Time-optimal control problem in the space of probability measures

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Abstract. We are going to define a time optimal control problem in the space of probability measures. Our aim is to model situations in which the initial position of a particle is not exactly known, even if the evolution is assumed to be deterministic. We will study some natural generalization of objects commonly used in control theory, proving some interesting properties. In particular we will focus on a comparison result between the classical minimum time function and its natural generalization to the probability measures setting.

Keywords: optimal transport, differential inclusions, time optimal control

1 Introduction

Usual finite-dimensional time optimal control problem can be stated as follows: given a set-valued map $F:\mathbb{R}^d \rightrightarrows \mathbb{R}^d$ satisfying some structural assumptions, and a nonempty closed subset $S\subseteq\mathbb{R}^d$, we consider the solutions of differential inclusion starting from a given point $x_0\in\mathbb{R}^d$, namely

$$\begin{cases}\dot{x}(t)\in F(x(t)), & t>0,\\ x(0)=x_0\in\mathbb{R}^d.\end{cases} \tag{1}$$

Then we can define the *minimum time function* $T:\mathbb{R}^d\to[0,+\infty]$ by setting for every $x_0\in\mathbb{R}^d$

$$T(x_0):=\inf\{T>0:\ \exists x(\cdot) \text{ solving (1) such that } x(T)\in S\}. \tag{2}$$

The study of the minimum time function and of its properties is a central topic in control theory, and the related literature is huge.

* **Acknowledgements:** The authors have been supported by INdAM - GNAMPA Project 2015: *Set-valued Analysis and Optimal Transportation Theory Methods in Deterministic and Stochastics Models of Financial Markets with Transaction Costs.*

Our starting remark is that in many reald-world applications the *starting position* x_0 of the moving particle is known only up to some uncertainties: for example it can be obtained only by an averaging of many measurement processes. It is worth of noticing that this situation can happen even if we assume a pure deterministic evolution of the system.

A natural choice to model the (possible) imperfect knowledge we possess about the particle's starting position is to consider it as a probability measure $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$. The case in which μ_0 is a Dirac delta concentrated at a point x_0 corresponds of course to the classical case in which perfect knowledge of the starting position is assumed.

This fact leads us to formulate directly our problem as regarding time-dependent measures, i.e. curves in $\mathscr{P}(\mathbb{R}^d)$. In this sense, the evolution of the starting *measure* μ_0 gives us a *macroscopic* point of view on the system, while the single (classical) trajectory corresponds to a *microscopic* point of view.

A natural requirement for the evolving measure $t \mapsto \mu_t$ is that at every time $t \in [0,T]$ we must have $\displaystyle\int_{\mathbb{R}^d} d\mu_t = 1$, since the probability to find *somewhere* the classical particle must be always equal to 1. This lead us to consider the evolution of the measure ruled by the following *continuity equation*, to be understood in the distributional sense

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0, \quad t > 0, \\ \mu_{|t=0} = \mu_0, \end{cases} \tag{3}$$

where $v_t(\cdot)$ is a time-depending Borel vector field belonging to $L^1_{\mu_t}(\mathbb{R}^d;\mathbb{R}^d)$ for a.e. $t \in [0,T]$.

It is well known that if $v_t(\cdot)$ is Lipschitz continuous, we can consider the *characteristics system*

$$\begin{cases} \dfrac{d}{dt} T_t(x) = v_t(T_t(x)), \\ T_0(x) = x, \end{cases} \tag{4}$$

and the *unique* solution of (3) can be expressed by the *push-forward* of the initial measure μ_0 by the time-depending vector field $T_t(\cdot)$ solving (4), i.e., $\mu_t = T_t \sharp \mu_0$, where the push-forward $X\sharp\mu$ of a measure μ by a Borel vector field X is defined as

$$\int_{\mathbb{R}^d} \varphi(x)\, d(X\sharp\mu) = \int_{\mathbb{R}^d} \varphi \circ X(x)\, d\mu, \quad \forall \varphi : \mathbb{R}^d \to \mathbb{R} \text{ bounded Borel function.}$$

However (3) has been proven to be well-posed even in situations in which the vector field v_t lacks of such a regularity to have uniqueness of the solutions of (4). Heuristically, this is due to the fact that the evolution of the measure is not affected by singularities in a μ_t-negligible set. Following [2], we recall that the integrability assumption $\|v_t\|_{L^p_\mu(\mathbb{R}^d)} \in L^1([0,T])$ yields the existence of a solution of (3) in the sense of a continuous curve $t \mapsto \mu_t$ in the space of probability measures endowed with the weak* topology induced by the duality with continuous and bounded functions $\varphi \in C^0_b(\mathbb{R}^d)$ (i.e., a *narrowly continuous curve* in the space of probability measures).

In many cases, the solutions of (3) can be constructed as *superpositions* of characteristics in the following sense: every probability measure $\boldsymbol{\eta}$ on the product space $\mathbb{R}^d \times \Gamma_T$, where Γ_T is the space of continuous curves in $\mathbb{R}^d$ defined on $[0,T]$, concentrated on the integral solutios of (4) (without assuming any uniqueness of the latter), can be used to define a solution of (3). Conversely, also every solution of (3) admits such a representation. We refer to [1] and [2] for this kinds of results (see also Theorem 5.8 in [3] and Theorem 8.2.1 in [2]).

In a control-theoretic framework, in order to find a proper generalization of (1), it seems a natural choice to couple the dynamic (3) with the nonholonomic constraint $v_t(x) \in F(x)$ for a.e. $t \in [0,T]$ and μ_t-a.e. $x \in \mathbb{R}^d$, i.e., to ask that the driving vector field for the time-dependent measure μ_t is a suitable Borel selection of the set-valued map F. This is motivated also from the fact that in this case for *smooth* vector field v_t, the solutions of the characteristics system (4) turns out to be admissible trajectories of (1).

The link between the solutions of (3) and (4) has been extensively studied in the last years, we refer to [1] for a detailed presentation of the related issues. In particular, there are provided sufficient conditions in order to grant existence and uniqueness in special classes of measures of the solutions of (3) also in cases where the corresponding (4) fails to provide uniqueness of the solutions. Moreover, also the strict relationships between (3) and optimal transport theory has been already studied by many authors, and we refer to [2], [3], and [5] for further details.

If we restrict our attention to the set $\mathscr{P}_p(\mathbb{R}^d)$ of Borel probability measures with finite *p-moment*, i.e. measures μ satisfying $|\cdot| \in L^p_\mu(\mathbb{R}^d)$, we can consider also the metric structure induced by the *p-Wasserstein distance* $W_p(\cdot,\cdot)$ between measures. We refer the reader to [2] for all the details on Wasserstein distance.

In order to state our time-optimal control problem, we need also a convenient generalization of the target set S of the classical case. To introduce it, we consider the following heuristic argument (closely related to some interpretation of quantum mechanics), which follows the probabilistic motivation which leaded us to consider the controlled continuity equation as a good replacement for the differential inclusion.

Suppose to have an observer making some measurements on the system. The only quantity which we can consider is an average of the results of the measurements. From a mathematical point of view, we can model the measurement as a continuous map $\phi \in C^0(\mathbb{R}^d)$, thus the average result of the measurement of a system whose state is described by $\mu \in \mathscr{P}(\mathbb{R}^d)$ is given by the expected value of ϕ, namely $\displaystyle\int_{\mathbb{R}^d} \phi(x)\, d\mu(x)$.

A natural choice for the target set is to fix a threshold for each measurement and try to steer the system into states where the results of such measurements is below that threshold. Without loss of generality, we can fix the threshold to be 0 for all the measurements in which we are interested, thus the generalized target can be defined as follows: fix a subset $\Phi \subseteq C^0(\mathbb{R}^d;\mathbb{R})$ (which corresponds to the measurements in which we are interested) and define the generalized target to be

$$\tilde{S}^{\Phi} := \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \leq 0 \text{ for all } \phi \in \Phi \right\}.$$

In general some additional requirements on Φ are needed in order to have a good definition of the integral. We will deal mainly with the case in which for all $\phi \in \Phi$ there exist constants $A, B > 0$ and $p \geq 1$ such that $\phi(x) \geq A|x|^p - B$. An important example of this situation is given by fixing $S \subseteq \mathbb{R}^d$ and considering $\Phi = \{d_S(\cdot)\}$, i.e., we are going to measure the average distance from S. With this definition we have

$$\tilde{S}^{\{d_S\}} := \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} d_S(x)\, d\mu(x) \leq 0 \right\} = \{\mu \in \mathscr{P}(\mathbb{R}^d) : \operatorname{supp} \mu \subseteq S\}.$$

Another interpretation of our framework in this case can be given in terms of *pedestrian dynamics*: suppose that initially we have a crowd of people represented by a (normalized) probability measure μ_0 and that we can identify a *safety zone* $S \subseteq \mathbb{R}^d$, while $F(\cdot)$ represents some (possible) nonholonomic constraints to the motion. Then if our aim in case of danger is to steer all the crowd to the safety zone in the minimum time possible, we can choose $\Phi = \{d_S(\cdot)\}$. In a more realistic situation, it may be not possible to steer *all* the crowd to S. If we fix $\alpha \in [0,1]$ and choose $\Phi = \{d_S(\cdot) - \alpha\}$, we are still satisfied for example if the ratio between the number of people in the safe zone and all the people is above $1 - \alpha$, or if we can take the people sufficiently near to the safe zone.

Having defined the set of admissible trajectories and the target in the space of probability measures, the definition of generalized minimum time function at a probability measure μ_0 is the straigthforwardly generalization of the classical one, i.e., the infimum of all the times T for which there exists an admissible trajectory defined on $[0,T]$ and satisfying $\mu_T \in \tilde{S}^{\Phi}$.

The paper is structured as follows: in Section 2 we introduce precise definition of the generalized object we are going to study, together with some of their properties. In Section 3 we are going to prove the main results of the paper, finally in Section 4 we will give some insight of the current work.

2 Generalized objects and their properties

Definition 1 (Standing Assumption). *We will say that a set-valued function $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ satisfies the assumption (F_j), $j = 0, 1$ if the following hold true*

(F_0) $F(x) \neq \emptyset$ *is compact and convex for every* $x \in \mathbb{R}^d$, *moreover* $F(\cdot)$ *is continuous with respect to the Hausdorff metric, i.e. given* $x \in X$, *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|y - x| \leq \delta$ *implies* $F(y) \subseteq F(x) + B(0, \varepsilon)$ *and* $F(x) \subseteq F(y) + B(0, \varepsilon)$.

(F_1) $F(\cdot)$ *has linear growth, i.e. there exist nonnegative constants* L_1 *and* L_2 *such that* $F(x) \subseteq \overline{B(0, L_1|x| + L_2)}$ *for every* $x \in \mathbb{R}^d$.

Definition 2 (Generalized targets). *Let* $p \geq 1$, $\Phi \subseteq C^0(\mathbb{R}^d, \mathbb{R})$ *such that the following property holds*

(T_E) *there exists* $x_0 \in \mathbb{R}^d$ *with* $\phi(x_0) \leq 0$ *for all* $\phi \in \Phi$.

We define the generalized targets $\tilde{S}^\Phi$ *and* $\tilde{S}^\Phi_p$ *as follows*

$$\tilde{S}^\Phi := \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \leq 0 \text{ for all } \phi \in \Phi \right\},\ \tilde{S}^\Phi_p := \quad \tilde{S}^\Phi \cap \mathscr{P}_p(\mathbb{R}^d).$$

We define also the generalized distance *from* $\tilde{S}^\Phi_p$ *as* $\tilde{d}_{\tilde{S}^\Phi_p}(\cdot) := \inf_{\mu \in \tilde{S}^\Phi_p} W_p(\cdot, \mu)$.

For further use, we will say that Φ *satisfies property* (T_p) *with* $p \geq 0$ *if the following holds true*

(T_p) *for all* $\phi \in \Phi$ *there exist* $A_\phi, C_\phi > 0$ *such that* $\phi(x) \geq A_\phi |x|^p - C_\phi$.

The following proposition establishes some straightforward properties of the generalized targets. Its proof is immediate from the definition of generalized target.

Proposition 1 (Properties of the generalized targets). *Let* $p \geq 0$ *and* $\Phi \subseteq C^0(\mathbb{R}^d, \mathbb{R})$ *be such that* (T_E) *and* (T_0) *holds. Then* $\tilde{S}^\Phi$ *and* $\tilde{S}^\Phi_p$ *are convex, moreover* $\tilde{S}^\Phi$ *is* w^**-closed in* $\mathscr{P}(\mathbb{R}^d)$, *while* $\tilde{S}^\Phi_p$ *is closed in* $\mathscr{P}_p(\mathbb{R}^d)$ *endowed with the* p*-Wasserstein metric* $W_p(\cdot, \cdot)$. *If moreover* (T_p) *holds for some* $p \geq 1$, *then* $\tilde{S}^\Phi = \tilde{S}^\Phi_p$ *is compact in the* w^**-topology and in the* W_p*-topology.*

Definition 3 (Admissible curves). *Let* $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ *be a set-valued function,* $I = [a, b]$ *a compact interval of* $\mathbb{R}$, $\alpha, \beta \in \mathscr{P}(\mathbb{R}^d)$. *We say that a Borel family of probability measures* $\boldsymbol{\mu} = \{\mu_t\}_{t \in I}$ *is an* admissible trajectory (curve) defined in I for the system joining α and β, *if there exists a family of Borel vector fields* $v = \{v_t(\cdot)\}_{t \in I}$ *such that*

1. $\boldsymbol{\mu}$ *is a narrowly continuous solution in the distributional sense of the continuity equation* $\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$, *with* $\mu_{|t=a} = \alpha$ *and* $\mu_{|t=b} = \beta$.

2. $J_F(\boldsymbol{\mu}, v) < +\infty$, *where* $J_F(\cdot)$ *is defined as*

$$J_F(\boldsymbol{\mu}, v) := \begin{cases} \displaystyle\int_I \int_{\mathbb{R}^d} \left(1 + I_{F(x)}(v_t(x))\right) d\mu_t(x)\, dt, & \text{if } \|v_t\|_{L^1_{\mu_t}} \in L^1([0, T]), \\ +\infty, & \text{otherwise}, \end{cases} \tag{5}$$

where $I_{F(x)}$ *is the indicator function of the set* $F(x)$, *i.e.,* $I_{F(x)}(\xi) = 0$ *for all* $\xi \in F(x)$ *and* $I_{F(x)}(\xi) = +\infty$ *for all* $\xi \notin F(x)$.

In this case, we will also shortly say that $\boldsymbol{\mu}$ *is* driven *by* v.

When $J_F(\cdot)$ is finite, this value expresses the time needed by the system to steer α to β along the trajectory $\boldsymbol{\mu}$ with family of velocity vector fields v.

Definition 4 (Generalized minimum time). *Let* $\Phi \in C^0(\mathbb{R}^d;\mathbb{R})$ *and* $\tilde{S}^\Phi$, $\tilde{S}^\Phi_p$ $(p \geq 1)$ *be the corresponding generalized targets defined in Definition 2. In analogy with the classical case, we define the* generalized minimum time function $\tilde{T}^\Phi : \mathscr{P}(\mathbb{R}^d) \to [0,+\infty]$ *by setting*

$$\tilde{T}^\Phi(\mu_0) := \inf \{ J_F(\boldsymbol{\mu}, v) : \boldsymbol{\mu} \textit{ is an admissible curve in } [0,T] , \\ \textit{driven by } v, \textit{ with } \mu_{|t=0} = \mu_0,\, \mu_{|t=T} \in \tilde{S}^\Phi \Big\}, \tag{6}$$

where, by convention, $\inf \emptyset = +\infty$.

Given $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, *an admissible curve* $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tilde{T}^\Phi(\mu_0)]} \subseteq \mathscr{P}(\mathbb{R}^d)$, *driven by a time depending Borel vector-field* $v = \{v_t\}_{t\in[0,\tilde{T}^\Phi(\mu_0)]}$ *and satisfying* $\mu_{|t=0} = \mu_0$ *and* $\mu_{|t=\tilde{T}^\Phi(\mu_0)} \in \tilde{S}^\Phi$ *is* optimal for μ_0 *if* $\tilde{T}^\Phi(\mu_0) = J_F(\boldsymbol{\mu}, v)$.

Given $p \geq 1$, *we define also a generalized minimum time function* $\tilde{T}^\Phi_p$: $\mathscr{P}_p(\mathbb{R}^d) \to [0,+\infty]$ *by replacing in the above definitions* $\tilde{S}^\Phi$ *by* $\tilde{S}^\Phi_p$ *and* $\mathscr{P}(\mathbb{R}^d)$ *by* $\mathscr{P}_p(\mathbb{R}^d)$. *Since* $\tilde{S}_p \subseteq \tilde{S}$, *it is clear that* $\tilde{T}^\Phi(\mu_0) \leq \tilde{T}^\Phi_p(\mu_0)$.

3 Main results

Theorem 1 (First comparison between $\tilde{T}^\Phi$ and T). *Consider the generalized minimum time problem as in Definition 4 assuming* (F_0), (F_1), *and suppose that there exists* $S \subseteq \mathbb{R}^d$ *such that* $\tilde{S}^\Phi = \tilde{S}^{\{d_S\}}$. *Then for all* $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$ *we have* $\tilde{T}^\Phi(\mu_0) \geq \|T\|_{L^\infty_{\mu_0}}$, *where* $T : \mathbb{R}^d \to [0,+\infty]$ *is the classical minimum time function for the system* $\dot{x}(t) \in F(x(t))$ *with target* S.

Proof. For sake of clarity, in this proof we will simply write $\tilde{T}$ and $\tilde{S}$, thus omitting Φ.

If $\tilde{T}(\mu_0) = +\infty$ there is nothing to prove, so assume $\tilde{T}(\mu_0) < +\infty$. Fix $\varepsilon > 0$ and let $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \subseteq \mathscr{P}(\mathbb{R}^d)$ be an admissible curve starting from μ_0, driven by $v = \{v_t\}_{t\in[0,T]}$ such that $T = J_F(\boldsymbol{\mu}, v) < \tilde{T}(\mu_0) + \varepsilon$ and $\mu_{|t=T} \in \tilde{S}$. In particular, we have that $v_t(x) \in F(x)$ for μ_t-a.e. $x \in \mathbb{R}^d$ and a.e. $t \in [0,T]$, hence $|v_t(x)| \leq (L_1 + L_2)(1 + |x|)$ for μ_t-a.e $x \in \mathbb{R}^d$. Accordingly,

$$\int_0^T \int_{\mathbb{R}^d} \frac{|v_t(x)|}{1+|x|}\, d\mu_t\, dt \leq T(L_1 + L_2) < +\infty.$$

By the superposition principle (Theorem 5.8 in [3] and Theorem 8.2.1 in [2]), we have that there exists a probability measure $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ satisfying

1. $\boldsymbol{\eta}$ is concentrated on the pairs $(x,\gamma) \in \mathbb{R}^d \times \Gamma_T$ such that γ is absolutely continuous and
$$\gamma(t) = x + \int_0^t v_t(\gamma(s))\, ds$$

2. for all $t \in [0,T]$ and all $\varphi \in C_b^0(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x)d\mu_t(x) = \iint_{\mathbb{R}^d\times\Gamma_T} \varphi(\gamma(t))\,d\boldsymbol{\eta}(x,\gamma).$$

Evaluating the above formula at $t=0$, we have that if $x \notin \operatorname{supp}\mu_0$ or $\gamma(0)\neq x$, then $(x,\gamma)\notin \operatorname{supp}\boldsymbol{\eta}$.

Let $\{\psi_n\}_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^d;[0,1])$ with $\psi_n(x)=0$ if $x\notin B(0,n+1)$ and $\psi_n(x)=1$ if $x\in \overline{B(0,n)}$. By Monotone Convergent Theorem, since $\{\psi_n(\cdot)d_S(\cdot)\}_{n\in\mathbb{N}} \subseteq C_b^0(\mathbb{R}^d)$ is an increasing sequence of nonnegative functions pointwise convergent to $d_S(\cdot)$, we have for every $t\in[0,T]$

$$\begin{aligned}\iint_{\mathbb{R}^d\times\Gamma_T} d_S(\gamma(t))\,d\boldsymbol{\eta}(x,\gamma) &= \lim_{n\to\infty}\iint_{\mathbb{R}^d\times\Gamma_T} \psi_n(\gamma(t))d_S(\gamma(t))\,d\boldsymbol{\eta}(x,\gamma)\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} \psi_n(x)d_S(x)\,d\mu_t(x)\end{aligned}$$

By taking $t=T$, we have that the last term vanishes because $\mu_{|t=T}\in\tilde{S}$ and so $\operatorname{supp}\mu_{|t=T}\subseteq S$, therefore

$$\iint_{\mathbb{R}^d\times\Gamma_T} d_S(\gamma(T))\,d\boldsymbol{\eta}(x,\gamma)=0.$$

In particular, we necessarily have that $\gamma(T)\in S$ and $\gamma(0)=x$ for $\boldsymbol{\eta}$-a.e. $(x,\gamma)\in\mathscr{P}(\mathbb{R}^d\times\Gamma_T)$, whence $T\geq T(x)$ for μ_0-a.e. $x\in\mathbb{R}^d$, since $T(x)$ is the infimum of the times needed to steer x to S along trajectories of the system. Thus, $\tilde{T}(\mu_0)+\varepsilon\geq T(x)$ for μ_0-a.e. $x\in\mathbb{R}^d$ and, by letting $\varepsilon\to0$, we conclude that $\tilde{T}(\mu_0)\geq \|T\|_{L^\infty_{\mu_0}}$. □

It can be shown that the inequality appearing in Theorem 1 may be strict without further assumptions, however the following result state a relevant case in which equality holds, justifying also the name of *generalized minimum time problem* we gave.

Lemma 1 (Second comparison result). *Assume the same hypotheses and notation as in Theorem 1. Then, for every $x_0\in\mathbb{R}^d$ we have $\tilde{T}^\Phi(\delta_{x_0})=\tilde{T}_p^\Phi(\delta_{x_0})=T(x_0)$ for all $p\geq 1$.*

Proof. Let us use the same notation as before, thus omitting Φ.

By Theorem 1 we have $\tilde{T}(\delta_{x_0})\geq\|T\|_{L^\infty_{\delta_{x_0}}}=T(x_0)$. Conversely, let $\gamma_\varepsilon(\cdot)$ be a solution of $\dot{x}(t)\in F(x(t))$ such that $\gamma_\varepsilon(0)=x_0$ and $\gamma_\varepsilon(T(x_0)+\varepsilon)\in S$. Set $\mu_t^\varepsilon=\gamma_\varepsilon(t)\sharp\delta_{x_0}$ and $\boldsymbol{\mu}^\varepsilon=\{\mu_t^\varepsilon\}_{t\in[0,T(x_0)+\varepsilon]}$. By Theorem 8.3.1 in [2], we have that there exists a Borel vector field $v_t^\varepsilon:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ such that $\partial_t\mu_t^\varepsilon+\operatorname{div}(v_t^\varepsilon\mu_t^\varepsilon)=0$. Moreover, by construction we have that $\dot{\gamma}_\varepsilon(t)=v_t^\varepsilon(\gamma_\varepsilon(t))\in F(\gamma_\varepsilon(t))$, thus $v_t^\varepsilon(x)\in F(x)$, for μ_t^ε-a.e. $x\in\mathbb{R}^d$ and a.e. $t\in[0,T]$. We conclude that μ_t^ε is an admissible curve steering δ_{x_0} to $\tilde{S}$ in time $T(x_0)+\varepsilon$, hence $\tilde{T}(\delta_{x_0})\leq T(x_0)+\varepsilon$. By letting $\varepsilon\to0^+$, we obtain the desired equality. □

4 Conclusion

The study of generalized minimum time function in the space of probability measures is still largely in progress. In the forthcoming paper [4], more general cases will be treated, together with a dynamic programming principle and a result of existence of optimal trajectories in the space of probability measures.

We plan also to extend the definition of minimum time by possibly adding some terms in the functional penalizing the concentration of the mass, in order to treat more realistic problems coming from pedestrian dynamics.

Finally, the characterization of the generalized minimum time as solution of a suitable infinite-dimensional Hamilton-Jacobi-Bellmann equation seems to be quite hard, as well as to state a result comparable to Pontryagin Maximum Principle for this kind of problems. In [6] a similar problem was addressed, trying to characterize the infimum in the space of curves on $\mathscr{P}_2(\mathbb{R}^d)$ of an action-like functional (without control) starting from a given measure. In [6] was obtained only that this value is a viscosity subsolution of a suitable HJB equation, while the supersolution part was proved only in dimension 1 using special representation of the optimal transport map on $\mathbb{R}$.

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