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RESEARCH ARTICLE

A class of individual expenditure functions

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Abstract This study examines the regularity properties ensuring that individual expenditure functions are legitimate individual cost functions in the context of collective household models. The structure of collective household models entails a scaling of income through a function that describes how resources are shared within the household. This modified income function defines expenditure functions at the individual level. Our study completes previous work on modifying functions by Barten, Gorman, and Lewbel that was limited to the investigation of the scaling of prices and the translation of income without considering the scaling of incomes. We find that the product of the modifying function and the household expenditure function maintains the regularity properties of expenditure functions if the modifying function is positive, homogeneous of degree zero and at least quasi-concave. We also examine how changes in prices affect the curvature of the modified income function and, in turn, inequality in the distribution of resources within the household. An example shows how our results can be used to test the curvature properties of individual expenditure functions as well as to measure the inequality within the household.

Keywords Individual expenditure function · Collective household model · Concavity · Homogeneity · Intra-household inequality

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1 Introduction

For long time the behavior of multi-person households has been described within the fiction of a unitary utility function, although neoclassical theory of demand explains constrained optimal choices of individuals. The unitary approach has been extensively applied in theoretical and empirical works because it allows testing restrictions on household behavior. However, a number of empirical evidences find that the unitary model fails in representing the behavior of multi-person households and, moreover, is unable to measure the welfare of individuals. The collective approach (Apps and Rees 1988, 1997; Blundell et al. 2007; Browning and Chiappori 1998; Chiappori 1988; Donni 2003; Donni and Chiappori 2011; Lundberg and Pollak 1996) allows the identification of the sharing rule governing the intra-household allocation of resources and the recovery of individual preferences and welfare functions. The structure of collective models entails a modification of income, or non-labor income in a labor supply context, through a function that scales income describing how resources are shared within the household. This modified income function is in general expressed in terms of individual levels of income and defines expenditure functions at the individual level.

This study examines the regularity properties ensuring that individual expenditure functions (Browning et al. 2014; Chiappori et al. 2002; Dunbar et al. 2013; Menon and Perali 2012) are plausible individual cost functions within the collective context. We extend the results of Lewbel to the non-differentiable case using relatively simpler proofs. Unlike Lewbel's (1985) seminal work, which focused mainly on a price transformation *à la* Barten (1964) and an income translation *à la* Gorman (1976), we study the transformation of income with a scaling function as traditionally adopted in the collective theory. We find that the product of the modifying function and the household expenditure function maintains the regularity properties of expenditure functions if the modifying function is positive, homogeneous of degree zero and at least quasi-concave. We also examine the curvature properties of the income scaling function to introduce new measures of inequality (Peluso and Trannoy 2007) describing how differently each member of the household responds to price changes. An example of modified individual expenditure functions that shows the relevance of our results for testing the curvature properties of individual expenditure functions as well as for the measurement of inequality within the household is also presented.

Preliminaries Lewbel (1985) describes within a unitary framework the properties of the following general transformation of the household cost function that maintains integrability of a demand system

$$e(p, u, d) = f[e^*(h(p, d), u), p, d], \tag{1}$$

where p is an ℓ vector of prices, d a vector of demographic variables and $e^*(h(p, d), u)$ is a legitimate cost function corresponding to the minimum expenditure necessary to attain utility level u at scaled prices $p^* = h(p, d)$. The transformation functions f and h are continuous and twice differentiable. The f function describes interactions of d with e^*, while both the f and h modifying functions allow interactions of d with p in an almost unlimited variety of interesting forms. Modifying functions can be interpreted as household technologies. They generate demographically varying intermediate goods where p^* is the shadow price vector of the intermediate goods.

Before the introduction of the collective household theory by Chiappori (1988), the literature on modifying functions of demand systems focused mainly on a general transformation encompassing both demographic scaling of prices (Barten 1964) given by $h_i(p,d) = p_i^* = p_i s_i(d)$ for the ith good and demographic translating of expenditure (Gorman 1976) $f(p,u,d) = e^*(h(p,d),u) + P(p,d)$, where $P(p,d)$ represents fixed overheads for necessary quantities (Pollak and Wales 1981; Lewbel 1985; Perali 2008). Lewbel's contribution (1985) mainly focuses on a price scaling transformation h *à la* Barten and an income translation f *à la* Gorman within the unitary context. We complete his seminal work by studying the properties of the individual expenditure function obtained from the transformation of income with a scaling function.[1] The extension of our interest is, therefore, the scaling of expenditure through the function $m(p,d)$

$$f(p,u,d) = e^*(p,u,d)m(p,d), \tag{2}$$

which has been used in empirical work to identify the rule governing the intra-household allocation of resources (Dunbar et al. 2013; Lewbel and Pendakur 2008; Menon and Perali 2012). Further, while Lewbel (1985) examines the restrictions of the modifying functions h and f, we are interested in analyzing the properties of m guaranteeing that the product $e^*(p,u,d)\,m(p,d)$ maintains the properties of f.

From unitary to collective models Our interest in the scaling function $m(p,d)$ is motivated by its importance in the context of the collective theory. Interestingly, the collective approach shares intriguing similarities with the theory of modifying functions. Two-stage budgeting is a special case of modifying functions where in the first stage allocation on an aggregate good is observed and then the unobserved second stage allocation on intermediate goods is deduced using demographic information. In a collective context, the difference is in the second stage where the budget is not allocated among unobserved intermediate goods, but rather among observable goods assignable to specific household members. The observed second stage is then used to identify how the household budget is allocated among the K household members $e(p,u,d) = e_1(p,u_1,d) + \cdots + e_K(p,u_K,d)$, where $e_k(p,u_k,d)$ is the individual expenditure function[2] and $u = (u_1,\ldots,u_K)$ with u_k being the utility level of individual k. Then, each member separately chooses her/his unobserved consumption subject

[1] The scaling modification of expenditure has been introduced by Lewbel (1985, Theorem 8), more for mathematical completeness rather than for its economic relevance in the unitary household context. Lewbel defines the function $\beta(d)$ that scales the cost function e^*, but, unlike the present analysis, the function $\beta(d)$ does not depend on prices and is assumed to be $\beta(d) = 1$, neglecting to signal that the product of functions may generate undesirable characteristics of f.

[2] We call e_k individual expenditure and is also known as sharing rule in the collective literature, often denoted by ϕ_k.

to the corresponding individual budget constraint

$$\max_{c_k}\{U_k(c_k, d)|e_k = p^T c_k = e_k^*(p, u_k, d)\, m(p, d)\}. \tag{3}$$

We thus define the f_k transformation in this way

$$e_k(p, u_k, d) = f_k[e_k^*(p, u_k, d), p, d] = e_k^*(p, u_k, d)m(p, d), \tag{4}$$

where $m(p, d)$ is the income scaling function. For example, suppose that the household budget $e(p, u, d)$ of a family composed by a couple and one child is entirely spent to buy food c_f at price p_f and clothing c_k for adults and children at prices p_{c_k} with $k = a, c$. Following Dunbar et al. (2013) and Menon and Perali (2012), this information is sufficient to derive a resource share as $\eta_k = \frac{a_k(p_f c_f)+p_{c_k}c_k}{y}$, where a_k is a known measure of publicness of good f with $0 \leq a_k \leq 1$.[3] The individual-specific resource share η_k is used to approximate the partially observed individual expenditure function $e_k^*(p, u_k, d)$ that is corrected by the income scaling function $m(p, d)$ describing how assignable goods are allocated in each household to obtain the best estimate of $e_k(p, u_k, d)$.

The regularity properties of individual expenditure functions $e_k(p, u_k, d)$ and the income scaling function $m(p, d)$ have not been characterized yet, though estimates of the sharing rule are extensively used in the empirical literature. This characterization is of general interest because it extends microeconomics theory to host the fact that the relevant decision unit is the individual. The assumption that household and individual behavior is the same introduces a significant aggregation bias. As stressed in Browning et al. (2013:3) "what is relevant is not the "preferences" of a given household, but rather the preferences of the individuals that compose it." Browning and Chiappori (1998) study a general collective model that includes household public goods and externalities. Their aim is to test whether household data support the conditions derived by the unitary and collective model. Our contribution is to study the testable properties that must be satisfied by individual expenditure functions to be theoretically plausible cost functions.

We now proceed to Sect. 2 by extending the theory of legitimate cost functions of Lewbel (1985) to individual expenditure functions. The conditions for individual expenditure functions to be legitimate functions are derived for specific functional forms. Section 3 analyzes the curvature properties of the individual expenditure function describing individual aversion to changes in prices. An example is provided in Sect. 4. The conclusive section discusses implications of our results lending special emphasis to the importance of testing the curvature properties of the estimated sharing rule in empirical studies.

[3] When goods are necessities, such as food, the range $0 < a_k < 1$ does not include the limits.

2 Modified individual expenditure functions in a collective context

Consider a family of two persons $k = 1, 2$.[4] Each member k privately consumes a bundle of market goods $c_k \in \mathbb{R}_+^\ell$ and faces a vector of prices $p \in \mathbb{R}_{++}^\ell$. Note that purchases of individuals may include also leisure time and thus the vector of prices would include the corresponding wages. In the analysis, we omit the consumption of public goods or the presence of externalities within families.[5] We also abstract from the consumption of domestically produced goods (Apps and Rees 1997; Chiappori 1997).

The collective model relies on the following assumptions. Firstly, preferences of each family member over the consumption of c_k are represented by an individual quasi-concave utility function $U_k\,(c_k, d)$ twice differentiable and strictly increasing in c_k, where $d \in \mathbb{R}^n$ describes observable heterogeneity both at the individual level, such as age or education of individual k, and at the family level, such as quality of the living area. Secondly, outcomes of the decision problem are assumed to be Pareto-efficient. Pareto-efficiency implies that the consumption equilibrium will be on the Pareto frontier of the family. Further, when all goods are privately consumed and there are no consumption externalities, these assumptions allow describing family behavior using a two-stage process: first, the family agrees on a rule to share resources among its members, then, each member maximizes her individual utility function subject to her individual share of income.[6]

In the dual representation of individual consumption choices, each person minimizes her share of family resources to achieve a given level of individual utility. Formally, we define the collective individual expenditure function of member k as

$$e_k\,(p, u_k, d) = \min_{c_k}\{p^T c_k \mid U_k\,(c_k, d) \geq u_k\}, \tag{5}$$

where $e_k(p, u_k, d)\colon \mathbb{R}_{++}^\ell \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{++}$ represents the minimum level of expenditure needed to individual k to achieve the level of utility u_k at given prices p.

Following Lewbel's (1985) specification of the *household* transformed expenditure function $e(p, u, d) = f[e^*(h(p, d)u), p, d]$, we specify the *individual* expenditure function e_k as follows

$$e_k\,(p, u_k, d) = f_k[e_k^*\,(p, u_k, d)\,, p, d], \tag{6}$$

where $e_k^*\,(p, u_k, d)$ is a legitimate cost function if it is: (a) homogeneous of degree 1 in p; (b) positive, strictly increasing in u_k and non-decreasing in $p_i,\ i = 1, \ldots, \ell$;

4 Our results can be straightforwardly extended to larger family units.

5 The inclusion of public goods would imply the derivation of Lindhal shadow prices described by a specific transformation as shown in Browning et al. (2013). However, because in this work we concentrate on a transformation of income rather than prices, we prefer to leave the treatment of public goods aside to avoid a potential source of confusion. This will be the object of future research.

6 For a complete appraisal of the collective household theory see for instance Browning et al. (2014).

(c) concave in p; (d) continuous in p and u_k.[7] We note that the function f_k, which is assumed continuous, allows interaction of demographic variables and prices with the expenditure function. Our interest is to derive restrictions on f_k guaranteeing that e_k is also a theoretically plausible cost function for which properties (a)–(d) hold. The following, though simple, are useful to verify the properties of the individual expenditure function in empirical applications.[8]

Proposition 1 (Homogeneity of degree 1 in p) *Let e_k^* be a legitimate cost function and let $e_k(p, u_k, d) = f_k[e_k^*(p, u_k, d), p, d]$. If f_k is homogeneous of degree 1 in (e_k^*, p), then e_k is homogeneous of degree 1 in p.*

Proof Because e_k^* is homogeneous of degree 1 in p and f_k is homogeneous of degree 1 in (e_k^*, p), by definition of homogeneous function we have $f_k[e_k^*(tp, u_k, d), tp, d] = f_k[te_k^*(p, u_k, d), tp, d] = tf_k[e_k^*(p, u_k, d), p, d]$. □

Proposition 2 (Positive, strictly increasing in u_k and non-decreasing in p) *Let $e_k(p, u_k, d) = f_k[e_k^*(p, u_k, d), p, d]$, with being a legitimate cost function. If $f_k[e_k^*(p, u_k, d), p, d] > 0$, f_k strictly increasing in e_k^* and non-decreasing in p_i, $\forall i = 1, \ldots, \ell$, then $e_k(p, u_k, d)$ is positive, strictly increasing in u_k and non-decreasing in p.*

Proof The condition $f_k[e_k^*(p, u_k, d), p, d] > 0$ implies that $e_k > 0$. Because e_k^* is a legitimate cost function, it is strictly increasing in u_k and non-decreasing in p_i, $\forall i = 1, \ldots, \ell$. The former condition implies that taking two different values of u_k, $u_{k1} < u_{k2}$, then $e_k^*(p, u_{k1}, d) < e_k^*(p, u_{k2}, d)$. Because f_k is strictly increasing in e_k^*, we have $f_k[e_k^*(p, u_{k1}, d), p, d] < f_k[e_k^*(p, u_{k2}, d), p, d]$, i.e. the strict monotonicity of f_k with respect to u_k. For two different vectors of prices $p^1 = (p_1^1, \ldots, p_l^1)$ and $p^2 = (p_1^2, \ldots, p_l^2)$, the monotonicity of e_k^* with respect to p_i implies that if $p_i^1 < p_i^2$, $e_k^*(p_i^1, u_k, d) \leq e_k^*(p_i^2, u_k, d)$, with $i = 1, \ldots, \ell$. Then, the monotonicity of f_k with respect to e_k^* implies that $f_k[e_k^*(p_i^1, u_k, d), p, d] \leq f_k[e_k^*(p_i^2, u_k, d), p, d]$, and, finally, from the monotonicity of f_k with respect to p_i, follows that $f_k[e_k^*(p_i^1, u_k, d), p_i^1, d] \leq f_k[e_k^*(p_i^2, u_k, d), p_i^1, d] \leq f_k[e_k^*(p_i^2, u_k, d), p_i^2, d]$, guaranteeing the monotonicity of e_k also with respect to p_i. □

Proposition 3 (Concavity in p) *Let e_k^* be a legitimate cost function and $e_k(p, u_k, d) = f_k[e_k^*(p, u_k, d), p, d]$. Assume f_k concave in (e_k^*, p) and increasing in e_k^*. Then $e_k(p, u_k, d)$ is concave in p.*

Proof For $e_k(p, u_k, d)$ to be concave in p, it must be that $\forall \alpha \in [0, 1]$ and $\forall p^1, p^2 \in \mathbb{R}_{++}^{\ell}$, $e_k(\alpha p^1 + (1-\alpha)p^2, u_k, d) \geq \alpha e_k(p^1, u_k, d) + (1-\alpha)e_k(p^2, u_k, d)$. We start

[7] The individual expenditure function may depend also on distribution factors that are variables affecting the household decision process without influencing either individual preferences or the budget constraint. Distribution factors are helpful in recovering the structure of the collective model and play an important role in empirical applications (Chiappori and Ekeland 2009; Chiappori et al. 2002; Menon and Perali 2012). Here, without loss of generality, distribution factors are not explicitly modeled but are considered elements of the vector of exogenous characteristics d.

[8] Unlike Lewbel (1985), these propositions allow for non-differentiable functions.

from the hypothesis of concavity of e_k^*, $e_k^*(\alpha p^1+(1-\alpha)p^2, u_k, d) \geq \alpha e_k^*(p^1, u_k, d)+(1-\alpha)e_k^*(p^2, u_k, d)$. Then, from the monotonicity of f_k we obtain $f_k[e_k^*(\alpha p^1+(1-\alpha)p^2, u_k, d), \alpha p^1+(1-\alpha)p^2, d] \geq f[\alpha e_k^*(p^1, u_k, d)+(1-\alpha)e_k^*(p^2, u_k, d), \alpha p^1+(1-\alpha)p^2, d]$ and, because f_k is concave in (e_k^*, p), then $f_k[\alpha e_k^*(p^1, u_k, d)+(1-\alpha)e_k^*(p^2, u_k, d), \alpha p^1+(1-\alpha)p^2, d] \geq \alpha f_k[e_k^*(p^1, u_k, d), p^1, d]+(1-\alpha)f_k[e_k^*(p^2, u_k, d), p^2, d]$. □

The above propositions describe the properties that f_k must have to guarantee that the function $e_k\,(p, u_k, d) = f_k[e_k^*\,(p, u_k, d)\,, p, d]$ is also legitimate for any legitimate cost function e_k^*.

We now complete the characterization of the function f_k to include the product between a legitimate cost function and an income scaling function that depends on prices and exogenous factors. The modifying function f_k describing how prices and demographic factors interact with $e_k^*\,(p, u_k, d)$ is as follows

$$\begin{aligned} e_k\,(p, u_k, d) &= f_k[e_k^*\,(p, u_k, d)\,, p, d] \\ &= e_k^*\,(p, u_k, d)\,m(p, d) \quad \text{with } 0 < m(p, d) < \frac{e}{e_k^*}, \end{aligned} \tag{7}$$

where $e_k^*\,(p, u_k, d)$ is a legitimate individual cost function and $e\,(p, u, d)$ is the cost function at the household level.[9] Note that the function f_k corrects an individual income $e_k^*\,(p, u_k, d)$ measured with error according to the index $0 < m(p, d) < \frac{e}{e_k^*}$ that may correct towards the bottom or the top depending on whether $m(p, d) \lesseqgtr 1$.

We now investigate the properties of the scaling function $m(p, d)$ needed to preserve a regular and theoretically plausible $e_k\,(p, u_k, d)$.

Proposition 4 (Homogeneity of degree 0 in p) *If the scaling function $m(p, d)$ is homogeneous of degree zero in p and $e_k^*(p, u_k, d)$ is a legitimate cost function, then $e_k(p, u_k, d) = e_k^*(p, u_k, d)m(p, d)$ is homogeneous of degree 1 in p.*

Proof By the definition of homogeneity $e_k(tp, u_k, d) = e_k^*(tp, u_k, d)m(tp, d) = te_k^*(p, u_k, d)t^\alpha m(p, d) = t^{\alpha+1}e_k^*(p, u_k, d)m(p, d) = t^{\alpha+1}e_k(p, u_k, d)$, thus e_k is homogeneous of degree 1 in p if and only if $m(p, d)$ is homogeneous of degree 0 in p. □

Proposition 5 (Positive and non-decreasing in p_i) *Let $e_k(p, u_k, d) = e_k^*(p, u_k, d)$ $m(p, d)$. If $m(p, d)$ is positive and $e_k^*(p, u_k, d)$ is a legitimate cost function, then $e_k(p, u_k, d)$ is positive and strictly increasing in u_k. If $m(p, d)$ is non-decreasing in p_i, $\forall i = 1, \ldots, \ell$, then $e_k(p, u_k, d)$ is also non-decreasing in p_i, $\forall i = 1, \ldots, \ell$.*

Proof The positivity of $e_k(p, u_k, d)$ follows from the definition. Then, taking two different values of u_k with $u_{k1} < u_{k2}$, we have $e_k^*(p, u_{k1}, d)m(p, d) < e_k^*(p, u_{k2}, d)m(p, d)$, because e_k^* is a cost function and $m(p, d)$ is positive,

9 A similar scaling transformation of a plausible cost function is described in Lewbel (1985), Theorem 8. This specification is adopted in Dunbar et al. (2013), Menon and Perali (2012) and Menon et al. (2012) to estimate the sharing rule.

$e_k(p, u_k, d)$ is increasing in u_k. Further, taking two different vectors of prices $p^1 = (p_1^1, \ldots, p_l^1)$ and $p^2 = (p_1^2, \ldots, p_l^2)$, with $p_i^1 \leq p_i^2$, we have $e_k^*(p_i^1, u_k, d)m(p_i^1, d) \leq e_k^*(p_i^2, u_{k2}, d)m(p_i^2, d)$, because both $e_k^*(p, u_k, d)$ and $m(p, d)$ are non-decreasing in $p_i, \forall i = 1, \ldots, \ell$. □

If e_k is non-decreasing in $p_i, \forall i = 1, \ldots, \ell$, and we suppose that $e_k^*(p, u_k, d)$ and $m(p, d)$ are differentiable with respect to $p_i, \forall i = 1, \ldots, \ell$, then

$$\begin{cases} \frac{\partial e_k(p_k, u_k, d)}{\partial p_1} = \frac{\partial e_k^*}{\partial p_1}(p, u_k, d)m(p, d) + e_k^*(p, u_k, d)\frac{\partial m}{\partial p_1}(p, d) \geq 0, \\ \vdots \\ \frac{\partial e_k(p_k, u_k, d)}{\partial p_\ell} = \frac{\partial e_k^*}{\partial p_\ell}(p, u_k, d)m(p, d) + e_k^*(p, u_k, d)\frac{\partial m}{\partial p_\ell}(p, d) \geq 0. \end{cases} \tag{8}$$

This result follows from the assumptions on $m(p, d)$ and recalling that $e_k^*(p, u_k, d)$ is a legitimate cost function.

Remark 1 Note that if $\frac{\partial m}{\partial p_i} \geq 0, \forall i = 1, \ldots, \ell$, then the ℓ inequalities of system (8) are satisfied, but Proposition 5 gives only a sufficient condition. Therefore, the system is satisfied even if

$$\frac{\partial e_k^*}{\partial p_i}\frac{1}{e_k^*} \geq -\frac{\partial m}{\partial p_i}\frac{1}{m}. \tag{9}$$

Multiplying each side of this equation by p_i, we obtain

$$\frac{\partial e_k^*}{\partial p_i}\frac{p_i}{e_k^*} \geq -\frac{\partial m}{\partial p_i}\frac{p_i}{m}, \tag{10}$$

describing the relationship between the elasticity of the expenditure $e_k^*(p, u_k, d)$ and the elasticity of the scaling function $m(p, d)$ with respect to the ith price.

Remark 2 (Continuity) To maintain property (d) of $e_k(p, u_k, d)$, $m(p, d)$ has to be continuous with respect to p.

The concavity requirement for the expenditure function $e_k(p, u_k, d)$ is as follows. Propositions 6–8 together correspond to Proposition 3.

Proposition 6 (Corollary 5.18 of Avriel et al. 1988) *Let $e_k^*\colon X \subseteq \mathbb{R}^s \to \mathbb{R}$ and $m\colon X \subseteq \mathbb{R}^s \to \mathbb{R}$ be non-negative and concave functions. Then, the function $e_k(x) = e_k^*(x)m(x)$ is semi-strictly quasi-concave on X, with respect to x.*

Observe that we need the same domain for the functions e_k^* and $m(p, d)$. This implies that to verify concavity these functions are supposed to vary only in prices.

Proposition 7 (Proposition 3.30 of Avriel et al. 1988) *If $e_k\colon X \subseteq \mathbb{R}^s \to \mathbb{R}$ is an upper semicontinuous semi-strictly quasi-concave function, defined on a convex set X, then it is also quasi-concave.*

Further, the following result holds (Theorem 21.15 of Simon and Blume 1994).

Proposition 8 *Suppose that $e_k : X \subset \mathbb{R}^s \to \mathbb{R}$ be positive and X is a convex cone. If e_k is homogeneous of degree one and quasi-concave on X, then it is concave on X.*

Our results have important implications both for providing testable hypotheses about the curvature of the sharing rule that are not usually tested in the empirical collective literature and for the measurement of inequality within the household as shown in the next section.

3 The curvature of the income scaling function and intra-household inequality

The inequality in the distribution of household resources depends on the concavity of $m(p, d)$. As the concavity of $m(p, d)$ increases due to a price change also the level of inequality within the household increases. A variation in prices may affect the allocation of resources and the well-being of each family member.

Let us assume that the income scaling function $m : \mathbb{R}^{\ell}_{++} \times \mathbb{R}^n \to \mathbb{R}_{++}$ is twice continuously differentiable with respect to prices. Concavity in p of $m(p, d)$ implies that the Hessian matrix H_m is negative semidefinite $v^T H_m v \leq 0, \forall v \in \mathbb{R}^s$. Recall the definitions of ultramodularity and supermodularity (Marinacci and Montrucchio 2005)

$$\begin{aligned} \frac{\partial^2 m}{\partial p_i \partial p_j} &\geq 0, \quad \text{for any } 1 \leq i \leq j \leq \ell \quad \text{and} \\ \frac{\partial^2 m}{\partial p_i \partial p_j} &\geq 0, \quad \text{for any } 1 \leq i < j \leq \ell. \end{aligned} \tag{11}$$

Ultramodular and supermodular functions are used in social sciences to analyze how one agent's decision affect the incentives of others. In the present case, we may refer to incentives of sharing household resources with other members of the household.

We can then recover an absolute and a relative inequality aversion measure analogous to the risk aversion coefficient

$$\rho_{a_i}(m) = -\frac{\dfrac{\partial^2 m}{\partial p_i^2}}{\dfrac{\partial m}{\partial p_i}} \quad \text{and} \quad \rho_{r_i}(m) = -\frac{\dfrac{\partial^2 m}{\partial p_i \partial p_j}}{\dfrac{\partial m}{\partial p_i}}. \tag{12}$$

The inequality aversion coefficient $\rho(m)$ describes the changes in the income scaling function $m(p, d)$ in response to a change in the price of one good $\rho_{a_i}(m)$, or two goods $\rho_{r_i}(m)$.

One can see that if m is non-decreasing in p_i and $-m$ is also ultramodular with respect to prices, then $\rho_{a_i}(m) \geq 0$ and also $\rho_{r_i}(m) \geq 0$, while if $-m$ is supermodular, then $\rho_{r_i}(m) \geq 0$. If $-m$ is neither ultra nor supermodular, then the kth individual price change generates a relatively less unequal distribution of the resources.

In general, the product of two (quasi-)concave functions is not (quasi-)concave. However, as stated in Prékopa et al. (2011), the product of two uniformly quasi-concave functions is quasi-concave. Two functions $f_{1,}$ f_2, are said uniformly quasi-concave if and only if

$$\min\{f_i(x), f_i(y)\} = f_i(x) \quad \text{or} \quad f_i(y), \quad \forall i = 1, 2, \ \forall x, y \in R^{\ell}. \tag{13}$$

Therefore, if m and e_k^* are uniformly quasi-concave, then e_k is also quasi-concave, and moreover it is concave if Proposition 8 holds. Note that if the scaling function $m(p, d)$ is not concave in p, $m(p, d)$ can be transformed in a concave function. To this end, we may use the class of concave transformable (or transconcave) functions (Avriel et al. 1988) by adopting a one to one transformation of their domain. The function $m\colon A \subseteq \mathbb{R}^{\ell}_{++} \times \mathbb{R}^n \to C \subseteq \mathbb{R}_{++}$ is said to be G-concave if there exists a continuous real-valued increasing function G defined on C such that $G(m(p, d))$ is concave over A. Alternatively, letting G^{-1} denote the inverse of G,

$$m(\alpha p^1 + (1-\alpha)p^2, d) \geq G^{-1}[\alpha G(m(p^1, d)) + (1-\alpha)G(m(p^2, d))] \tag{14}$$

holds $\forall p^1, p^2 \in A$ and $\alpha \in [0, 1]$.

To be concavifiable $m(p, d)$ must be at least quasi-concave. We observe also that in the twice continuously differentiable case, necessary and sufficient conditions lie between the properties of pseudoconcavity and strong pseudoconcavity.[10]

Additional assumptions on G may be required to obtain a scaling function $G\,(m(p, d))$ satisfying Propositions 4 and 5. If m is at least quasi-concave, and G is any positive real-valued increasing function, then the composition $G\,(m(p, d))$ is positive, homogeneous of degree 0, non-decreasing in prices and concave.

4 Example

The following example illustrates how to test the regularity conditions of the individual expenditure function (sharing rule) derived in Propositions 4 (homogeneity), 5 (positive and non-decreasing in prices), 6 and 7 (concavity) of Sect. 2 and the aversion coefficients introduced in Sect. 3.[11] The example uses the estimates of the individual expenditure e_k obtained in Menon and Perali (2012), where a general collective consumption specification as in Eq. (7) is estimated. Using the information of clothing expenditures for adults and children, in Menon and Perali (2012) observed individual income e_k^* is scaled by the modifying function $m(p, d)$ as in (7)

$$e_k = e_k^* \, m(p, d) \quad \text{with } 0 < m(p, d) < \tfrac{e}{e_k^*}. \tag{15}$$

[10] To establish conditions under which a twice continuously differentiable function is transformable concave for some transformation see Avriel et al. (1988). For results referring to functions not necessarily twice differentiable see Crouzeix (1977).

[11] We are thankful to an anonymous referee for suggesting to explain our results by means of an empirical example.

In Menon and Perali (2012) the function that scales individual income is specified as a Cobb–Douglas

$$m(p,d) = \Pi_{i=1}^{\ell} p_i^{\phi_{p_i}} \Pi_{j=1}^{n} d_j^{\phi_{s_j}}, \tag{16}$$

where $j = 1, \ldots, n$ indexes the exogenous factors $d = \{\text{age ratio, education ratio}\}$ with associated parameters ϕ_{s_i} and $i = 1, \ldots, \ell$ indexes the prices $p = \{\text{adult clothing, children clothing}\}$ with associated parameters ϕ_{p_i}. The estimated parameters and associated variable means, the value of the resource share, income scaling function, the sharing rule in level and proportion are reported in Table 1. The average resource share $\eta_k = 0.67$ is obtained as the ratio between observed individual expenditure e_k^* and total household expenditure e. The predicted value of the income scaling function $m\,(p,d)$ is 0.726 and it satisfies the bounds of Eq. (7), $0 < m < 1.499$. The function $m\,(p,d)$ captures how assignable goods are allocated within each household. In our case, the sharing rule in levels $e_k(p, u_k, d) = e_k^* m$ is scaled down from about 1238 to about 899 Euros. On average the observed resource share η_k is allocated about 2/3 to the parent and 1/3 to the child component, but the share $e_k/e = 0.484$ shows that parents actually keep for themselves almost 1/2 rather than 2/3 of the household resources. As described in Propositions 4–7, if $m\,(p,d)$ is positive, non-decreasing in prices, homogeneous of degree 0 and concave, then e_k is a legitimate cost function. Table 2 shows the tests of these regularity properties. Given the Cobb–Douglas form of $m(p,d)$, the function is positive in its domain. Because the gradient with respect to prices of the modifying function $m(p,d)$

$$\nabla m_p = \begin{bmatrix} m & \dfrac{\phi_{p_1}}{p_1} \\ m & \dfrac{\phi_{p_2}}{p_2} \end{bmatrix} \tag{17}$$

is positive for each element, then the income scaling function is non-decreasing in prices. The income scaling function estimated in Menon and Perali (2012) is not homogeneous of degree zero because both price parameters are positive and significantly different from zero. Imposing the homogeneity property ($\phi_{p_1} + \phi_{p_2} = 0$), the function cannot be non-decreasing in each price. However, the expenditure function is non-decreasing if the inequality in Eq. (10) is satisfied, that is, if $\phi_{p_i} \geq -\dfrac{\partial e_k^*}{\partial p_i}\dfrac{p_i}{e_k^*}$, $\forall i$.

Further, $m(p,d)$ is concave if and only if $v^T H_m v \leq 0$, where the Hessian matrix H_m is as follows

$$H_m = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix} = \begin{bmatrix} m\,\dfrac{\phi_{p_1}(\phi_{p_1}-1)}{p_1^2} & m\,\dfrac{\phi_{p_1}\phi_{p_2}}{p_1 p_2} \\ m\,\dfrac{\phi_{p_1}\phi_{p_2}}{p_2 p_1} & m\,\dfrac{\phi_{p_2}(\phi_{p_2}-1)}{p_2^2} \end{bmatrix}. \tag{18}$$

As shown in Table 2, the Hessian matrix is negative definite. Note that if m is a Cobb–Douglas function, then m is concave when $\phi_{p_i} \geq 0$, $\forall i$ and the degree of homogeneity is not greater than 1 as in our example. Also note that m cannot be both homogeneous

Table 1 Estimated parameters, income scaling function and sharing rule

	Estimated parameters and variables	Parameter	Mean
Clothing price—adults	ϕ_{p_a}	0.039	0.436
		0.015	
Clothing price—children	ϕ_{p_c}	0.159	0.181
		0.030	
Father/mother age ratio	ϕ_{s_1}	−0.061	1.613
		0.053	
Father/mother education ratio	ϕ_{s_2}	0.027	1.629
		0.021	
Household expenditure (Euro)	e		1856.780
Individual (adults) expenditure (Euro)	e_k^*		1237.860
Resource share of adults	$\eta_k = \frac{e_k^*}{e}$		0.670
	Predicted functions	**Function value**	
Income scaling function	$m(p,d) = \Pi_{i=1}^{n} d_i^{\phi_{s_i}} \Pi_{j=1}^{\ell} p_j^{\phi_{p_j}}$	0.726	
Sharing rule of adults in levels (Euro)	$e_k(p,d) = e_k^* m(p,d)$	898.684	
Sharing rule of adults in proportion	e_k/e	0.484	

Variable means and estimated parameters are from Menon and Perali (2012) as reported in Tables 1 and 2, respectively, noting that here variable means are in anti-logs. Standard errors are in italics

Table 2 Test of the regularity conditions of the sharing rule and inequality aversion

	Test	Outcome
Homogeneity of degree zero	$\phi_{pa} = -\phi_{pc}$	Not homogeneous in p
Non decreasing in p	$\nabla m_p > 0$	Non decreasing in p
Concavity in p	$H_m = \begin{bmatrix} -0.1434 & 0.057 \\ 0.057 & -2.9 \end{bmatrix}$	H_m is negative definite
Aversion to inequality	$\rho(m) = \begin{bmatrix} 2.21 & -0.877 \\ -0.089 & 4.64 \end{bmatrix}$	$-m$ neither ultra nor supermodular

Tests and outcomes are based on the empirical analysis by Menon and Perali (2012)

of degree 0 and concave. Definitions in (13) and (14) guarantee the quasi-concavity of the expenditure function even if m is not concave.

The income scaling function estimated in Menon and Perali (2012) is not homogeneous of degree 0 in prices. This implies that the individual expenditure function is not homogeneous of degree 1. However, the income scaling function passes all other regularity tests requiring m to be positive, non-decreasing in prices and concave. In empirical applications homogeneity of degree 0 of the income scaling function would have to be imposed to maintain the homogeneity of degree 1 of the individual expenditure function and to guarantee that the function is at least quasi-concave by Propositions 6 and 7.

From expressions in Eq. (12), the absolute and relative inequality aversion coefficients are

$$\rho(m) = \begin{bmatrix} \rho_{a_1}(m) & \rho_{r_1}(m) \\ \rho_{r_2}(m) & \rho_{a_2}(m) \end{bmatrix} = \begin{bmatrix} \frac{1-\phi_{p_1}}{p_1} & -\frac{\phi_{p_1}}{p_1} \\ -\frac{\phi_{p_2}}{p_2} & \frac{1-\phi_{p_2}}{p_2} \end{bmatrix} \tag{19}$$

for $\phi_{p_i} \neq 1$ and $\phi_{p_i} \neq 0,\ \forall i$. Because $-m$ is neither ultra nor supermodular, then both prices of assignable goods generate a less unequal distribution of household resources. The Hessian and inequality aversion matrices take the following values

$$H_m = \begin{bmatrix} -0.143 & 0.057 \\ 0.057 & -2.9 \end{bmatrix} \quad \text{and} \quad \rho(m) = \begin{bmatrix} 2.21 & -0.877 \\ -0.089 & 4.64 \end{bmatrix}$$

showing that m is more concave in the dimension that refers to the distribution of resources towards children than towards adults because both the own derivative, in absolute terms, and the absolute inequality aversion coefficient of the child are larger than those of the adults, that is $|m_{2,2}| > |m_{1,1}|$ and $\rho_{a_2}(m) > \rho_{a_1}(m)$. If there are disposable resources in the family, when the parents hold more and more resources for themselves, they also distribute more to their children because $m_{1,2}$ is a positive cross derivative. Because $\rho_{r_1}(m)$ is in absolute terms greater than $\rho_{r_2}(m)$, the percentage change in the marginal redistribution of resources towards the adults, when the price of the children change, is greater than the percentage change in the redistribution of resources towards the children, when the price of the adults change.

5 Conclusions

This study contributes to the literature of collective household models by completing Lewbel's (1985) seminal work with the study of the properties of collective individual expenditure functions necessary to be theoretically plausible after the scaling modification of income. We show that the income scaling function must be positive, non-decreasing, homogenous of degree zero and concave in prices.

The characterization of the collective class of individual expenditure functions presented here allows testing the regularity properties of the sharing rule as illustrated in the example. We also examine the curvature properties of the income scaling function to introduce new measures of inequality aversion describing how differently individual price changes affect the distribution of household resources.

As a final recommendation, applications of collective models should verify the regularity properties on a standard basis so that more evidence can be gathered about the behavioral structure of the sharing rule and welfare analysis can be robustly performed.

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