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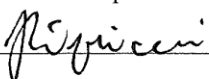
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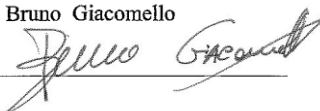
**A new mathematical framework for the balance sheet dynamic
modeling applied to CFaR and LaR**

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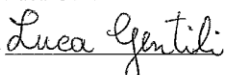
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A New Mathematical Framework for the Balance Sheet Dynamic Modeling applied to CFaR and LaR

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Abstract

In this thesis we intend to introduce a new theoretical framework that will allow us to define a new set of balance sheet mathematical models. The elements of this class will enable their user to produce a dynamical representation of an entire balance sheet through finite difference linear systems characterized by the possibility to be presented as a function with a closed form formula. In the first chapter after a historical introduction about the long and close relationship between mathematics and accounting, with special regard to the link between the balance sheet modeling and linear algebra, we will focus on the in depth mathematical analysis, present in literature, of the double-entry book-keeping tool. As a matter of fact, to this day, the double entry bookkeeping system can be considered as the reference mathematical theoretical framework of the accounting practice and thinking. The result of this analysis, combined with the results already present in literature about the dynamic representation of the balance sheet through first order finite difference linear systems, will enable us to introduce a new idea relating to a vector specifically built to describe the link between a single accounting item and the liquidity. We called it brick-vector since we can build a balance sheet model merging into an algebraic system the brick-vectors of all the accounting items chosen for our modelization. The brick-vector concept, combined with the theoretical framework previously introduced, will allow us to reach a new modelization presenting the aforementioned characteristics, i.e. a dynamic representation of an entire balance sheet, constituted by accounting items of our choosing, that can also be presented through a closed form formula. We think these peculiar traits can turn this class of models into an agile tool useful for different applications in several areas of research. As a matter of fact we presume they will enable the researcher to exploit both the possibilities offered by an algorithmic implementation of the model as well as the possibilities presented by pure mathematics through its closed form formula. In order to start applying this class of models we close the first chapter presenting an averaging procedure (based on the concept of functional mean according to Chisini) that allows us to reduce the impact of the inevitably high number of variables that a balance sheet model time series brings with itself. In the second chapter we intend to show some of the possibilities offered by the brick vector formalization applying it to the problem of the cash flow risk assessment. Firstly we present a medium firm balance sheet model and we explore its closed form solution. Then we perform on the model our Chisini averaging procedure during which we present its relative mathematical shape. Finally after the introduction of a sensitivity analysis, in order to show some of the descriptive capabilities of the model, we apply it to the problem of cash flow risk assessment. We present the approaches proposed so far toward the issue of the computation of CFaR (Cash Flow at Risk) and then we propose our new methodology. It has the goal to overcome some of the main shortcomings of the previous approaches through the creation of a link between the accounting data, summarizing the firm's business structure, and some macroeconomic drivers of particular importance. We end the chapter presenting a case study relating to Alitalia airlines where we apply the model to its balance sheet data and we perform our CFaR eval-

uation. In the third chapter we intend to keep on exploring the potential of the brick vector formalization applying it to the problem of the liquidity risk assessment in the banking sector. After an introduction to the issue of liquidity risk as well as that of the bank's balance sheet modeling we present a commercial bank balance sheet model. Then we show its closed form solution and we perform our averaging procedure. We display the commercial bank balance sheet model evolution through a simulation aiming to portray the behavior of medium sized Italian commercial bank. Finally we discuss the problem of the liquidity risk assessment and we propose a new liquidity risk measure, tailored on the issue of funding liquidity, which is based on the CFaR methodology presented in the previous chapter. We called this new measure FLaR (Funding Liquidity at Risk) and through its medium-term time perspective it is meant to complement the role performed by the LaR (Liquidity at Risk) instrument in a short-term temporal perspective. We close the chapter presenting some future possible developments in the application of the brick vector framework to the liquidity risk assessment issue. Finally we conclude our thesis reviewing its content in relation to its goal to try to bridge the gap between the accounting field and other research areas of the economic science as well as the world of economic theory with that of economic practice.

Keywords: Balance sheet, Mathematical model, Difference equation system, Closed form solution, Dynamical balance sheet model, Cash Flow, Liquidity, Risk, CFaR, LaR.

A New Mathematical Framework for the
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Introduction

When we started this research several years ago, we wanted to create a Budgeting Model that could work in a dynamic framework capable of providing information about the financial position, income, assets and liabilities of an enterprise. Using the results obtained by Castagnoli et al. (1995), which represented the principal accounting items of a balance sheet through difference equations, we decided to describe an entire balance sheet by means of a set of difference equations. Then this single attempt led to a comprehensive mathematical framework having the aim of formalizing balance sheet models according to the need felt by the researcher.

In order to achieve that goal we started from the analysis of the historical relationship between mathematics and accounting, so to understand not only which one is the current mathematical framework of the accounting practice as well as its motifs, but also to obtain a thorough knowledge of the previous attempts at accounting formalization, as well as their results. This historical relationship is part of the stream of accounting research that in the English speaking academia is referred to as analytical accounting, dealing with the formalization of the accounting field. Unfortunately the different lines of research that we identified didn't seem, to this day, to have a strong and lasting impact on the accounting world both the academic one as well as the practitioner one (see Mattesich (2000), (2005)). As a matter of fact one of the many reasons of complaint among the accounting world about accounting research is related to the existence of a huge gap between the academic research and the accounting practice (see ACCA (2010), Guthrie et al. (2011), Laughlin et al. (2011), Unerman et al. (2009)). One of the main hopes, related to the research we are presenting, is that of contributing to the lessening of the aforementioned gap as well as to the possible creation of a link between the accounting field, with all its capability of providing descriptive data, and other areas of research in the economic and financial academic world.

Given the previous conditions we identified, as the current mathematical framework of the accounting discipline, the one implied by the double entry book keeping system whose best formalization attempts (see Ellerman (1984)) identify its foundations on two principles. The former dealing with the group of couples of positive numbers associated to the Debit and Credit shape of the system, while the latter related to the encoding of the transactions into the Balance Sheet equation.

Our mathematical framework moves from the mutation of the two aforementioned characteristics in order to focus on the streams of wealth in and out of the firm associated with the transaction records. In so doing it tries to achieve a formalization that not only can be implemented via computer but can also be presented in pure mathematical shape through a closed form formula. In order to achieve that goal we will introduce a new concept, that of a vector dealing with the relationship between an accounting item and the liquidity item. We called it brick vector since upon it we can build any balance sheet modelization that we are in the need of building. The final model will be a simple combination of the brick vectors related to the accounting items we choose to be present in our system. Once we have achieved the formalization we intended, with its closed form formula, given the high number of variables and parameters involved in any balance sheet model, we will briefly address the problem of the modelization of the time series related to those variables and parameters. So we will present an averaging procedure, coherent with Chisini's definition (see Iurato 2012), in order to achieve the simplest possible shape for our modelizations.

Once the modelization theory is fully presented we will give two examples of its feasible applications, examples that we intended as a hint of the possibilities implicit in this mathematical framework. Since the liquidity item plays such a key role in the brick vector approach we thought reasonable to start applying it to cash flow related issues. Firstly we will present the modelization of an industrial firm balance sheet and we will use it to produce a new approach to the cash flow at risk (CFaR) measure. The CFaR moves from the Value at Risk measure (VaR) and it is intended to be applied to industrial firms in order to assess the probability distribution of their cash flow generating capability. To this day three main approaches have been proposed but none of them has still gained general consensus (see Yan 2014, Andrn 2005, RiskMetrics 1999, Stein 2001). After presenting those approaches, with their main critical issues, we will introduce our CFaR measure, which tries to link the historical balance sheet data, via the Chisini-like averaging procedure, to some macro variables (about which there is a relative general consensus on how to model them). Our CFaR approach will be proposed through an application to a real life case study, namely the last seven years of the Alitalia company balance sheet, before its default.

The second example on which we will apply our new mathematical framework will be about the construction of a formalization for a commercial bank balance sheet that we will use to present a measure of Liquidity distress. The problem of risk related to liquidity in the case of a bank, given the inner workings of the financial sector, takes very specific and complex features. The development of modelling bank liquidity has thus been rather slow despite bankers ranking liquidity risk as one of the top five risks to consider (DCSFI 2010). As recently as 2007 Fiedler (2007) argued about the lack of a sophisticated method to capture a bank's liquidity position. The topic has raised in importance and urgency after the 2007/2008 financial crisis. Nonetheless the management of the liquidity risk is handled internally by each banking firm according to private models and approaches (see Castagna et al. 2013, Matz et al. 2006). Even

the regulator position, enforced through stress testing exercises coherently with the Basel III regulations, is not free of diverse shortcomings pointed out in recent literature, especially for their static approach (see Halaj 2016, Henry et al. 2013, RTF 2015). Here we just want to propose, through a simulation, an approach to the liquidity probability distribution measurement, tailored to the issue of funding liquidity risk, that in a dynamic fashion could be able to relate the history of the balance sheet dynamics to one or more important stochastic macroeconomic factors.

Finally we conclude the present work remarking our hope that the mathematical framework we have developed could present itself as a useful and agile tool to bring together the expertise of the accounting field (as well as the huge amount of economic and financial data present in the balance sheet which remains the most informative instrument in the economic practice) with the world of the economic and financial scientific research.

Chapter 1

A New Mathematical Framework for the Balance Sheet Dynamic Modeling

1.1 Introduction

Through the present work we intend to introduce a new theoretical mathematical framework that enables us to define a family of models aiming to describe the balance sheet dynamically. The representation, that this class of models wants to introduce, presents the following features: it aims to a modelization of the entire balance sheet, it wants to model the balance sheet dynamically, it uses the mathematical tool of the linear finite difference system of the first order, and more importantly it aspires to be a mathematical modelization that can be presented as a function with a closed form formula.

In order to introduce such new theoretical framework it is important to start from the study of the long and close relationship between mathematics and accounting. We think this should be done for two main reasons. The first one is to show the streams of research, with their related literature, from which our new theoretical framework stems out. There has been a good deal of research aiming to the mathematical formalization of the accounting discipline, to the use of matrix algebra in order to describe the evolution of the balance sheet as an affine transformation as well as to the modelization of the accounting practice through the use of information technology, among many other instances. The mathematical framework we intend to present has the objective to move from those works with the hope of providing a formalization that could be felt as useful primarily for being at the same time utilizable via computer as well as

via pure mathematics.

But there's a second reason for which we think it is important to analyze, as much as we can here, the history of accounting and mathematics, a more general and significant one. Because, apart from giving the notion of the many brilliant attempts made through the course of history at employing the mathematical technology at disposal of the researchers in order to improve the accounting field and the economic knowledge of the firm in general, it is significant to review this history in order to get the idea of the evolution of the general theoretical framework through which scholars and researchers have seen the accounting discipline. Because that framework represents the lenses through which scholars and practitioners see accounting, talk about it and more importantly think about it. And it is interesting to notice that while on the one hand academia complains about the existence of a gap between the research world and the practitioner world, on the other hand, although there has been a tumultuous change in the way accounting is thought and practiced in the last fifty years (especially because of IT), from a mathematical point of view the theoretical framework is still the one of the double-entry book-keeping system, introduced five hundred years ago. Here we do not enter into the discussion about the reasons at the core of this condition, we just want to convey the idea of why we think a sound analysis of the history of the relationship between mathematics and accounting represents the foundation of a work like the one that we are presenting. Naturally we can just propose a summary of this analysis, presenting only the aspects of such review that are more related to the modelization that we intend to pursue. We have divided this history into two parts, the first one from ancient times to the 20th century and the second one from the 20th century to the present day.

1.1.1 History of accounting and mathematics from ancient times to the 20th century

The relationship between mathematics and bookkeeping or accounting is ancient and more than once, especially in antiquity, proved itself to be very close and very deep indeed.

As a matter of fact the first known formalized accounting system is considered by many to be the precursor of the writing of abstract counting and even of writing itself. We are talking about the Sumerian “token system” based on clay tokens representing individual assets and their impressions on clay envelopes as an inseparable depiction of the corresponding equity (Mattessich 2000), established by the Sumerians even before what is considered as the actual dawn of their civilization.

Another example of accounting giving an essential contribution to mathematics (Colebrook 1973) was the idea and justification of using not only positive but also negative numbers, by Indian mathematicians as Brahmagupta, in the 7th century, or Bhaskara, in the 12th century, in order to devise a mathematical

formalization for the accounting practice. They considered debt claims (or debits) as positive items and liabilities (or credits) as negative claims and so they were the first to conceive negative numerals as legitimate mathematical notions, hundreds of years before the Europeans. Actually ancient Indian mathematics not seldom found itself closely linked to the accounting discipline developing brilliant sophisticated ideas much ahead of their times, as for example the notion of purchasing power fluctuations that can be found in Kautilia's Arthashastra as early as in the IV century B.C.

And we definitely must not forget that the decimal system, one of the greatest, if not the greatest scientific development of the XIII century western world, a product of the Indian mathematical genius delivered to us through the cultural mediation of the Arab world, was introduced and spread through Europe for accounting purposes. The great Fibonacci's "Liber Abaci" (1202) itself was also a treatise on bookkeeping and accounting, that probably relied on some preceding Arab text (Antonini 1996) at a time when the Arab scientific culture was conveying much of the Indian tradition on the field.

So it is not surprising that the first published treatise on double entry bookkeeping ("Particularis de computis et scripturis") appeared in a mathematical text. Naturally we refer to Luca Pacioli's famous "Summa de Arithmetica, geometria, proportioni et proportionalità" (Part I, Section 9, Treatise 11) published in 1494. There Pacioli presents, not in a theoretical way but in a very practical one, a new method for taking record in the accounting books of every economic and/or financial transaction into which a firm can be involved. We will see an in depth analysis of the mathematical features and implications of the double entry theoretical framework in the next section. Here we would like to underline how successful such method has proved to be over the course of the last five hundred years. Today it still represents the framework into which practitioner accountants, everywhere in the world, develop their work, be it through the help of accounting software or be it still through the use of a simple pen and a paper sheet. It has come to represent the universal language through which every accountant in the world thinks and speaks about any firm, and from a mathematical point of view it is an arithmetic language.

For what we have seen so far in classical and medieval times, without forgetting the Indian and Arab world as well as during the Renaissance, it was arithmetic that dominated the scene. From the 16th century onwards valuation issues, based on discounted debts annuity methods and present value approaches, occasionally touched the accounting discipline (Mattessich 2005), as in the work of Stevin (also deemed as the inventor of the income statement) or in some of the work by Leibniz.

But for centuries the major mathematical aspects of accounting were limited to arithmetic. Finally in the 19th century algebra started to play an increasingly important role in the academic teaching of bookkeeping and in conveying different classification schemes. It was mainly in the 19th century that algebra was employed to express more general accounting relations starting a minor revolution that may be difficult to comprehend by modern accountants and scholars. The differences between the algebraic relations that were debated at the time

may seem to us so simplistic that they border on the trivial, yet accounting academia of the 19th century didn't see it that way. An example of this attitude can be traced back to the many theories of accounts classes that flourished during that period and the harsh controversies that they sparked. Many authors tried to prove that their classification scheme was better than the others or even the only correct one. Usually these schemes were presented in the form of simple algebraic equations, each one a variation or re-arrangement of the equilibrium equation (expressing that the value of all assets (or debits) equal the value of all equities (credits)). So from a mathematical point of view as well as from a contemporary mindset standpoint, these schemes would all be considered equivalent one to the other (Mattessich 2000). While one author pleaded for the equation: "Assets = Liabilities + Owner's Equity", his opponent would argue in favor of the mathematically equivalent relation: "Assets - Liabilities = Owner's Equity", intending to emphasize a different kind of classification and introducing an extra-mathematical element that a mathematician may find trivial as for him the two equations are equivalent. Actually in this debate we could see an early, still timid, example of the issue of the relationship between the mathematical information and the economic information contained in the balance sheet, an issue that we will have to examine when we will present differences and similarities between the mathematical framework we are proposing and the one implied by the double entry book keeping system.

Another important innovation, for the scope of the present work, introduced in the 19th century by several accounting scholars, on the topic of the modeling of the accounting theory and practice, is the use of matrices. Although their use was substantially different from the employment of matrix algebra we are going to do in the mathematical framework we are about to present, we think it's important to mention at least the original contributions by August De Morgan and Giovanni Rossi. De Morgan (1846) was the first to introduce the idea of an accounting matrix and Rossi (1889) offered, in his book on the double entry chessboard, dozens of examples in which accounting matrices played a decisive role but neither he nor de Morgan used formally matrix algebra. Rossi did suggest that the accounting matrix ("lo scacchiere a schede" as he called it) could be converted into a sophisticated computing device. This might be regarded as an anticipation of the computerized spreadsheets though Rossi, like Charles Babbage (1791-1871) the inventor of the first theoretical 'digital' computer, had a mechanical device in mind instead of an electronic one. Rossi pointed at the future of computerized accounting spreadsheets which developed into a fundamental branch of analytical accounting.

For the sake of completeness we feel compelled to mention another couple of developments occurred in the 19th century. They represent remarkable steps forward with regard to some of the most important issues of the history of analytical accounting in general and of the relationship between accounting and mathematics in particular. Although they do not deal with the specific task of modelling the balance sheet through the use of mathematical tools such as affine transformations or first order finite difference systems. The first one is the use of compound interest calculations that began to be felt of high importance in the

insurance business of the time. The other was the reappearance of the present value concept. As an example Seicht (1970) points out that the application of the present value approach (the kapitaltheoretische Bilanz) played an important role in the German Railway Statutes of 1863 and subsequent legislations.

Anyway although the classification algebraic schemes of the 19th and early 20th century as well as the impact of inflation calculations and of present value approaches¹ helped to improve the spread of algebraic thinking in accounting, until the first half of the 20th century systematic mathematical models were rarely used. During the first half of the new century even the present value approach was by no means readily accepted for accounting purposes. An illustration of such attitude can be seen in the ideas of the great scholar Schmalenbach who rejected the present value approach for statement presentation and admitted it only for the evaluation of firms in special cases, like assessments or liquidations and only outside the balance sheet. However some voices at the time were in favor of employing present values in accounting such as Canning (1929), Zappa (1937) and more importantly Preinreich (1933, 1936, 1937, 1938, 1939), who produced a series of brilliantly original publications that were unfortunately neglected at the time until they were rediscovered towards the end of the century. Indeed Gabriel Preinreich may be considered a precursor with regard to several of the issues, including the clean surplus theory, that modern analytical accounting tackled in the second half of the 20th century, a period through which the relationship between mathematics and accounting flourished as never before.

1.1.2 History of accounting and mathematics through the 20th century until now

The second half of the 20th century introduced a great variety of mathematical approaches and techniques to the study field of accounting. We will divide this enormous research work into four main areas and we shall briefly address each one of them, starting from the ones less relevant to the scope of our work and ending with those which show the closest relation towards it.

The first area of research that can be identified deals with the statistical methods, particularly sampling techniques that were introduced to accounting and auditing. The application of statistical sampling methods to accounting and particularly to auditing (audit risk models) has become one of the most successful mathematical tools of the public accounting profession. It is important to underline that while for the other areas of research that tried to merge mathematics and accounting the results have been quite limited, mostly due to a resistance from the practitioner world to employ new ideas that require

¹In Austria, Germany and France as well as in America by Sweeney.

a challenging mathematical culture, this stream of research proved to be spectacularly successful, particularly as far as auditing is concerned. The pioneer in this field has been Laurence Vance (1950), yet his work has been eclipsed by more sophisticated sampling techniques as presented in such texts as Trueblood and Cyert (1954), Trueblood and Cooper (1955), Cyert and Trueblood (1957), Stringer (1961), Arkin (1984) and many others, as well as in research papers like Ijiri and Kaplan (1971). To those efforts might be added the innumerable empirical accounting publications employing hypotheses testing and other statistical techniques that are indispensable to pursue this kind of direction. However we think it should be underlined that while the foundations of hypotheses testing are mathematical, so still analytical, the application of those tools is on the contrary inductive-empirical. So if those works, in theory, are still dealing with an aspect of the relationship between mathematics and accounting, in essence they fall beyond the very analytical nature of mathematics and more importantly beyond that part of the accounting theory, referred to as analytical accounting, that explores the rigorous formalization of the field, often (but not always) through mathematics.

The second area of research concerns the employment and development of the present value approach and its evolution into the clean surplus theory. During the second half of the century the influence of economics and the emergence of finance as a subject independent of accounting gave a decisive boost to a further exploration of the present value approach for accounting theory and practice, including statement presentation. The “synthetic balance sheet” theory of Albach (1965) and the “capital-theoretic balance sheet” theory of Seicht (1970) were the two major comprehensive accounting theories, both based on the present value approach, coming out of the continental Europe academic environment. In the English speaking academic world, the American paper by Alexander (1948) as well as the one by Corbin (1962) promoted the present value approach and gained wide notoriety. Moreover Gordon (1960) first and Peasnell (1982) later took up Preinreich’s ideas and developed them into the theory of clean surplus. However the major breakthrough in this area came with a series of publications by Ohlson, Feltham and others. The notion of clean surplus refers to an income derived from a comparison of the book value of a firm’s owners’ equity at the beginning of the accounting period with the firm’s value at the end of this period (eliminating new investments or withdrawals by owners). It results from the all-inclusive income statement where total sum of all the annual earnings, from the firm birth to its liquidation, equals the total sum of all annual incomes of the firm’s lifetime. It can be expressed by the following equation which is a simplified version of the “modern clean surplus theory” as presented by Ohlson (1995) and Feltham and Ohlson (1995).

Clean surplus relation (CSR):

$$Bookvalue(t) = Bookvalue(t - 1) + income(t) - dividends(t) \quad (1.1.1)$$

The conventional current operating income does not provide such a clean surplus (or “income”), because the sum of all the annual income figures do not necessarily add up to the firm’s life-time earnings. Those “not clean” income statements exclude many kind of extraordinary items (like capital and non-operating gains or losses, residuals from past periods, etc.) relegating them to separate surplus statements and they allow income smoothing over several accounting periods revealing the long-term profit trend. A second notion constitutes the cornerstone of this theory, that of the residual income (also called “excess income” or “abnormal earnings”). It is defined, under the CSR, as the net income minus a capital charge, which equals the riskless interest rate i . According to Christensen and Feltham (2003).

Residual Income Relation (RIR):

$$Marketvalue(t) = Bookvalue(t) + All\ future\ “residual\ income”\ discounted\ at\ rate\ i \quad (1.1.2)$$

Hence the residual income can be regarded as the difference between the firm’s income and the cost of capital. This theory can be seen as an elegant extension, in several directions, of the present value approach in determining the value of a firm and its shares. However the original clean surplus relation (CSR) as shown above still has an important limitation. Its income or dividends refer to future expectations and not to the past figures of accounting statements. To overcome this problem Ohlson and Feltham extended the theory in several ways, of which the most important was to relate past figures to future expectations. They created a linear information dynamics by adding to the right hand side of the RIR equation a set of stochastic variables and error terms that imposed a series of restrictions (as risk neutrality of investors) and linearity assumptions about the probabilistic time series behaviour of abnormal earnings and of information about other than abnormal earnings (for example innovations). Such a modification, apart from being an example of the use of more sophisticated stochastic mathematical tools, attempts to project accounting figures of the past into the future. To what extent these or similar regressions will prove satisfactory has still to be fully tested empirically. There are many analytical as well as empirical attempts to develop sufficiently accurate means of predicting future earnings, being such prediction pivotal for any modern value theory of the firm. The extended clean surplus theory of Feltham and Ohlson has definitely been one of the most popular among the many attempts. Even commercial variations of this model have been widely marketed showing an interest that goes beyond academia. One of the ultimate goals and hopes of our present work is that

of building a new mathematical framework for the balance sheet modelization capable of stimulating the flourishing of new ideas about issues such as future earnings prediction and firm valuation, with special regard to the link between the firm accounting items dynamics and their most significant macroeconomic drivers.

The third area of research refers to the attempts made to axiomatize accounting, using set-theory and similar mathematical devices in order to attain rigorous formulations of the accounting principles. It all started in the 1950s and early 1960s when the relationship among mathematics and accounting took a new direction bringing to a lot of attempts to rebuild the accounting discipline on more rigorous foundations, given the dissatisfaction with the traditional framework of accounting rules and loosely connected principles. At a time when related disciplines (such as economics, finance, operations research etc.) reached for more sophisticated mathematical methods and tools, young scholars felt the need for a more analytical and systematic approach in the construction of a conceptual framework for business accounting. This direction was actually pursued by two groups that partly competed and partly cooperated with each other. The first group engaged in an approach that could be qualified as “postulational” since although it aimed at founding the accounting discipline on a limited and sound number of hypotheses and postulates, it tended to avoid rigorous mathematical concepts. The other group instead followed a line of thought that can be defined as axiomatic since it sought a more rigorous methodology with clear assumptions, mathematical theorems and corresponding proofs. It is revealing of an attitude still present today how the approach of this last group was heavily attacked, at the time, for being too mathematically sophisticated while in comparison to the more recent trends of stochastic-analytical accounting, those earlier applications of mathematical concepts were relatively moderate. The first group, composed by scholars like Moonitz (1961), Sprouse (1962), Givens (1966), got its impetus from Chambers (1955, 1957, 1966), while the second one, Winborne (1962) and especially Ijiri (1965, 1967, 1971, 1975), have been stimulated by experiments in Mattessich (1957, 1964).

Although some publications, as Zeff (1982) or Slaymaker (1996) and others, indicate that some of those endeavours may ultimately have influenced the Financial Accounting Standards Board, those conceptualizations were occasionally criticized (as Archer (1993)) and experienced quite limited academic success. This may be attributed, on the one hand to the fact that the foundations of such conceptual frameworks were probably not formulated rigorously enough, while on the other to the fact that those mathematical and axiomatic accounting formalizations were probably premature at a time when practitioners were not even ready for the much less demanding mathematics of inflation accounting (Mattessich (1995)). Axiomatization and related analytical efforts resumed in America and Great Britain in the late 1970s, as in Orbach (1978) and Tippet (1978), and continued in general into the 1980s, 1990s, and even beyond, though by that time this area no longer occupied centre-stage. Anyway it is important to underline that on a purely theoretical level the second (the mathematical) approach influenced a wide range of scholars worldwide in exam-

ining in a fairly rigorous way the foundations of several aspects of the accounting field. The present work, apart from borrowing some of the specific concepts of the axiomatization literature, can be considered to be in line with that trend.

The fourth area of research is very important for the scope of our work, since it concerns the use of accounting matrices that led to computerized spreadsheets, and more importantly the application of matrix algebra to the accounting theory and practice. Broadly speaking we could divide this area of research into two paths, which anyway in more than one occasion happened to overlap. The first one is represented by the use of matrices in order to formalize and present the practice of the accounting work, coherently with the double-entry book-keeping system, so that it could be efficiently processed through computerized spreadsheets and a relationship between accounting and the new emerging field of information technology could be established. The second path consists in the attempt by some scholars to describe the firm dynamics and in particular the balance sheet dynamics through the use of matrix algebra both theoretically and via computer. As for the first path, as we have previously hinted, accounting matrices were already known in the 19th century in a way that sometimes seemed to anticipate the use of information technology, but the first suggestion of an electronic spreadsheet applied for accounting purposes is represented by the seminal work by Mattessich (1961) on budgeting models and system simulations. The subsequent elaboration of this idea consisted in presenting as a prototype a mathematical budget model of an entire firm, in Mattessich (1964), as well as a complete computer program (with sub-budgets for cash flows, labour costs, material costs, purchases, sales, overhead expenses with proper allocations, as well as a projected income statement and balance sheet). That is universally considered the first example of an electronic spreadsheet (wrote in Fortran IV) and more importantly the forerunner of such best-selling spreadsheet programs for personal computer such as VisiCalc, SuperCalc, Lotus 1-2-3 and Excel.

It is important to notice that although the use of computers allows to reach the goal of system simulation, in this first branch of research matrix algebra is not used for the modelization of the firm dynamics in a mathematical way. Through the use of information technology the objective of presenting the firm in a dynamic way is obtained, but there is still no mathematically formalized description of its accounting dynamics. This is one of the most important byproducts of the second path of research in this area, especially with respect to the scope of the present work.

Matrix algebra seems to have been used first in macro-accounting by Leontieff (1951) and later by Fuerst (1955), then in cost accounting by two Germans Pichler (1953) and Wenke (1956, 1959) and in general accounting theory by Mattessich (1957, 1964). Subsequently a flood of pertinent publications in the application of matrix algebra, linear and non-linear programming and other mathematical techniques appeared as in Rosenblatt (1957,1960). Among the books in this area the following must be emphasized: Ijiri's (1965) dissertation on goal-oriented models and the publications by William and Griffin (1964) and Corcoran (1968).

For the scope of the present research it is definitely important also to cite,

the aforementioned Mattessich aside, the work by Butterworth (1972) for his modelization of the firm balance sheet in a dynamic way, the work by Melse (2006) as well as the work by Tippet (2000) that not only describes the firm balance sheet through a finite difference system of the first order but also models its dynamics with the aid of stochastic mathematical tools.

Lastly we think particularly compelled to address the work by Arya et al. (2004) for its links to the bases of the framework that we are about to introduce. In his work, gathering especially from Mattessich and Ijiri, he and his colleagues prove that it is possible to model the evolution of a balance sheet through a linear finite difference system and then they draw a parallel between the linear algebra modelization and the double-entry bookkeeping framework in order to establish an algebraic interpretation of the error checking capability of the double-entry bookkeeping system.

1.1.3 History of accounting and mathematics: some remarks and conclusions.

Our mathematical framework moves from a linear algebra interpretation of the balance sheet dynamics in line to that devised by Arya, but instead of looking at the similarities between the linear algebra description and the double entry bookkeeping approach, it starts to build from the differences between the two, analyzing the reasons behind those differences.

In order to do that we have to begin from a thorough scrutiny of the double-entry bookkeeping system from a mathematical point of view. In our summary of the history of mathematics and accounting naturally we had to leave out a lot of research areas, among those we didn't address the attempts to mathematically analyze the double entry bookkeeping system but we made that choice only because we are going to do it in the next section.

Before that we would like to close this section with a couple of remarks on the results of all this research through history and the general state of analytical accounting, remarks that we think should guide us in the way we conduct and present our research work.

The first thing to notice when we talk about accounting research in general, it is the existence of a universally perceived gap between the scientific world of academia and the practitioner world as well as general feeling that accounting research is too self-referential (see Unerman et al. (2009)). So it is not surprising that even a giant like Mattessich (2005) has an uncertain attitude when he is called to assess the state of analytical accounting as a whole, today.

The case of the research area on axiomatization is emblematic since the postulational approach failed because it wasn't rigorous enough while the second approach failed because it was perceived by many as using too much mathematics.

Moreover the case of the research area about the use of statistical tools and the sampling techniques, which is an example of success, gives us the idea that it is important to present a research which proposes something felt as useful and attempts to formalize it using the most accessible mathematical tools capable of delivering the result.

As a matter of fact the work from another area of research, that we had to leave out of our history summary, the one trying to merge agency theory and accounting, that has given results which are by many regarded as some of the most formidable intellectual achievements in the analytical accounting field, by many others has received wide critiques. Because it is a line of work that is felt more as “economics of accounting” than accounting in the traditional meaning, since it requires considerable prerequisites of mathematics, finance and economics and so it is inaccessible not only to the majority of practitioners but also to many accounting academics (Mattessich (2005)).

So, as we can see, another general problem of the accounting research is that, especially when it uses mathematical tools, it often tries to reach goals into other areas of the economic science beyond what are strictly felt by the accounting community as the field’s boundaries.

Basically we can assess that the relationship between mathematics and accounting, very close and deep in ancient times, in the last fifty years, though still fruitful, has produced mixed results. If on the one hand it has proved to be very lively, sparking ideas in many different directions, on the other hand it has encountered several obstacles, in part due to shortcomings in the accounting research itself, in part due to the lack of a widespread advanced mathematical culture which would be required in order to comply with the mathematical tools that this relationship has come to imply.

Consequently to this day no new general mathematical framework can be deemed as established beyond the one implied by the double entry book keeping system, so in order to present a new one we should start from an analysis of the main double entry mathematical features. Finally this historic review seems to advise us on some of the characteristics that our new mathematical framework should possess: that of presenting itself as useful as it can, with special regard to its possible links with other areas of the economic research, while trying to retain the simplest mathematical shape capable of delivering the desired results.

1.2 The theoretical mathematical framework of the double entry bookkeeping system

In order to try to establish a new theoretical framework for the balance sheet modeling, it is important to start from the analysis of the current theoretical mathematical framework into which the balance sheet is written and thought, namely the double entry bookkeeping system. This mathematical analysis will have to tell us the reasons behind the main features of the double entry approach and the goals that the double entry approach reaches through these characteristics. We think this should be done to understand which one of those features could be modified so that our new mathematical modelization could reach its own goals. So let's start from this analysis and more generally from a history of the encounters between mathematics and the double entry book-keeping system.

1.2.1 Mathematical analysis of the double entry book-keeping system

A revealing example of the problematic attitude between the accounting academic world and the mathematical one can be represented by the history of the mathematical analysis of the double entry bookkeeping system. To this day it is little known in mathematics and it is even virtually unknown in accounting that the double entry system is based on a mathematical construction of undergraduate algebra, the group of differences, in which the integers are represented as equivalence classes of ordered pairs of natural numbers. The T-accounts of double entry bookkeeping are precisely the ordered pairs of the group of differences construction. With the exception of a paragraph by D.E. Littlewood, until the fundamental work by Ellerman (1984) unfortunately even to this day not widely known, there is not a single mathematics book which notes that this construction is the theoretical basis of a mathematical technique applied, everyday everywhere, in the mundane world of business for over five centuries. And even though that construction is standard fare in an undergraduate modern algebra course, it is a relevant thing to notice that its connection with double entry bookkeeping is totally absent in the accounting literature.

Through the course of history the encounters between mathematics and double have been so sparse that the highlights can be easily specified. A description of double entry bookkeeping was first published by the Italian mathematician Luca Pacioli in 1494. The method had been developed in Italy during the fourteenth century. Although Pacioli's system was governed by precise rules, his presentation was in practical and non-mathematical form. Let's keep in mind

that as an abstract mathematical construction the group of differences seems to have been first published by Sir William Rowan Hamilton in 1837. He made no mention of bookkeeping although accountants, at the time, had been using an intuitive algebra of the ordered pairs, by them called T-accounts, for about four centuries.

Arthur Cayley (1821-1895) was one of the few later mathematicians who wrote about double entry bookkeeping. In the year before his death he published a small pamphlet entitled “The Principles of Book-keeping by Double Entry” in which he wrote:

“The Principles of Book-keeping by Double Entry constitute a theory which is mathematically by no means uninteresting: it is in fact like Euclid’s theory of ratios an absolutely perfect one, and it is only its extreme simplicity which prevents it from being as interesting as it would otherwise be.”

In the pamphlet, Cayley did not present a mathematical formulation, but only described double entry bookkeeping in the practical informal terms familiar to Cayley from his fourteen years of work as a lawyer. However, in his presidential address to the British Association for Advancement of Science, Cayley hinted that the “notion of a negative magnitude” is “used in a very refined manner in bookkeeping by double entry”.

Another brief but insightful observation was made in a semi popular work by D. E. Littlewood in which he noted that the ordered pairs in the group of differences construction function like the debit and credit balances in a bank account:

“The bank account associates two totals with each customer’s account, the total of moneys credited and the total of moneys withdrawn. The net balance is then regarded as the same if, for example, the credit amounts of £102 and the debit £100, as if the credit was £52 and the debit £50. If the debit exceeds the credit the balance is negative.

This model is adopted in the definition of signed integers. Consider pairs of cardinal numbers (a, b) in which the first number corresponds to debit, and the second to the credit. A definition of equality is adopted such that $(a, b) = (c, d)$ if and only if $a + d = b + c$. “

Some modern accounting theorists believe that the mathematical treatment of the double entry bookkeeping must involve transaction matrices. This is not totally correct, since this transaction matrices represent only a good formal way of representation for the transactions described otherwise through double entry bookkeeping. As a matter of fact the presentation of transactions involving scalars can be facilitated using a square array or table of scalars usually called “transaction matrix”. These transaction tables were first used by the English

mathematician August DeMorgan and have been popularized through history in the ways we presented in the previous section.

Transaction tables have, however in a way, retarded the development of a mathematical formulation of double entry bookkeeping. We will see shortly that double entry bookkeeping lives in group theory, not in matrix algebra. As we will see onward in this work matrix algebra is the best mathematical tool in order to describe the balance sheet dynamics, not the mathematical essence of double entry bookkeeping. And the seminal work that thoroughly analyzed the double entry bookkeeping system through the instrument of group theory is the 1984 work by Ellerman: “The mathematics of Double Entry Bookkeeping”. His analysis revolves around two main ideas.

1.2.1.1 The double entry bookkeeping system and the Pacioli group

The first cornerstone of a mathematical formalization of the double entry bookkeeping system, according to Ellerman, is the acknowledgement that the double entry is based on the construction of the integers (positive and negative) as equivalence classes of ordered pairs of natural numbers (so only positive).

The ordered pairs of this construction correspond to the T-accounts of the double entry bookkeeping, the left hand entry in an ordered pair corresponds to the *debit* side of the T-account and the right hand to the *credit* side.

We can borrow the notation $[d//c]$ from Pacioli himself as the transposition of the following T-account

$$\begin{array}{cc} \textit{Debits} & \textit{Credits} \\ d & c \end{array},$$

and we can start to show how the algebraic structure of an additive group can easily be built over the set of those ordered couples provided of an equivalence relation. Since the label T-account will be used lately in specific accounting contexts, as long as we are dealing with the algebraic structure of this set of ordered couples we will refer to the elements of this set as T-terms.

As last preliminary remark we would like to stress the fact that the numbers in the ordered pairs, in the T-terms, are all positive numbers, in the original idea of the double entry they would have been all natural numbers.

We can define the sum between two T-terms $[a//b]$ and $[e//f]$ as follows:

$$[a//b] + [e//f] = [(a + e)//(b + f)] \quad (1.2.1)$$

The result is internal to the original set, since the two numbers $(a+e)$ and $(b+f)$ will be positive numbers, and it carries the algebraic properties of the usual sum.

With a sum described as above it is easy to verify that $[0//0]$ can be defined as the zero T-term since adding it to any T-term makes no difference. Formally speaking:

$$[a//b] + [0//0] = [0//0] + [a//b] = [a//b] \quad (1.2.2)$$

An equivalence relation between two T-terms is defined by Ellerman in the following way: $[a//b]R[e//f]$ if and only if the cross sums, $a + f = b + e$, namely the sums of a couple debit with the other credit and vice-versa, are equal.

This definition represents one of the features at the core of the double entry mathematical formalization since basically it states that two T-terms, or two T-accounts in the accounting application, are equivalent if they represent the same amount of wealth flowing into the account, or away from it, depending on which side of the equilibrium equation the account is located

$$[a//b]R[e//f] \text{ iff } (a - b) = (e - f) \text{ or } (b - a) = (f - e) \quad (1.2.3)$$

This equivalence relation is compatible with the addition defined above in the sense that if $[a//b]R[A//B]$ and $[e//f]R[E//F]$ then also the respective sums are equivalent, namely $[(a + e)//(b + f)]R[(A + E)//(B + F)]$.

The last step in order to define an additive group on the set of ordered pairs provided of the above equivalence relation, that we will call the Pacioli group, is of course to verify the existence of an inverse term for every T-term in the set. The numbers occurring in a T-term can never be negative, but we can still define the negative of a T-term without negative numbers. Hence we define the negative or inverse of a T-term $[a//b]$ as its reverse $[b//a]$ since the result of the sum of the two terms is a zero T-term, namely:

$$[a//b] + [b//a] = [(a + b)//(b + a)]R[0//0] \quad (1.2.4)$$

And this completes the definition of the ordered pairs construction of the integers from the natural numbers, following Ellerman, we call this group of ordered couples the Pacioli group.

As a matter of fact there is a one to one relationship between the classes of ordered pairs in the Pacioli group and the integers (positive and negative), as well as, if we allow the elements of the couples (a, b, e, f, etc. etc.) to be positive real numbers there would be a one to one relationship between the classes of ordered pairs and the real numbers (positive and negative). So we can see that the Pacioli group can be interpreted as a way of expressing positive and negative numbers, only through the use of positive quantities.

1.2.1.2 The double entry bookkeeping system and the equilibrium equation of the balance sheet

The second cornerstone of the in depth analysis performed by Ellerman on the double entry bookkeeping system is represented by the formalization of the relationship between the Pacioli group and the balance sheet equilibrium equation, starting from the realization that the double entry method uses the Pacioli group to perform additive algebraic operations on equations.

We start the analysis of this relationship by describing a method through which we can translate or encode equations into the Pacioli group. We call zero-term a T-term that is equal to the zero T-term $[0//0]$, so that basically $[b//b]$, for every b , will be a zero-term. The translation of equations into the Pacioli group is very simple: equations between nonnegative numbers correspond to zero-terms. As a matter of fact, given any equation where all numbers are nonnegative such as $a + + b = e + + f$ we encode each left-hand-side number as a debit balance T-term such as $[a//0]$, we encode every right-hand-side number as a credit balance $[0//e]$ and we sum them all. With this translation the original equation holds if and only if the result is a zero-term, in the case of our example the equation holds if and only if $[a//0] + + [b//0] + [0//e] + + [0//f]$ is a zero-term.

In double entry bookkeeping, transactions must be recorded in such a way as to maintain the truth of the equilibrium equation, or the balance sheet equation:

$$Assets = Liabilities + NetWorth \quad (1.2.5)$$

Namely transactions must be recorded by valid algebraic operations which transform equations into equations. In the Pacioli group we have just seen that an equation translates into a zero-term, so a valid algebraic operation of that sort would be an operation that transforms zero-terms (equations) into zero-terms (equations). But there is only one such operation: add a zero-term. Thus

a transaction must be represented by a zero-term to be added to the zero-term representing the balance sheet equation.

In bookkeeping the double entry principle is that each transaction must be recorded with equal debits and credits. The mathematical basis for this principle is that transactions are represented by zero-terms so the debits must equal the credits (in every transaction) since a zero-term is a translation of an equation. More specifically the zero-terms arising as the representation of equations, as the balance sheet equation or the profit and loss equation, will be called equational zero-terms, while the zero-terms arising from transactions will be called transactional zero-terms. The additive algebraic operations on an equation will work according to the following scheme:

$$(original\ equational\ zero-term) + (transactional\ zero-term) = (final\ equational\ zero-term) \quad (1.2.6)$$

Actually there are situations in which an equational zero term is translated into a transactional zero term, since once that scheme is used with respect to the profit and loss equation (dealing with one specific period) the final equation will give us pieces of information to be used as transactional zero-terms in the scheme of the balance sheet equational zero term (which will give us the economic history of the firm).

The last thing to specify is how to reverse the translation process, how to decode zero-terms into equations. An equational zero term is a sum of T-terms, it is not itself an equation with a left and right hand side. Indeed the T-terms can be shuffled around in any order. To decode a zero-term into an equation, one can use any criterion one wishes to divide the T-terms into two sets, left (L) and right (R), and then construct an equation according to the following principles: if a T-term is in the set L , for example $[a//b]$, then decode it as the number $(a - b)$ on the left-hand side of the equation, while if $[a//b]$ is in the set R , decode it as $(b - a)$ on the right-hand side of the equation. This procedure will always produce a valid equation given a zero-term.

Naturally in bookkeeping the T-accounts in the final equational zero-term would be put in the sets L and R according to the side of the initial balance sheet equation from which the accounts were originally encoded.

The last thing to underline about this second main feature of the double entry bookkeeping method is that while the first one, seen before, allowed to describe positive and negative quantities in terms only of positive numbers, in the case of the zero-terms or the equivalent equations into or from which they are translated, they provide an instant checking for the validity of the transaction they are referring to, as we are going to see in this next section.

1.2.2 Conclusions and remarks on the main features of the double entry bookkeeping system mathematical framework

The two main features of the double entry bookkeeping system mathematical framework, that Ellerman has identified, are probably at the basis of the method's success. As a matter of fact they give to the employer of the system two very useful benefits, especially from the point of view of the historical period in which it was invented as well as, at least, the first four centuries of its use.

The Pacioli group gives the opportunity to express the whole accounting of the firm, positive as well as negative flows of wealth, only through the use of positive numbers, which especially in the 15th century when the system was devised, was definitely a good thing. Let's see more in detail this characteristic with a very simple exemplification. We have a balance sheet where the simplest of the balance sheet equations is updated:

$$Assets = Liabilities + NetWorth \quad (1.2.7)$$

and where we do not have temporary or flow accounts such as revenues or expenses, so basically we do not have a profit and loss statement. In this situation we will have only three accounting items, namely *Assets*, *Liabilities* and *NetWorth*, to which T-terms will refer, and that when they have attached the accounting labels Assets, etc., can properly be called T-accounts. It is the position of the account in the all-positive equation above that identifies the account as a left-hand side (*LHS*) or debit-balance account or as a right-hand side (*RHS*) or credit-balance account. Now in general debiting any x to an account means adding the debit T-term $[x//0]$ to the T-account, while crediting x to an account means adding the credit T-term $[0//x]$ to the T-account.

It is a common mistake of non-accountants to think that debit means negative. But it all depends if the account is a *LHS* account or a *RHS* account, which, as in our case, can be easily assessed looking at the balance sheet equation. As a matter of fact to debit an account does not necessarily mean to subtract from the balance in the account, that is only true for credit-balance accounts, debiting a debit-balance account, like in our case Assets, means adding to the account's balance.

The second feature of the double entry bookkeeping system, the fact that the zero-term representation of the balance sheet equilibrium equation gives a quick check of the plausibility of the transaction, for each transaction, is even more important. It remains very useful today in the everyday accounting practice, and it was definitely more so in a period when there were no computers or

electronic calculators, especially in relation to the recording of a high number of transactions.

Of course this checking opportunity would be present in any kind of transaction recording method that would be based on updating a complete accounting equation. The double entry system is a system of recording transactions and its feature relating to the zero-term encoding of equations makes it immediate after some practice. For example if, in relation to the above balance sheet, an event was formulated as the transaction of adding 200 \$ to both Liabilities and NetWorth, some thought would be required to see that this formulation of a business event could not possibly be correct and much more would be required for a multiple entry transaction in an accounting system with a lot of accounts. Yet the check is immediate in the double entry system, Liabilities and NetWorth are both credit-balance accounts so the proposed transaction is a double credit transaction in violation of the double entry principle.

1.3 From the double entry bookkeeping system to a linear algebra modelization of the balance sheet dynamics

So far we have seen the characteristics of the double entry bookkeeping system mathematical framework, now let's approach the linear algebra modelization of the balance sheet dynamics and establish how the features previously discussed can be modified in order to achieve different goals.

We can summarize this approach presenting an example in line with a work of Arya et al. (2004) in which he states, among other things, that the dynamics of a balance sheet representation can be modeled through an affine transformation.

Let's consider a balance sheet that has all its accounting items at zero at the beginning of the period and then is subjected to the following transactions. Equipment is purchased for \$80, let the depreciation expense be \$20, the stock is sold for \$100, and cash revenue is \$30.

The Balance sheet at the end of the period will be the following:

<i>Assets</i>		<i>Liabilities</i>	
<i>Cash</i>	50	<i>Acc. Depreciation</i>	20
<i>Equipment</i>	80	<i>Capital Stock</i>	100
		<i>Income</i>	10
<i>Assets</i>	130	<i>Liabilities + Equity</i>	130

We are in an accounting situation that can be modeled as follows. We can define a Balance sheet vector with the above accounting items, each one having value 0 at the beginning of the period

$$[Cash_0, Equipment_0, Acc. Depreciation_0, Capital Stock_0] = [0, 0, 0, 0] \quad (1.3.1)$$

and we can define a vector of transactions \vec{C}_1 (expressing the transactions taking place from the time step 0 to the time step 1) with values as above (equipment purchased for \$80, depreciation expense of \$20, stock sold for \$100, cash revenue of \$30) namely

$$\vec{C}_1 = [ca_1, cb_1, cc_1, cd_1] = [80, 20, 100, 30] \quad (1.3.2)$$

So at the end of the accounting period the Balance sheet vector will be:

$$\left\{ \begin{array}{l} Cash_1 = -(Cash_0 + ca_1) + (Acc. Depreciation_0 + cc_1) + \\ \quad + (Capital Stock_0 + cd_1) = \\ \quad = -(0 + ca_1) + (0 + cc_1) + 0 + (cd_1) = \\ \quad = -80 + 10 + 30 = 50 \\ Equipment_1 = (Cash_0 + ca_1) = (0 + ca_1) = 80 \\ Acc. Depreciation_1 = (Equipment_0 + cb_1) = (0 + cb_1) = 20 \\ Capital Stock_1 = (Acc. Depreciation_0 + cc_1) = (0 + cc_1) = 100 \end{array} \right. \quad (1.3.3)$$

Naturally a system of first order linear difference equations can be expressed through a matrix representation. So considering the Balance sheet vector at the beginning of the period and the vector of transactions defined above, through

the action of the following accounting matrix the Balance sheet vector at the end of the accounting period will be:

$$\begin{bmatrix} Cash_1 \\ Equipment_1 \\ Acc. Depreciation_1 \\ Capital Stock_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 + 80 \\ 0 + 20 \\ 0 + 100 \\ 0 + 30 \end{bmatrix} = \begin{bmatrix} 50 \\ 80 \\ 20 \\ 100 \end{bmatrix} \quad (1.3.4)$$

Basically we have just established that we can generally formalize the dynamics of the balance sheet mathematically as follows:

$$S_n = f(S_{n-1}, C_n) \quad (1.3.5)$$

where S_n is the vector of the balance sheet items at time n , C_n represents the vector of the economic and financial transactions occurring (or recorded) between time $(n - 1)$ and time n , and f is a linear affine function on both arguments.

Expressing the function above through a matrix representation, applied to the previous example, we will have:

$$\begin{bmatrix} Cash_n \\ Equipment_n \\ Acc. Depreciation_n \\ Capital Stock_n \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} Cash_{n-1} + ca_n \\ Equipment_{n-1} + cb_n \\ Acc. Depreciation_{n-1} + cc_n \\ Capital Stock_{n-1} + cd_n \end{bmatrix} \quad (1.3.6)$$

In the example above we have built the balance sheet vector using only four accounting items, but in general we could theoretically choose as much accounting items as we need for the particular formalization that we have in mind.

Now the first thing that we have to underline, in the formalization that we have just presented, is that we have already modified one of the two main characteristics of the double entry mathematical framework, and rightfully so,

namely the equilibrium balance sheet equation. As we have seen in the previous section the motivations behind the double entry framework can be seen as useful in the everyday accounting practice, especially at the time when it was devised, but here we are more interested in the mathematical possibility of the formalization we are presenting, as well as its IT applications.

So when we define the balance sheet vector, we do not present in the vector every accounting item that can be found in an equilibrium equation because one of the items, exactly because of the equation, is necessarily linearly dependent on the others.

The item that we choose not to present is the NetWorth, since its value can be devised at every moment summing up the values of all the accounting items that are present in the balance sheet vector, taking into account their sign.

We chose not to present the NetWorth, among all the items that we could have chosen, for a reason. The main purpose of the balance sheet is to record and present an amount of information about the firm, rightfully considered important. And naturally from an economic perspective the information of the net worth is of paramount importance but from a mathematical point of view is the result of all the streams of wealth going in and out of the firm. So if we track the records of the accounting items related to those flows of wealth we will have, not only already the information with respect to the net worth, but hopefully also a better mathematical description of the firm accounting dynamics.

The same will happen for the vectors describing the economic and financial transactions. Anytime we will build a system as a model of a balance sheet we will try to formalize the elements of the transaction vector linking them to accounting items, or aggregations of them, that will mostly be accounting items relating to the profit and loss statement. Again the profit (or loss) item won't be present, since it will be linearly dependent on the other transaction vector elements values.

As for the other characteristic of the double entry bookkeeping framework, meaning the fact of expressing every flow of wealth positive or negative always through the use of positive numbers, actually in the example above we did not modify it completely, since we expressed streams of negative wealth through positive numbers and then we subtracted them from the cash-flow, but nothing constraints us to simply represent every flow of wealth going to the firm through a positive number and every stream of wealth going out of the firm through a negative number.

In this case the accounting item of the equilibrium equation that we choose not to be present in our balance sheet modelization will be given by the exact sum of all the other accounting items, since everyone will appear on the summation with its proper sign.

Now the last aspect of this formalization that we should analyze is related to the fundamental fact that one of our main goals is to express the value of the balance sheet at time n through a closed form formula.

In order to get to a workable closed form formula for the balance sheet vector, especially in the general case in which it is formed by a reasonably high number of accounting items, the modelization shown above it is not useful. If we will

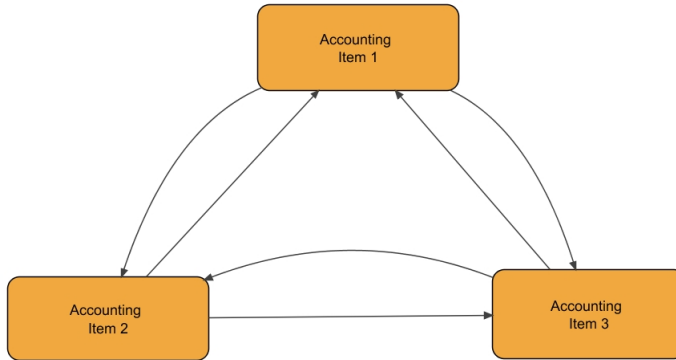
be dealing with a balance sheet vector with k accounting items, we will have to work with a $k \times k$ matrix, and in order to reach the closed form formula we will have to multiply and sum several of this matrices.

Now let's analyze what is the meaning of the matrix elements, using again the example above. If we multiply the matrix and take a look just at the first equation of the resulting system:

$$\begin{aligned}
 Cash_n = & m_{1,1}(Cash_{n-1} + ca_n) + m_{1,2}(Equipment_{n-1} + cb_n) + \\
 & + m_{1,3}(Acc. Depreciation_{n-1} + cc_n) + m_{1,4}(Capital Stock_{n-1} + cd_n) \\
 & \dots\dots\dots \\
 & \dots\dots\dots
 \end{aligned}
 \tag{1.3.7}$$

we can realize that the element $m_{i,j}$ regulates the flow of wealth (positive or negative) from the accounting item of position j at time $(n - 1)$, varied by the transaction of position j (happened between time $n - 1$ and time n), to the accounting item of position i .

Basically if we are in a situation where we have modeled the balance sheet using only three accounting items:



we can see, for example, that the element $m_{2,3}$ will regulate the flow of wealth, happening in the period, from the accounting item 3 to the accounting item 2 (naturally if $m_{2,3}$ assumes a negative value it will regulate a negative flow of wealth from item 3 to item 2):

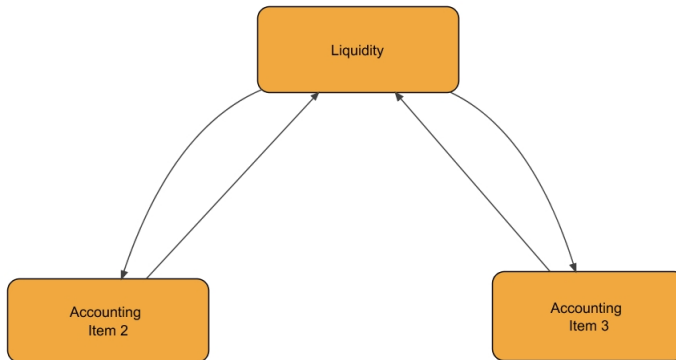
$$AccountingItem3 \longrightarrow AccountingItem2$$

and of course the element $m_{3,2}$ will regulate the flow of wealth in the other direction, namely the stream that during the time period will go from the accounting item 2 to the accounting item 3:

$$AccountingItem3 \longleftarrow AccountingItem2$$

So if we could express all this streams of wealth in a simpler way, from a mathematical standpoint, we could reach a modelization described by a system with a mathematical shape that will enable us to present it in a closed form formula as workable as possible.

We start from an idea expressed in the work by Ijiri (1965) on the axiomatization of the accounting discipline, the concept of u-measure. According to this idea if we express, as it is normally done, the value of every accounting item present in a balance sheet through money, specifically through a certain currency, we can consider money, that currency, as the universal unit of measurement of our balance sheet. Consequently we can identify a particular accounting item that can operate as a mediator between any two accounting items, i.e. the liquidity. Instead of having streams of wealth that directly move from one accounting item to another, we can model the streams of wealth so that in the same time period they all move accordingly to the following scheme: from the starting account to the liquidity and from the liquidity to the ending account:

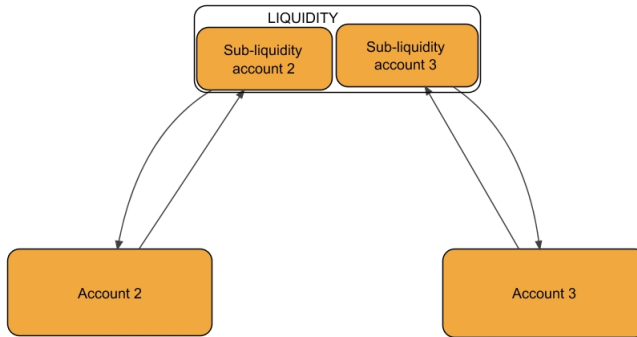


so that basically all the accounting items will be connected one another only through the liquidity item. If we consider a single flow of wealth starting from the item A and ending into the item B, during the time window that we are facing, the fact that the modelization of this flow of wealth is made so that it passes through the liquidity item, since naturally it doesn't change the end result but only the calculation process through which this end result is reached, can be interpreted just as an evaluation procedure, coherently with the u-measure concept seen above.

Once the modelization proceeds following the above principles it is clear how it becomes of pivotal importance the way in which we formalize the relationship of each accounting item with the liquidity. We called this formalization brick-vector, since combining the brick-vectors relating to each and every accounting item present in the balance sheet vector (other than the liquidity of course) we can build the system that describes our balance sheet.

1.3.1 The concept of the brick-vector

For the reasons seen in the previous subsection the brick-vector will be the atomic element upon which we will build the linear system that will model the dynamics of the balance sheet we are interested in. Since every account is connected to the others only through the liquidity item we can think the liquidity as a sum of a number of sub-liquidities each one related to its specific accounting item:



So that basically we can consider the brick-vector as the modelization of the relationship between each account and its specific sub-liquidity.

In the most general terms the brick-vector will assume the following shape, where, for reasons seen before, the value of every accounting item and vector variable will be expressed in money (so everything except the elements of the array) :

$$\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} (1 - \alpha_n) & \beta_n \\ \alpha_n & (1 - \beta_n) \end{bmatrix} \begin{bmatrix} LI_{n-1} + A_n^{LI} \\ I_{n-1} + A_n^I \end{bmatrix} + \begin{bmatrix} P_n^{LI} \\ P_n^I \end{bmatrix} \quad (1.3.8)$$

In (1.3.8) I_n will be the value of a specific accounting item at time n , and LI_n will be the value of the sub-liquidity related to the account I at time n . As we can see they depend on their values at time $n - 1$, and on a couple of vectors describing the transactions happened in the time window between $n - 1$ and n , the vector \vec{A} and the vector \vec{P} . The difference between the two vectors is that on \vec{A} the accounting matrix acts immediately, while the vector \vec{P} represents transactions whose values we want to put on certain accounts on which the accounting matrix will operate its redistribution effect the next time step. Naturally the variable A_n^I will represent the value of the transactions in \vec{A} that it is due to the account I , while A_n^{LI} will represent the one due to the liquidity (via the sub-liquidity), and the same will happen with respect to the vector \vec{P} . Finally the parameters in the accounting matrix α_n and β_n regulate the flows of wealth between the item and the sub-liquidity and vice-versa. The first parameter α_n represents the percentage of wealth that (in the time window between $n - 1$ and n) goes from the sub-liquidity, augmented by the transaction A_n^{LI} , to the account I , while β_n carries out the same task for the flow of wealth going from the item I to the liquidity.

The shape seen above for the brick vector is the most general one, a lot of times several of the above variables or parameters won't be needed and they will simply carry the value zero. Let's see as example the shape of a brick-vector describing the relationship between the Receivables from clients and its relative sub-liquidity, for an industrial firm:

$$\begin{bmatrix} LT_n \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & \eta_n \\ 0 & (1 - \eta_n) \end{bmatrix} \begin{bmatrix} LT_{n-1} + 0 \\ T_{n-1} + Z_n \end{bmatrix} \quad (1.3.9)$$

where T will be the accounting item modeling the Receivables from clients, LT will be its sub-liquidity, Z_n will be the variable representing the sales and the parameter η_n will be the percentage of the past Receivables plus the sales, happened in the time window, that are liquidated during that time window.

In this example we can see an important characteristic of the brick-vector namely the fact that it allows to represent the flows of wealth between the two items in two different ways, since in theory we could simply avoid the use of the accounting matrix and express the event only through the transaction vector:

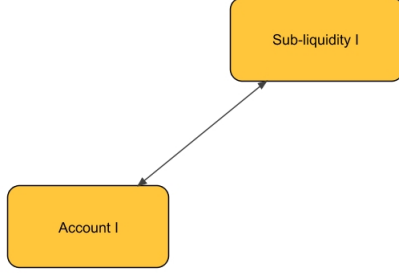
$$\begin{bmatrix} LT_n \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} LT_{n-1} + Z_n^{LT} \\ T_{n-1} + Z_n^T \end{bmatrix} \quad (1.3.10)$$

where in this example Z_n^T will be the part of the sales happened in the accounting period which are not liquidated, while Z_n^{LT} will be the value of the part of the sales in the accounting period that becomes cash-flow. The choice between the use of one of the two different ways, or a mix of the two, for each brick-vector (so for each accounting item), will naturally depend on what kind of information we are interested in (or is at our disposal), combined with the mathematical tools that the modelization will have to interact with, hence the mathematical goals it is set to achieve.

An example of this kind of attitude could be even seen in the way we are about to reach the aim of getting to a closed form formula for our balance sheet vector at time n .

As a matter of fact the major shortcoming of the formalization (1.3.8) is that in order to combine the vectors into a single balance sheet system we have to deal with a number of extra-parameters (as much as the number of the brick-vectors), at every time step. These extra parameters should weight the contribution of each sub-liquidity on the total of the liquidity. This would happen in a situation where in order to find a workable closed form solution one of the most serious obstacles is represented by the sheer number of variables and parameters that inevitably any balance sheet modelization brings with itself.

A solution would be once again represented by an attempt to limit the ways through which the streams of wealth can travel between the accounting item and its sub-liquidity. The previous problem arises from the parameter α_n regulating the flow of wealth from the sub-liquidity to the accounting item. Then we should try to find ways so that the only path through which the stream can happen, in both directions, would be the one from the item to the sub-liquidity:



The stream in the different direction (from the sub-liquidity to the account) then could be modeled in two different ways. The first is to consider it always as a flow of wealth going from the account to the sub-liquidity but of negative value, namely allowing β_n to take negative values. The second is through the use of the transaction vectors as seen in the example (1.3.10). In a situation of that kind the brick vector would get the following pivotal upper triangular general shape:

$$\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & \beta_n \\ 0 & (1 - \beta_n) \end{bmatrix} \begin{bmatrix} LI_{n-1} + A_n^{LI} \\ I_{n-1} + A_n^I \end{bmatrix} + \begin{bmatrix} P_n^{LI} \\ P_n^I \end{bmatrix} \quad (1.3.11)$$

where β_n would be in the condition of assuming negative values. And where the modelization of the vectors \vec{A} and (or) \vec{P} could be devised not only to model accounting items, or aggregations of accounting items coming from the profit and loss statement, but also a possible stream of wealth occurring from the sub-liquidity to the account I .

1.4 An example of a balance sheet modelization through a linear finite difference system

Now following the theoretical mathematical framework that we have presented so far we would like to build an example of a balance sheet modelization with its closed form formula representation. Naturally, as we have already hinted before, one of the greatest problems of the closed form setting is the high number of variables that any balance sheet imply. In the last section of this chapter we

will discuss briefly some of the paths that we think this research should follow in order to alleviate the impact of that issue. So every time someone should approach a formalization according to the present framework, the first step should always be a careful assessment of the accounting items to use (through a work of aggregation of different items). In every mathematical modelization there is a heavy trade off between the need for detail and the need for simplicity but here more than ever.

So, in order to provide a first example and for the sake of clarity, we prefer to develop one of the simplest formalizations allowed by our framework. One in which the balance sheet vector will be represented only by three items: naturally the liquidity L , then one accounting item for the assets other than the liquidity, we will name it S , and one for the liabilities I . In this situation we will have to develop only two brick vectors. The first will be the one expressing the relationship between our assets account and the liquidity, and it will be defined as follows:

$$\begin{bmatrix} LS_n \\ S_n \end{bmatrix} = \begin{bmatrix} 1 & \pi_n \\ 0 & (1 - \pi_n) \end{bmatrix} \begin{bmatrix} LS_{n-1} + 0 \\ S_{n-1} + P_n \end{bmatrix} \quad (1.4.1)$$

where the variable P_n represents the aggregation of all the accounting items and records in the profit and loss statement bringing a positive stream of wealth from the outside world to the firm and the parameter π_n represents the percentage of that wealth that is turned into cash flow in the time window considered.

The second one will be the brick vector representing the relationship between the liabilities and the liquidity and it will take this form:

$$\begin{bmatrix} LI_n \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & \nu_n \\ 0 & (1 - \nu_n) \end{bmatrix} \begin{bmatrix} LI_{n-1} + 0 \\ I_{n-1} + N_n \end{bmatrix} \quad (1.4.2)$$

where in this case the variable N_n will represent the aggregation of all the accounting items and records in the profit and loss statement bringing a negative stream of wealth from the outside to the firm, and the parameter ν_n will be the percentage of that negative wealth that is turned into negative cash-flow in the time window considered.

As we can see, in the formalizations (1.4.1) and (1.4.2) mathematically speaking the flow of wealth goes always from the accounting item to the liquidity.

Simply, in the case of the liabilities I and of the variable N the values will always be negative. This choice will also present the positive trait that it will enable us to obtain the net worth just by summing up all the accounting items in the balance sheet vector.

The closed form formula expressing the first brick-vector at time n will be the following one:

$$\begin{cases} LS_n = \sum_{l=1}^n \pi_l (\prod_{i=1}^{l-1} (1 - \pi_i) S_0) + \sum_{l=1}^n \pi_l (\sum_{h=1}^l P_h (\prod_{j=h}^{l-1} (1 - \pi_j))) \\ S_n = \prod_{i=1}^n (1 - \pi_i) S_0 + \sum_{h=1}^n (\prod_{j=h}^n (1 - \pi_j)) P_h \end{cases} \quad (1.4.3)$$

while the second brick-vector will have at time n a closed form formula as follows²:

$$\begin{cases} LI_n = \sum_{l=1}^n \nu_l (\prod_{i=1}^{l-1} (1 - \nu_i) I_0) + \sum_{l=1}^n \nu_l (\sum_{h=1}^l N_h (\prod_{j=h}^{l-1} (1 - \nu_j))) \\ I_n = \prod_{i=1}^n (1 - \nu_i) I_0 + \sum_{h=1}^n (\prod_{j=h}^n (1 - \nu_j)) N_h \end{cases} \quad (1.4.4)$$

It is worth noting that the shape of the two formulas is exactly the same, as it would have been expected since the systems from which they stem have the same mathematical shape.

Now we combine the two brick-vectors into one single system expressing the whole balance sheet vector at time n . The close form formulas of the items S_n and I_n will be the same formulas seen in (1.4.3) and (1.4.4), while that of the liquidity will be the sum of the formulas of the sub-liquidities in (1.4.3) and (1.4.4):

$$\begin{cases} L_n = LS_n + LI_n \\ S_n = S_n \\ I_n = I_n \end{cases} \quad (1.4.5)$$

²Everywhere will hold the convention that any time $a > b$ we will have $\prod_a^b = 1$ and $\sum_a^b = 0$.

As we have previously stated, one of the inevitable problems of a mathematical modelization of a balance sheet is the huge number of variables and parameters that even the simplest of modelizations necessarily involves. The framework related to the idea of the brick-vector makes no exception (and it couldn't be otherwise) but it attempts to alleviate the consequences of this issue by breaking down the final formula into smaller ones so that even the most complex expression, the one of the liquidity, presents itself as a repetition of the same general formula combined by the sum operation.

1.4.1 Remarks on the model presented and on its closed form formula

Inevitably in a balance sheet formalization there is a heavy trade off between the need of detail and the need of a workable mathematical modelization. The brick vector framework tries to mitigate this issue through the link constituted by the Liquidity item whose closed form solution is represented by a summation of very similar formulas. Naturally this is not enough since, in the kind of balance sheet modelizations that we have proposed so far, we are stuck, mathematically speaking, with a dependency on the whole history of the firm that we are formalizing. So we have to device methods in order to reduce the impact of this issue and in the next section we will discuss about that matter.

1.5 Time series modelizations and averaging procedure

As we have just seen the main problem for a workable shape of our balance sheet modelization is constituted by the high number of variables implied by the model. Since the number of items is chosen at the beginning of the formalization, the only choice left to address this issue is to reduce the dependency of the system on the time sequences of the variables and parameters involved in the closed form formula. This aim can be achieved by a mathematical modelization of those time series so that they are expressed by a function (on the time variable or a recursive one). Here we will start discussing about the simplest functional form that can be used for this purpose which is the constant one, equivalent to an averaging procedure.

1.5.1 An averaging procedure according to the Chisini definition

In order to model the time series involved in our formalization, moving from the premise that they are all constant in time, the first step will be to express the closed form formulas seen above accordingly. Then we will try to replicate the general value of the balance sheet vector at a certain time n (so that, in possible model applications, we could project the future life of the firm from that point in time under the assumption that it is stationary). To operate this replication method we will basically use for each and every brick vector an averaging process according to the Chisini meaning of the term (Iurato 2012), namely we will look for the constant value b such that:

$$g(b, b, \dots, b) = g(a_1, a_2, \dots, a_n) \quad (1.5.1)$$

although, technically speaking, in the brick vector case we will deal with a couple of those functions g embedded in the closed form formula of the vector. Let's look at this averaging method more in detail applying it to a brick vector. It formalizes the Receivables from Clients in a possible model for an industrial firm and has the following shape:

$$\begin{bmatrix} LT_n \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & \eta_n \\ 0 & (1 - \eta_n) \end{bmatrix} \begin{bmatrix} LT_{n-1} + 0 \\ T_{n-1} + Z_n \end{bmatrix} \quad (1.5.2)$$

where Z_n represents the amount of sales sold by the firm between the time step $n - 1$ and the time step n and η_n is the weighted average percentage of those sales that becomes cash-flow in the period. Its closed form formula will be:

$$\begin{cases} LT_n = LT_0 + \sum_{l=1}^n \eta_l (\prod_{i=1}^{l-1} (1 - \eta_i) T_0) + \sum_{l=1}^n \eta_l (\sum_{h=1}^l Z_h (\prod_{j=h}^{l-1} (1 - \eta_j))) \\ T_n = \prod_{i=1}^n (1 - \eta_i) T_0 + \sum_{h=1}^n (\prod_{j=h}^n (1 - \eta_j)) Z_h \end{cases} \quad (1.5.3)$$

When all the parameters η_i and the variables Z_h are constant in time that closed form formula takes the following shape:

$$\begin{cases} LT_n = LT_0 + T_0(1 - (1 - \eta)^n) + Z[n - (1 - \eta)(\frac{1 - (1 - \eta)^n}{\eta})] \\ T_n = (1 - \eta)^n T_0 + Z(1 - \eta)\frac{(1 - (1 - \eta)^n)}{\eta} \end{cases} \quad (1.5.4)$$

Now all that is left to do is to find the value of Z and the value of η that give us the values of LT_n and T_n , namely we have to solve the system above in the variables Z and η . Their values will be the following, for Z it will simply be:

$$\bar{Z} = \frac{\sum_{i=1}^n Z_i}{n} \quad (1.5.5)$$

while for η we will have to plug the value of \bar{Z} in the expression of T_n so that we will obtain the equation:

$$T_n = (1 - \eta)^n T_0 + \bar{Z}(1 - \eta)\frac{(1 - (1 - \eta)^n)}{\eta} \quad (1.5.6)$$

which is an n -degree polynomial equation in η that can be solved, in order to get the value $\bar{\eta}$ we are looking for, only using standard numerical procedures.

1.5.2 The Chisini averaging procedure applied to the previous model

As we have just seen in the previous subsection, our Chisini-like averaging procedure will give us average values for the parameters and the variables of a brick

vector according to the shape we have chosen for it. For the model (1.4.5) they will simply be the result of the following procedure. Starting from the first brick vector, when all the parameters π_i and the variables P_h are constant in time, we will have that its closed form formula will take the following shape:

$$\begin{cases} LS_n = LS_0 + S_0(1 - (1 - \pi)^n) + P[n - (1 - \pi)(\frac{1 - (1 - \pi)^n}{\pi})] \\ S_n = (1 - \pi)^n S_0 + P(1 - \pi) \frac{(1 - (1 - \pi)^n)}{\pi} \end{cases} \quad (1.5.7)$$

Now all that is left to do is to find the value of P and the value of π that give us the values of LS_n and P_n , namely we have to solve the system above in the variables P and π . Their values will be the following, for P it will simply be:

$$\bar{P} = \frac{\sum_{i=1}^n P_i}{n} \quad (1.5.8)$$

while for π we will have to plug the value of \bar{P} in the expression of S_n so that we will obtain the equation:

$$S_n = (1 - \pi)^n S_0 + \bar{P}(1 - \pi) \frac{(1 - (1 - \pi)^n)}{\pi} \quad (1.5.9)$$

which is an n -degree polynomial equation in π that can be solved, in order to get the value $\bar{\pi}$ we are looking for, only using standard numerical procedures.

For the second brick vector, when all the parameters ν_i and the variables N_h are constant in time, we will have a closed form formula that takes the following shape:

$$\begin{cases} LI_n = LI_0 + I_0(1 - (1 - \nu)^n) + N[n - (1 - \nu)(\frac{1 - (1 - \nu)^n}{\nu})] \\ I_n = (1 - \nu)^n I_0 + N(1 - \nu) \frac{(1 - (1 - \nu)^n)}{\nu} \end{cases} \quad (1.5.10)$$

Now we will have to find the value of N and the value of ν that will give us the values of LI_n and I_n so we will have to solve the system above in the variables N and ν . Their values will be the following, for N it will be:

$$\bar{N} = \frac{\sum_{i=1}^n N_i}{n} \quad (1.5.11)$$

while for ν we will have to plug the value of \bar{N} in the expression of I_n so that we will get the equation:

$$I_n = (1 - \nu)^n S_0 + \bar{N}(1 - \nu) \frac{(1 - (1 - \nu)^n)}{\nu} \quad (1.5.12)$$

which is an n -degree polynomial equation in ν that can be solved, in order to get the value $\bar{\nu}$ we are looking for, only using standard numerical procedures.

Finally the system will take the following form:

$$\begin{cases} L_n = L_0 + S_0(1 - (1 - \pi)^n) + P[n - (1 - \pi)(\frac{1 - (1 - \pi)^n}{\pi})] + \\ + I_0(1 - (1 - \nu)^n) + N[n - (1 - \nu)(\frac{1 - (1 - \nu)^n}{\nu})] \\ S_n = (1 - \pi)^n S_0 + P(1 - \pi) \frac{(1 - (1 - \pi)^n)}{\pi} \\ I_n = (1 - \nu)^n I_0 + N(1 - \nu) \frac{(1 - (1 - \nu)^n)}{\nu} \end{cases} \quad (1.5.13)$$

into which we can plug the values previously found of the parameters and variables and this will enable us to obtain the value of the balance sheet at time n .

1.6 Conclusions and remarks

In the present chapter we have introduced a new theoretical framework for the mathematical interpretation of the balance sheet that allows to build a class of models describing its evolution through the tool of the linear first order finite difference system. Moreover this framework is devised so that the modelizations would possess the feature of not only being able to be implemented through the

use of information technology but also being described by a closed form formula, allowing the use of tools typical of pure mathematics.

We started from a summary of the history of the relationship between accounting and mathematics, giving a particular attention to what has happened in the last fifty years. Then we moved to an in depth analysis of the mathematical characteristics of the double-entry bookkeeping system, since it represents the current theoretical framework under which the mathematical interpretation of the balance sheet is implicitly done every day by the economic agents. It could be described as the “lens” through which accountants, everywhere in the world, deal with the balance sheet, speak of it, and more importantly think about it.

Following the ideas by Ellerman (1984) we identified two main features of the double-entry mathematical framework, which are linked to the two main reasons behind its overwhelming success and diffusion over the last five centuries. The use only of positive numbers to describe streams of wealth both positive and negative, and the automatic checking capability of the correctness of each transaction record, coherently with the fundamental balance sheet equilibrium equation. Both characteristics appeared to be of pivotal importance in a condition where the accounting work would need to be done by the largest number of agents, sometimes with the smallest possible training, without the use of any sort of calculator. Such as the world in which the double entry system was devised that, as far as those traits are concerned, remained nearly unaltered until fifty years ago.

Of course everything changes in a situation like ours where the accounting procedures need to be modeled in order to achieve mathematical purposes so that the model obtained can be used, via computer or pure mathematics, in the most different areas of the accounting and economic research such as corporate finance, budgeting simulation or risk management, just to name a few.

So, starting from the idea of formalizing the dynamics of the balance sheet through linear affine transformations and matrix algebra, we decided to change the features discussed above, in order to model the balance sheet as a system. Consequently linear dependence among the items cannot be allowed, and in order to follow as closely as possible the streams of wealth in and out of the firm (as well as among different accounting items), the use of negative values may become important.

In order to achieve the goal of expressing the value of the system at time n through a closed form formula, we attempted to model the flow of wealth among the accounting items through the introduction of the brick-vector concept and then we presented the reasons behind the need to model the flows of wealth within the brick-vector itself in different ways. This completed the introduction of our theoretical mathematical framework. Finally we gave the example of one simple model built according to our approach, one of the infinite number of models belonging to the set of systems that can be defined coherently with this work.

As we have stated more than once during this chapter, one of the main goals (if not the main goal of this work) is to reach a class of models that not only can

be expressed through a closed form formula but that can be expressed through a workable one. Of course this would be the main problem of any balance sheet formalization since the evolution of the balance sheet depends on a huge number of variables, especially when the number n of time steps increases. Naturally this is a trait that can't be avoided by any formalization. What our formalization tries to do about that issue is to break down the problem into smaller ones, through the brick-vector concept, and then combine them in a simple way so that the most complex formula (the one of the liquidity) will be a summation of sub-formulas (the ones of the sub-liquidity) all having roughly the same shape.

The present thesis, as in general the brick vector formalization, would like to provide an attempt at linking the accounting field more closely to different areas of research, such as corporate finance, finance engineering, risk management, corporate risk management etc. In order to foster employments of that sort we had to try to overcome what we think is the most problematic issue of this formalization namely its dependency on a high number of variables and parameters. This matter can be alleviated through an appropriate modelization of the time series of those variables and parameters. So that at this stage of our theory these variables and parameters should be seen mainly as a template on which to develop further formalizations. Here we proposed the mathematically simplest possible one, in which all the variables and parameters are constant in time, through an averaging procedure coherent with the functional average definition by Chisini. In this case a desirable next step for the present research would be a modelization of the sequences according to a linear function, for example an autoregressive of the first order for the variables while the parameters continue to be constant. Naturally in such a situation there would definitely be the need to address the issue of a stochastic definition of the modelization we have presented. The problem just discussed intertwines with the direction that we intend to pursue with the present research, namely the use of the proposed models in cases of budgeting simulation, cash flow at risk, optimization etc. In cases such as these we hope this modelization could result to be useful not only for its possible IT implementations but also because of its pure mathematical shape.

Actually we hope that this mathematical framework and its class of models could prove to be useful in several areas of research, starting from the risk management area as well as the corporate finance field. With proper time series modelizations and proper averaging procedures, we hope the model could produce a good proxy of the dynamics of the main accounting items linked, through the time series modelizations, to the main macroeconomic drivers of the most important accounts. This would prove particularly useful in a situation where, as for the clean surplus theory, the major problem of the analytical accounting research, as well as its applications, seems to be the linking of future projections of the values of the accounting items to their past values, which are the accounting data in our possession.

Another remark we want to convey about the present work is instead related to the relatively simple shape that we have chosen for the brick vector. We think that a relatively simple template (on which to operate in a second moment for

more details or constructions) could result more appropriate in an attempt to bridge the gap between the world of the accounting academia and the world of the accounting practice. Finally we hope that the broad spectrum of models and formalizations that this theoretical mathematical framework allows could give space to researchers from different academic paths, especially from accounting and finance, to come together for, on the one hand, having a more fruitful use of the incredible amount of data which constitutes the essence of the accounting discipline, and on the other hand, reaching a more integrated approach thanks to the deep mathematical methods that fields like the financial one can offer.

Chapter 2

A Dynamic Model for Cash Flow at Risk

In this second chapter we introduce a new quantitative instrument for a firm, in line with the brick vector framework previously seen, to integrate the set of information available from its accounting records. We start from the building up of a quantitative model based on accounting data that is able to represent and simulate the relevant dynamics related to the company. This application is based on the idea of representing the dynamics of a balance sheet using a mathematical formalization inspired by that of Butterworth (1972), Melse (2006), Cooke and Tippet (2000). Then we define a new risk measure based on the Cash-Flow-at-Risk (CFaR) measure, tailored on our new balance sheet modeling approach, which takes full advantage of its focus on the liquidity process. CFaR is an extension of Value-at-Risk (VaR): while the latter focuses on market risk by forecasting changes in the overall value of an asset or portfolio, CFaR deals with variations in cash flow during a given period. In addition, we will show a concrete application of the CFaR in a case study based on a real data set, in order to illustrate the potentiality of the new measure in providing risk management information.

This work represents the follow up of Arya et al. (2004) and Girardi et al. (2010) who created a flexible and dynamic budgeting model capable of providing information about the financial position, cash flow, assets and liabilities of a generic industrial firm. By analyzing the dynamics of the financial statement, Girardi et al. (2010) formalized the dynamics of the principal balance sheet items, using a set of difference equations. The goal of their research was to create a dynamic mathematical model capable of formalizing the cash flow budget of a firm, by showing the evolution of the main accounting items affecting the liquidity, in the spirit of (Mattessich 1961). With appropriate inputs in time

series form, the model is then able to calculate, at any period, the liquidity level and the value of key accounting items affecting the cash flows. The budgeting approach of Girardi et al. (2010) can be considered as the starting point for our new class of models. In fact, their formalization can be extended by considering further balance sheet items, in order to analyze different firms, according to the brick vector framework. This can be considered as a first attempt to create a unified framework for the firm analysis, in line with Ohlson (1983), who linked variations in the financial statement and other information variables to the underlying equity price, and also with Cooke and Tippet (2000), who extended the analysis of Ohlson (1983) to a stochastic framework. It is important to remark that the new direction introduced by Girardi et al. (2010) follows the need, suggested by Black (1980, 1993), to provide a conventional interplay between the income statement and the balance sheet, as the latter is not capable in itself to give enough information about the real firm's value.

In order to build up a dynamic budgeting model, one has to provide a mathematical formalism for the dynamics of the balance sheet, which involves the analysis of both the double entry book-keeping procedures previously seen as well as the relationship with the income statement. More generally, one has to introduce some mathematical techniques into the field of accounting. During the last decades, many economists and accountants have worked to find such formalism, see e.g. Butterworth (1972), Melse (2006), Ohlson (1983), Cooke and Tippet (2000) and a great variety of mathematical approaches and techniques have been introduced in the accounting literature. However, a clear and complete approach has not yet been provided, partly because people were mostly interested in specific goals, thus giving only a partial picture of the topic. The work on matrix representation of accounting introduced by Mattessich more than five decades ago, see e.g. Mattessich (1957, 1958, 1961), was a fundamental source of inspiration for us. The accounting measurement proposed by Ijiri (1965), who stressed the importance of cash as a building block for measuring all relevant quantities in the entire balance sheet, and the results of Melse (2006), who defined the fundamental accounting relations in a temporal perspective, were also crucial for our research. In particular, the latter work represents an essential step in order to abstract the accounting system from merely a tool to the level of a general purpose system for strategic planning and management control, in line with Leitch and Chen (1999).

Let us now introduce the mathematical formalism of the dynamics of the relevant accounting items presented in the balance sheet. We assume that the balance sheet at time $n = 1, 2, \dots$ can be represented as a vector S , containing the state variables, which evolves according to the following linear-affine dynamics:

$$S_n = M_n(S_{n-1} + C_n) + F_n, \quad (2.0.1)$$

where the *accounting (square real) matrix* M reflects the linear transition between times $n-1$ and n , while C, F are the vectors of economic/financial transactions in the period $[n-1, n]$ (note that the presence of C is quite redundant

in the dynamics of S : however, as it denotes the purely economic transactions carried out in the reference period, we prefer to include explicitly its presence in the equation in order to obtain a system with more information power).

The system (2.0.1) should involve a parsimonious number of (observable) variables, like in Girardi et al. (2010), who used six state variables that can be found in the balance sheet. In this chapter, we will adopt their approach in the particular case where the parameters of the system (2.0.1) are constant. Under this assumption, we shall see that the liquidity process can be explicitly determined by the previous system, using the averaging procedure discussed in chapter one, and we will be able to investigate its properties. More generally, the constant parameter assumption is also motivated by the possibility to get additional information on the dynamic structure of the balance sheet, especially in a forecast perspective. Note that this assumption does not restrain the dynamic nature of the model: in fact, the parameter set is calibrated on real data in order to perfectly replicate the results of the general model that has been previously introduced and investigated by Girardi (2010). We will accomplish this task with a two step procedure. The first step consists in dividing the general case into a system of difference equations involving the balance sheet and the liquidity process (positive or negative and without active interests), namely into what we have defined in the previous chapter as brick vectors. We shall see that this formalism will allow us to get some crucial information for our analysis. In the second step, we will provide a closed form solution for the difference equation system and we will focus on a special version of the general solution by applying the notion of average parameter in the sense of Chisini (1929) (see also de Finetti and Mura (1995)). In other words, we look for the constant parameters that replicate the behavior of the balance sheet for a given time horizon $n = 1, 2, \dots$. We then perform a sensitivity analysis with respect to the relevant parameters, in order to test the stability of the liquidity process. This is of course extremely important in view of the sustainability of the firm. In this perspective, our approach represents a first attempt in the computation of a new measure of CFaR, which is defined as the maximum shortfall of cash the firm is willing to tolerate with a given confidence level (see Andrn et al. (2005), Yan et al. (2014)). This new measure combines the bottom-up VaR approach of Riskmetrics and the exposure-based CFaR approach (see e.g. Andrn et al. (2005)), as we consider how the company exposure, in terms of liquidity risk, is affected by some macroeconomic variables dynamics. We first estimate, from the past financial statements, the average dynamic of the main accounting items, then we forecast the dynamics of the macroeconomic variables by simulating multiple paths. Finally, we bring all together and we simulate the cash flow statement by considering both the average dynamics and the multiple paths of the macroeconomic variables, thus evaluating the cash flow at risk.

We emphasize that our representation provides a natural relationship between the firm supply chain and the liquidity flow. In fact we will show that the model gives a great information about the internal processes of the firm

with the possibility to analyze separately the needs of operating, investing and financing liquidity, in the spirit of what suggested by e.g. Tsai (2008, 2011).

2.1 The model

In order to give firstly an intuition and introduce the reader into the topic, we begin by providing a stylized example, inspired by Arya et al. (2004) and rearranged in order to better fit to the modelization we are about to propose and to a more realistic situation, of a balance sheet whose evolution can be expressed by mean of an affine transformation as in (2.0.1).

Consider a firm that at time $n = 0$ starts with an initial capital stock of \$80 divided in cash for \$40 and equipment for \$40. During the first financial year it purchases equipment for \$40, with an accumulated depreciation expense of \$20, and it sells stock for \$20. The cash revenue is \$30, so that the total income is \$10¹. Now we consider the typical representation of the balance sheet at the beginning and at the end of the period. The balance sheet at the beginning can be represented as follows:

<i>Assets</i> ($n = 0$)		<i>Liabilities + Equities</i> ($n = 0$)	
<i>Cash</i>	\$40	<i>Depreciation</i>	\$0
<i>Equipment</i>	\$40	<i>Equity</i>	\$80
<i>Total Asset</i>	\$80	<i>Total Liabilities + Equity</i>	\$80

At the end $n = 1$ we have the following situation:

<i>Assets</i> ($n = 1$)		<i>Liabilities + Equities</i> ($n = 1$)	
<i>Cash</i>	\$50	<i>Depreciation</i>	\$20
<i>Equipment</i>	\$80	<i>Equity</i>	\$110
<i>Total Assets</i>	\$130	<i>Total Liabilities + Equity</i>	\$130

where the equity at time $n = 1$ includes the income.

In order to describe the dynamic of the balance sheet, we then introduce the balance sheet vector S as

$$S = \begin{pmatrix} L \\ K \\ DD\&A \\ E \end{pmatrix},$$

where L denotes the liquidity (cash or cash equivalent), K stands for the fixed asset (tangible and intangible assets, in our example equipment), $DD\&A$

¹The income corresponds to the difference between the revenue (\$30) and the depreciation (\$20).

for depreciation, depletion and amortization (that is the variation of asset value due to their use), while E denotes the equity or the stock.

We can now write the balance sheet at time $n = 1$ as the result of a matrix transformation:

$$S_1 = \begin{pmatrix} 1 & -1 & 0 & 1/4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 100/80 \end{pmatrix} \left[\begin{pmatrix} \$40 \\ \$40 \\ \$0 \\ \$80 \end{pmatrix} + \begin{pmatrix} \$0 \\ \$0 \\ \$0 \\ \$0 \end{pmatrix} \right] + \begin{pmatrix} \$30 \\ \$0 \\ \$20 \\ \$10 \end{pmatrix} = \begin{pmatrix} \$50 \\ \$80 \\ \$20 \\ \$110 \end{pmatrix}.$$

Therefore, the evolution of the balance sheet vector can be written as

$$S_1 = \begin{pmatrix} 1 & -1 & 0 & 1/4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 100/80 \end{pmatrix} (S_0 + C_1) + F_1$$

where the initial vector of S is given by

$$S_0 = (\$40, \$40, \$0, \$80)^\top.$$

The economic transactions occurred in the accounting period $[0, 1]$ are included in the (trivial) vector $C_1 = (\$0, \$0, \$0, \$0)^\top$, while $F_1 = (\$30, \$0, \$20, \$10)^\top$.

The previous matrix representation (2.0.1) is a flexible and parsimonious tool in order to catch the general features of the evolution of the balance sheet. Of course the dimension of the state variable vector S is crucial for the model to be efficient: in the general model we are going to introduce in the next subsection, we will need to increase the dimension of the balance sheet vector from four to six, however we shall show that the system still remains fully tractable.

2.2 Model Specification

Motivated by the previous example, in this section we provide an analytical overview of the dynamics of the accounting items, specifying their contribution in generating cash flows. Following Girardi (2010), and in order to keep the model parsimonious, we assume that the balance sheet can be fully explained through a six-dimensional vector S of accounting items as follows:

$$S_n = (L_n, T_n, R_n, K_n, D_n, B_n)^\top, \quad (2.2.1)$$

where

- L_n = Cash and Cash equivalent (bank and post office accounts, cheques, cash on hand);

- T_n = Account Receivables (from Trade, Subsidiaries, Associated companies and other);
- R_n = Inventory (Work in process WIP, Finished Goods);
- K_n = Tangible and Financial Fixed assets (PPE and other, Financial Assets);
- D_n = Account Payables (Trade accounts and other payables);
- B_n = Financial Debt (long-term and short-term).

In the balance sheet evolution there will be some accounting and financial variables that are typically random and can be modelled as (discrete time) stochastic processes. They include the following items:

- Z_n = Revenues (revenues from sales and other income);
- G_n = Costs (raw materials, service expenses, lease and rental costs, employee expenses);
- E_n = Principal repayments;
- A_n = Amortization;
- X_n = Total Non-Operating Income/Expenses plus Financial Income.

These items are generally exogenous, and the model is able to well represent different types of (industrial) companies according to the functional form of the corresponding dynamic equations. Of course, for firms working in the financial/banking industry, we should adapt the model in order to consider the cash flows due to the financial assets and financial yields.

In the following we shall introduce the dynamics of the state variables of the balance sheet vector S . Let us begin with T , the outstanding trade receivables. At any time n , a fixed proportion $\eta_n \geq 0$ of outstanding trade receivables (including the *sales* denoted by the process Z_n) are settled immediately and transformed in liquidity (i.e. in cash, or cash equivalents), while the remaining proportion $(1 - \eta_n)$ is postponed to the subsequent period, so that the dynamics of the outstanding receivables from clients is given by the following difference equation:

$$T_n = (1 - \eta_n)(T_{n-1} + Z_n), \quad (2.2.2)$$

where T_{n-1} are the outstanding trade receivables at the beginning of the period $[n - 1, n]$ ². In particular, the part of outstanding receivables transformed into

²The percentage η_n of receivables settled immediately is usually computed as a weighted average of the *earnings* in the period $[n - 1, n]$, denoted by F_n , with weight η_n^1 , and the outstanding credits at the beginning of the period (that is, T_{n-1}), with weight η_n^2 . In formulas, $\eta_n = \frac{\eta_n^1 T_{n-1} + \eta_n^2 F_n}{T_{n-1} + F_n}$. In other words, η_n can be written as the percentage of receivables from clients at time $n - 1$ and the earnings from the n -th period received at time n .

liquidity is given by $\eta_n(T_{n-1} + Z_n)$. We emphasize that at this stage we do not make any assumption on the stochastic process Z describing the sales. However, observe that Z_n is stochastic and exogenous, while η_n is a parameter depending on the characteristics of the customers.

Let us now consider the inventories R . We denote with V_n the *unitary average value of unsold inventories*, while Q_n denotes the amount of *unsold inventories* at time n and $\Delta Q_n = Q_n - Q_{n-1}$, so that the difference equation satisfied by the unsold inventories reads as follows:

$$R_n = \phi_n R_{n-1} + \Delta Q_n V_n, \quad (2.2.3)$$

where $\phi_n \geq 0$ represents the ratio between the unitary value of unsold inventories at times $n-1$ and n , that is

$$\phi_n = \frac{V_n}{V_{n-1}}. \quad (2.2.4)$$

Here we assume that inventories can not generate liquidity. Typically, ϕ_n (or V_n and V_{n-1}) depends on both business decisions and the market trend, while ΔQ_n can be considered as exogenous as it depends on the production cycle.

Let us now consider the evolution of the fixed assets K . We denote with γ_n the percentage of the part of liquidity, relative to the value of the assets, used to pay the acquisition of new assets and with A_n the *depreciation cost*, so that

$$K_n = (1 + \gamma_n)K_{n-1} - A_n. \quad (2.2.5)$$

Here K_n and A_n depend on business decisions, while γ_n is the rate of change in assets, typically negative.

The evolution of trade/operating payables D is parametrized by ω_n ³, representing the proportion of the debt that is settled at each period and by G_n , denoting the *total costs* occurred in the period, so that

$$D_n = (1 - \omega_n)(D_{n-1} - G_n). \quad (2.2.6)$$

Then, the part of trade/operating payables transformed into liquidity is given by $\omega_n(D_{n-1} - G_n)$. Here, G_n depends on the cost structure of the firm, while ω_n is the rate of change in the payables, typically positive.

Finally, the financial debt (including both short and long-term borrowings) B , evolves according to

$$B_n = B_{n-1} - E_n, \quad (2.2.7)$$

where E_n represents the repayment of the principal, i.e. the *capital mortgage reimbursed* from financial payables (if negative), or *new external financing* (if

³As for the parameter η_n , also ω_n can be computed as a weighted average $\omega_n = \frac{\omega_n^1 T_{n-1} + \omega_n^2 F_n}{T_{n-1} + F_n}$, with weights given by debt at time $n-1$ and costs faced in the n -th period that are liquidated at time n .

positive). Typically, E_n depends on the company and mostly on the choice of the lenders. Note however that we assume that interest expenses are systematically paid at the end of each period.

Now we have all the ingredients to write the difference equation describing the evolution of the liquidity process L , which involves all previous quantities (but the inventories R), in line with Girardi (2010):

$$L_n = L_{n-1} + \eta_n(T_{n-1} + Z_n) - \gamma_n K_{n-1} + \omega_n(D_{n-1} - G_n) + d_n B_{n-1} - E_n + X_n \quad (2.2.8)$$

where d_n represents the rate of (passive) *interest for loans* and X_n is the liquidity variation (plus or minus) due to *extra-ordinary transactions*⁴.

Remark: In the presence of a significant income tax, it could be necessary to include a tax rate, together with the positive gross and net income processes in the system. In this case one should introduce the net liquidity process that is related to (2.2.8) through $(Net\ Liquidity)_n = L_n - Taxes_n$. We do not enter into details as it is not the main focus of the paper, however we emphasize that our framework is flexible enough to easily include also taxation.

Summing up the equations (2.2.2)-(2.2.8), we can write the evolution of the balance sheet vector S by the following system of difference equations:

$$\begin{cases} L_n = L_{n-1} + \eta_n(T_{n-1} + Z_n) - \gamma_n K_{n-1} + \omega_n(D_{n-1} - G_n) + d_n B_{n-1} - E_n + X_n \\ T_n = (1 - \eta_n)(T_{n-1} + Z_n) \\ R_n = \phi_n R_{n-1} + \Delta Q_n V_n \\ K_n = (1 + \gamma_n)K_{n-1} - A_n \\ D_n = (1 - \omega_n)(D_{n-1} - G_n) \\ B_n = B_{n-1} - E_n, \end{cases} \quad (2.2.9)$$

which is easily seen to be of the form (2.0.1), with the transition matrix M_n given by

$$M_n = \begin{pmatrix} 1 & \eta_n & 0 & -\gamma_n & \omega_n & d_n \\ 0 & 1 - \eta_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \gamma_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - \omega_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

once the transaction vector C is defined as $C = (0, Z, 0, 0, -G, 0)^\top$ and the vector F is defined by $F = (-E + X, 0, (\Delta Q)V, -A, 0, -E)^\top$.

⁴The absence of active interests on the liquidity process can be considered to be restrictive only for large financial firms. Of course, this assumption could be easily removed by introducing a specific variable relative to financial investments and related active interests.

2.3 Description of the system equations in the brick vector framework

As we have already hinted in section 1.5 of the previous chapter, one of the greatest problems of any balance sheet modelization is the high number of variables that it inevitably implies. So every time someone would approach a modelization according to the brick vector framework the first step should always be a careful assessment of the accounting items to use since, as always for mathematical modelizations, there is a heavy trade off between the need for detail and the need for simplicity. As far as a general outlook on this matter could be of help, we tend to think that, especially in the case of industrial firms, a good point of balance can be represented by the model we have introduced in the previous paragraph (in line with Girardi et al. (2010), following Mattessich (1961)) where we decided to choose the next accounting items for the balance sheet vector:

1. L_n Current Liquidity;
2. T_n Receivables from Clients;
3. R_n Inventory;
4. K_n Fixed Assets;
5. D_n Payables to Suppliers;
6. B_n Financial Payables.

and employing mainly the vector \vec{A} and rarely the vector \vec{P} ⁵ we tried to reach a sound formalization for the interpretation of the variables in the transaction vector as aggregations of accounts taken from the profit and loss statement.

In this case the brick-vectors will have the shape that we are about to present. For the Receivables from Clients we will have:

$$\begin{bmatrix} L_n^T \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & \eta_n \\ 0 & (1 - \eta_n) \end{bmatrix} \begin{bmatrix} L_{n-1}^T + 0 \\ T_{n-1} + Z_n \end{bmatrix} \quad (2.3.1)$$

where Z_n represents the amount of sales sold by the firm between the time step $n - 1$ and the time step n and η_n is the weighted average percentage of

⁵See section 1.3.1 in the previous chapter.

those sales that becomes cash-flow in the period. For the Inventory the brick vector will present this simple form:

$$\begin{bmatrix} L_n^R \\ R_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \phi_n \end{bmatrix} \begin{bmatrix} L_{n-1}^R + 0 \\ R_{n-1} + \Delta Q_n \cdot V_n \end{bmatrix} \quad (2.3.2)$$

where $\Delta Q_n = Q_n - Q_{n-1}$ is the variation in inventory between $n - 1$ and n , while V_n is average unit evaluation of inventory at time n and $\phi_n = \frac{V_n}{V_{n-1}}$ is ratio between unit values of inventory at dates n and $n - 1$. The brick vector relative to the Fixed Assets instead will be as follows:

$$\begin{bmatrix} L_n^K \\ K_n \end{bmatrix} = \begin{bmatrix} 1 & -\gamma_n \\ 0 & (1 + \gamma_n) \end{bmatrix} \begin{bmatrix} L_{n-1}^K \\ K_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -A_n \end{bmatrix} \quad (2.3.3)$$

whose parameter γ_n will be the percentage of new investments in tangible fixed assets and the variable A_n is the value of amortization between time $n - 1$ and time n . Relatively to the Payables to Suppliers Item its brick vector will be the following:

$$\begin{bmatrix} L_n^D \\ D_n \end{bmatrix} = \begin{bmatrix} 1 & \omega_n \\ 0 & (1 - \omega_n) \end{bmatrix} \begin{bmatrix} L_{n-1}^D \\ D_{n-1} + G_n \end{bmatrix} \quad (2.3.4)$$

and in the above formulas ω_n will be the weighted average percentage of payments in the period n , while G_n will be the costs in the period n expressed through a negative number, since they represent a stream of wealth from the firm to the outside world. Finally the brick vector B_n relative to the Financial Payables will be:

$$\begin{bmatrix} L_n^B \\ B_n \end{bmatrix} = \begin{bmatrix} 1 & d_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} L_{n-1}^B \\ B_{n-1} \end{bmatrix} + \begin{bmatrix} E_n \\ E_n \end{bmatrix} \quad (2.3.5)$$

where d_n is the rate of interest on the Financial Payables and E_n represents the amount of capital repaid about the existing loans (if positive), or new loans from third parties (if negative). In this situation usually B_0 will be different from zero and it will have a negative value since it will represent the financial debt at the beginning of the life of the firm.

Now if we combine the previous brick vectors we will obtain the following system:

$$\begin{bmatrix} L_n \\ C_n \\ R_n \\ K_n \\ D_n \\ Df_n \end{bmatrix} = \begin{bmatrix} 1 & \eta_n & 0 & -\gamma_n & \omega_n & d_n \\ 0 & 1 - \eta_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \gamma_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - \omega_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\times \left\{ \begin{bmatrix} L_{n-1} \\ C_{n-1} \\ R_{n-1} \\ K_{n-1} \\ D_{n-1} \\ Df_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ Z_n \\ \Delta Q_n \cdot V_n \\ 0 \\ G_n \\ 0 \end{bmatrix} \right\} + \begin{bmatrix} E_n + X_n \\ 0 \\ 0 \\ -A_n \\ 0 \\ E_n \end{bmatrix} \quad (2.3.6)$$

where we have added in the first position of the vector \vec{P} the variable X_n representing extraordinary financings/dividends plus taxes happened between the time step $n - 1$ and the time step n .

2.4 Solution in Closed Form

As we have seen in the first chapter a closed form solution can be obtained easily for the balance sheet system by developing the closed form formula of each brick vector and then combine them. The procedure will be as follows.

2.4.1 Brick vectors

The equation (2.2.8) describing the time evolution of the liquidity process has been disentangled as a sum of different terms associated to sub-models⁶ for the liquidity. This decomposition can be formally written as follows:

$$L_n = L_0 + L_n^T + L_n^R + L_n^K + L_n^D + L_n^{B,X},$$

with initial condition $L_0^T = L_0^R = L_0^K = L_0^D = L_0^B = 0$, where:

- L_n^T denotes the liquidity coming from the outstanding trade receivables T , typically generating positive cash flows;
- L_n^R denotes the liquidity coming from the inventories: as we assumed no direct impact on the liquidity process it turns out that L_n^R satisfies a trivial equation and it does not generate any cash flow;
- L_n^K denotes the liquidity coming from the fluctuations of the balance sheet properties, which can be positive or negative;
- L_n^D denotes the liquidity coming from the variation of the payables to suppliers, which typically generates negative cash flows (for example in the case of non-performing loans);
- $L_n^{B,X}$ denoted the liquidity coming from the financial business and it can generate positive or negative cash flows⁷.

The previous decomposition gives us the possibility to calculate separately each class of sub-liquidity terms, namely each brick vector, as we are going to do in the next subsections.

2.4.2 Brick vector 1: Liquidity from Operating Cash Flows (inflows only)

Let us consider the liquidity coming from the evolution of the outstanding trade receivables, also called *operating cash flows*, which represents the positive core business of the liquidity evolution. Denoting with L_n^T this term, we can introduce the following sub-model:

$$\begin{cases} L_n^T = L_{n-1}^T + \eta_n(T_{n-1} + Z_n) \\ T_n = (1 - \eta_n)(T_{n-1} + Z_n). \end{cases} \quad (2.4.1)$$

This system describes the evolution of the part of liquidity coming from sales and credits that are immediately available (as cash or cash equivalents), together with receivables which will be available in the future. In order to solve

⁶ Another way to indicate the brick vectors.

⁷ We decided to apply the vector X to the brick vector L^B but we could have easily decomposed the liquidity in this fashion $L_n - X_n = L_0 + L_n^T + L_n^R + L_n^K + L_n^D + L_n^B$

the system, let us consider the second equation which admits the following solution:

$$T_n = \prod_{i=0}^{n-1} (1 - \eta_{n-i}) T_0 + \sum_{h=0}^{n-1} \left(\prod_{i=0}^h (1 - \eta_{n-i}) Z_{n-h} \right), \quad (2.4.2)$$

while the dynamics of L_n^T gives

$$L_n^T = \sum_{h=0}^{n-1} \eta_{n-h} Z_{n-h} + \sum_{j=0}^{n-1} \eta_{n-j} T_{n-1-j}. \quad (2.4.3)$$

Now we plug the expression (2.4.2) into (2.4.3) and we arrive to the closed form solution for the liquidity coming from the operating cash flows ⁸:

$$\begin{aligned} L_n^T &= \sum_{h=0}^{n-1} \eta_{n-h} Z_{n-h} + \\ &+ \sum_{j=0}^{n-1} \eta_{n-j} \left[\prod_{i=0}^{n-2-j} (1 - \eta_{n-1-i-j}) T_0 + \sum_{h=0}^{n-j-2} \left(\prod_{i=0}^h (1 - \eta_{n-1-j-i}) \right) Z_{n-1-h-j} \right] \end{aligned}$$

It is useful for the sequel to get some recursive relations for the sub-models. For the positive core business system, it is easily seen that summing up the equations we obtain

$$L_n^T + T_n - (L_{n-1}^T + T_{n-1}) = Z_n, \quad (2.4.4)$$

from which we deduce

$$L_n^T + T_n - (L_0^T + T_0) = \sum_{i=1}^n Z_i, \quad (2.4.5)$$

meaning that in the positive core business, between two periods the difference in value is due only to earning from sales, in line with any basic principle of cash flow statement, see e.g. Mulford and Comiskey (2005).

2.4.3 Brick vector 2: Liquidity from Inventories (no direct flows)

We saw that unsold inventories have no direct impact on the liquidity process. However, it is important to notice that they have anyway an indirect impact as

⁸We use hereafter the convention that $\prod_a^b = 1$ and $\sum_a^b = 0$ whenever $a > b$, then for $j = n-1$ we have $\prod_{i=0}^{n-2-j} (1 - \eta_{n-1-i-j}) = 1$ and $\sum_{h=0}^{n-j-2} [\prod_{i=0}^h (1 - \eta_{n-1-j-i}) Z_{n-1-h-j}] = 0$.

they can affect the possible future borrowings. In fact, unsold inventories can be considered as additional guarantees for external financings. We shall investigate this crucial aspect in future research. For the moment, we limit to associate a trivial sub-model to the part of liquidity coming from the inventories. Denoting with L_n^R this term, we obtain the following system:

$$\begin{cases} L_n^R = 0 \\ R_n = \phi_n R_{n-1} + \Delta Q_n V_n, \end{cases} \quad (2.4.6)$$

which admits the following solution:

$$L_n^R = L_0^R, \\ R_n = \prod_{i=0}^{n-1} (\phi_{n-i}) R_0 + \sum_{h=1}^{n-1} \left[\prod_{i=0}^{h-1} (\phi_{n-i}) \Delta Q_{n-h} V_{n-h} \right] + \Delta Q_n V_n.$$

2.4.4 Brick vector 3: Liquidity from Investing Cash Flows (inflows/outflows)

Let us now consider the part of liquidity L_n^K coming from the fluctuations of the balance sheet properties, which can be positive or negative:

$$\begin{cases} L_n^K = L_{n-1}^K - \gamma_n K_{n-1} \\ K_n = (1 + \gamma_n) K_{n-1} - A_n. \end{cases} \quad (2.4.7)$$

To solve this system, we develop the equation for K_n ⁹:

$$K_n = \prod_{i=0}^{n-1} (1 + \gamma_{n-i}) K_0 - \sum_{i=1}^{n-1} \left[\prod_{j=0}^{i-1} (1 + \gamma_{n-j}) A_{n-i} \right] - A_n,$$

while the dynamics for L_n^K gives

$$L_n^K = -\gamma_1 K_0 - \sum_{i=1}^{n-1} \gamma_{i+1} K_i.$$

Assuming that for $i = 1$ we have $\sum_{h=1}^{i-1} \left(\prod_{j=0}^{i-1} (1 + \gamma_{n-j}) A_{n-h} \right) = 0$, we obtain

$$L_n^K = -\gamma_1 K_0 - \sum_{i=1}^{n-1} \gamma_{i+1} \left[\prod_{h=0}^{i-1} (1 + \gamma_{i-h}) K_0 - \sum_{h=1}^{i-1} \left(\prod_{j=0}^{h-1} (1 + \gamma_{i-j}) A_{i-h} \right) - A_i \right].$$

⁹We use the convention that for $n = 1$ we have $\sum_{i=1}^{n-1} \left[\prod_{j=0}^i (1 + \gamma_{n-j}) A_{n-i} \right] = 0$ and $\sum_{i=1}^{n-1} \gamma_{i+1} K_i = 0$.

Let us now consider the recursive relation coming from the system defining L_n^K : we have

$$L_n^K + K_n - (L_{n-1}^K + K_{n-1}) = -A_n, \quad (2.4.8)$$

so that

$$L_n^K + K_n - (L_0^K + K_0) = -\sum_{i=1}^n A_i,$$

meaning that the difference in value between two periods are given only by mortgages, according to the basic rules of cash flow statement, see e.g. Mulford and Comiskey (2005).

2.4.5 Brick vector 4: Liquidity from Payables to Suppliers (outflows only)

Consider now the liquidity L_n^D coming from the variation of the payables to suppliers, that is the sub-model describing the impact of the costs in both current and future liquidity:

$$\begin{cases} L_n^D = L_{n-1}^D + \omega_n(D_{n-1} - G_n) \\ D_n = (1 - \omega_n)(D_{n-1} - G_n). \end{cases} \quad (2.4.9)$$

Also in this case the solution is simple: from the dynamics of the debt (2.2.6) we get

$$D_n = \prod_{i=0}^{n-1} (1 - \omega_{n-i}) D_0 - \sum_{h=0}^{n-1} \left(\prod_{i=0}^h (1 - \omega_{n-i}) \right) G_{n-h},$$

while for the corresponding liquidity L_n^D we have

$$L_n^D = \sum_{j=0}^{n-1} \omega_{n-j} D_{n-1-j} - \sum_{h=0}^{n-1} \omega_{n-h} G_{n-h}.$$

Replacing the expression of D_n gives us:

$$\begin{aligned} L_n^D &= \sum_{j=0}^{n-1} \omega_{n-j} \left[\prod_{i=0}^{n-2-j} (1 - \omega_{n-1-i-j}) \right] D_0 + \\ &- \sum_{h=0}^{n-2-j} \left(\prod_{i=0}^h (1 - \omega_{n-1-j-i}) \right) G_{n-1-h-j} \Big] - \sum_{h=0}^{n-1} \omega_{n-h} G_{n-h} \end{aligned}$$

Finally, the recursive relation driving the system for L_n^D is

$$L_n^D + D_n - (L_{n-1}^D + D_{n-1}) = -G_n, \quad (2.4.10)$$

that is

$$L_n^D + D_n - (L_0^D + D_0) = -\sum_{i=1}^n G_i,$$

which states that the difference in value between two times is only given by costs occurred in that period. This is also in line with the basic principles of cash flow statement, see e.g. Mulford and Comiskey (2005).

2.4.6 Brick vector 5: Liquidity from Financing Cash Flows (inflows/outflows)

We consider now $L_n^{B,X}$, that is the part of the liquidity coming from the financial business plus the total non operating income/expenses. We obtain the following system:

$$\begin{cases} L_n^{B,X} = L_{n-1}^{B,X} + d_n B_{n-1} - E_n + X_n \\ B_n = B_{n-1} - E_n. \end{cases} \quad (2.4.11)$$

From the dynamics of B_n , given in (2.2.7), we obtain

$$B_n = B_0 - \sum_{h=1}^n E_h,$$

and

$$L_n^{B,X} = \sum_{i=1}^n d_i B_{i-1} + \sum_{i=1}^n (-E_i + X_i).$$

Replacing in $L_n^{B,X}$ the expression of B_n yields

$$L_n^{B,X} = \sum_{i=1}^n d_i (B_0 - \sum_{j=1}^{i-1} E_j) + \sum_{i=1}^n (-E_i + X_i).$$

2.5 Model Specification with Constant Parameters

In this section we will apply the averaging procedure we have discussed in the first chapter and we will focus on the special case where all the parameters in the accounting matrix M in (2.0.1) and the major exogenous items (such as $\Delta Q, Z$ and X) are constant, i.e. we consider the system (2.2.9) in the following specification:

$$\begin{cases} L_n = L_{n-1} + \eta(T_{n-1} + Z) - \gamma K_{n-1} + \omega(D_{n-1} - G) + dB_{n-1} - E + X \\ T_n = (1 - \eta)(T_{n-1} + Z) \\ R_n = R_{n-1} + \Delta Q \cdot V \\ K_n = (1 + \gamma)K_{n-1} - A \\ D_n = (1 - \omega)(D_{n-1} - G) \\ B_n = B_{n-1} - E, \end{cases} \quad (2.5.1)$$

with a given initial condition $(L_0, T_0, R_0, K_0, D_0, B_0)^{10}$ where $L_0 = 0$ as well as all the decomposed liquidity items.

In the constant parameters case the solution for the sub-models simplify greatly as well as the recursive relations: we briefly illustrate the results below. The first sub-model associated with the system (2.4.1) becomes:

$$\begin{cases} L_n^T = T_0(1 - (1 - \eta)^n) + Z \left(n - (1 - \eta) \left(\frac{1 - (1 - \eta)^n}{\eta} \right) \right) \\ T_n = (1 - \eta)^n T_0 + Z(1 - \eta) \frac{1 - (1 - \eta)^n}{\eta}, \end{cases}$$

while the recursive relation (2.4.4) reads

$$L_n^T + T_n - (L_{n-1}^T + T_{n-1}) = Z,$$

from which we deduce

$$L_n^T + T_n - (L_0^T + T_0) = nZ. \quad (2.5.2)$$

For the second sub-model associated with the system (2.4.6) we have

$$R_n = R_0 + n\Delta Q \cdot V \quad (2.5.3)$$

and, of course, $L_n^R = L_0^R$.

For the third sub-model related to (2.4.7), we obtain

$$\begin{cases} L_n^K = K_0(1 - (1 + \gamma)^n) - A \left(n + \frac{1 - (1 + \gamma)^n}{\gamma} \right) \\ K_n = (1 + \gamma)^n K_0 + \frac{1 - (1 + \gamma)^n}{\gamma} A, \end{cases} \quad (2.5.4)$$

¹⁰Note that in the steady state scenario we are considering, we have $\phi = 1$.

while the recursive relation (2.4.8) reads

$$L_n^K + K_n - (L_{n-1}^K + K_{n-1}) = -A,$$

which leads to

$$L_n^K + K_n - (L_0^K + K_0) = -nA. \quad (2.5.5)$$

For the forth sub-model, related to (2.4.9), we get

$$L_n^D = (1 - (1 - \omega)^n)D_0 - \left(n - (1 - \omega) \left(\frac{1 - (1 - \omega)^n}{\omega} \right) \right) G,$$

which leads to

$$D_n = (1 - \omega)^n D_0 - (1 - \omega) \frac{(1 - (1 - \omega)^n)}{\omega} G. \quad (2.5.6)$$

The recursive relation (2.4.10) becomes

$$L_n^D + D_n - (L_{n-1}^D + D_{n-1}) = -G,$$

from which it follows

$$L_n^D + T_n - (L_0^D + D_0) = -nG. \quad (2.5.7)$$

Finally, for the fifth sub-model, associated with (2.4.11), we get

$$\begin{cases} L_n^{B,X} = ndB_0 - d^{\frac{n(n-1)}{2}}E + n(-E + X) \\ B_n = B_0 - nE. \end{cases} \quad (2.5.8)$$

2.5.1 Parameter Averaging

We now focus on the main result of the paper, namely the interpretation of the constant parameters as averages in the sense of Chisini (1929). We will use the recursive relations we found in the previous subsection in order to apply the definition of mean given by Chisini to the constant parameter setting. That is, the average parameters can be obtained by equating the constant-parameter recursive system with the result coming from the general model with general parameters. The average parameters obviously depend on the choice for the time interval $[0, n]$.

Let us now investigate separately any sub-model.

2.5.1.1 Operating Cash Flows

According to the average in the sense of Chisini, we are looking for the constant sales process Z and the constant parameter η that replicate, at a given fixed time horizon $[0, n]$, the behavior of the sub-model with general parameter η_n and general sales process Z_n ¹¹. Note that this assumption is not equivalent to

¹¹For sake of notational simplicity we skip the obvious dependence of Z and η on the time window n .

assume that the system is stationary. As we can also equate the differences of the sub-models, we use the recursive relation (2.5.2) and we arrive to the following equation:

$$nZ = \sum_{i=1}^n Z_i$$

from which we can then define the Chisini mean value Z as

$$Z = \frac{\sum_{i=1}^n Z_i}{n}$$

Now we plug this value in the equation defining the receivables T_n and we deduce the corresponding average value for η that replicates the value of the liquidity in the sub-model:

$$T_n = (1 - \eta)^n T_0 + Z(1 - \eta) \left(\frac{1 - (1 - \eta)^n}{\eta} \right).$$

This is a polynomial equation in η that can be solved using standard numerical procedures.

2.5.1.2 Inventories

From (2.5.3) and assuming that unsold inventories are equi-distributed along time, i.e.

$$\Delta Q = \frac{\sum_{i=1}^n \Delta Q_i}{n}$$

we can determine the average value for V :

$$V = \frac{R_n - R_0}{\sum_{i=1}^n \Delta Q_i}. \quad (2.5.9)$$

2.5.1.3 Investing Cash Flow (Inflows/Outflows)

From the expression (2.5.3), giving L_n^K , we get immediately

$$nA = \sum_{i=1}^n A_i,$$

that is

$$A = \frac{\sum_{i=1}^n A_i}{n},$$

and replacing this value in the expression defining K_n we get

$$K_n = (1 + \gamma)^n K_0 + \frac{1 - (1 + \gamma)^n}{\gamma} A$$

which is a polynomial implicitly giving the average parameter γ .

2.5.1.4 Payables to Suppliers (Outflows)

From (2.5.7) it follows immediately

$$nG = \sum_{i=1}^n G_i$$

that is

$$G = \frac{\sum_{i=1}^n G_i}{n}.$$

Again, we replace this value into the expression (2.5.6) giving D_n and obtain the following polynomial equation defining the average parameter ω :

$$D_n = (1 - \omega)^n D_0 - (1 - \omega) \frac{(1 - (1 - \omega)^n)}{\omega} G.$$

2.5.1.5 Financing Cash Flows (Inflows/Cashflows)

Assuming that minus and plus values are equi-distributed along time, i.e.

$$nX = \sum_{i=1}^n X_i,$$

and using relations (2.5.8) , we obtain the following average parameters:

$$E = \frac{B_0 - B_n}{n},$$

$$d = \frac{2(L_n^{B,X} + B_0 - B_n - nX)}{(n+1)B_0 + (n-1)B_n}.$$

In conclusion, we provided explicit expressions for the constant parameters replicating the behavior of the balance sheet at a fixed time n , in terms of observable quantities involving past financial statements. In the next subsection, we perform a sensitivity analysis in order to capture the relevance of each parameter and their impact on the fluctuations of the liquidity process.

2.6 Sensitivity Analysis

In this section we investigate the sensitivity of the liquidity process with respect to the average parameters. This study is important for two reasons: first, this allows us to understand the impact of the physical supply chain and the firm structure on the cash flows, in line with Tsai (2011). Secondly, it naturally opens the door to the Cash-Flow-at-Risk approach that we shall introduce in

the next section.

We first consider the parameters associated to the percentage η of receivables that are converted immediately in liquidity and the amount Z of earning from sales giving the liquidity from operating cash flows L_n^T . We set the following ranges for the parameters:

$$T_0 = 100, \quad n = 10, \quad Z \in [0, 100], \quad \eta \in (0, 1].$$

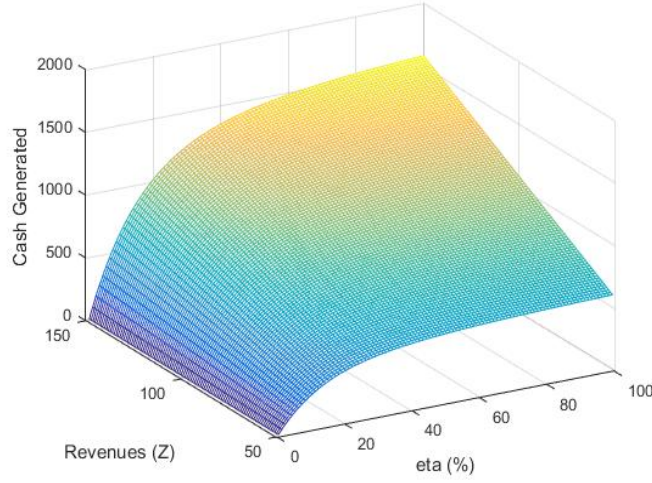


Figure 2.6.1: Dependence of the liquidity L_n^T from Operating Cash Flows as a function of Z and η .

Figure 2.6.1 shows that the liquidity process has a linear dependence only on Z but not on η . The liquidity L_n^T is highly sensitive when η is small, and this effect is more relevant for high values of Z . That is, a small positive variation of the inflows has a strong positive impact on the liquidity generated by credits when $\eta \in [0.01, 0.5]$. The effect is less relevant when η is greater. This is intuitive because when η is close to one, almost all receivables are converted immediately in liquidity, so that the income policy becomes less relevant.

Let us now consider the sensitivity of the liquidity process L_n^D with respect to the percentage of debt payments (ω) and current costs (G). We use the expression giving L_n^D with the following values of the parameters:

$$D_0 = 100, \quad n = 10, \quad G \in [0, 100], \quad \omega \in (0, 1].$$

Figure 2.6.2 shows a similar dependence of the liquidity with respect to the previous case. In fact, while the dependence is linear in G , we observe a non linear behavior with respect to ω . The impact on liquidity is similar to the one in η but in the opposite way: a positive high variation on the payment policy

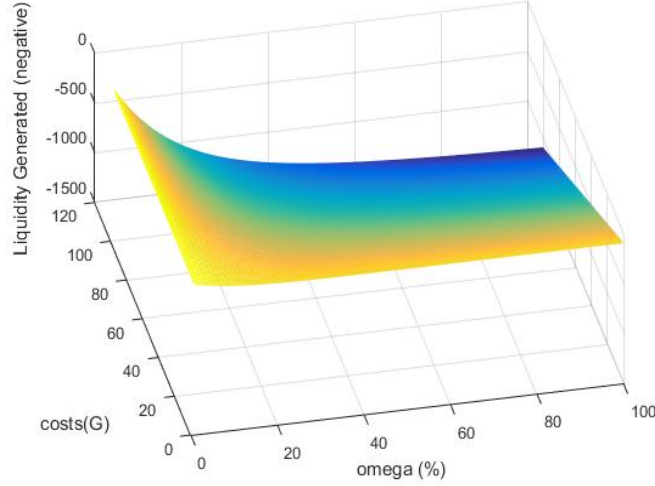


Figure 2.6.2: Dependence of the liquidity L_n^D as a function of ω and G .

has a strong negative impact on the liquidity, mostly for $\omega \in [0.01, 0.5]$, while this effect is less evident for ω close to one. Again, this is intuitive because when ω is close to one all the debt is immediately subtracted from the current liquidity.

We now proceed our analysis by assuming that

- Costs (G) are linear affine functions of earnings, i.e. $G = \lambda Z + Y$, where $\lambda \in [0, 1)$ and Y represent fixed costs;
- ω is a linear function of η , i.e. $\omega = \xi \eta$ for some $\xi \in \mathbb{R}$.

We consider the following set of parameters:

$$T_0 = D_0 = 100, \quad n = 10, \quad Z \in [50, 150], \quad \eta \in (0, 1], \quad G = 0.5Z + 10, \quad \omega = 0.7\eta.$$

Assuming that earnings are always greater than costs (meaning that the firm has a sustainable economic structure), we see from Figure 2.6.3 that the liquidity process is always positive, as expected. This scenario remains stable when taking $\xi = 1$: the only difference relies on the fact that now the liquidity could reach lower values, as results from Figure 2.6.4

Now we consider a different scenario and assume that η is strictly less than ω , but we still assume that earnings are greater than costs. For example, we consider the range $\eta \in (0, 0.4]$ and we take $\xi = 2.5$, in order to satisfy the constraint $\omega \in [0, 1]$. In this case Figure 2.6.5 shows a completely different behavior, as the liquidity can even become negative.

This is not surprising as $\eta < \omega$ means that payments are settled quicker than earnings, and for low values of Z it is not possible to absorb fixed costs:

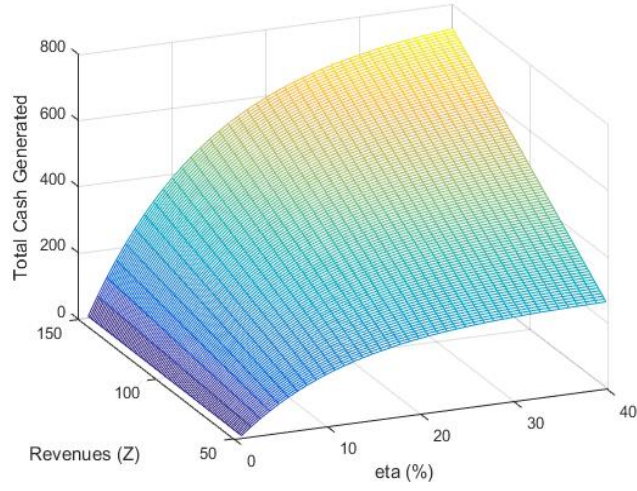


Figure 2.6.3: Dependence of the liquidity L_n^D as a function of Z and η under the assumption that $G = \lambda Z + Y$ and $\omega = \xi\eta$ with $\lambda = 0.5$, $Y = 10$, $\xi = 0.7$.

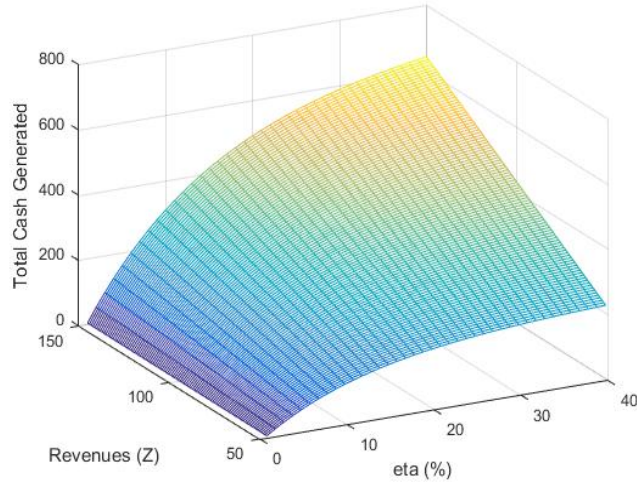


Figure 2.6.4: Dependence of the liquidity L_n^D as a function of Z and η under the assumption that $G = \lambda Z + Y$ and $\omega = \xi\eta$ with $\lambda = 0.5$, $Y = 10$, $\xi = 1$.

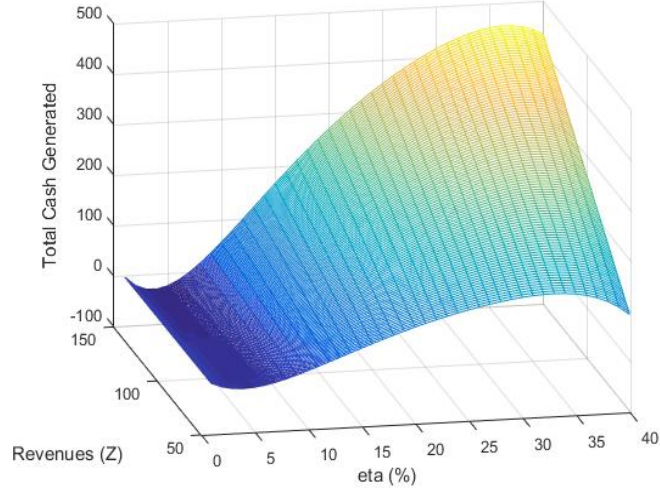


Figure 2.6.5: Dependence of the liquidity L_n^D as a function of Z and η under the assumption that $G = \lambda Z + Y$ and $\omega = \xi\eta$ with $\eta \in (0, 0.4]$, $Y = 10$, $\xi = 2.5$.

in other words, the firm has no more a sustainable structure.

Last, but not least, we focus on the relation between financing costs and the corresponding liquidity. We assume that the average parameters of the financial business lead to a situation which is well described in Figure 2.6.5. Moreover, we assume that the level of financing, if positive, is proportional to the difference $L_n^T - L_n^D$, that is

$$E = \max[\alpha(L_n^T - L_n^D), 0],$$

where α denotes the proportionality coefficient. Let us consider the following values:

$$T_0 = D_0 = 100, \quad n = 10, \quad Z \in [50, 150],$$

$$\eta \in (0, 0.4], \quad G = 0.5Z + 10, \quad \omega = 2.5\eta, \quad \alpha = 1.$$

We obtain the situation described in Figure 2.6.6 : the level of financing is proportional to the ability of the firm to generate liquidity, while it is zero when the liquidity associated to credits is less than the (negative) liquidity associated

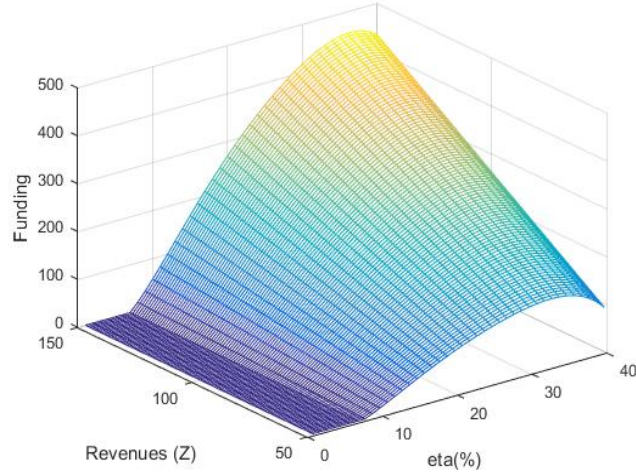


Figure 2.6.6: Dependence of the liquidity L_n^D as a function of Z and η under the assumption that $G = \lambda Z + Y$ and $\omega = \xi\eta$ with $\eta \in (0, 0.4]$, $Y = 10$, $\omega = 2.5\eta$, $\xi = 2.5$, $\alpha = 1$.

to payments. Of course, in this case it is also possible to find the corresponding average passive interest rate d .

2.7 Cash Flow at Risk

Cash Flow at Risk (CFaR) determines the maximum shortfall of cash the firm is willing to tolerate with a given confidence level. CFaR is calculated in the same way as VaR, but on cash flow rather than value. Following Shimko (1998) and Yan et al. (2014), in absence of liquidity risk CFaR has been viewed as to be equivalent to VaR as soon as financial institutions are not particularly interested in cash flows over decades, in fact any gain or loss in value immediately affects reported earnings and cash flow. This is no more true in thin markets or in presence of liquidity risk, where it may happen that a well-capitalized bank would be forced into bankruptcy, because illiquid markets would not allow banks to transfer marketable securities into cash in time. In this perspective, CFaR can be more useful than VaR in terms of measuring liquidity risk. Despite VaR being a method to determine capital requirements for absorbing investment loss, it has nothing to do with estimating sufficient cash holdings for financial

institutions. Besides, VaR typically computes over days or weeks, whereas CFaR is measured over quarters or years. We refer to Wiedemann et al. (2003), Stein et al. (2001b), Andren et al. (2005), Andren et al. (2010), Yan et al. (2014) for a detailed explanation of the CFaR methodology, as well as for a review of related literature.

2.7.1 Methods for computing Cash-Flow-at-Risk

Since the last financial crisis, firms faced a liquidity distress when borrowing proved to be much more expensive than ever before. The calculation of the single risk statistics requires a forecast of the probability distribution of the cash flow at some future point in time.

There are different methods to simulate such distribution. The first method, employed by RiskMetrics (1999) in line with the VaR, is a bottom-up approach, and consists in assuming that production volumes, prices and costs are the key factors that determine the future cash flows. The distribution of the conditional value of cash flow can be calculated by random prices and rates generating their own variance-covariance matrix. This method doesn't seem to be suitable for capturing the macroeconomic effects that can also have an impact on the future cash flows, see e.g. Andren et al. (2005).

The second method is a top-down approach, see e.g. Stein et al. (2001a) : this is based on the assumption that total cash flow volatility is the ultimate variable of interest, so that one estimates such volatility from historical cash flows of a company (when such data exist), or from data taken from clustering of similar firms. Of course, this method shares the same shortcomings of the rating models, where the representative firm can have a very different behavior than any other element of the same class.

Given the limitations of both bottom-up and top-down methods, Andren et al. (2005) use a third approach, called exposure-based CFaR. After a comprehensive assessment on the cash flow risk factors that should be fully considered, this approach estimates a firm's cash flow volatility dealing with its own macroeconomic exposure, by allowing for a fundamental analysis of the company's exposure to changes in the macro economy.

2.7.2 Our CFaR Approach

The methodology we are introducing here combines the bottom-up approach of RiskMetrics (1999) and the exposure-based CFaR approach of Andren et al. (2005): we evaluate how a cash flow of the company can be affected by changes in no value-adding macroeconomic variables, i.e. variables that are not directly connected with the core business of the company. This technique leads to a rigorous valuation of cash flows, based on both the dynamics of the main accounting items and the main macroeconomic variables. The first step of our approach consists in estimating, from the past financial statements, a

set of average parameters and variables that describe the dynamics of the main accounting items, according to the dynamic definition of the liquidity process of the company.

The previous section highlighted the possibility to investigate the sensibility of the cash flows on the average parameters, which reflect the fundamental quantities of the financial statement that are specific of the firm. In this respect, thanks to our dynamic model we are improving the CFaR approach of Andren et al. (2005), as we are able to explain endogenously such sensibilities instead of using a regression analysis on external (macro or micro economic) risk factors.

The second step consists in forecasting the dynamics of the macroeconomic variables that mainly affect the financial statement, by simulating multiple paths. At this stage we limit ourselves to consider just no adding-value variables, but it could be possible to imagine a more complete scenario analysis involving also adding-value variables like for example sales, earnings, etc.. Finally, we bring all together and we simulate the cash flow statement considering both the average dynamics and the macroeconomic variables, thus evaluating the distribution of Cash Flow at Risk. At this point, we can evaluate the CFaR as any risk measure such as VaR or expected shortfall by introducing a particular confidence level.

2.8 A Case Study: Alitalia - Italian Air Lines

In order to illustrate in a concrete case the approach introduced so far, we apply our methodology to “Alitalia - Linee Aeree Italiane S.p.A.”, the former flag company that declared bankruptcy in 2008. Alitalia transported passengers and cargos throughout the world since 1947 and its services included the typical activities carried out by airline companies such as flight and ground operations, marketing, business strategies, and sales. Alitalia’s shares were traded in the “Borsa Italiana” stock market till 2008 and it has reported only one year of profit (1998) since its foundation in 1946, while it reported net losses of more than 3.7 billion euros between 1999 and 2008.

We examined the consolidated financial statements of Alitalia concerning the fiscal years from 2000 to 2007, that is till the last financial year before the default¹² and we aggregate the sum of the exposures in the usual accounting items involving the 6-dimensional vector $S_n = (L_n, T_n, R_n, K_n, D_n, B_n)$. together with the realization of the exogenous risk factors $(Z_n, G_n, E_n, A_n, X_n)$.

Here below we show the values corresponding to the starting point 2000 and 2007¹³:

¹²All the data refer to the firm annual financial statements.

¹³Values are in thousands euro.

2000	2007
$L_{00} = 91,913$	$L_{07} = 329,532$
$T_{00} = 1,340,520$	$T_{07} = 666,935$
$R_{00} = 231,440$	$R_{07} = 41,100$
$K_{00} = 2,326,504$	$K_{07} = 2,009,487$
$D_{00} = 1,186,800$	$D_{07} = 1,183,873$
$B_{00} = 989,087$	$B_{07} = 1,534,181$

Using the formulae presented in Section 2.5 we get the average parameters η, γ, ω, d and the average processes V, Z, G, E, A, X replicating the values of the accounting items on the whole time window [2000,2007]¹⁴.

Parameters : $\eta \cong 0.88$; $\gamma \cong -0.14$; $\omega \cong 0.81$; $d \cong 0.10$

Variables : $V \cong 27,191$; $Z \cong 4,929,159$; $G \cong 4,894,454$;

$E \cong 77,870$; $A \cong 359,379$; $X \cong 14,889$.

By analyzing the series of the financial statement, we realize a high degree of sensitivity of cash flows with respect to the payables and to financial debt. In particular, we observe that the main (non value adding) macroeconomic variables affecting the cash flows are the cost for raw materials (aviation gasoline) and the interest rate on financial debt. In fact, Figure 2.8.1 shows that the 16% of the total cost of production was composed in average by gasoline, while Figure 2.8.2 shows that about the 8% of the total debt of Alitalia was constituted by the passive interest rate.

The previous analysis gives us the ingredients to investigate the sensitivity of the cash flows with respect to the fluctuations of the macroeconomic variables. For simplicity, we consider the crude oil as a proxy for the price of gasoline, and we model its evolution by a geometric Brownian motion (see e.g. Mostafei et al. (2013)), while the (passive) interest rate dynamics is assumed to follow a mean reverting (Vasicek) process, see e.g. Bjork (1998). Of course one can allow for more sophisticated stochastic models that fit better the mean reversion and the peculiarities of these macroeconomic variables: in this case study we limit ourselves to simple models that are anyway sufficient to illustrate our CFaR based methodology. Considering the daily Euribor interest rate and the daily Brent prices (CL3) from 2000 to 2007 we estimated the following parameters:

¹⁴Here V embeds the value of ΔQ .

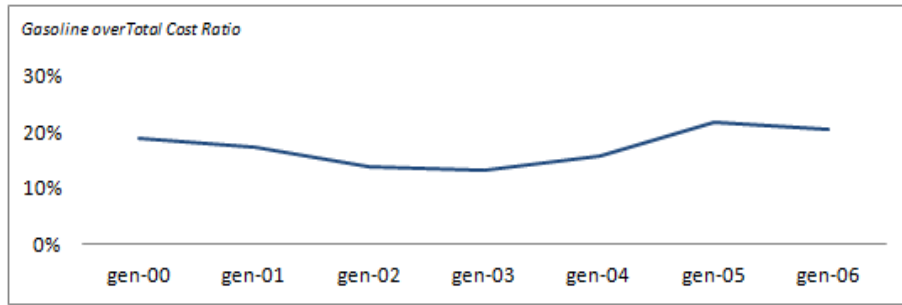


Figure 2.8.1: Impact of the aviation gasoline on the total cost of production for Alitalia during the period [2000,2007].

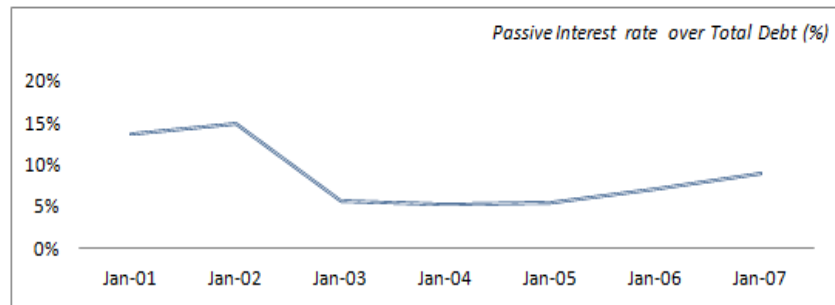


Figure 2.8.2: Impact of the passive interest rate debt over the total debt of Alitalia during the period [2000,2007].

<i>Oil market</i>	<i>Interest rate market</i>
<i>Oil price : euro 90,09</i>	<i>Interest rate : 4.745%</i>
<i>Drift : 17.31</i>	<i>Mean reversion : 3.8%</i>
<i>Volatility : 31.38%</i>	<i>Equilibrium rate : 2.8%</i>
	<i>Volatility : 3.73%</i>

The correlation between the Euribor and the Brent has been estimated in $\rho = 5.19\%$. Then, using a MonteCarlo approach, we simulated 10^5 paths for both diffusions and for each path we estimated the total cash flow.

We measured the impact of the oil price on the total cost through a linear regression and we found the following relation:

$$\Delta_{cost} = 0,8966 * \Delta_{oil} - 0,1167, \quad (2.8.1)$$

with a $R^2 = 0,7386$. Whereas for the passive interest rate, we evaluated an average spread over the Euribor from the accounting data (assumed to be around 5.6% in line with the CDS market).

Finally, we evaluated the Cash flow at Risk (CFaR), that is the amount of potential cash flow decrease, associated to one and two years, and we found the values (in thousands of euros) described in the Figures 2.8.3 and 2.8.4, respectively.

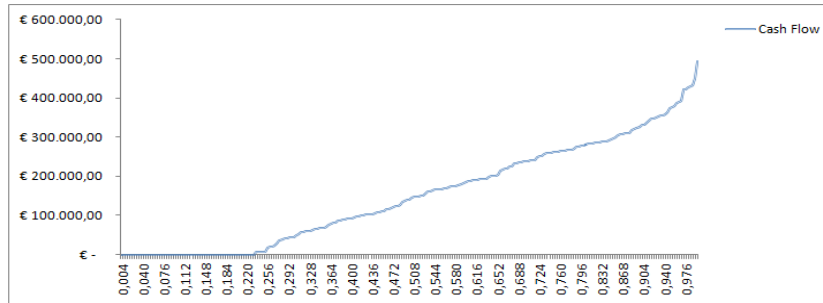


Figure 2.8.3: Cash Flow at Risk for the Alitalia company with 1 year horizon. In the y axis there are thousands of Euros. In the x axis the corresponding probability.

For example, there is about 22% probability that the liquidity process will go below zero in 2008, thus giving a strong negative message about the distress of the company. This probability is dramatically larger (about 50%) when projecting the simulation in two years. Moreover, there is about 10% probability that the liquidity in 2008 will be greater than the one in 2007 (around \$ 370 millions), thus confirming that the firm tends to burn financial resources. Overall, the previous results show the high degree of sensitivity of the firm cash flow to relatively small variations of the macro variables, and, what is more, they reveal the fragility of the financial structure of Alitalia.

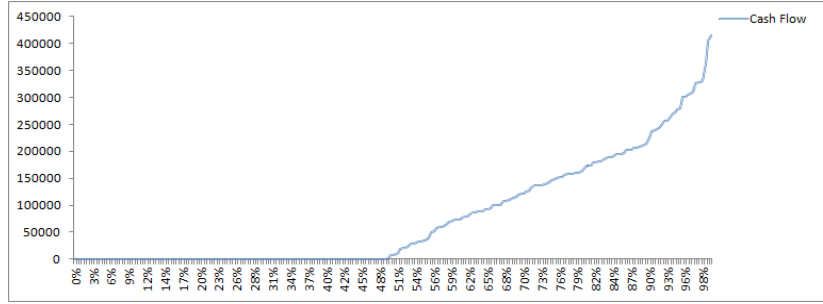


Figure 2.8.4: Cash Flow at Risk for the Alitalia company with 2 years horizon. In the y axis there are thousands of Euros. In the x axis the corresponding probability.

2.9 Conclusion

We presented a general framework that allows to compute a liquidity risk measure for unlisted firms, namely the Cash Flow at Risk, just by using the information coming from the balance sheets and some related macroeconomic variables. A case study based on real data highlighted the great flexibility of the approach, which leads to straight and clear conclusions on the financial stability of the firm. Notice that during the analyzed period [2000, 2007], Alitalia received two financial supports from the Italian Government that avoided its default. Although in our dynamic model this information is already included, the fragility of the financial structure appears in its full evidence through our CFaR measure when adopting the model in order to predict default. In this sense, in our projections we implicitly include in the balance sheet a future (average) income from the Government according to the past accounting data and we observe the evolution of the firm. However, one could also reverse things and try to look for the financial support policy to apply in order to reach some target level for the liquidity of the firm in the future. Of course, this opens the door to new promising credit risk models that we are going to diffusely investigate in the future. At this stage, our main purpose is to convince the reader about the applicability of the proposed accounting model to general corporate problems and in particular risk management analysis.

Chapter 3

A Commercial Bank Balance Sheet Model for Liquidity Risk Measurement

3.1 Introduction

In this chapter we will apply the brick vector framework to a possible modelization of a commercial bank balance sheet and then we will use this formalization to introduce a new measure of Liquidity distress. The problem of modeling a bank balance sheet presents very specific and complex features in itself (see Halaj 2013, 2016) and the issue of liquidity risk assessment in the case of a financial firm even more so. Banks' balance sheets vary their composition according to market conditions implied by a growth of credit and debt in the economy as well as valuation changes but also bank's own strategic actions. The regulatory regime imposes on banks the need to retain sufficient amount of capital to endure shocks and enough liquid assets to meet obligations in the majority of plausible future scenarios. For example it defines rigid limits to the expansion of the balance sheet as well as the concentration of exposures over a certain capital base and liquidity buffers. All these conditions make the handling of banks' balance sheets a challenging task in general and consequently also challenging from a modeling perspective. Nonetheless practitioners and the research community have been seeking for the right framework to effectively model the balance sheet according to its sensitivity to the market conditions (see Balasubramanian et al. 2013, Birge et al. 2013, Halaj 2013). With the present work we hope to provide the preliminary stage of a new approach more rooted in the data stemming from the accounting records.

As far as the liquidity risk assessment is concerned, even though liquid-

ity management is a core activity of the banking business, it has not received much attention in the last decades mainly because liquidity wasn't perceived as scarce. Unfortunately this idea proved to be utterly wrong during the global financial crisis of 2008 where the consequences of this preconception manifested in the most ominous way. Both financial regulators and academic researchers realized that the worst financial crisis since 1929 was primarily due to liquidity issues. However, still to this day, simple unsophisticated reports constitute the standard fare for banks' liquidity management disclosing, such as statistically calculating the funding gap between assets and liabilities under different maturity ladders, or listing funding channels which banks can put forward without having to hold virtually any test on the quality of those resources in crisis situations. In fact the development of bank liquidity modelling proved to be rather slow, despite bankers feeling liquidity risk as one of the top five risks to consider (DCSFI 2010) in the financial business. Differently from other kinds of risk, as credit or market risk where the use of advanced methodology is the common standard, Fiedler (Fiedler 2007) complains about the lack of any sort of sophisticated technique in order to summarize a bank's liquidity position by testing if it will have sufficient cash to pay future bills. This is partially due to the fact that researchers typically have chosen Value at Risk (VaR) as the basis for the management of risk in the financial sector, leaving Cash-Flow at Risk (CFaR) as the proper tool to assess risk management among non-financial firms. As a matter of fact there is an argument that a financial firm's VaR represents also its CFaR (in so being a measure of its liquidity risk probability), since the portfolio holdings (see Shimko 1998) are marked-to-market by financial firms. But VaR unlike liquidity related risk measures, will capture only a small part of the bank's total risk exposure, since it doesn't tackle the hazard linked to the underlying commercial cash-flow. More importantly it cannot fully take into account the volatility pertaining to the liquidity and in so doing VaR doesn't represent an efficient and satisfactory tool to manage and assess liquidity risk. Therefore the banking sector, as well as the risk management academic research world, should develop more advanced liquidity models to control its related hazard. Banks face a serious liquidity risk when their net cash flows cannot meet their liabilities as they fall due. But taking market liquidity for granted, financial institutions proved to be not particularly interested in cash flow issues over the last decades. Shimko, among others, argued that a bank's VaR is also its Liquidity risk measure since banks' marked-to-market portfolios can be converted into cash at short notice and any change of its value immediately affects reported earnings and cash flow. However this argument doesn't hold in 'thin' markets and challenges fundamental accounting principles. In conditions of moderately illiquid markets assets would become less marketable and would not be readily converted into cash (Lippman and McCall 1986). A liquidity crisis, differently by other crisis, can make the markets become even thinner, possibly for months or years. A well-capitalized bank could be forced into bankruptcy because very illiquid markets would not allow banks to turn marketable securities into cash on time. In addition, under accounting theory, for a bank that has to make contractual payments during a particular period,

the drying up of cash flow income might put the bank at risk of default, even though its net worth remains relatively stable. Moreover returns on a bank's assets and liabilities (or Net incomes), as the key VaR matrix, cannot provide an accurate picture of the bank's current cash holding without taking account of non-cash expense items, while variations in a bank's profit and loss might not always capture the changes in liquidity especially during stressed periods. In the study of those assessments LaR (Liquidity at Risk) would be more useful than VaR, in terms of measuring liquidity risk, since the latter has nothing to do with estimating cash holdings for financial institutions.

The aforementioned issues pertaining the relationship between VaR and liquidity have very deep consequences when they intertwine with the work of the financial regulatory bodies since VaR-based capital adequacy measures have been increasingly adopted in the last twenty years by regulators and supervisors. As a matter of fact the Basel Committee on Banking Supervision in 1995 allowed commercial banks, subject to certain safeguards, to use their own internal VaR estimates to determine their capital requirements for market risk under an amendment to the Basel Capital Accord (Holton 2002). In later years, especially after the 2008 financial crisis, the various shortcomings of that approach have been fully realized and the need for a specific liquidity risk assessment has been acutely perceived by the financial regulatory bodies. Nowadays solvency and liquidity limits are modelled in a 'worst case scenario' manner. It means that the only admissible strategies, employed by the banks, are those that guarantee with a very high probability that the bank remains solvent and liquid. This methodology for the risk measurement, as well as its limits, is reflected in the existent regulatory regime both in the solvency context, by the VaR concept (Basel II VaR capital constraint or measurement of banks' economic capital), and in the liquidity context, whose risk assessment is instead left to stress testing exercises (LaR internal models of banks' cash-flow distributions are not mandatory although usually performed for inner purposes) (see Matz et al. 2006, Castagnoli et al. 2013). The main shortcoming of the current approach is related to the fact that the banks' balance sheet natural tendency to evolve dynamically proves to be particularly problematic for financial analysts and banking regulators. A usual static balance sheet assumption, as taken for the stress testing exercises enforced on the European banking sector by the ECB, may be valid only in some very special cases of shocks of the low magnitude and in relatively short periods (especially in instances where it is reasonable to assume that an adjustment may need time for preparation and coordination of actions by the management of the bank). Those implicit limits of the stress testing exercises have been pointed out in works such as those by Henry (2013) and RTF (2015). One of the most relevant aspects of the relationship between investment strategies and funding conditions, which is one of the most sensible issue pertaining the current regulation, deals with the effectiveness of the liquidity management both by the bank leadership as well as by the macro policies enforced on those leadership by the regulator. As we have seen risk based capital regulation has a strong and well established position in the traditional banking regulation while this is not the case for liquidity rules

imposed on banks. In the Basel II framework it has been decided to uphold a form of code of good practice rather than well-defined liquidity indicators and benchmarks. Such an approach was justified by the belief that setting liquidity limits based on very aggregate supervisory data cannot be an effective tool to control banks liquidity conditions. Contrary to solvency measures, aggregation for liquidity purposes delivers a very imprecise picture. Therefore Basel II as far as liquidity standards were concerned decided only to plead with the banking sector to start the development of internal liquidity management systems satisfying some high level principles. Basel III departed from that approach and imposed minimal liquidity ratios (short-term Liquidity Coverage Ratio (LCR) and longer-term Net Stable Funding Ratio (NSFR)). This regulation has been devised in order to attempt to address the causes of the crisis erupted in 2007 and it tries to capture the characteristics of the balance sheet items that are felt as particularly important from a liquidity perspective (e.g. stability of funding sources, generally expected haircuts on certain asset classes or operational relationships with customers). Nonetheless the LCR and the NSFR liquidity rules simplify the liquidity risk measurement issue and cannot replace a fully-fledged statistical, stochastic and/or behavioural approach to cash management, collateral management and sustainability of funding sources. The LaR approach attempts to overcome at least in part those issues and its philosophy is similar to that of the VaR used for the solvency regulations (Matz et al. 2006). Its aim is to limit bank's exposure to the liquidity risk by allowing for taking only those liquidity positions that with high probability ensure that the bank can meet its obligations in the near future. It does that starting from the assessment of a probability distribution for the liquidity in the next future as VaR does for the value of the bank portfolio. There is no general consensus on the most effective way to compute a LaR measure and every banking firm approaches this task through internal models and for its own purposes. As we have seen this happens mainly because the regulator enforces only tests founded on static balance sheet ratios although the limits of this methodology have been not seldom matter of debate in the last years. With the LaR approach that we are about to present we propose a first step on a possible path that tries to address three of the limitations that we have seen to be perceived in the regulations and in the academic work stemming from the regulatory debate. The first one is the lack of a dynamic perspective in the current liquidity risk assessment methodology with the consequences briefly mentioned above. The second is the absence of a liquidity risk measurement with a quarterly or yearly perspective relating specifically to funding liquidity issues, since LaR as well as VaR is generally assumed to compute only over days or weeks.

The third one concerns a characteristic of the bank balance sheet dynamic models proposed so far (Halaj 2016, Birge et al. 2013) which inevitably present a very high level of aggregation on the accounting items. The brick vector framework allows in principle to select the level of aggregation (in the accounting items of the model) that the researcher deems more appropriate for his purposes and in the next section we will propose what we think is a model presenting a suitable trade-off between the need of detail and the need of mathematical tractability.

In the past bank balance sheet problems have been remarkably tackled with methodologies of optimal portfolio choice with transaction costs (see Davis and Norman 1990), but the theory of portfolio choice with transaction costs easily produces computationally intractable problems. Here we want to propose, through a simulation, a first approach to the liquidity probability distribution measurement that, in a dynamic fashion and with a convenient mathematical tractability, could be able to relate the history of the balance sheet dynamics and the accounting data to some macroeconomic factors deemed of particular importance for the bank's business structure.

3.2 A proposal for the balance sheet dynamical modelization of a commercial bank

The proposition for a balance sheet formalization we will now present is about a commercial bank and in the case of this model, inspired by works like that of Halaj (see Halaj 2013, 2016, Birge et al. 2013), we intend to propose its closed form solution as well. Naturally in this case the problem of the high number of parameters and variables in the formalization presents itself once more, and the goal of finding an equilibrium between the need of detail and that of mathematical tractability constitutes a delicate task. With this aim in mind we decided to choose for our model the following accounting Items, in accordance with the IFRS¹ principles :

1. C_n Current Liquidity;
2. L_n Loans and receivables towards clients;
3. T_n Loans and receivables towards banks;
4. S_n Securities portfolio;
5. K_n Fixed assets;
6. Y_n Deposits from clients;
7. H_n Deposits from banks;
8. O_n Financial liabilities;
9. D_n Account Payables.

¹International Financial Reporting Standards

Now we will present how to build each one of the eight brick vectors that will constitute our bank balance sheet model. The first one L_n will be representing the “Loans and receivables towards clients” constituted by non-derivatives financial assets with fixed or determinable repayment scheme²:

$$\begin{bmatrix} CL_n \\ L_n \end{bmatrix} = \begin{bmatrix} 1 & [\rho l_n(1 - \pi_n)] \\ 0 & (1 - \pi_n) \end{bmatrix} \begin{bmatrix} CL_{n-1} - \Delta l_n \\ L_{n-1} + \Delta l_n \end{bmatrix} \quad (3.2.1)$$

where π_n is the weighted average of the percentage of defaulted loans between time $n - 1$ and n , ρl_n is the weighted average interest rate collected from Loans and Δl_n will be the variation on the Loans Item in the period³.

The second brick vector deals with T_n the Loans and receivables towards banks, central bank included. This item considers interbank positions including loans and receivables towards banks and minimum reserves on central bank. It takes this shape:

$$\begin{bmatrix} CT_n \\ T_n \end{bmatrix} = \begin{bmatrix} 1 & [\rho t_n] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} CT_{n-1} - \Delta t_n \\ T_{n-1} + \Delta t_n \end{bmatrix} \quad (3.2.2)$$

in this vector ρt_n will be the weighted average of the interest rate paid to our bank on this kind of loans while Δt_n will be the increment in the period of this asset.

The variable Securities S_n includes all the financial assets, except loans, considering both Held for trading and Held to maturity⁴.

$$\begin{bmatrix} CS_n \\ S_n \end{bmatrix} = \begin{bmatrix} 1 & (\rho s_n - \alpha_n \cdot \frac{\bar{P}_n}{P_{n-1}}) \\ 0 & (1 + \alpha_n) \cdot \frac{\bar{P}_n}{P_{n-1}} \end{bmatrix} \begin{bmatrix} CS_{n-1} \\ S_{n-1} \end{bmatrix} \quad (3.2.3)$$

²This type of asset can be both long term and short term.

³Difference between new loans and principal payments from time step $n - 1$ to time step n among the not defaulted loans

⁴The liquidity, generated by this item, may arrive from buying and selling stocks or from the coupons and dividends

The parameter $\phi_n = \frac{\bar{P}_n}{P_{n-1}}$ acts the revaluation procedure of the securities according to the adopted accounting principles since \bar{P}_n will be the weighted average of the prices of all the securities in the balance sheet at time n . Then, while ρs_n will deal with the interests or dividends generated by the securities, the parameter α_n will describe the increase or decrease of investments on securities.

The next brick vector will be the one about K_n the fixed assets and it will have the same dynamics seen in the case of the industrial firm, namely:

$$\begin{bmatrix} LK_n \\ K_n \end{bmatrix} = \begin{bmatrix} 1 & -\gamma_n \\ 0 & (1 + \gamma_n) \end{bmatrix} \begin{bmatrix} LK_{n-1} \\ K_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -A_n \end{bmatrix} \quad (3.2.4)$$

so the parameters will have the exact same meaning previously seen in chapter 2. All the brick vectors presented until now will be on the credit side of the balance sheet so according to the brick vector theory expressed in the first chapter all the external variables will be expressed through the use of positive numbers. On the contrary the four brick vectors we are about to propose will be on the debit side of the balance sheet, consequently the external variables in their formalization will be expressed by default through the use of negative numbers.

The brick vector Y_n dealing with the account representing deposits from clients will be formalized as follows:

$$\begin{bmatrix} LY_n \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & \theta y_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} LY_{n-1} - \Delta y_n \\ Y_{n-1} + \Delta y_n \end{bmatrix} \quad (3.2.5)$$

where θy_n will be the deposits interest rate (the weighted average as usual) paid by our bank to its clients and Δy_n will be the variation of this item in the period between $n - 1$ and n , namely new deposits by clients in the bank⁵.

The accounting item H_n describing the debts towards other banks can be modeled basically as the item Y_n just seen above. It takes this shape:

⁵ Δy_n will be negative by default, namely if the deposits will increase in the period, while it will be positive if the clients will withdraw their money from the bank

$$\begin{bmatrix} LH_n \\ H_n \end{bmatrix} = \begin{bmatrix} 1 & \theta h_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} LH_{n-1} - \Delta h_n \\ H_{n-1} + \Delta h_n \end{bmatrix} \quad (3.2.6)$$

in this case θh_n will be the deposit interest rate paid by our bank to other banks and Δh_n will represent the variation of this item in the period with the same conventions about its sign discussed above.

The item related to the financial liabilities O_n considers different securities, both listed and non-listed, and other financing lines with contractual repayment plans (mainly bonds issued by our bank). It takes this shape:

$$\begin{bmatrix} LO_n \\ O_n \end{bmatrix} = \begin{bmatrix} 1 & (\theta o_n + \beta_n) \\ 0 & (1 - \beta_n) \end{bmatrix} \begin{bmatrix} LO_{n-1} \\ O_{n-1} \end{bmatrix} \quad (3.2.7)$$

and in this brick vector θo_n will be the weighted average of the interest rates paid by our bank on these financial liabilities, while β_n will regulate the increase or decrease of this item⁶. Finally we will have the brick vector dealing with the account payables D_n which will be very similar to the one in the model of the small/medium industrial firm:

$$\begin{bmatrix} LD_n \\ D_n \end{bmatrix} = \begin{bmatrix} 1 & \omega_n \\ 0 & (1 - \omega_n) \end{bmatrix} \begin{bmatrix} LD_{n-1} \\ D_{n-1} + G_n \end{bmatrix} \quad (3.2.8)$$

As seen in the previous chapter, in the formulas above ω_n will be the weighted average percentage of payments in the period n , while G_n will be the costs in the period n expressed through a negative number according to the brick vector theory⁷.

⁶ β_n will be positive if there is a decrease while it will be negative in the case of an increase (if for example new bonds are issued)

⁷This is the only difference with the industrial firm model presented in the previous chapter where we expressed G_n by default through the use of positive numbers. This is due to the fact that we chose to keep on using the double-entry book keeping framework convention according to which any item, be it an asset or a liability, is reported by default through the use positive numbers. In the commercial bank case we decided to follow the brick vector theory to its full extent.

Now the last step will obviously be the combination of all the brick vectors just presented, which will happen according to the principles seen in the previous chapters:

$$\left\{ \begin{array}{l} C_n = (CL_n + CT_n + CS_n + CK_n + CY_n + CH_n + CO_n + CD_n) + X_n \\ L_n = L_n \\ T_n = T_n \\ S_n = S_n \\ K_n = K_n \\ Y_n = Y_n \\ H_n = H_n \\ O_n = O_n \\ D_n = D_n \end{array} \right. \quad (3.2.9)$$

where the value of each variable is the one in the relative brick vector defined until now. As seen before the variable X_n represents the extraordinary financings/dividends plus taxes happened between the time step $n - 1$ and the time step n .

2.2.1 The closed form formula of the commercial bank model

Now we will provide the closed form formula of the system just seen above and using the brick vector framework we will start from presenting the close form formula of each brick vector. One important thing to underline is that for every formula that we are about to write the following convention will stand, that any time $a > b$ we will have $\prod_a^b = 1$ and $\sum_a^b = 0$ for any product and any summation. The first brick vector than will take the following shape:

$$\left\{ \begin{array}{l} CL_n = CL_0 + \sum_{l=1}^n [\rho_l(1 - \pi_l)((\prod_{i=1}^{l-1}(1 - \pi_i))L_0)] + \\ + \sum_{l=1}^n [\rho_l(1 - \pi_l)[\sum_{h=1}^l \Delta l_h (\prod_{j=h}^{l-1}(1 - \pi_j))]] - \sum_{l=1}^n \Delta l_l \\ L_n = \prod_{i=1}^n (1 - \pi_i)L_0 + \sum_{h=1}^n (\prod_{j=h}^n (1 - \pi_j))\Delta l_h \end{array} \right. \quad (3.2.10)$$

while the second one will be such as:

$$\begin{cases} CT_n = CT_0 + \sum_{l=1}^n [\rho t_l(T_0)] + \\ + \sum_{l=1}^n [\rho t_l(\sum_{h=1}^l \Delta t_h)] - \sum_{l=1}^n \Delta t_l \\ T_n = T_0 + \sum_{h=1}^n \Delta t_h \end{cases} \quad (3.2.11)$$

The brick vector relating to the securities S_n will be expressed by the following formula:

$$\begin{cases} CS_n = CS_0 + \sum_{l=1}^n [(\rho s_l - \alpha_l \cdot \frac{\bar{P}_l}{P_{l-1}})((\prod_{i=1}^{l-1} ((1 + \alpha_i) \cdot \frac{\bar{P}_i}{P_{i-1}})) S_0)] \\ S_n = [\prod_{i=1}^n ((1 + \alpha_i) \cdot \frac{\bar{P}_i}{P_{i-1}})] S_0 \end{cases} \quad (3.2.12)$$

while the brick vector for the fixed assets will be:

$$\begin{cases} CK_n = - \sum_{l=1}^n \gamma_l (\prod_{i=1}^{l-1} (1 + \gamma_i) K_0) + \sum_{l=1}^n \gamma_l (\sum_{h=1}^{l-1} A_h (\prod_{j=h+1}^{l-1} (1 + \gamma_j))) \\ K_n = \prod_{i=1}^n (1 + \gamma_i) K_0 - \sum_{h=1}^n (\prod_{j=h+1}^n (1 + \gamma_j)) A_h \end{cases} \quad (3.2.13)$$

The fifth brick vector relating to the Y_n deposits from clients will have the next formula:

$$\begin{cases} CY_n = CY_0 + \sum_{l=1}^n (\theta y_l) Y_0 + \\ + \sum_{l=1}^n [(\theta y_l) [\sum_{h=1}^l \Delta y_h]] - \sum_{l=1}^n \Delta y_l \\ Y_n = Y_0 + \sum_{h=1}^n \Delta y_h \end{cases} \quad (3.2.14)$$

which is virtually the same as the one of the brick vector of the item H_n describing the debts towards other banks:

$$\begin{cases} CH_n = CH_0 + \sum_{l=1}^n (\theta h_l) H_0 + \\ + \sum_{l=1}^n [(\theta h_l) [\sum_{h=1}^l \Delta h_h]] - \sum_{l=1}^n \Delta h_l \\ H_n = H_0 + \sum_{h=1}^n \Delta h_h \end{cases} \quad (3.2.15)$$

Then we have the formula relating to the brick vector of the O_n Bonds/Loans item which will be:

$$\begin{cases} CO_n = CO_0 + \sum_{l=1}^n [(\theta o_l + \beta_l) ((\prod_{i=1}^{l-1} (1 - \beta_i)) O_0)] \\ O_n = [\prod_{i=1}^n (1 - \beta_i)] O_0 \end{cases} \quad (3.2.16)$$

Finally we will have the formula relating to the brick vector of the item dealing with the D_n costs:

$$\begin{cases} LD_n = \sum_{l=1}^n \omega_l (\prod_{i=1}^{l-1} (1 - \omega_i) D_0) + \sum_{l=1}^n \omega_l (\sum_{h=1}^l G_h (\prod_{j=h}^{l-1} (1 - \omega_j))) \\ D_n = \prod_{i=1}^n (1 - \omega_i) D_0 + \sum_{h=1}^n (\prod_{j=h}^n (1 - \omega_j)) G_h \end{cases} \quad (3.2.17)$$

Then naturally we can combine all the brick vectors in order to obtain a balance sheet model system, which will be as the next one:

$$\begin{cases} C_n = (CL_n + CT_n + CS_n + CK_n + CY_n + CH_n + CO_n + CD_n) + \sum_{h=1}^n X_h \\ L_n = L_n \\ T_n = T_n \\ S_n = S_n \\ K_n = K_n \\ Y_n = Y_n \\ H_n = H_n \\ O_n = O_n \\ D_n = D_n \end{cases} \quad (3.2.18)$$

where the value of each variable is the one in the relative brick vector defined until now. As usual the variable X_h represents the extraordinary financings/dividends plus taxes happened between the time step $h - 1$ and the time step h .

3.2.1 The Chisini averaging procedure applied to the commercial bank model

As we have just seen in the previous chapters our Chisini like averaging procedure, applied to brick vectors, will naturally give us average values for the parameters and the variables according to the shape of the brick vector it-self. So now let's have a look at the results of this procedure once we apply it to the commercial bank model we have proposed in the previous section. The first brick vector, with all its parameters and variables constant, will take the following shape:

$$\begin{cases} CL_n = CL_0 + (\rho l) \frac{(1-\pi)}{\pi} [1 - (1-\pi)^n] L_0 + \\ + (\rho l) \frac{(1-\pi)}{\pi} (\Delta l) [n - \frac{(1-\pi)[1-(1-\pi)^n]}{\pi}] - n(\Delta l) \\ L_n = (1-\pi)^n L_0 + (\Delta l)(1-\pi) [\frac{1-(1-\pi)^n}{\pi}] \end{cases} \quad (3.2.19)$$

where, taking the parameter π as exogenous, we will have for the parameter ρl and the variable Δl the following solutions:

$$\begin{cases} \overline{\Delta l} = \frac{L_n - (1-\pi)^n L_0}{(1-\pi) [\frac{1-(1-\pi)^n}{\pi}]} \\ \overline{\rho l} = \frac{CL_n - CL_0 + n(\overline{\Delta l})}{\frac{(1-\pi)}{\pi} [1 - (1-\pi)^n] L_0 + \frac{(1-\pi)}{\pi} (\overline{\Delta l}) [n - \frac{(1-\pi)[1-(1-\pi)^n]}{\pi}]} \end{cases} \quad (3.2.20)$$

For the second brick vector of the model we will have the next closed form formula, for the case of everything constant:

$$\begin{cases} CT_n = CT_0 + (\rho t)n[T_0 + \frac{(n+1)(\Delta t)}{2}] - n(\Delta t) \\ T_n = T_0 + n(\Delta t) \end{cases} \quad (3.2.21)$$

and the variable Δt and the parameter ρt will be as follows:

$$\begin{cases} \overline{\Delta t} = \frac{T_n - T_0}{n} \\ \overline{\rho t} = \frac{(CT_n - CT_0) + (T_n - T_0)}{nT_0 + \frac{(n+1)}{2}(T_n - T_0)} \end{cases} \quad (3.2.22)$$

The brick vector relating to the securities S_n will be expressed by the following formula, where we note $\phi = \sqrt[n]{(\prod_{h=1}^n \frac{P_h}{P_{h-1}})}$ and the value of this parameter will be exogenous:

$$\begin{cases} CS_n = CS_0 + (\rho - \alpha \cdot \phi)S_0[\frac{1 - [(1+\alpha) \cdot (\phi)]^n}{1 - [(1+\alpha) \cdot (\phi)]}] \\ S_n = [(1 + \alpha) \cdot (\phi)]^n \cdot S_0 \end{cases} \quad (3.2.23)$$

while the parameter α and the parameter ρ will be expressed by the next formulas, remembering that the value of ϕ will be considered as exogenous:

$$\begin{cases} \bar{\alpha} = \left(\frac{\sqrt[n]{(\frac{S_n}{S_0})}}{\phi} \right) - 1 \\ \bar{\rho s} = \left[\frac{(CS_n - CS_0)(1 - \sqrt[n]{(\frac{S_n}{S_0})})}{(S_0 - S_n)} \right] + \bar{\alpha}\phi \end{cases} \quad (3.2.24)$$

The brick vector for the fixed assets will be:

$$\begin{cases} CK_n = K_0[1 - (1 + \gamma)^n] - A[n + \frac{1 - (1 + \gamma)^n}{\gamma}] \\ K_n = (1 + \gamma)^n K_0 + A[\frac{1 - (1 + \gamma)^n}{\gamma}] \end{cases} \quad (3.2.25)$$

and we will have

$$\bar{A} = \frac{\sum_{i=1}^n A_i}{n} \quad (3.2.26)$$

,

then replacing this value in the expression defining K_n we get

$$K_n = (1 + \gamma)^n K_0 - \left[\frac{1 - (1 + \gamma)^n}{\gamma} \right] \bar{A} \quad (3.2.27)$$

which is a polynomial equation giving the average value $\bar{\gamma}$ of the parameter γ through standard numerical procedures.

The fifth brick vector relating to the Y_n deposits from clients, in the case of everything constant, will have the next formula:

$$\begin{cases} CY_n = CY_0 - n(\Delta y) + \\ + n(\theta y)[Y_0 + \frac{(n+1)}{2}\Delta y] \\ Y_n = Y_0 + n(\Delta y) \end{cases} \quad (3.2.28)$$

that will give us the following expressions for the average values $\overline{\Delta y}$ and $\overline{\theta y}$ of the variable Δy and the parameter θy :

$$\begin{cases} \overline{\Delta y} = \frac{Y_n - Y_0}{n} \\ \overline{\theta y} = \frac{(CY_n - CY_0) + (Y_n - Y_0)}{nY_0 + \frac{(n+1)}{2}(Y_n - Y_0)} \end{cases} \quad (3.2.29)$$

The brick vector just seen behaves virtually identically as the one of the item H_n describing the deposits from other banks:

$$\begin{cases} CH_n = CH_0 - n(\Delta h) + \\ + n(\theta h)[H_0 + \frac{(n+1)}{2}\Delta h] \\ H_n = H_0 + n(\Delta h) \end{cases} \quad (3.2.30)$$

this situation will give us the following expressions for the average values $\overline{\Delta h}$ and $\overline{\theta h}$ of the variable Δh and the parameter θh :

$$\begin{cases} \overline{\Delta h} = \frac{H_n - H_0}{n} \\ \overline{\theta h} = \frac{(CH_n - CH_0) + (H_n - H_0)}{nH_0 + \frac{(n+1)}{2}(H_n - H_0)} \end{cases} \quad (3.2.31)$$

Then we have the formula relating to the brick vector of the O_n Bonds/Loans item which will be:

$$\begin{cases} CO_n = CO_0 + [((\theta o) + \beta)(O_0)[\frac{1-(1-\beta)^n}{\beta}]] \\ O_n = (1 - \beta)^n O_0 \end{cases} \quad (3.2.32)$$

that, for the average values $\overline{\beta}$ and $\overline{\theta o}$ of the parameters β and θo , will give us the following formulas :

$$\begin{cases} \overline{\beta} = 1 - \left[\sqrt[n]{\frac{O_n}{O_0}} \right] \\ \overline{\theta o} = \frac{(CO_n - CO_0) \cdot \left\{ 1 - \left[\sqrt[n]{\frac{O_n}{O_0}} \right] \right\}}{(O_0 - O_n)} - \left\{ 1 - \left[\sqrt[n]{\frac{O_n}{O_0}} \right] \right\} \end{cases} \quad (3.2.33)$$

Finally we will have the formula relating to the brick vector of the item dealing with the D_n costs:

$$\begin{cases} LD_n = (1 - (1 - \omega)^n)D_0 + [n - (1 - \omega)(\frac{1-(1-\omega)^n}{\omega})]G \\ D_n = (1 - \omega)^n D_0 + (1 - \omega)\frac{(1-(1-\omega)^n)}{\omega}G \end{cases} \quad (3.2.34)$$

that will give us:

$$\bar{G} = \frac{\sum_{i=1}^n G_i}{n} \quad (3.2.35)$$

and again, we replace this value into the expression giving D_n and obtain the following polynomial equation defining the average value $\bar{\omega}$ of the parameter ω :

$$D_n = (1 - \omega)^n D_0 + (1 - \omega) \frac{(1 - (1 - \omega)^n)}{\omega} \bar{G} \quad (3.2.36)$$

value that as usual in cases such as this one must be pursued through standard numerical methods. Naturally in the case of the variable X_n of the external vector of the combined system, its averaging procedure will simply be given by the formula $\bar{X} = \sum_{h=1}^n X_h$.

3.3 Simulation of the model behaviour

Now we will present an eight-year evolution of a simulated balance sheet of a commercial bank according to the previous model. We would like to point out that the data that we are about to present are in line with that of a real medium size commercial bank of the north west of Italy (a Banca Popolare) effectively managed. We prefer to present them as a simulation since in the case of the commercial bank balance sheet model we do not possess critical pieces of information, that were not equally important in the case of the industrial firm balance sheet model, as the ones regarding securities portfolio composition, management credit policy as well as other relevant management policies. As far as the new LaR⁸ measure we want to propose is concerned, those pieces of information are fully available to the bank management as well as the regulatory bodies. With regard to their interactions with our model, they could also be mathematically inferred by the accounting information that a listed bank has to provide to the public as well as other econometric data, but their exact definition would require a double checking procedure that at this stage of our research goes beyond the scope of the present work. Consequently we prefer to present the

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following data as a simulated bank although we feel the need to underline the fact that they will exhibit a behaviour in line with that of a real bank with the characteristics discussed above. We will present firstly each one of the brick vectors constituting the model then the balance sheet in its entirety⁹. The first brick vector will evolve as the following:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CL(n)$	-148.681	-1.926.173	1.890.075	-1.098.984	-1.806.643	-1.199.478	-1.118.893	
$L(n)$	23.996.543	23.211.925	20.843.557	22.390.951	20.606.832	18.247.861	16.711.080	14.936.103
Parameters and Variables								
$\Delta l(n)$	929.466	2.842.080	-1.034.847	1.964.691	2.567.120	1.814.667	1.791.704	
$\rho l(n)$	0,0325	0,0394	0,0410	0,0386	0,0369	0,0337	0,0402	
π	0,006	0,020	0,024	0,008	0,010	0,015	0,0010	

As we can see, remembering equation (3.2.1), these data will represent the evolution of the brick vector describing L_n the loans and receivables towards clients. According to the brick vector theory put forward in the first chapter we can see how, being this accounting item on the credit side of the balance sheet equation, it will be expressed through the use of positive numbers. We refer to the second paragraph of this chapter for the accounting interpretation of the parameters and variables constituting this brick vector as well as the others we will present later. We would like to underline the fact that instead of the liquidity generated from the item we calculated the delta of that liquidity $\Delta CL(n) = [CL(n) - CL(n - 1)]$ since we don't know the brick vector decomposition of the general liquidity $C(0)$ in the starting year (2008) of the balance sheet time series. This doesn't invalidate the mathematical tractability of our model since we can retrieve information about the parameters and the variables through balance sheet data. The accounting technical details about the procedures required in order to obtain those pieces of information and to double-check them go beyond the scope of the present work. Here we intend primarily to focus on the mathematical aspects of the formalization of the balance sheet we have provided as well as on its possible applications. Coherently with our mathematical framework we can perform the Chisini averaging procedure discussed in the previous chapters applying (3.2.20), that will give us the following values:

$\overline{\Delta l}$	$\overline{\rho l}$	$\overline{\pi}$
1.469.044	0,0345449	0,0085918

⁹The data are expressed in thousands of Euros.

It is important to notice that we consider π as an exogenous parameter so in the equations (3.2.20) giving the average values $\overline{\Delta l}$ and $\overline{\rho l}$ we decided to use as $\overline{\pi}$ the geometric mean of the values of π in the time series.

The second brick vector, dealing with the accounting item T_n describing loans and receivables towards other banks (Central Bank included), will move accordingly with (3.2.2) following the next dynamics:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CT(n)$	119.129	401.859	381.629	359.749	-714.495	-159.105	-154.037	
$T(n)$	980.339	1.088.388	1.481.714	1.855.235	2.200.794	1.465.507	1.294.214	1.117.463
Parameters and Variables								
$\Delta t(n)$	-108.049	-393.326	-373.521	-345.559	735.287	171.293	176.751	
$\rho t(n)$	0,0113	0,0078	0,0055	0,0076	0,0094	0,0083	0,0176	

As we can see the item is quite stable and more importantly it has a much lower impact on the assets side of the bank's balance sheet than the previous one. This is coherent on the one hand with the business model of the kind of bank we are representing and on the other hand with the fact that we have chosen a time series which evolves after the 2007 financial crisis (see Castagna et al. 2013). On the brick vector data we can perform our averaging procedure through which we get the following results:

$\overline{\Delta t}$	$\overline{\rho t}$
-19589,14	0,0134

Looking at the mean value $\overline{\Delta t}$ of the variable Δt it is interesting to notice that on average the loans towards other banks in the time frame we consider have decreased.

The third brick vector deals with the accounting item S_n describing our bank's portfolio of securities, it will evolve following (3.2.3) and its dynamic will be the next one:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CS(n)$	756.408	-2.168.676	-1.875.466	-1.447.125	-89.246	163.183	730.591	
$S(n)$	8.180.458	8.837.235	6.540.905	4.535.466	2.889.312	2.907.376	2.991.053	3.585.207
Parameters								
$\alpha(n)$	-0,08	0,34	0,42	0,51	0,04	-0,05	-0,18	
$\rho s(n)$	0,00508	0,00905	0,01220	0,02909	0,00998	0,00721	0,01882	
$\varphi(n)$	1,0062	1,0105	1,0165	1,0398	0,9531	1,0194	1,0192	

The S_n item has a much broader impact than the previous one on the total assets, although not comparable with the importance of L_n , since as we will see in detail later its weight on the balance sheet assets side in the time window considered range between 11% and 24%. Looking at the time series of the parameter φ ¹⁰ we can notice that it expresses a volatility that it is not comparable with that of any real market index in the time frame examined, this could be due to the portfolio composition or to the accounting evaluation principles that the management opted for. The Chisini procedure will provide us with the following values for the mean parameters:

$\bar{\alpha}$	$\bar{\rho s}$	$\bar{\varphi}$
0,1151	0,0092	1,0089

As for the case of the first brick vector we consider one of the three parameters as exogenous¹¹, that parameter is φ and in the equations (3.2.23), giving the average values $\bar{\alpha}$ and $\bar{\rho s}$, we decided to use as $\bar{\varphi}$ the geometric mean of the values of φ in the time series.

For the last item in the asset side of the balance sheet, among the ones used in our modelization, the one K_n describing the fixed assets, the data we present will evolve as represented by the following table:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta C/K(n)$	-87.686	-123.679	-41.452	-28.998	-23.104	-12.675	-11.031	
$K(n)$	346.426	275.875	167.299	149.317	143.750	142.493	150.343	157.836
Parameters and Variables								
$A(n)$	17.135	15.103	23.470	23.431	21.847	20.525	18.524	
$\gamma(n)$	0,32	0,74	0,28	0,20	0,16	0,08	0,07	

Naturally the brick vector observes equation (3.2.4) and the averaging process will give us these values:

\bar{A}	$\bar{\gamma}$
20005	0,22

through the use of equations (3.2.25) and (3.2.26) as well as the standard numerical procedures implied by equation (3.2.27).

¹⁰See the second paragraph of this chapter for its financial interpretation

¹¹Since we have only two equations, from a mathematical standpoint we have to

From the following brick vector onward we will deal with items which reside on the liabilities side of the balance sheet and accordingly with the brick vector theory put forward in the first chapter they will be expressed through the use of negative numbers¹². The first brick vector of the liabilities, representing the fifth brick vector of our bank balance sheet model, is the one relating to the item Y_n describing the evolution of the deposits from clients. It will behave, according to (3.2.5), as follows:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CY(n)$	689.111	5.277.168	115.237	2.527.316	747.266	1.025.261	625.706	
$Y(n)$	-27.647.209	-26.810.842	-21.280.780	-20.844.527	-17.984.401	-17.035.101	-15.896.505	-15.094.791
Parameters and Variables								
$\Delta y(n)$	-836.367	-5.530.061	-436.253	-2.860.126	-949.300	-1.138.596	-801.714	
$\theta y(n)$	0,00533	0,00943	0,01508	0,01597	0,01123	0,00665	0,01107	

We will see in detail in the next paragraph that it represents the main component of the total liabilities, as we can sense from a quick comparison of the magnitude of this item with the ones of the items presented so far. The average values of this brick vector parameters and variables will be the next ones:

$\overline{\Delta y}$	$\overline{\theta y}$
-1.793.202	0,009914

resulting from the application of (3.2.29). We think it is worth noting that, coherently with the item time series, the averaged $\overline{\Delta y}$ tells us that the deposits, on average, increase in the time frame considered. The negative value of $\overline{\Delta y}$ describing this growth depends on the fact that the item is on the liabilities side of the balance sheet.

The second brick vector among the liabilities, representing the sixth brick vector of our bank balance sheet model, is the one relating to the item H_n describing the evolution of the deposits from other banks (including the Central Bank) accordingly with equation (3.2.6). It will evolve as the following:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CH(n)$	-18.306	-1.392.757	-155.451	96.300	1.382.176	2.222	380.185	
$H(n)$	-2.302.136	-2.314.035	-3.692.634	-3.826.808	-3.693.891	-2.276.296	-2.260.274	-1.857.018
Parameters and Variables								
$\Delta h(n)$	11.899	1.378.599	134.174	-132.917	-1.417.595	-16.022	-403.256	
$\theta h(n)$	0,00278	0,00612	0,00576	0,00957	0,00959	0,00606	0,01021	

¹²Since these items are on the debit side of the balance sheet equation

This item is quite stable and more importantly it has a much lower impact on the liabilities side of the bank balance sheet than the Y_n one. As for its counterpart in the assets side, the account T_n , its behaviour is coherent on the one hand with the business model of the kind of bank we are representing and on the other hand with the fact that we are dealing with a scenario post 2007 financial crisis (see Castagna et al. 2013). On the brick vector data we can perform our averaging procedure through which we get the following results:

$\overline{\Delta h}$	$\overline{\theta h}$
-63.588	0,0102

Looking at the mean value $\overline{\Delta h}$ of the variable Δh (keeping in mind our brick vector theory convention on the signs of the variables) it is interesting to notice that on average the deposits from other banks, in the time frame we consider, have increased.

The third brick vector among the liabilities, representing the seventh brick vector of our bank balance sheet model, is the one relating to the item O_n describing the behaviour of the financial liabilities (mainly securities issued by our bank). It will move according to the next path:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CO(n)$	-316.656	460.096	-81.779	92.249	645.104	271.319	-131.158	
$O(n)$	-3.229.895	-3.462.334	-2.910.945	-2.904.347	-2.727.109	-2.022.593	-1.697.022	-1.778.726
Parameters								
$\beta(n)$	0,067	-0,189	-0,002	-0,065	-0,348	-0,192	0,046	
$\theta o(n)$	0,024	0,031	0,030	0,031	0,029	0,032	0,028	

Applying our averaging procedure to those brick vector data we get the following results:

$\overline{\beta}$	$\overline{\theta o}$
-0,089	0,031

We would like to underline the fact that, keeping in mind the brick vector equation (3.2.7), a negative parameter β stands for an increase in the financial liabilities item (e.g. the bank is issuing new securities) , as we can see looking

at the value of the averaged parameter whose result is coherent with the item time series.

The fourth brick vector among the liabilities, representing the last brick vector of our bank balance sheet model, is the one relating to the item D_n describing the evolution of the Account Payables. It will evolve as the following:

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Accounting Items								
$\Delta CD(n)$	-513.002	-357.529	-227.839	-464.488	-144.083	-89.598	-324.092	
$D(n)$	-678.166	-722.835	-668.480	-578.961	-727.637	-557.740	-335.095	-360.656
Parameters and Variables								
$G(n)$	-468.333	-411.884	-317.358	-315.812	-313.980	-312.243	-298.531	
$\omega(n)$	0,43	0,33	0,25	0,45	0,17	0,14	0,49	

Naturally the brick vector observes equation (3.2.8) and the averaging process will give us the next values:

\overline{G}	$\overline{\omega}$
-348.305	0,33

through the use of equations (3.2.34) and (3.2.35) as well as the standard numerical procedures implied by equation (3.2.36).

Finally we can present the evolution the system describing our entire balance sheet model using (3.2.18) rearranged in the following way:

$$\left\{ \begin{array}{l} C_n = C_{n-1} + (\Delta CL_n + \Delta CT_n + \Delta CS_n + \Delta CK_n + \Delta CY_n + \Delta CH_n + \Delta CO_n + \Delta CD_n) \\ L_n = L_n \\ T_n = T_n \\ S_n = S_n \\ K_n = K_n \\ Y_n = Y_n \\ H_n = H_n \\ O_n = O_n \\ D_n = D_n \end{array} \right. \quad (3.3.1)$$

that will give us these results

Year	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
<i>C</i>	769.719	289.401	119.092	114.139	78.120	81.144	80.016	82.745
<i>L</i>	23.996.543	23.211.925	20.843.557	22.390.951	20.606.832	18.247.861	16.711.080	14.936.103
<i>T</i>	980.339	1.088.388	1.481.714	1.855.235	2.200.794	1.465.507	1.294.214	1.117.463
<i>S</i>	8.180.458	8.837.235	6.540.905	4.535.466	2.889.312	2.907.376	2.991.053	3.585.207
<i>K</i>	346.426	275.875	167.299	149.316	143.750	142.493	150.343	157.836
<i>Y</i>	-27.647.209	-26.810.842	-21.280.780	-20.844.527	-17.984.401	-17.035.101	-15.896.505	-15.094.791
<i>H</i>	-2.302.136	-2.314.035	-3.692.634	-3.826.808	-3.693.891	-2.276.296	-2.260.274	-1.857.018
<i>O</i>	-3.229.895	-3.462.334	-2.910.945	-2.904.347	-2.727.109	-2.022.593	-1.697.022	-1.778.726
<i>D</i>	-678.166	-722.835	-668.480	-578.961	-727.637	-557.740	-335.095	-360.656

The last thing we think it is important to present, about this simulation concerning our commercial bank balance sheet model behaviour, is a checking of the results brought us by our averaging procedures. So if we recall the dynamic of our balance sheet formalization where the Chisini averaging procedure has been applied, i.e.

$$\vec{S}_n = M * [\vec{S}_{n-1} + \vec{A}] + \vec{P}$$

and we consider as S_0 the balance sheet vector at the end of year 2008, namely

$$\vec{S}_0 = \begin{pmatrix} C_0 \\ L_0 \\ T_0 \\ S_0 \\ K_0 \\ Y_0 \\ H_0 \\ O_0 \\ D_0 \end{pmatrix} = \begin{pmatrix} 82.745 \\ 14.936.103 \\ 1.117.463 \\ 3.585.207 \\ 157.836 \\ -15.094.791 \\ -1.857.018 \\ -1.778.726 \\ -360.656 \end{pmatrix}$$

having as vector \vec{A} the values of the averaged variables belonging to the anticipated vector

$$\vec{A} = \begin{pmatrix} (-\Delta l - \Delta t - \Delta y - \Delta h) \\ \Delta l \\ \Delta t \\ 0 \\ 0 \\ \Delta y \\ \Delta h \\ 0 \\ G \end{pmatrix} = \begin{pmatrix} 407.336 \\ 1.469.043 \\ -19.589 \\ 0 \\ 0 \\ -1.793.202 \\ -63.588 \\ 0 \\ -348.305 \end{pmatrix}$$

and as vector \vec{P} the values of the averaged pertaining variables

$$\vec{P} = \begin{pmatrix} X \\ 0 \\ 0 \\ 0 \\ (-A) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20.005 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

if we apply the matrix M of the values of the averaged parameters

$$M = \begin{pmatrix} 1 & \rho l(1-\pi) & \rho t & (\rho s - \alpha \varphi) & -\gamma & \theta y & \theta h & (\theta o + \beta) & \omega \\ 0 & (1-\pi) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+\alpha)\varphi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-\omega) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0,0342 & 0,0134 & -0,1069 & -0,2150 & 0,0099 & 0,0101 & -0,0575 & 0,3328 \\ 0 & 0,9914 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1,1250 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1,2150 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1,0889 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0,6671 \end{pmatrix}$$

and we reiterate the procedure for seven times, we obtain the following values for the balance sheet vector at the end of the year 2015

$$\vec{S}_7 = \begin{pmatrix} C_7 \\ L_7 \\ T_7 \\ S_7 \\ K_7 \\ Y_7 \\ H_7 \\ O_7 \\ D_7 \end{pmatrix} = \begin{pmatrix} 769.720 \\ 23.996.543 \\ 980.339 \\ 8.180.458 \\ 346.426 \\ -27.647.209 \\ -2.302.136 \\ -3.229.895 \\ -678.167 \end{pmatrix}$$

values that match nearly perfectly those of our simulated balance sheet at time 31/12/2015, except for a couple of minor discrepancies¹³ given by approximations in the calculation procedures.

3.4 A proposal for a new LaR measure applied to the simulated bank balance sheet

The last financial crisis has revealed shortcomings in risk management, with special regard to the management of liquidity risk, that have imperiled the financial service industry and the economy at large. Since then banks and regulators have begun to pay close attention to the concept of liquidity risk and undertook significant changes under the so-called Basel III framework. In order to address the aforementioned concept we have to start from a general definition of the term liquidity. The idea of liquidity represents a much broader notion than that of cash flow that we have encountered in the previous chapter and there is still some debate over its proper bounds. It has been said that “liquidity is easier to recognize than define” (Crockett 2008) and that it can be an elusive concept. Generally speaking liquidity is about having access to cash flow when you need it (Armstrong et al. 2008) and in the case of the financial sector a specific notion has been proposed, namely that it represents the capacity of a bank to fund increases in assets and meet obligations as they come due, without incurring unacceptable losses¹⁴ (Basel Committee 2008a). Here we will refer to the unified and consistent approach to the concept of financial liquidity and liquidity risk proposed by Nikolau (Nikolau 2009) which has become a standard fare in liquidity risk related literature. Nikolau identifies three main types of liquidity, pertaining to the liquidity analysis of the financial system, as well as their respective risks: central bank liquidity, market liquidity and funding liquidity.

¹³Their order of magnitude is of one unit in the last digit of the Current Liquidity item and the Account Payables item

¹⁴Currently much of the debate surrounding the liquidity concept revolves around term “unacceptable”.

Central bank liquidity can be defined as the ability of the central bank to supply to the financial system the liquidity needed and the risk associated to this notion, although in theory not equal to zero, is normally assumed as practically non-existent. A totally different story is the one about the concept of market liquidity which a number of different studies define as the ability to trade an asset at short notice, at low cost and with little impact on its price. It incorporates (Fernandez 1999) the key elements of volume, time and transaction costs so that it relates to depth¹⁵, tightness¹⁶ and resiliency¹⁷ of the market. Its related risk can be generally defined as the inability to trade at a fair price with immediacy and among the three kinds of liquidity, with their related risk, is the one about which literature is more abundant. An in depth analysis of this kind of risk is beyond the scope of the present work, here we deem important to underline two key aspects among its features. First of all market liquidity risk is in most cases low and stable, it is rare and episodic (see Pastor and Staumbaugh 2003) and its episodic nature results from downward spirals due to mutually reinforcing funding and market illiquidity (see Brunnemeier and Pedersen 2005, 2007) so that it is deeply related to the funding liquidity risk. Secondly it is significant to notice that even in its very own definition it is implied a short-term time perspective on the issue of liquidity risk. This is of primary importance since due to the complex nature of the liquidity and liquidity risk concepts, as well as the deep linkages between the three different kinds of liquidity, the problem needs to be addressed from different temporal points of view (see Neu and Vogt 2012), one short-term and the other more medium long-term. This can be seen also in the approach followed by the regulatory bodies since in December 2009 the BCBS¹⁸ has proposed two new global measures for managing liquidity risk. They have become the internationally adopted pillars of new liquidity regulations and reflect the two aforementioned time perspectives. Moving from the previous qualitative (Basel II regulations) towards a more quantitative approach, the Basel III international regulatory framework has devised measures based on two liquidity ratios: a stressed 1-month liquidity coverage ratio (LCR) and a structural (more than 1 year) net stable funding ratio (NSFR) (see Castagna et al. 2013). The first ratio has been conceived to immunize banks against short-term liquidity shocks while the latter limits the refinancing risk and the maturity transformation in funding (see the BIS¹⁹ document “International framework for liquidity risk measurement, standards and monitoring” from December 2009 and the update from July 2010).

This double approach on the time perspective is also implicit in the definition of the different kinds of liquidity and liquidity risk. So if market liquidity and its associated risk are concepts more focused on the short-term, funding

¹⁵A market is deep when a large number of transactions can occur without affecting the price i.e. the number of buyers and sellers is large.

¹⁶A market is tight when transaction prices do not diverge from mid-market prices.

¹⁷A market is resilient when price fluctuations from trades and imbalances in order flows are quickly adjusted.

¹⁸Basel Committee on Banking Supervision

¹⁹Bank for International Settlements

liquidity and its relative risk are more related to a medium long-term point of view. Nikolau defines funding liquidity as the ability of solvent institutions to make agreed upon payments in a timely fashion (see Drehmann and Nikolau 2008). In practice funding liquidity, being a flow concept, can be understood in terms of a budget constraint, namely an entity is liquid as long as its inflows are bigger or equal to its outflows. In its determination it is of primary importance to define the liquidity sources available to the financial firm we are analysing. In the case of banks the sources can be classified (see Nikolau 2009) into four categories: the depositors, the market²⁰, the interbank market and finally the central bank. Funding liquidity risk, according to the IMF (2008), captures the inability of a financial intermediary to service their liabilities as they fall due. Typically funding liquidity risk depends on the availability of the four liquidity sources mentioned above and the consequent ability to satisfy the budget constraint over a certain period of time. Measuring funding liquidity risk is not a trivial task and academic evidence on the properties of funding liquidity risk, to this day, is meager (Nikolau 2009). In most cases practitioners construct funding liquidity ratios, as proxies for funding liquidity risk, that reveal different aspects in the availability of funds within a given time horizon. They can be produced through static balance sheet analysis or by dynamic stress testing techniques and scenario analysis. In general the probability of becoming illiquid is typically measured for a given period ahead and can differ significantly according to the length of the period (Matz and Neu 2006, Drehmann and Nikolau 2008). Within this framework we think the CFaR approach we presented in the previous chapter can represent the basis for a new dynamic liquidity risk measure, tailored on the funding liquidity issue and its implicit medium/long-term time perspective. The LaR measure, being in many respects a byproduct of VaR, has a short-term temporal point of view since generally it computes only over days or weeks. On the other hand the CFaR necessarily implies a medium-term temporal perspective for the simple fact that it is measured over fiscal years or at least semesters. Moreover the LaR measure, as in the VaR case, presents a purely quantitative top-down approach, in a situation where both practitioners and the academic community think that in the occurrence of the last financial crisis deep problems originated from banks' reliance on purely quantitative approaches which lacked business judgment (see Castagna et al. 2013). In the case of liquidity risk qualitative business judgement turns out to be crucial, and during the 2007 financial crisis an emphasis on quantitative probabilistic methods compromised the capacity of risk management to perceive the implicit liquidity risk in their banks' business models (see Neu and Vogt 2012). Consequently we think that our CFaR approach, stemming from a balance sheet analysis, may hopefully represent the first step of a methodology that merges a quantitative perspective with a more business structure oriented point of view. Naturally we have to reshape our CFaR approach in order to serve the purpose of the funding liquidity risk assessment issue, and in this spirit we will refer to

²⁰ A bank can sell its assets or generate liquidity through securitization, loan syndication and the secondary market for loans.

this new Liquidity at Risk measure as Funding Liquidity at Risk or FLaR. To reach our goal, firstly we have to focus on the stochastic modelization of some macroeconomic drivers able to describe the four categories of banks' liquidity sources that we have previously identified. At this stage of our research the full extent of a stochastic modelization as the one just proposed goes beyond the scope of the present work. Here we just want to introduce the proposal of a new liquidity risk measure, to highlight the new ground of possibilities laid down by the mathematical formalization we presented in the first chapter. Consequently we will choose only one macroeconomic driver related to the funding source we can detect as the most important, accordingly with a balance sheet analysis. The following table describes the weights of each item on the total assets or the total liabilities²¹ during the course of the time frame we considered:

Anno	31/12/2015	31/12/2014	31/12/2013	31/12/2012	31/12/2011	31/12/2010	31/12/2009	31/12/2008
Assets								
Weight of L on the total assets	0,70	0,69	0,71	0,77	0,80	0,80	0,79	0,75
Weight of T on the total assets	0,03	0,03	0,05	0,06	0,08	0,06	0,06	0,06
Weight of S on the total assets	0,24	0,26	0,22	0,16	0,11	0,13	0,14	0,18
Weight of K on the total assets	0,01	0,01	0,01	0,01	0,01	0,01	0,01	0,01
Liabilities								
Weight of Y on the total liabilities	0,82	0,80	0,75	0,74	0,72	0,78	0,79	0,79
Weight of H on the total liabilities	0,07	0,07	0,13	0,14	0,15	0,10	0,11	0,10
Weight of O on the total liabilities	0,10	0,10	0,10	0,10	0,11	0,09	0,08	0,09
Weight of D on the total liabilities	0,02	0,02	0,02	0,02	0,03	0,03	0,02	0,02

As we can infer from the analysis in the above table, the main source of funding for our bank is represented by the item *Y* describing the deposits from clients, whose weight on the total liabilities ranges from 72% in 2011 to 82% in 2015. The interbank market source and the central bank source combined, represented by the item *H* describing the deposits from other banks, range from a weight of 7% to 15% on the total liabilities. Finally the market source, mainly described²² by the item *O* reporting the financial liabilities, varies between 8% and 10%. We can conclude that our bank, since, differently from bigger institutions, its market exposure is contained, presents a business model that we could describe as a classic banking business model, based on the management of the temporal mismatch between deposits and loans.

Given the importance of the deposits from clients in its funding policy we decided to stochastically model a macroeconomic driver that could expose the sensibility of the balance sheet structure (representing our bank's business model) to changes in the environment that could affect this funding source, namely the value of the averaged interest rate $\bar{\theta}_y$ paid on the deposits.

The interest rate dynamics is assumed to follow a mean reverting (Vasicek) process, see e.g. Bjork (1998). We considered the six-month Euribor interest rate from 2009 to 2015 and through a MLE²³ procedure we estimated the following parameters:

²¹ Depending on the side of the balance sheet equation where it belongs

²² Actually also the item *S* plays a varying role

²³ Maximum likelihood estimation

Interest rate market

Interest rate : -0.04%

Mean reversion : 187.8%

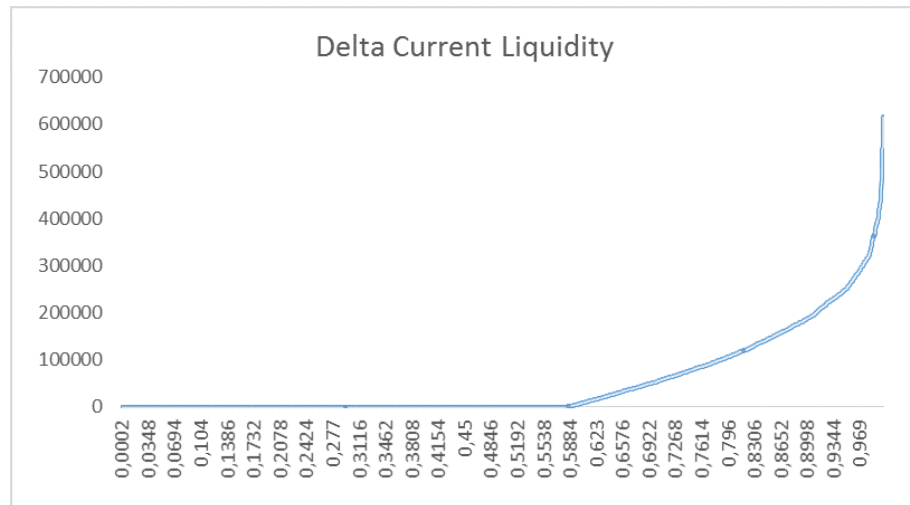
Equilibrium rate : 0.81%

Volatility : 1.23%

On our 2009-2015 Euribor time series we performed the following statistics:

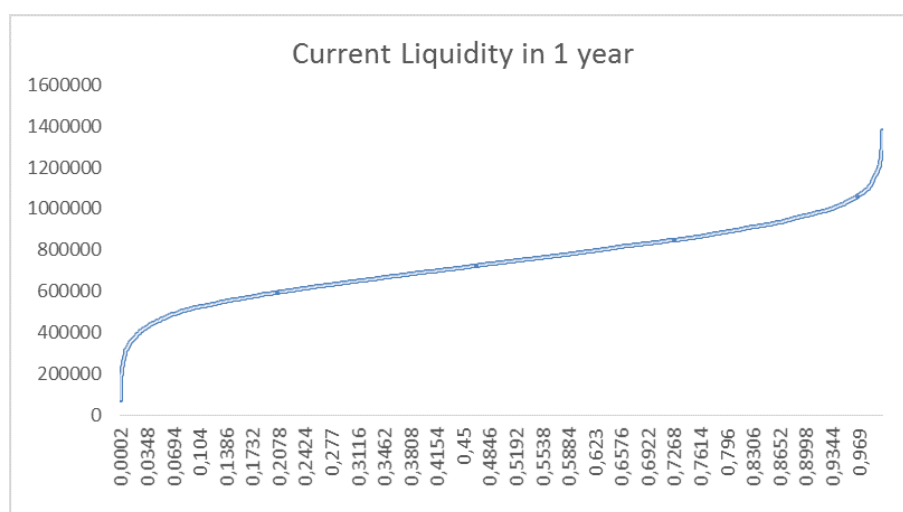
Sample statistics:	
Mean	0,0082
Volatility	0,0064

and since the approximated value of our averaged parameter was $\overline{\theta y} = 0,0100$ we assumed a spread over the Euribor of 0,0018. Then, using a MonteCarlo approach, we simulated 5000 paths for the diffusion process representing $\overline{\theta y}$ and for each path²⁴ we estimated the future value of the Current Liquidity item associated to one and two years. Coherently with the balance sheet time series we proposed, representing an effectively and cautiously managed bank, although there's a very high probability, around 58,5%, that in the year 2016 our bank will deplete its Current Liquidity reserves, as we can see in the next chart

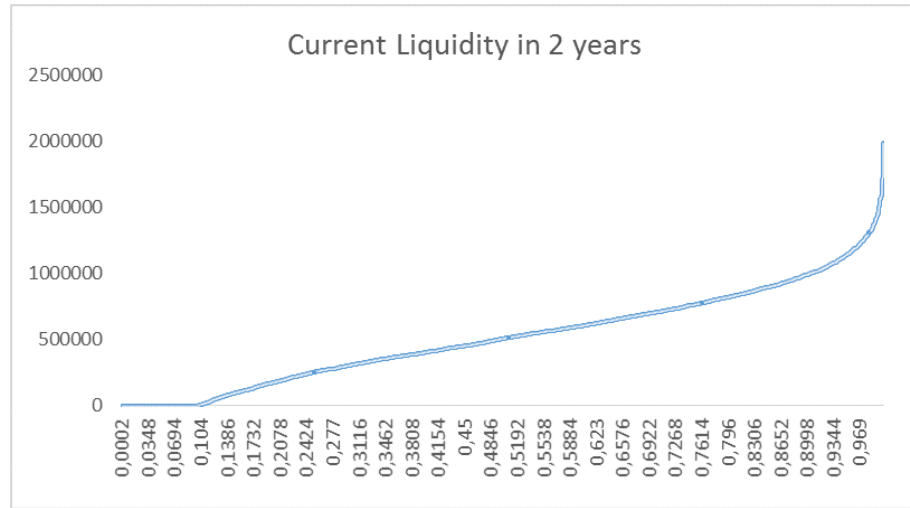


²⁴Using the median of each path from a time standpoint

in a one year perspective the bank's balance sheet structure appears really solid. As a matter of fact our FLaR measure tells us that there's only a 1% probability that the current liquidity reserves will fall under 332.650 thousands of Euros ($FLaR_{1\%} = 332.650$) at the end of 2016, and a 5% probability that they will fall under 458.460 thousands of Euros ($FLaR_{5\%} = 458.460$) , as we can see through the following chart



The situation changes in a two-year time frame since as we can see with the aid of the next chart



our bank is in a situation where there is 10,06% probability that its Current Liquidity reserves will fall below zero at the end of 2017. In the time window considered the management would have different options to respond to the situation, not lastly by putting into action the funding liquidity contingency plan that it is required to devise and test periodically by the regulatory bodies. This kind of scenario analysis could imply a possible further path of development for the FLaR measure in order to propose a dynamic stress testing approach, once the measure would be more properly defined. As we have stated before here we just wanted to propose a new liquidity risk measure referring to a medium/long-term time perspective, never tackled before in a dynamic manner, mainly in order to highlight the scientific prospects given by the brick vector mathematical accounting formalization, which is the central topic of the present work. In the ensuing chapter conclusion we will briefly discuss some plausible paths of development for the FLaR measure, possibly in light of the opportunities offered by the brick vector framework.

3.5 Conclusions and remarks

In this chapter after presenting some of the main issues concerning the topic of liquidity and liquidity risk management, we tried to show how the brick vector mathematical framework could make a contribution in order to confront the issue. Firstly we presented a commercial bank balance sheet model, devised in accordance with the IFRS²⁵ accounting principles, as well as the model shape once the Chisini averaging procedure is performed. Secondly we displayed a simulation of the model behaviour deeply rooted in real balance sheet data so

²⁵International Financial Reporting Standards

that we could regard our example as in line with the performance of an Italian medium commercial bank, effectively and cautiously managed in the time frame considered. Finally we proposed a new liquidity risk measure, the FLaR (Funding Liquidity at Risk), specifically tailored on the funding liquidity issue and its medium/long-term temporal perspective. We have seen how the regulators as well as the banking world pay a specific attention on the addressing of the liquidity risk topic from two different temporal perspectives, one short-term, through the implementation of the LCR²⁶, and the other long-term, through the NSFR²⁷. The LaR measure can be considered as a version of VaR specifically moulded for the liquidity problem, given that it is basically a VaR performed on the time series of the various inflows and outflows. Consequently it is representative of a short term point of view as well as a purely quantitative top-down approach. So we deemed fit to propose a CFaR version specifically tailored on the liquidity topic, since the CFaR measure is medium/long-term oriented as it is the problem of the funding liquidity risk assessment. We named this version FLaR (Funding Liquidity at Risk) and we believe that its bottom-up approach can be suitable for the specific challenges presented by the funding liquidity issue. Moreover we believe that the FLaR measure can be an example of the contributions that the brick vector mathematical framework can provide to various areas of research in the finance field, through its ability to merge accounting data insights with the formalization of mathematical finance. Therefore we would like to close the present chapter discussing interesting plausible further developments that we think our research could pursue concerning the FLaR measure, as well as the linkages that some of this developments could have with the possibilities offered by the accounting formalization presented in the first part of the present thesis.

Primarily we believe that our research should explore the stochastic modelization of different macro-drivers in order to achieve the FLaR full potential in describing funding liquidity risk. In so doing we should find a way to take into account simultaneously the four primary sources of funding liquidity identified in the previous paragraph. This is particularly true in the case of a banking business model like the one developed by the international banking system in the last decades, that presents a balance sheet exposure to market instruments not comparable with previous standards. In this regard the simulation we presented is indicative of a more classical banking business model, principally focused on maturity transformation. In order to achieve this goal the primary obstacle would definitely be the stochastic modelization of a synthetic macroeconomic driver that could represent a proxy for the source of funding liquidity represented by the market. That's because it should subsume a lot of the concepts, well defined in literature, concerning the deep relationship between funding liquidity risk and market liquidity risk. It should take into account the three main dimensions regarding the idea of market liquidity risk, namely market depth, tightness and resilience, and reshape them through a funding liquidity

²⁶Liquidity coverage ratio

²⁷Net stable funding ratio

centered perspective. In this regard the brick vector framework could result as particularly helpful since theoretically it is able to formalize any balance sheet reclassification, and the pursue of the aforementioned endeavour with all probability would imply a balance sheet reclassification specifically designed for the purpose.

This descriptive capability would definitely prove useful in the case of what we think should be the second path that our future research on the FLaR measure could take. The NSFR is defined as the ratio between Available Stable Funding over Required Stable Funding and naturally its outcome has to be greater than one²⁸. What must be assumed as Required Stable Funding is defined by the regulator and it represents the major component of the stress testing exercise element embedded in the funding liquidity measure implied by the ratio. On the other hand the numerator of the fraction, the Available Stable Funding, is basically a weighted balance sheet reclassification according to rules and weights (ranging from 100% to 0%) defined by the regulator. We have previously discussed how both the LCR and the NSFR are static measures and how, not only in the academic world, this feature raises concerns, on the one hand about the reliability of the ratios as proper stress testing exercises and on the other hand about their possible pro-cyclical effects at the macro level. To put it simply, as an old adage of the banking sector says: “A lack of liquidity can kill a bank quickly whereas too much liquidity can kill a bank slowly” and it can kill the whole economic environment as well. In this situation we could think to try to use the dynamic nature of the brick vector mathematical framework in order to make the NSFR a dynamic measure. Moreover we could use the FLaR approach on the reclassified balance sheet on the numerator of the ratio to assess the bank’s probability of passing the stress testing exercise, in relation to its exposure to some macro-drivers felt as crucial for the management decision making process.

Finally we hope that both the FLaR approach and especially the brick vector framework could be felt as having potential for contributions to established braches of research such as the ones dealing with stress testing exercises or asset-liabilities management. Possibly even in cases where an issue is felt to be better addressed through the lens of an optimization procedure applied to a mathematical model deeply rooted in the economic underlyings of the banking firm.

²⁸Greater than 100% since it is expressed as a percentage.

Conclusion

Nowadays one of the most interesting challenges in the economic research is related to the establishing of new informative frameworks enabling us not only to better comprehend key subjects of the modern economic system but that will also integrate with other models or informative systems already present in literature. For that reason we developed a new mathematical framework for the creation of accounting models that tries to represent the firm and the firm's dynamics through the most important source of information of an enterprise, namely the balance sheet. Moreover we decided to create this mathematical modeling framework because the accounting research world considers itself and is believed to be not enough properly integrated with other economic research fields. At the same time given that most of the financial/economic applications use data coming from balance sheets, this instrument certainly plays a key role in the economic environment.

In order to create this framework we started from underlining, in the accounting history and literature, the many attempts at providing a mathematical formalization of the transaction recording procedures. We classified the attempts made in the last fifty years into four main research strands that have not led to a completely defined framework because of the many reasons we thoroughly summarized in the first chapter of the present work. Therefore we identified the current mathematical framework as the one implied by the double-entry bookkeeping system and we analyzed its main mathematical features, as well as their relationship with the accounting practice. Modifying those characteristics in order to obtain a dynamic finite difference system description of the balance sheet we arrived at our new mathematical framework for the accounting field (that we named brick vector mathematical framework). Finally we closed the first chapter discussing the problem of the inevitable high number of parameters and variables that a balance sheet modelization implies, and we started to address the issue proposing an averaging procedure on the balance sheet time series, inspired by the Chisini concept of functional average.

In the second chapter we started to show how we think the brick vector mathematical framework could result useful in bringing together accounting data economic insights with the research opportunities offered by established mathematical modelization approaches, present in other economic research areas. Given the key role that the liquidity accounting item plays in the brick vector concept, we decided to apply the framework to the issue of determining

a risk measure for the Cash Flow generation capability of an industrial firm. So we created an industrial firm balance sheet model and, after a review of the main attempts at estimating a CFaR (Cash-Flow at Risk) measure with their relative shortcomings, we proposed our CFaR approach and we applied it to a case study. In so doing we touched the subject of how our method could limit the shortcomings of previous approaches, because of its roots stemming from an holistic view of the firm's accounting data summarizing its general economic exposure.

In the third and final chapter we decided to show the potential utility of the brick vector framework through its application to a problem that, after the last financial crisis, has become poignant to the financial sector, i.e. the assessment of liquidity risk. So we created a commercial bank balance sheet model coherent with the IFRS accounting standards and we simulated its behaviour, offering an example in line with the data of a medium sized Italian commercial bank effectively managed. Finally we proposed a new risk measure for the assessment of the funding liquidity risk (coherently we named it FLar) discussing how its dynamic approach, its long-term time perspective and its link to the banking firm's accounting data could contribute to the general debate on the issue of liquidity risk. Then we concluded the chapter presenting some possible future paths of research on the liquidity risk assessment topic, concerning the FLar measure and how the brick vector framework could result useful in their development.

We hope that in the course of the present thesis we resulted convincing in proving what we consider to be the main point of this work, namely how the new mathematical framework that we have devised could result useful in several areas of the economic research, because of its inherent potentials to merge informative accounting data with mathematical modeling, the real world of economic practice with the academic world of economic science.

In so doing giving a contribution to stimulate new approaches in the accounting research towards the development of tools that could be more informative towards the economic and financial research world as well as the general public, as it is requested by many in the accounting academic community.

But here we would like to close the present work discussing some main issues relating to the mathematical formalization aspect of the brick vector framework. Namely the fundamental general matter of the growing use of mathematical models by the financial world and the academic community as well as their reliance on the statistical science.

One of the major shortcomings of the CFaR top-down approaches, as well as the exposure-based methodology, lies in the lack of a number of data that could be perceived as statistically significant. This, we think, depends on a fundamental difference between a firm of industrial nature and a purely financial firm, for example a hedge fund. The first one must possess a business structure while the other mainly implements investment strategies. This implies a major dissimilarity between the time horizon embedded in the two kinds of economic activity, since the business structure of a production firm must be assessed in a medium long-term perspective (quarterly, yearly or even more) while the

investment strategies of a purely financial firm have to be reevaluated, at least partially, on a daily basis. Therefore the use of statistical instruments are not as significant as they are in the financial world when it comes to the CFaR assessment of production firms. Even if we had a number of data statistically significant, for example if we had access to a century long balance sheet data time series or we had Cash Flow data collected on a daily basis, they would most likely be not as significant as they would be in a financial context. In the first case because in a decades long balance sheet time series most of the data would no more be in any way indicative of the business environment in which the firm is operating at the moment. In the second case because the data would most likely be tainted by exogenous noise producing shocks (for example seasonality) that would add useless or even damaging information to the purpose of capturing the firm's business structure. We think that this business structure is expressed in a synthetic way by the information carried by the balance sheet, which as a matter of fact, it is used in the real economic world, among other things, in order to signal just that. So our CFaR method, rooted in the accounting data, has the aim to create a tool that allows for a quantitative approach on top of qualitative analyses of real economic data, in a field, the Cash Flow risk assessment of production firms, where purely quantitative techniques are faced with the limits discussed above.

The liquidity risk assessment issue, on the other hand, presents very specific features, because, in light of what we have just expressed, we could look at the banking firm as having some sort of double nature, especially since the distinction between investment banking and commercial banking has been fading. So the banking enterprise shows the traits of a purely financial firm, which implements investment strategies and is more focused on a short-term temporal perspective, but at the same time can't be lacking a business model, whose structure implies a medium long-term approach. Hence the double temporal perspective needed to address the liquidity risk issue, which is expressed also in the two liquidity ratios that are in the process of being enforced by the regulators: the liquidity coverage ratio, more short-term sighted, and the net stable funding ratio, more long-term oriented, as we have amply discussed in chapter three. The proposal of a FLaR measure, whose approach is deeply linked to our CFaR methodology, has the aspiration to provide a fresh point of view, in the liquidity risk assessment debate, presenting all the characteristics just discussed above for the CFaR. Especially since we find ourselves in a point in time where concerns are expressed, both in the academic and the practitioner world, about an increasingly over-reliance on quantitative models by the financial environment at large.

Those concerns have a long history, since they have closely followed the development of quantitative finance in a period, the last thirty years, where it has completely reshaped the financial world, becoming the dominant force in the field nonetheless. An example of that attitude can be seen in the criticism that surrounded VaR since it started to move from trading desks into the public eye in 1994 thanks to the publication by J.P. Morgan of their variance-covariance methodology, still to this day the most employed method to compute

the measure. In a famous 1997 debate Nassim Taleb set out the major points of contention and among other things he declared that VaR was in the end charlatanism because it claimed to estimate the risk of rare events, which is impossible. Later he popularized the concept in his famous 2007 book “The Black Swan: The Impact of the Highly Improbable”.

The main problem with the top-down models we use relates to the fact that they all rely in one way or another on the central limit theorem (as well as on the assumption that the best predictor of our future is our past). At the same time we know that in the economic and financial arena unlikely events are much more common than what our models would predict according to their assumptions based on the Gaussian distribution, the well know fat tails argument.

Moreover this spreading reliance on mathematical and statistical models²⁹ have led in recent years to the awareness, in the risk management community, of the growing importance of model risk management. Model risk is defined as the risk of loss that could result from the use or misuse of mathematical models. One of the acknowledged sources of model risk is the complexity of the model itself, leading to an incorrect identification of its risk factors because of the difficult task of becoming aware of the hidden hypotheses implied by the model. This factor is often cited as an example of unrecognized risk in relation to the problems experienced by the owners of mortgage backed securities portfolios during the 2007 financial crisis.

Our brick vector framework would like to be seen as an attempt to mitigate the issue since its last goal is to try to incorporate some business judgement into our mathematical models.

On an even broader perspective the problem of the relationship between mathematical formalization and the real world affects the economic science as a whole, the issue has always been debated but again the last financial crisis made it more poignant³⁰. The economists didn’t fully realize the importance of finance and financiers put too much faith in the models produced by economists. Moreover the rational agent hypothesis implied by the concept of homo economicus is increasingly struggling to describe an economic environment always more complex because of globalization and the digital revolution. Many argue that we should try to merge well established economic theories with the most pressing results coming from the behavioural economics school, in order to take more into account man’s herd behaviour. Naturally the open question relates to the issue of how to do that.

More in general it is widely felt that academia and the economic sciences should try to find ways to incorporate aspects of the real world into their mathematical formalizations and we hope that the present work could be perceived as a contribution, be it small or not, towards that goal.

²⁹In an article of the February of this year Crespo et al., senior McKinsey analysts, report that the number of models rises 10 to 25 percent annually in large institutions

³⁰A summary of the evolution of this discussion in the last years can be found in the article “What’s wrong with finance”, The Economist, (May 1 of 2015).

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