\; 0\,. \end{cases} \tag{38}$$

3. Next, we prove that the monotonicity properties of $V(t,\cdot)$ and $p(t,\cdot)$ are preserved in time. Differentiating w.r.t. x one obtains

$$V_{xt} = -rV_x + H_x + H_\xi V_{xx} + H_p p_x + \sigma^2 x V_{xx} + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xxx}\,, \tag{39}$$

$$p_{xt} = -(r+\lambda)p_x + \left(\frac{d}{dx} H_\xi(x, V_x, p)\right) p_x + H_\xi p_{xx} + \sigma^2 x p_{xx} + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) p_{xxx}\,. \tag{40}$$

By (26), for every x, p one has $H_x(x,0,p) = H_p(x,0,p) = 0$. Hence $V_x \equiv 0$ is a subsolution of (39) and $p_x \equiv 0$ is a supersolution of (40). In view of (38), we obtain

$$p_x(t,x) \le 0 \le V_x(t,x) \qquad \text{for all} \quad t \ge 0, \quad x \in [0, x^*].$$

This concludes the proof that the set $\mathcal{D}$ in (35) is positively invariant for the system (32)–(33).

4. Thanks to the bounds (24)–(25), we can now apply Theorem 3 in [2] and obtain the existence of a steady state $(V^\varepsilon, p^\varepsilon) \in \mathcal{D}$ for the system (32)–(33).
We recall the main argument in [2]. For every $T > 0$ the map $(V_0, p_0) \mapsto S_T(V_0, p_0)$ is a compact transformation of the closed convex domain $\mathcal{D}$ into itself in $\mathcal{C}^2(\mathbb{R}^2)$. By Schauder's theorem it has a fixed point. This yields a periodic solution of the parabolic system (32)–(33) with period T. Letting $T \to 0$, one obtains a steady state.

5. It now remains to derive a priori estimates on this stationary solution, which will allow us to take the limit as $\varepsilon \to 0$. Consider any solution to

$$\begin{cases} -rV + H(x, V', p) + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right) V'' &= 0\,, \\ (r+\lambda)(1-p) + H_\xi(x, V', p) p' + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right) p'' &= 0 \end{cases} \tag{41}$$

with V increasing, p decreasing, and satisfying the boundary conditions (23).
By the properties of H derived in Lemma 2.3, we can find $\delta > 0$ small enough and $\xi_0 > 0$ such that the following implication holds:

$$x \in [0, \delta], \quad p \in [\theta(x^*), 1], \quad \xi \ge \xi_0 \qquad \Longrightarrow \qquad H(x, \xi, p) \le 0\,.$$

As a consequence, if $V'(x) > \xi_0$ for some $x \in [0, \delta]$, then the first equation in (41) implies $V''(x) \ge 0$. We conclude that either $V'(x) \le \xi_0$ for all $x \in [0, \delta]$ or else V' attains its maximum on the subinterval $[\delta, x^*]$.
By the intermediate value theorem, there exists a point $\hat{x} \in [\delta, x^*]$ where

$$V'(\hat{x}) = \frac{V(x^*) - V(\delta)}{x^* - \delta} \le \frac{B}{x^* - \delta}\,. \tag{42}$$

By (24), the derivative V' satisfies a differential inequality of the form

$$|V''| \le c_1 |V'| + c_2\,, \qquad x \in [\delta, x^*], \tag{43}$$

for suitable constants c_1, c_2. By Gronwall's lemma, from the differential inequality (43) and the estimate (42) one obtains a uniform bound on $V'(x)$ for all $x \in [\delta, \hat{x}] \cup [\hat{x}, x^*]$. Relying on the first equation of (41), we also obtain a uniform bound on $V''(x)$ for all $x \in [\delta, x^*]$.

6. Similar arguments apply to p'. By (25), the term $H_\xi(x, V', p)$ in (41) is uniformly bounded. For every $\delta > 0$, (41) shows that p' satisfies a linear ODE whose coefficients remain bounded on $[\delta, x^*]$, uniformly w.r.t. ε. This yields the bound

$$|p'(x)| \leq C_\delta \qquad \text{for all } x \in [\delta, x^*] \tag{44}$$

for some constant C_δ, uniformly valid as $\varepsilon \to 0$. Relying on the second equation of (41), we also obtain a uniform bound on $p''(x)$ for all $x \in [\delta, x^*]$

To make sure that, as $\varepsilon \to 0$, the limit satisfies the boundary value $p(0) = 1$, one needs to provide a lower bound on p also in a neighborhood of $x = 0$, independent of ε. Introduce the constant

$$\gamma \doteq \min\left\{1\,,\ (r+\lambda)\left(\frac{\lambda+r}{\theta(x^*)} - \lambda + \sigma^2 - \mu\right)^{-1}\right\}.$$

Then define

$$p^-(x) \doteq 1 - cx^\gamma,$$

choosing $c > 0$ so that $p^-(x^*) = \theta(x^*)$. We claim that the convex function p^- is a lower solution of the second equation in (41). Indeed, by (41), one has

$$(r+\lambda)cx^\gamma - H_\xi(x, V', p)\, c\gamma x^{\gamma-1} \geq \left[(r+\gamma) - \left(\frac{\lambda+r}{\theta(x^*)} - \lambda + \sigma^2 - \mu\right)\gamma\right] cx^\gamma \geq 0.$$

7. Letting $\varepsilon \to 0$, we now consider a sequence $(V^\varepsilon, p^\varepsilon)$ of solutions to (41) with boundary conditions (23). Thanks to the previous estimates, the functions $(V^\varepsilon)'$ and $(p^\varepsilon)'$ are uniformly bounded by some constant $C_{1,\delta} > 0$ on $[\delta, x^*]$, and p^ε satisty

$$p^-(x) \leq p^\varepsilon(x) \leq 1 \qquad \text{for all } \ x \in [0, x^*], \quad \varepsilon > 0.$$

On the other hand, since H and H_ξ are uniformly bounded and uniformly Lipschitz on $[\delta, x^*] \times [-C_{1,\delta}, C_{1,\delta}] \times [\theta(x^*), 1]$, the functions

$$(V^\varepsilon)'' = \frac{2}{2\varepsilon + \sigma^2 x^2} \cdot \left[rV^\varepsilon - H(x, (V^\varepsilon)', p^\varepsilon)\right]$$

and

$$(p^\varepsilon)'' = \frac{2}{2\varepsilon + \sigma^2 x^2} \cdot \left[(r+\lambda)\cdot(p^\varepsilon - 1) - H_\xi(x, (V^\varepsilon)', p^\varepsilon)(p^\varepsilon)'\right]$$

are also uniformly bounded and uniformly Lipschitz on $[\delta, x^*]$.

By choosing a suitable subsequence, we achieve the uniform convergence $(V^\varepsilon, p^\varepsilon) \to (V, p)$, where V, p are twice continuously differentiable on the open interval $]0, x^*[$, and satisfy the boundary conditions (23). ∎

Having constructed a solution (V, p) to the boundary value problem (22)–(23), a standard result in the theory of stochastic optimization implies that the feedback control $u^*(\cdot)$ in (19) is optimal for the problem (10) with dynamics (7). For a proof of this "verification theorem," see Theorem 4.1, p. 149, in [14] or Theorem 11.2.2, p. 241, in [17]. As a consequence of Theorem 3.1 we thus obtain the following.

Corollary 3.2. *Under the same assumptions as in Theorem 3.1, the debt management problem (7)–(11) admits an optimal solution.*

4. The deterministic case. If $\sigma = 0$, then the stochastic equation (7) reduces to the deterministic control system

$$\dot{x} \;=\; \left(\frac{\lambda + r}{p} - \lambda - \mu\right) x - \frac{u}{p}\,. \tag{45}$$

Throughout the paper, we always assume $r > \mu$. The deterministic debt management problem can be formulated as follows:

(DMP) Given an initial value $x(0) = x_0 \in [0, x^*]$ of the debt, minimize

$$\int_0^{T_b} e^{-rt} L(u(t))\, dt + e^{-rT_b} B\,, \tag{46}$$

subject to the dynamics (45), where the bankruptcy time T_b is defined as in (9), while

(47)

$$p(t) \;=\; \int_t^{T_b} (r+\lambda) e^{-(r+\lambda)s} ds + e^{-(r+\lambda)(T_b - t)} \cdot \theta(x^*) \;=\; 1 - (1 - \theta(x^*))\, e^{-(r+\lambda)(T_b - t)}\,.$$

Since in this case the optimal feedback control u^* and the corresponding functions V, p may not be smooth, a concept of equilibrium solution should be more carefully defined.

Definition 4.1 (deterministic optimal feedback solution). *A couple of piecewise Lipschitz continuous functions $u = u^*(x)$ and $p = p^*(x)$ provide an equilibrium solution to the debt management problem* (DMP)*, with continuous value function V^*, if the following hold:*

(i) *For every $x_0 \in [0, x^*]$, V^* is the minimum cost for the optimal control problem*

$$\textit{minimize:} \quad \int_0^{T_b} e^{-rt} L(u(x(t)))\, dt + e^{-rT_b} B \tag{48}$$

subject to

$$\dot{x}(t) \;=\; \left(\frac{\lambda + r}{p^*(x(t))} - \lambda - \mu\right) x(t) - \frac{u(t)}{p^*(x(t))}\,, \qquad x(0) \;=\; x_0\,. \tag{49}$$

Moreover, every Carathéodory solution of (49) with $u(t) = u^*(x(t))$ is optimal.

(ii) *For every $x_0 \in [0, x^*]$, there exists at least one solution $t \mapsto x(t)$ of the Cauchy problem*

$$\dot{x} \;=\; \left(\frac{\lambda + r}{p^*(x)} - \lambda - \mu\right) x - \frac{u^*(x)}{p^*(x)}\,, \qquad x(0) \;=\; x_0, \tag{50}$$

such that

$$p^*(x_0) \;=\; \int_0^{T_b} (r+\lambda) e^{-(r+\lambda)t} dt + e^{(-r+\lambda)T_b}\,\theta(x^*) \;=\; 1-(1-\theta(x^*))\cdot e^{-(r+\lambda)T_b}\,, \tag{51}$$

with T_b as in (9).

In the deterministic case, (22) takes the form

$$\begin{cases} rV(x) &= H\big(x, V'(x), p(x)\big)\,, \\ (r+\lambda)(p(x)-1) &= H_\xi\big(x, V'(x), p(x)\big)\, p'(x) \end{cases} \tag{52}$$

with Hamiltonian function (see Figure 1)

$$H(x,\xi,p) \;=\; \min_{\omega\in[0,1]} \left\{ L(\omega) - \frac{\xi}{p}\omega \right\} + \left(\frac{\lambda+r}{p} - (\lambda+\mu) \right) x\,\xi\,. \tag{53}$$

We consider solutions to (52) with the boundary condition

$$\begin{cases} V(0) &= 0, \\ V(x^*) &= B, \end{cases} \qquad \begin{cases} p(0) &= 1, \\ p(x^*) &= \theta(x^*)\,. \end{cases} \tag{54}$$

Let's introduce two functions:

$$H^{\max}(x,p) \;\doteq\; \sup_{\xi\geq 0}\; H(x,\xi,p) \qquad \text{and} \qquad \xi^\sharp(x,p) \;\doteq\; \operatorname*{argmax}_{\xi\geq 0}\; H(x,\xi,p)\,.$$

Recalling (30), we have

$$H_\xi(x,\xi,p) \;=\; \frac{(\lambda+r)x - u^*(x,p)}{p} - (\lambda+\mu)x, \tag{55}$$

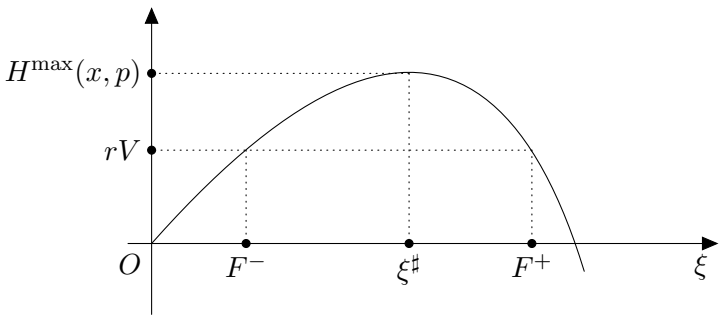

Figure 1. *In the case where $(\lambda+r)x-(\lambda+\mu)px<1$, the Hamiltonian function $\xi\mapsto H(x,\xi,p)$ has a global maximum $H^{max}(x,p)$. For $rV\leq H^{max}$, the values $F^-(x,V,p)\leq\xi^\sharp(x,p)\leq F^+(x,V,p)$ are well defined.*

where

$$u^*(\xi,p) \;=\; \operatorname*{arg\,min}_{w\in[0,1]}\left\{L(w)-\frac{\xi}{p}\cdot w\right\} \;=\; (L')^{-1}\left(\frac{\xi}{p}\right) \;<\; 1\,. \tag{56}$$

Two cases may occur:

- If $(\lambda+r)x-(\lambda+\mu)px \;\geq\; 1$, then the function $\xi\mapsto H(x,\xi,p)$ is monotone increasing and

$$H^{\max}(x,p) \;=\; \lim_{\xi\to\infty} H(x,\xi,p) \;=\; +\infty\,. \tag{57}$$

  In this case, we will define $\xi^\sharp(x,p)\doteq+\infty$.
- If $(\lambda+r)x-(\lambda+\mu)px<1$, we define

$$u^\sharp(x,p) \;=\; (\lambda+r)x-(\lambda+\mu)px\,. \tag{58}$$

  From (55) and (56), we have

$$\xi^\sharp(x,p) \;=\; p\cdot L'(u^\sharp(x,p)) \;=\; p\,L'\Big((\lambda+r)x-(\lambda+\mu)px\Big) \tag{59}$$

  and it yields

$$H^{\max}(x,p) \;=\; H\left(x,\xi^\sharp(x,p),p\right) \;=\; L\Big((\lambda+r)x-(\lambda+\mu)px\Big)\,.$$

  Notice that $u^\sharp$ is the control that keeps the debt x constant in time. This value $u^\sharp$ achieves the minimum in (53) when

$$L'\Big((\lambda+r)x-(\lambda+\mu)px\Big) \;=\; \frac{\xi}{p}\,.$$

Observe that

$$H_{\xi\xi}(x,\xi,p) \;\leq\; 0\,, \qquad \begin{cases} H_\xi(x,\xi,p) \;>\; 0 & \text{for all } 0\leq\xi<\xi^\sharp(x,p)\,,\\ H_\xi(x,\xi,p) \;<\; 0 & \text{for all } \xi>\xi^\sharp(x,p)\,. \end{cases} \tag{60}$$

We regard the first equation in (52) as an implicit ODE for the function V. For every $x\geq 0$ and $p\in[0,1]$, if $rV(x)>H^{\max}(x,p)$, then this equation has no solution. On the other hand, when

$$0 \;\leq\; rV(x) \;\leq\; H^{\max}(x,p),$$

the implicit ODE (52) can equivalently be written as a differential inclusion:

$$V'(x) \;\in\; \Big\{F^-(x,V,p)\,,\; F^+(x,V,p)\Big\}, \tag{61}$$

where $F^{\pm(x,V,p)}$ are denoted by

$$F^-(x,V,p) \;\leq\; \xi^\sharp(x,p) \;\leq\; F^+(x,V,p) \quad\text{and}\quad H(x,F^\pm(x,V,p),p) \;=\; rV\,.$$

Remark 4.2. Recalling (45), we observe that

- the value $V' = F^+(x,V,p) \geq \xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot{x} < 0$,
- the value $V' = F^-(x,V,p) \leq \xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot{x} > 0$,
- when $rV = H^{max}(x,p)$, then the value $V' = F^+(x,V,p) = F^-(x,V,p) = \xi^\sharp(x,p)$ corresponds to the unique control such that $\dot{x} = 0$.

Since $\xi \mapsto H(x,\xi,p)$ is concave down, the functions $F^\pm$ satisfy the following monotonicity properties (Figure 1):

(MP) For any fixed x,p, the map $V \mapsto F^+(x,V,p)$ is decreasing, while $V \mapsto F^-(x,V,p)$ is increasing.

For $V' = F^-$, the second ODE in (52) can be written as

$$p'(x) \;=\; G^-\big(x,V(x),p(x)\big),$$

where

$$G^-(x,V,p) \;\doteq\; \frac{(r+\lambda)(p-1)}{H_\xi\big(x,F^-(x,V,p),p\big)} \;\leq\; 0\,. \tag{62}$$

4.1. Construction of a solution. Consider the function

$$W(x) \;\doteq\; \frac{1}{r}L\big((r-\mu)x\big), \tag{63}$$

with the understanding that $W(x) = +\infty$ if $(r-\mu)x \geq 1$. Notice that $W(x)$ is the total cost of keeping the debt constantly equal to x (in which case there would be no bankruptcy and hence $p \equiv 1$).

Moreover, denote by $(V_B(\cdot), p_B(\cdot))$ the solution to the system of ODEs

$$\begin{cases} V'(x) &= F^-(x,V(x),p(x))\,, \\ p'(x) &= G^-(x,V(x),p(x))\,, \end{cases} \tag{64}$$

with terminal conditions

$$V(x^*) \;=\; B, \qquad\qquad p(x^*) \;=\; \theta(x^*)\,. \tag{65}$$

Notice that the ODE (64) admits a unique local solution around every point (x_0,p_0) with $V(x_0,p_0) = \eta_0$ provided that $H_\xi(x_0,F^-(x_0,\eta_0,p_0),p_0) \neq 0$, i.e., $F^-(x_0,\eta_0,p_0) < \xi^\sharp(x_0,p_0)$, or, equivalently, $r\eta_0 < H^{\max}(x_0,p_0)$. On the other hand, if $V_B(x) < W(x)$, then $H_\xi\,(x, V_B'(x), p_B(x)) > 0$. Assume by contradiction that

$$H_\xi\,\big(x,V_B'(x),p_B(x)\big) \;=\; 0\,.$$

Then we have

$$V_B(x) \;=\; \frac{1}{r}\cdot H^{\max}(x,p_B(x)) \;\geq\; \frac{1}{r}\cdot H^{\max}(x,1) \;=\; W(x)$$

and it yields a contradiction. Thus, (V_B, p_B) is uniquely defined on $[x_1, x^*]$ where the point

$$x_1 \doteq \inf\left\{x \in [0, x^*]\,;\quad V_B(x) < W(x)\right\}. \tag{66}$$

Call $V_1(\cdot)$ the solution to the backward Cauchy problem

$$\begin{cases} V'(x) &= F^-(x, V(x), 1)\,, & x \in [0, x_1], \\ V(x_1) &= W(x_1); \end{cases} \tag{67}$$

we will show that a feedback equilibrium solution to the debt management problem is obtained as follows (see Figure 2):

$$V^*(x) = \begin{cases} V_1(x) & \text{if} \quad x \in [0, x_1], \\ V_B(x) & \text{if} \quad x \in [x_1, x^*], \end{cases} \tag{68}$$

$$p^*(x) = \begin{cases} 1 & \text{if } x \in [0, x_1], \\ p_B(x) & \text{if } x \in]x_1, x^*], \end{cases} \tag{69}$$

$$u^*(x) = \begin{cases} \underset{\omega\in[0,1]}{\operatorname{argmin}} \left\{L(\omega) - \dfrac{(V^*)'(x)}{p^*(x)}\omega\right\} & \text{if } x \neq x_1, \\ (r-\mu)x_1 & \text{if } x = x_1\,. \end{cases} \tag{70}$$

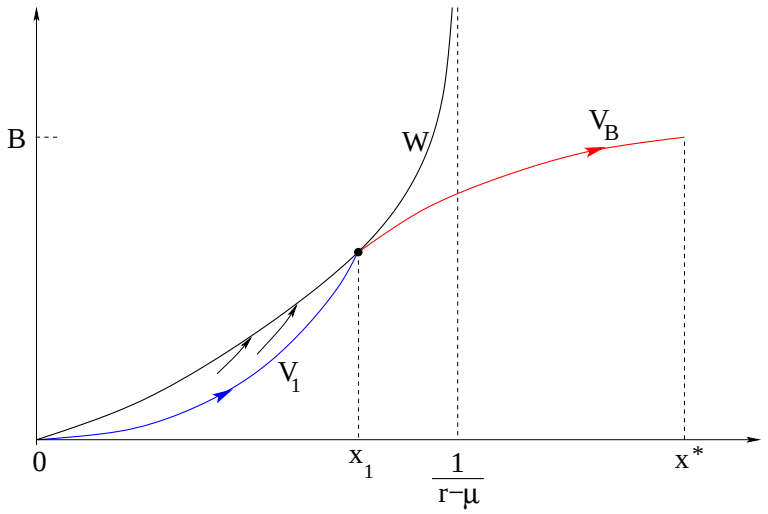

Figure 2. *Constructing the equilibrium solution in feedback form. For an initial value of the debt $x(0) \leq x_1$, the debt increases until it reaches x_1, then it is held at the constant value x_1. If the initial debt is $x(0) > x_1$, the debt keeps increasing until it reaches bankruptcy in finite time.*

Theorem 4.3. *Assume that the cost function L satisfies the assumptions* (A)*, and moreover $L((r-\mu)x^*) > rB$. Then the functions V^*, p^*, u^* in (68)–(70) are well defined and provide an equilibrium solution to the debt management problem, in feedback form.*

Proof.

1. The solution of (64)–(65) satisfies the obvious bounds

$$V' \geq 0, \qquad p' \leq 0, \qquad V(x) \ \leq \ B, \qquad p(x) \in [\theta(x^*), 1].$$

We begin by proving that the function V_B is well defined and strictly positive for $x \in]x_1, x^*]$.
To prove that

$$V_B(x) \ > \ 0 \quad \text{for all } x \in]x_1, x^*]\,,$$

assume, on the contrary, that $V_B(y) = 0$ for some $y > x_1 \geq 0$. From (59), it holds that

$$\xi^\sharp(x,p) \ \geq \ C_1 > 0 \qquad \text{for all } x \in [y, x^*], p \in [\theta(x^*), 1]$$

for some positive constant C_1. Recalling (60), we obtain that

$$H_\xi(x, \xi, p) \ \geq \ C_2 \qquad \text{for all } x \in [y, x^*], p \in [\theta(x^*), 1], \xi \in [0, C_1]$$

for some positive constant C_2. Since $H(x, 0, p) = 0$, the mean value theorem yields

$$H(x, F^-(x, V, p), p) \ \geq \ C_2 \cdot F^-(x, V, p) \quad \text{for all } x \in [y, x^*], p \in [\theta(x^*), 1]$$

provided by $F^-(x, V, p) \leq C_1$. The definition of F^- implies that there exists a constant $\delta_1 > 0$ small such that

$$F^-(x, V, p) \ \leq \ \frac{r}{C_2} \cdot V \tag{71}$$

for all $x \in [y, x^*]$, $p \in [\theta(x^*), 1]$, and $V \in [0, \delta_1]$. Hence, for any solution of (64), $V(y) = 0$ implies $V(x) = 0$ for all $x \geq y$, providing a contradiction.
Next, observe that the functions F^-, G^- are locally Lipschitz continuous as long as $0 \leq V < H^{max}(x, p)$. Moreover, $V(x) < W(x)$ implies

$$V(x) \ < \ W(x) \ = \ H^{max}(x, 1) \ \leq \ H^{max}(x, p(x)).$$

Therefore, the functions V_B, p_B are well defined on the interval $[x_1, x^*]$.

2. If $x_1 = 0$ the construction of the functions V^*, p^*, u^* is already completed in step 1. In the case where $x_1 > 0$, we claim that the function V_1 is well defined and satisfies

$$0 \ < \ V_1(x) \ < \ W(x) \qquad \text{for } 0 < x < x_1\,. \tag{72}$$

Indeed, if $V_1(y) = 0$ for some $y > 0$, the Lipschitz property (71) again implies that $V_1(x) = 0$ for all $x \geq y$. This contradicts the terminal condition in (67).
To complete the proof of our claim (72), it suffices to show that

$$W'(x) \ < \ F^-(x, W(x), 1) \qquad \text{for all } x \in]0, x_1]. \tag{73}$$

This is true because

$$W'(x) = \frac{r-\mu}{r} L'\big(r-\mu)x\big) = \frac{r-\mu}{r}\,\xi^\sharp(x,1) < \xi^\sharp(x,1)$$
$$= F^-\big(x, H^{max}(x,1),1\big) = F^-(x,W(x),1).$$

3. In the remaining steps, we show that V^*, p^*, u^* provide an equilibrium solution. Namely, they satisfy the properties (i)–(ii) in Definition 2.
To prove (i), call $V(\cdot)$ the value function for the optimal control problem (48)–(49). For any initial value, $x(0) = x_0$, in both cases $x_0 \in [0, x_1]$ and $x_0 \in]x_1, x^*]$, the feedback control u^* in (70) yields the cost $V^*(x_0)$. This implies

$$V(x_0) \leq V^*(x_0)\,.$$

To prove the converse inequality we need to show that, for any measurable control $u : [0, +\infty[\mapsto [0,1]$, calling $t \mapsto x(t)$ the solution to

$$\dot x = \left(\frac{\lambda+r}{p_{x_1}(x)} - \lambda - \mu\right) x - \frac{u(t)}{p_{x_1}(x)}\,, \qquad x(0) = x_0, \tag{74}$$

one has

$$\int_0^{T_b} e^{-rt} L(u(t))dt + e^{-rT_b} B \geq V^*(x_0), \tag{75}$$

where

$$T_b = \inf\big\{t \geq 0\,;\ \ x(t) = x^*\big\}$$

is the bankruptcy time (possibly with $T_b = +\infty$).
For $t \in [0, T_b]$, consider the absolutely continuous function

$$\phi^u(t) \doteq \int_0^t e^{-rs} L(u(s))ds + e^{-rt} V^*(x(t)).$$

At any Lebesgue point t of $u(\cdot)$, recalling that (V^*, p^*) solves the system (52), we compute

$$\frac{d}{dt}\phi^u(t) = e^{-rt}\Big[L(u(t)) - rV^*(x(t)) + (V^*)'(x(t))\cdot \dot x(t)\Big]$$
$$= e^{-rt}\left[L(u(t)) - rV^*(x(t)) + (V^*)'(x(t))\left(\left(\frac{\lambda+r}{p^*(x(t))} - \lambda - \mu\right)x(t) - \frac{u(t)}{p^*(x(t))}\right)\right]$$
$$\geq e^{-rt}\left[\min_{\omega\in[0,1]}\left\{L(\omega) - \frac{(V^*)'(x(t))}{p^*(x(t))}\,\omega\right\} + \left(\frac{\lambda+r}{p^*(x(t))} - \lambda - \mu\right)x(t)(V^*)'(x(t)) - rV^*(x(t))\right]$$
$$= e^{-rt}\Big[H\left(x(t), (V^*)'(x(t)), p^*(x(t))\right) - rV^*(x(t))\Big] = 0.$$

Therefore,

$$V^*(x_0) \;=\; \phi^u(0) \;\leq\; \lim_{t\to T_b-} \phi^u(t) \;=\; \int_0^{T_b} e^{-rt} L(u(t))dt + e^{-rT_b} B,$$

proving (75).

4. It remains to check (ii). The case $x_0 = 0$ is trivial. Two main cases will be considered.
Case 1. $x_0 \in]0, x_1]$. Then there exists a solution $t \mapsto x(t)$ of (50) such that $p(t) = 1$ and $x(t) \in]0, x_1]$ for all $t > 0$. Moreover,

$$\lim_{t\to+\infty} x(t) \;=\; x_1\,.$$

In this case, $T_b = +\infty$ and (51) holds.
Case 2. $x_0 \in]x_1, x^*]$. Then $x(t) > x_1$ for all $t \in [0, T_b]$. This implies

$$\dot x(t) \;=\; H_\xi(x(t), V_B(x(t)), p_B(x(t)))\,.$$

From the second equation in (52) it follows that

$$\frac{d}{dt}\, p(t) \;=\; p'(x(t))\dot x(t) = (r+\lambda)(p(t)-1),$$
$$\frac{d}{dt} \ln(1 - p(x(t))) = (r+\lambda)\,.$$

Therefore, for every $t \in [0, T_b]$ one has

$$p(x(0)) \;=\; 1 - (1 - p(x(t))) \cdot e^{-(r+\lambda)t}.$$

Letting $t \to T_b$ we obtain

$$p(x_0) \;=\; 1 - (1 - \theta(x^*))) \cdot e^{-(r+\lambda)T_b},$$

proving (51). ∎

Remark 4.4. In general, however, we cannot rule out the possibility that a second solution exists. Indeed, if the solution V_B, p_B of (64)–(65) can be prolonged backward to the entire interval $[0, x^*]$, then we could replace (68)–(69) simply by $V^*(x) = V_B(x)$, $p^*(x) = p_B(x)$ for all $x \in [0, x^*]$. This would yield a second solution.

We claim that no other solutions can exist. This is based on the fact that the graphs of W and V_B cannot have any other intersection, in addition to 0 and x_1. Indeed, assume on the contrary that $W(x_2) = V_B(x_2)$ for some $0 < x_2 < x_1$ (see Figure 3). Since $p_B(x_2) < 1$ and $W'(x_2) \leq V_B'(x_2)$, the inequalities

$$rW(x_2) \;=\; H(x_2, W'(x_2), 1) \;<\; H(x_2, W'(x_2), p_B(x_2)) \;\leq\; H(x_2, V_B'(x_2), p_B(x_2)) \;=\; rV_B(x_2)$$

yield a contradiction.

Next, let $V^\dagger, p^\dagger$ be any equilibrium solution and define

$$x^\dagger \;\doteq\; \sup\big\{x \in [0, x^*]\,;\;\; p(x) = 1\big\}.$$

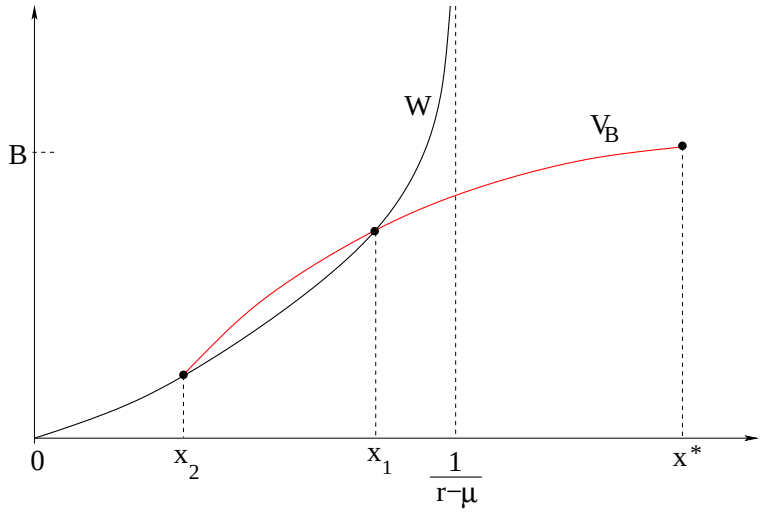

Figure 3. *By the monotonicity properties of the Hamiltonian function H in (53), the graphs of V_B and W cannot have a second intersection at a point $x_2 > 0$.*

Then

- on $]x^\dagger, x^*]$ the functions $V^\dagger, p^\dagger$ provide a solution to the backward Cauchy problem (64)–(65);
- on $]0, x^\dagger]$ the function $V^\dagger$ provides the value function for the optimal control problem

$$\text{minimize: } \int_0^\infty e^{-rt} L(u(t))\, dt$$

subject to the dynamics (with $p \equiv 1$)

$$\dot{x} \;=\; (r-\mu)x - u$$

and the state constraint $x(t) \in [0, x^\dagger]$ for all $t \geq 0$.

The above implies

$$\begin{cases} V^\dagger(x) = V_B(x) \text{ if } x \in [x^\dagger, x^*], \\ V^\dagger(x) \leq W(x) \text{ if } x \in [0, x^\dagger]. \end{cases}$$

Since $V^\dagger$ must be continuous at the point x_2, by the previous analysis this is possible only if $x_2 = 0$ or $x_2 = x_1$.

5. Dependence on the bankruptcy threshold x^*. In this section we study the behavior of the value function V_B when the maximum size x^* of the debt, at which bankruptcy is declared, becomes very large.

From a modeling point of view, this amounts to discussing the possibility of the optimality of a *Ponzi scheme*, in which the debt is serviced by initiating more and more new loans. We will show that under some natural assumptions on the function $\theta(\cdot)$ expressing the fraction recovered by lenders as a function of the debt-to-income ratio at the moment of bankruptcy.

For a given $x^* > 0$, we denote by $V_B(\cdot, x^*)$, $p_B(\cdot, x^*)$ the solution to the system (64) with terminal data (65). Letting $x^* \to \infty$, we wish to understand whether the value function V_B remains positive or approaches zero uniformly on bounded sets. Toward this goal, we introduce the constant

$$M_1 \doteq \frac{2}{r-\mu} \cdot \max\left\{1, \frac{rB}{L'(0)}\right\}. \tag{76}$$

Recalling Lemma 2.3 for $\sigma = 0$, we have

$$H(x,\xi,p) \geq \frac{(r-\mu)x-1}{p} \cdot \xi \qquad \text{for all } x \in [0,x^*], \xi \geq 0\,.$$

Thus, the first equation of (52) implies that

$$rB \geq rV_B(x,x^*) \geq \frac{(r-\mu)x-1}{p_B(x,x^*)} \cdot V_B'(x,x^*), \qquad x \in [0,x^*]\,.$$

In turn, if $x^* > M_1$, this yields

$$\frac{V_B'(x,x^*)}{p_B(x,x^*)} \leq L'(0) \qquad \text{for all } x \in [M_1, x^*]\,.$$

Calling $u = u^*(x)$ the optimal feedback control, by (19) we have

$$u^*(x) = 0 \qquad \text{for all } x \in \big[M_1,\, x^*\big]\,. \tag{77}$$

In this case, the Hamiltonian function takes a simpler form, namely,

$$\begin{aligned} H(x,V',p) &= \big[(\lambda+r) - -(\lambda+\mu)p\big] \cdot \frac{V'x}{p}\,, \\ H_\xi(x,V',p) &= \big[(\lambda+r) - (\lambda+\mu)p\big]x\,. \end{aligned}$$

Therefore, the system of ODEs (64) can be written as

$$\begin{cases} V' &= \dfrac{rp}{[(\lambda+r)-(\lambda+\mu)p]x}\,V\,, \\ p' &= (\lambda+r)\cdot\dfrac{p(p-1)}{[(\lambda+r)-(\lambda+\mu)p]\,x}\,. \end{cases} \tag{78}$$

The second ODE in (78) is equivalent to

$$\frac{d}{dx}\ln\left(\frac{(1-p(x))^{r-\mu}}{p(x)^{r+\lambda}}\right) = \frac{r+\lambda}{x}.$$

Solving backward the above ODE with the terminal data $p(x^*) = \theta(x^*)$, we obtain

$$p_B(x,x^*) = \frac{\theta(x^*)x^*}{x} \cdot \left(\frac{1-p_B(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}} \qquad \text{for all } x \in \big[M_1,\, x^*\big]\,. \tag{79}$$

Therefore,

$$p_B(x,x^*) \;\geq\; \frac{\left(\dfrac{\theta(x^*)x^*}{x}\right)^{\frac{r+\lambda}{r-\mu}}}{1+\left(\dfrac{\theta(x^*)x^*}{x}\right)^{\frac{r+\lambda}{r-\mu}}} \qquad \text{for all } x\in\left]M_1,\, x^*\right]. \tag{80}$$

Different cases will be considered, depending on the properties of the function $\theta(\cdot)$. By obvious modeling considerations, we shall always assume

$$\theta(x)\in[0,1], \qquad \theta'(x)\leq x \qquad \text{for all } x\geq 0.$$

We first study the case where θ has compact support. Recall that M_1 is the constant in (76).

Lemma 5.1. *Assume that*

$$\theta(x) \;=\; 0 \qquad \textit{for all } x\geq M_2 \tag{81}$$

for some constant $M_2\geq M_1$. Then, for any $x^*>M_2$, the solution $V_B(\cdot,x^*)$, $p_B(\cdot,x^*)$ of (64)–(65) satisfies

$$V_B(x,x^*) \;=\; B \qquad \text{and} \qquad p_B(x,x^*) \;=\; 0 \qquad \textit{for all } x\in\left]M_2,\, x^*\right].$$

Proof. By (79) and (81), for every $x^*>M_2$ one has

$$p_B(x,x^*) \;=0 \qquad \text{for all } x\in\left]M_2,\, x^*\right].$$

Inserting this into the first ODE in (78), we obtain

$$V_B'(x,x^*) \;=\; 0.$$

In turn, this yields $V_B(x,x^*) \;=\; B$ for all $x\in\left]M_2,\, x^*\right]$. This means that bankruptcy instantly occurs if the debt reaches M_2. ∎

Next, we now consider that case where $\theta(x)>0$ for all x.

$$\theta(x) \;>\; 0 \qquad \text{for all } x\in[0,\infty[. \tag{82}$$

Lemma 5.2. *If $x^*>M_1$ and $\theta(x^*)>0$, then*

$$V_B(x,x^*) \;=\; B\cdot\left(\frac{p_B(x,x^*)x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \qquad \textit{for all } x\in\left]M_1,\, x^*\right]. \tag{83}$$

In particular, for $x\in\left]M_1,\, x^*\right]$ one has

$$B\cdot\left(1+\left(\frac{\theta(x^*)x^*}{x}\right)^{\frac{r+\lambda}{r-\mu}}\right)^{-\frac{r}{r+\lambda}} \;\leq\; V_B(x,x^*) \;\leq\; B\cdot\left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}}. \tag{84}$$

Proof. Since $p_B(x, x^*)$ solves the second equation of (78) and $p_B(x^*, x^*) = \theta(x^*) \in (0, 1)$, we have that $x \mapsto p_B(x, x^*)$ is a strictly decreasing function of x. For a fixed value of x^*, let $p \mapsto \chi(p) : [\theta(x^*), 1[\mapsto [0, x^*]$ be the inverse function of $p_B(\cdot, x^*)$. From (78), a direct computation yields

$$\begin{cases} \dfrac{d}{dp} V_B(\chi(p), x^*) &= \dfrac{rp}{[(\lambda + r) - (\lambda + \mu)p]\,\chi(p)} \cdot V_B(\chi(p), x^*) \cdot \chi'(p)\,, \\ \dfrac{d}{dp} p_B(\chi(p), x^*) &= (\lambda + r) \cdot \dfrac{p(p-1)}{[(\lambda + r) - (\lambda + \mu)p] \cdot \chi(p)} \cdot \chi'(p) = 1\,. \end{cases} \tag{85}$$

From (85) it follows that

$$\frac{d}{dp} \ln V_B(\chi(p), x^*) \;=\; \frac{r}{\lambda + r} \cdot \frac{1}{p - 1}\,.$$

Solving the above ODE with the terminal data $V_B(x^*, x^*) = B$, $p_B(x^*, x^*) = \theta(x^*)$, we obtain

$$V_B(\chi(p), x^*) \;=\; \left(\frac{1-p}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}} B\,, \tag{86}$$

hence

$$V_B(x, x^*) \;=\; \left(\frac{1 - p_B(x, x^*)}{1 - \theta(x^*)}\right)^{\frac{r}{r+\lambda}} B.$$

Recalling (79), a direct computation yields (83). The upper and lower bounds for $V_B(x, x^*)$ in (84) now follow from (80) and the inequality $p_B(x, x^*) \leq 1$. ∎

Corollary 5.3. *Assume that*

$$\limsup_{x \to +\infty} \theta(x)\, x \;=\; +\infty. \tag{87}$$

Then the value function $V^* = V^*(x, x^*)$ satisfies

$$\lim_{x^* \to +\infty} V(x, x^*) \;=\; 0 \qquad \text{for all } x \geq 0\,. \tag{88}$$

Indeed, for $x \geq M_1$ we have $V(x, x^*) = V_B(x, x^*)$, and (88) follows from the second inequality in (84). When $x < M_1$, since the map $x \mapsto V(x, x^*)$ is nondecreasing, we have

$$0 \;\leq\; \lim_{x^* \to \infty} V(x, x^*) \;\leq\; \lim_{x^* \to \infty} V(M_1, x^*) \;=\; 0\,.$$

Corollary 5.4. *Assume that*

$$R \;\doteq\; \limsup_{x \to +\infty} \theta(x) \cdot x \;<\; +\infty. \tag{89}$$

Then

$$V_B(x, x^*) \;\geq\; B \cdot \left(1 + \left(\frac{R}{x}\right)^{\frac{r+\lambda}{r-\mu}}\right)^{-\frac{r}{r+\lambda}} \qquad \text{for all } \; x^* > x > M_1\,. \tag{90}$$

Moreover, the following holds:

(i) *If*

$$\frac{\theta'(x)}{\theta(x)} + \frac{1}{x} \;\geq\; 0 \qquad \textit{and} \qquad \theta'(x) \;\leq\; 0 \qquad \textit{for all} \;\; x > 0, \tag{91}$$

then

$$\inf_{x^*>0} V_B(x, x^*) \;=\; \lim_{x^*\to\infty} V_B(x, x^*) \;>\; 0 \qquad \textit{for all } x \geq M_1\,. \tag{92}$$

(ii) *Assume that there exist* $0 < \delta < 1$ *such that*

$$\delta \cdot \frac{\theta'(x)}{\theta(x)} + \frac{1}{x} \;<\; 0 \tag{93}$$

for all x *sufficiently large. Then, for each* $x > M_1$*, there exists an optimal value* $x^* = x^*(x)$ *such that*

$$V_B(x, x^*(x)) \;=\; \inf_{x^*\geq 0} V_B(x, x^*). \tag{94}$$

Proof. It is clear that (90) is a consequence of (84) and (89). We only need to prove (i) and (ii). For a fixed $x \geq M_1$, we consider the functions of the variable x^* alone:

$$Y(x^*) \;\doteq\; V_B(x, x^*), \qquad\qquad q(x^*) \;\doteq\; p_B(x, x^*).$$

Using (83) and (79), we obtain

$$\frac{Y'(x^*)}{Y(x^*)} \;=\; \frac{r}{r-\mu} \cdot \left(\frac{q'(x^*)}{q(x^*)} - \left[\frac{\theta'(x^*)}{\theta(x^*)} + \frac{1}{x^*} \right] \right) \tag{95}$$

and

$$\frac{q'(x^*)}{q(x^*)} \;=\; \frac{\theta'(x^*)x^* + \theta(x^*)}{\theta(x^*)x^*} + \frac{r-\mu}{r+\lambda} \cdot \left(\frac{-q'(x^*)}{1-q(x^*)} + \frac{\theta'(x^*)}{1-\theta(x^*)} \right). \tag{96}$$

This implies

$$\begin{aligned} \frac{q'(x^*)}{q(x^*)} - \left[\frac{\theta'(x^*)}{\theta(x^*)} + \frac{1}{x^*} \right] &= \left[\frac{1}{1 + \frac{r-\mu}{r+\lambda} \cdot \frac{q(x^*)}{1-q(x^*)}} - 1 \right] \cdot \left[\frac{\theta'(x^*)}{\theta(x^*)} + \frac{1}{x^*} \right] \\ &\quad + \frac{r-\mu}{(r+\lambda)\left(1 + \frac{r-\mu}{r+\lambda} \cdot \frac{q(x^*)}{1-q(x^*)}\right)} \cdot \frac{\theta'(x^*)}{1-\theta(x^*)}\,. \end{aligned} \tag{97}$$

If (91) holds, then (95) and (97) imply

$$\frac{Y'(x^*)}{Y(x^*)} \;=\; \frac{q'(x^*)}{q(x^*)} - \left[\frac{\theta'(x^*)}{\theta(x^*)} + \frac{1}{x^*} \right] \;\leq\; 0 \qquad \text{for all } x^* > x \geq M_1.$$

Hence the function Y is nonincreasing. This proves (92).

To prove (ii), we observe that

$$\limsup_{x^*\to\infty} \left(\frac{1}{1+\frac{r-\mu}{r+\lambda}\cdot\frac{q(x^*)}{1-q(x^*)}} - 1 \right) \;<\; 0\,, \qquad\qquad \lim_{x^*\to\infty} \theta(x^*) \;=\; 0\,.$$

Hence (93) and (97) imply

$$\frac{q'(x^*)}{q(x^*)} - \left[\frac{\theta'(x^*)}{\theta(x^*)} - \frac{1}{x^*}\right] \;>\; 0$$

for all x^* sufficiently large. By (95) this yields

$$\frac{Y'(x^*)}{Y(x^*)} \;>\; 0$$

for all x^* large enough. Hence there exists some particular value $x^*(x) \geq x$ where the function $x^* \mapsto Y(x^*) = V_B(x, x^*)$ attains its global minimum. This yields (94). ∎

6. Dependence on x^* in the stochastic case. In this section we study how the value function depends on the bankruptcy threshold x^* in the stochastic case where $\sigma > 0$.

Extensions of Corollaries 5.3 and 5.4 will be proved, constructing upper and lower bounds for a solution $V(\cdot, x^*)$, $p(\cdot, x^*)$ of the system (22)–(23), in the form

$$V_2(x) \;\leq\; V(x, x^*) \;\leq\; V_1(x), \qquad\qquad p_1(x) \;\leq\; p(x, x^*) \;\leq\; p_2(x), \tag{98}$$

where

(i) for any $V(\cdot,\cdot)$ with $V_x \geq 0$, the functions $p_1(\cdot)$ and $p_2(\cdot)$ are a subsolution and a supersolution of the second equation in (31), respectively;

(ii) for any $p(\cdot,\cdot)$ with $p \in [0,1]$ and $p_x \leq 0$, the functions $V_1(\cdot)$ and $V_2(\cdot)$ are a supersolution and a subsolution of the first equation in (31), respectively.

1. We begin by constructing a suitable pair of functions V_1, p_1. Let $(p_1, \widetilde{V}_1)$ be the solution to the backward Cauchy problem

$$\begin{cases} r\widetilde{V}_1(x) &= \left(\dfrac{\lambda+r}{p_1} + \sigma^2\right) x\widetilde{V}_1'\,, \\[2ex] (r+\lambda)(p_1-1) &= \left(\dfrac{\lambda+r}{p_1} + \sigma^2\right) xp_1'\,, \end{cases} \qquad \begin{cases} \widetilde{V}_1(x^*) &= B, \\[2ex] p_1(x^*) &= \theta(x^*). \end{cases} \tag{99}$$

This solution satisfies

$$p_1(x) \;=\; \frac{\theta(x^*)x^*}{x} \cdot \left(\frac{1-p_1(x)}{1-\theta(x^*)}\right)^{\frac{\sigma^2+\lambda+r}{\lambda+r}}, \qquad \lim_{x\to 0+} p_1(x) \;=\; 1\,, \tag{100}$$

$$\widetilde{V}_1(x) \;=\; B\cdot\left(\frac{1-p_1(x)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}, \qquad \lim_{x\to 0+} \widetilde{V}_1(x) \;=\; 0\,. \tag{101}$$

Using (99) and (100) one obtains

$$\begin{aligned} -1 = \; & p_1'(x) \cdot \left(\frac{x}{p_1(x)} + \frac{\sigma^2 + r + \lambda}{r + \lambda} \cdot \frac{x}{1 - p_1(x)} \right) \\ = \; & p_1'(x) \cdot \left(\frac{x}{p_1(x)} + \frac{\sigma^2 + r + \lambda}{r + \lambda} \cdot \frac{1 - \theta(x^*)}{(\theta(x^*)x^*)^{\frac{r+\lambda}{r+\lambda+\sigma^2}}} \cdot \frac{x^{\frac{\sigma^2}{r+\lambda+\sigma^2}}}{p_1(x)^{\frac{\lambda+r}{\lambda+r+\sigma^2}}} \right) . \end{aligned}$$

Since p_1 is monotone decreasing, it follows that $p_1''(x) > 0$ for all $x \in]0, x^*[$. In turn, this yields

$$(r + \lambda)(1 - p_1) + \Big(\frac{\lambda + r}{p_1} + \sigma^2 \Big) x p_1' + \frac{\sigma^2 x^2}{2} p_1'' \; > \; 0 \, .$$

Recalling (25), we have

$$(r + \lambda)(1 - p_1) + H_\xi(x, \xi, p_1) p_1' + \frac{\sigma^2 x^2}{2} p_1'' \; > \; 0 \qquad \text{for all } \xi \geq 0. \tag{102}$$

Next, differentiating both sides of the first ODE in (99), we obtain

$$\left(r - \sigma^2 - \frac{\lambda + r}{p_1} + \frac{(\lambda + r) p_1'}{p_1^2} x \right) \cdot \widetilde{V}_1' \; = \; \left(\frac{\lambda + r}{p_1} + \sigma^2 \right) x \widetilde{V}_1'' \qquad \text{for all } x \in]0, x^*[\, .$$

This implies

$$\widetilde{V}_1''(x) \; < \; 0 \qquad \text{for all } x \in]0, x^*[\, .$$

Recalling (24) and (99), we obtain

$$- r \widetilde{V}_1 + H(x, \widetilde{V}_1', p_1) + \frac{\sigma^2 x^2}{2} \widetilde{V}_1'' \; < \; 0 \, . \tag{103}$$

When $x \geq \frac{1}{\lambda + r}$, the map $p \mapsto H(x, \xi, p)$ is monotone decreasing. Defining

$$V_1(x) \; \doteq \; \begin{cases} \widetilde{V}(\frac{1}{r+\lambda}) & \text{for } \; x \in \left[0, \frac{1}{r+\lambda} \right], \\ \widetilde{V}(x) & \text{for } \; x \in \left[\frac{1}{r+\lambda}, x^* \right], \end{cases}$$

we thus have

$$- r V_1(x) + H(x, V_1'(x), q) + \frac{\sigma^2 x^2}{2} V_1''(x) \; \leq \; 0 \qquad \text{for all } q \geq p_1(x) \, . \tag{104}$$

2. We now construct the functions V_2, p_2. Defining

$$\tilde{p}_2(x) \; \doteq \; \frac{1}{x} \left(\theta(x^*) x^* + \frac{2}{r - \mu} \right) ,$$

a straightforward computation yields

$$\tilde{p}_2'(x) \;=\; -\frac{\tilde{p}_2(x)}{x} \;<\; 0\,, \qquad\qquad \tilde{p}_2''(x) \;=\; 2\cdot\frac{\tilde{p}_2(x)}{x^2}\,.$$

Set

$$x_2 \;\doteq\; \theta(x^*)x^* + \frac{2}{r-\mu}\,, \tag{105}$$

and consider the continuous function

$$p_2(x) \;=\; \min\,\big\{1,\,\tilde{p}_2(x)\big\}. \tag{106}$$

For $x\in[0,x_2[$ one has $p_2(x)=1$ and hence

$$(r+\lambda)(1-p_2)+H_\xi(x,\xi,p_2)p_2'+\frac{\sigma^2x^2}{2}p_2'' \;=\; 0\,.$$

On the other hand, for $x\in]x_2,x^*[$ and $\xi\geq 0$, one has $p_2(x)=\tilde{p}_2(x)<1$, and

$$\begin{aligned} H_\xi(x,\xi,p_2) \;&\geq\; \frac{(\lambda+r)x-1}{p_2(x)}+(\sigma^2-\lambda-\mu)x \;\geq\; (r-\mu)x_2-\frac{1}{\tilde{p}_2(x_2)} \\ &=\; x_2\cdot\left[(r-\mu)-\frac{1}{\theta(x^*)x^*+\frac{2}{r-\mu}}\right] \;\geq\; \frac{r-\mu}{2}\cdot x \;>\; 0\,. \end{aligned} \tag{107}$$

Recalling (25), we get

$$\begin{aligned} &(r+\lambda)(1-p_2)+H_\xi(x,\xi,p_2)p_2'+\frac{\sigma^2x^2}{2}p_2'' \\ &\qquad\leq\; (r+\lambda)(1-p_2)+\left[\frac{(\lambda+r)x-1}{p_2}+(\sigma^2-\lambda-\mu)x\right]p_2'(x)+\frac{\sigma^2x^2}{2}p_2'' \\ &\qquad=\; (r+\lambda)(1-p_2)-\left[\frac{(\lambda+r)x-1}{p_2}+(\sigma^2-\lambda-\mu)x\right]\cdot\frac{p_2(x)}{x}+\sigma^2p_2 \\ &\qquad=\; (r+\lambda)(1-p_2)-\left[(\lambda+r)-\frac{1}{x}+(\sigma^2-\lambda-\mu)p_2\right]+\sigma^2p_2 \\ &\qquad=\; \frac{1}{x}-(r-\mu)p_2=-\frac{(r-\mu)\theta(x^*)x^*}{x}-\frac{1}{x} \;<\; 0\,. \end{aligned}$$

In particular,

$$(r+\lambda)(1-p_2)+H_\xi(x,\xi,p_2)\cdot p_2'+\frac{\sigma^2x^2}{2}p_2'' \;\leq\; 0 \qquad \text{for all } x\in\,]0,x^*[,\;\xi\geq 0\,. \tag{108}$$

Next, define

$$V_2(x) \doteq (1 - p_2(x))\, B \qquad \text{for all } x \in [0, x^*]\,. \tag{109}$$

For all $x \in [0, x_2]$, we thus have $V_2(x) = 0$, and hence

$$-rV_2 + H(x, V_2', q) + \frac{\sigma^2 x^2}{2} V_2'' \;=\; H(x, 0, q) \;=\; 0 \qquad \text{for all } q \in]0, 1]\,. \tag{110}$$

On the other hand, for $x \in]x_2, x^*]$ we have

$$V_2'(x) \;=\; B \cdot \frac{p_2(x)}{x} \;>\; 0 \qquad \text{and} \qquad V_2''(x) \;=\; -2B \cdot \frac{p_2(x)}{x^2}\,.$$

Recalling (24), (106), (107), and (105), we estimate

$$\begin{aligned}
&-rV_2 + H(x, V_2', p_2) + \frac{\sigma^2 x^2}{2} V_2'' \\
&\quad\geq\; -rV_2 + \left(\frac{(\lambda + r)x - 1}{p_2} + (\sigma^2 - \lambda - \mu)x \right) V_2' + \frac{\sigma^2 x^2}{2} V_2'' \\
&\quad= B \cdot \left[rp_2 - r + \left(\lambda + r - \frac{1}{x} + (\sigma^2 - \lambda - \mu) p_2(x) \right) - \sigma^2 p_2 \right] \\
&\quad= B \cdot \left(\lambda - \frac{1}{x} - (\lambda + \mu - r) p_2 \right) = B \cdot \left[\lambda(1 - p_2(x)) + (r - \mu) p_2(x) - \frac{1}{x} \right] \;>\; 0
\end{aligned}$$

for all $x \in]x_2, x^*[$.
Recalling (105), one has

$$(\lambda + r)x \;>\; 1 \qquad \text{for all } x \in]x_2, x^*]\,.$$

Therefore the map $p \to H(x, V_2'(x), p)$ is monotone decreasing on $[0, 1]$ for all $x \in]x_2, x^*]$. This implies

$$-rV_2 + H(x, V_2', q) + \frac{\sigma^2 x^2}{2} V_2'' \;\geq\; 0 \qquad \text{for all } x \in]x_2, x^*], \quad q \in]0, p_2(x)]\,.$$

Together with (110), we finally obtain

$$-rV_2(x) + H(x, V_2'(x), q) + \frac{\sigma^2 x^2}{2} V_2''(x) \;\geq\; 0 \qquad \text{for all } x \in]0, x^*[\,, \;\; q \in]0, p_2(x)]\,. \tag{111}$$

Relying on (102), (103), (108), and (111) and using the same comparison argument as in the proof of Theorem 3.1 we now prove the following.

Theorem 6.1. *In addition to* (A1)*, assume that* $\sigma > 0$ *and* $\theta(x^*) > 0$*. Then the system* (22) *with boundary conditions* (23) *admits a solution* $(V(\cdot, x^*), p(\cdot, x^*))$ *satisfying the bounds* (98) *for all* $x \in [0, x^*]$.

Proof.

1. Recalling $\mathcal{D}$ in (35), we claim that the domain

$$\mathcal{D}^0 \;=\; \Big\{(V,p)\in\mathcal{D}\;\Big|\;(V(x),p(x))\in[V_2(x),V_1(x)]\times[p_1(x),p_2(x)]\quad\text{for all } x\in[0,x^*]\Big\} \tag{112}$$

is positively invariant for the semigroup $\{S_t\}_{t\geq 0}$, generated by the parabolic system (32)–(33). Namely,

$$S_t(\mathcal{D}^0)\;\subseteq\;\mathcal{D}^0\qquad\text{for all } t\geq 0\,.$$

Indeed, from the proof of Theorem 3.1, we have

$$p_x(t,x)\;\leq\;0\;\leq\;V_x(t,x)\qquad\text{for all } t>0, x\in]0,x^*[\,. \tag{113}$$

We now observe the following:

(i) For any $V(\cdot,\cdot)$ with $V_x\geq 0$, by (102) the function $p(t,x)=p_1(x)$ is a subsolution of the second equation in (31). Similarly, by (108), the function $p(t,x)=p_2(x)$ is a supersolution.

(ii) For any $p(\cdot,\cdot)$ with $p\in[0,1]$ and $p_x\leq 0$, by (104) the function $V(t,x)=V_1(x)$ is a supersolution of the first equation in (31). Similarly, by (111), the function $V(t,x)=V_2(x)$ is a subsolution.

Together, (i)–(ii) imply the positive invariance of the domain $\mathcal{D}^0$.

2. Using the same argument as in step 4 of the proof of Theorem 3.1, we conclude that the system (22)–(23) admits a solution $(V,P)\in\mathcal{D}^0$. ∎

Corollary 6.2. *Let the assumptions in Theorem 6.1 hold. If*

$$\limsup_{s\to+\infty}\;\theta(s)\,s\;=\;+\infty,$$

then, for all $x\geq 0$, the value function $V(\cdot,x^*)$ satisfies

$$\lim_{x^*\to\infty}V(x,x^*)\;=\;0. \tag{114}$$

Proof. Using (47), (101), and Theorem 6.1, we have the estimate

$$V(x,x^*)\;\leq\;V_1(x)\;=\;B\cdot\left(\frac{1-p_1(x)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}\;\leq\;B\cdot\left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{r+\lambda+\sigma^2}}$$

for all $x\geq\frac{1}{r+\lambda}$. This implies that (114) holds for all $x\geq\frac{1}{r+\lambda}$. Since $x\mapsto V(x,x^*)$ is monotone increasing, we then have

$$0\;\leq\;\lim_{x^*\to\infty}V(x,x^*)\;\leq\;\lim_{x^*\to\infty}V\left(\frac{1}{r+\lambda},\,x^*\right)\;=\;0\qquad\text{for all }\;x\in\left[0,\;\frac{1}{r+\lambda}\right].$$

This completes the proof of (114). ∎

Corollary 6.3. *Let the assumptions in Theorem 6.1 hold. If*

$$C_1 \doteq \limsup_{s\to+\infty} \theta(s)\, s < +\infty,$$

then

$$\liminf_{x^*\to\infty} V(x, x^*) \geq B \cdot \left(1 - \frac{C_2}{x}\right) \qquad \text{for all } x > M_2\,, \tag{115}$$

where the constants C_2, M_2 are defined as

$$C_2 \doteq C_1 + \frac{2}{r-\mu} \qquad \text{and} \qquad M_2 \doteq \frac{\lambda+\mu-r}{\lambda} C_1 + \frac{2\lambda+\mu-r}{\lambda(r-\mu)} + 1\,.$$

Proof. This follows from (106), (109), and Theorem 6.1. ∎

7. Concluding remarks. If the upper bound for the debt size (beyond which bankruptcy instantly occurs) is allowed to be $x^* = +\infty$, then the equations (52) admit the trivial solution $V(x) = 0$, $p(x) = 1$, for all $x \geq 0$. This corresponds to a Ponzi scheme, producing a debt whose size grows exponentially, without bounds. In practice, this is not realistic because there is a maximum amount of liquidity that the market can supply. It is interesting to understand what happens when this bankruptcy threshold x^* is very large.

Our analysis has shown that three cases can arise, depending on the fraction θ of outstanding capital that lenders can recover.

(i) If $\lim_{s\to+\infty} \theta(s)\, s = +\infty$, then for the borrower it is convenient to have x^* as large as possible. Indeed, the expected total cost for servicing the debt approaches zero as $x^* \to +\infty$.

(iii) If $\lim_{s\to+\infty} \theta(s)\, s < +\infty$ and (91) holds, then for the borrower it is still convenient to have x^* as large as possible. However, as $x^* \to +\infty$, the expected total cost for servicing the debt remains uniformly positive.

(iii) If $\lim_{s\to+\infty} \theta(s)\, s < +\infty$ and (93) holds, then for every initial value x_0 of the debt there is a value $x^*(x_0)$ of the bankruptcy threshold which is optimal for the borrower.

Examples corresponding to three cases (i)–(iii) are obtained by taking

$$\theta(s) = \min\left\{1, \frac{R_0}{s^\alpha}\right\} \tag{116}$$

with $0 < \alpha < 1$, $\alpha = 1$, or $\alpha > 1$, respectively.

In case (iii) we observe that even if the bankruptcy threshold x^* were not imposed by the external market but could be selected by the borrower in an optimal way, this choice of x^* could never "time consistent." Indeed, assume that at the initial time $t = 0$ the borrower announces that he will declare bankruptcy when the debt reaches size x^*. Based on this information, the lenders determine the discounted price of bonds. However, when the time T_b comes when $x(T_b) = x^*$, it is never convenient for the borrower to declare bankruptcy. It is the creditors, or an external authority, that must actually enforce bankruptcy.

To see this, assume that at time T_b when $x(T_b) = x^*$ the borrower announces that he has changed his mind and will declare bankruptcy only at the later time T_b' when the debt reaches $x(T_b') = 2x^*$. If he chooses a control $u(t) = 0$ for $t > T_b$, his discounted cost will be

$$e^{-(T_b'-T_b)r} B < B.$$

This new strategy is thus always convenient for the borrower. On the other hand, it can be much worse for the lenders. Indeed, consider an investor having a unit amount of outstanding capital at time T_b. If bankruptcy is declared at time T_b, he will recover the amount $\theta(x^*)$. However, if bankruptcy is declared at the later time T_b', his discounted payoff will be

$$\int_{T_b}^{T_b'} (r+\lambda) e^{-(r+\lambda)(t-T_b)}\, dt + e^{-(r+\lambda)(T_b'-T_b)}\theta(2x^*).$$

To appreciate the difference, consider the deterministic case, assuming that $\theta(\cdot)$ is the function in (116), with $\alpha \geq 1$, and that $x^* \geq M_1$. By the analysis at the beginning of section 5, we have $u^*(x) = 0$ for all $x \in [x^*, 2x^*]$. Replacing x^* with $2x^*$ in (79) we obtain that the solution to (78) with terminal data

$$p(2x^*) \;=\; \theta(2x^*) \;=\; \frac{R_0}{(2x^*)^\alpha}$$

satisfies

$$p_B(x^*, 2x^*) \;=\; 2\theta(2x^*)\cdot\left(\frac{1-p_B(x^*,2x^*)}{1-\theta(2x^*)}\right)^{\frac{r-\mu}{r+\lambda}} \;<\; 2\theta(2x^*) \;=\; 2^{1-\alpha}\theta(x^*) \;\leq\; \theta(x^*)\,.$$

If the investors had known in advance that bankruptcy is declared at $x = 2x^*$ (rather than at $x = x^*$), the bonds would have fetched a smaller price.

In conclusion, if the bankruptcy threshold x^* is chosen by the debtor, the only equilibrium can be $x^* = +\infty$. In this case, the model still allows bankruptcy to occur, when total debt approaches infinity in finite time.

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