



**Dipartimento di Informatica
Università degli Studi di Verona**

**Rapporto di ricerca
Research report**

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A weakening residuation in adjoining closures

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Abstract

In this paper we weaken the conditions for the existence of adjoint closure operators, going beyond the standard requirement of additivity/co-additivity. We move from the notion of join-uniform (lower) closure operators, introduced in computer science in order to model perfect lossless compression in transformations acting on complete lattices. Starting from Janowitz's characterisation of residuated closure operators, we show that join-uniformity perfectly weakens additivity in the construction of residuated closures, and this is indeed the weakest property for this to hold. We conclude by characterising the set of all join-uniform lower closure operators as fix-points of a function defined on the set of all lower closures of a complete lattice.

Keywords: Residuated closures, uniformity, adjoint functions

1 Introduction

In this paper we weaken the notion of residuated closure operator. We consider closure operators on complete lattices [15] and Janowitz's notion of residuated closures as introduced in [1, 11]. Residuated closures are defined on lattices and they become complete join-morphisms on complete lattices. We aim at studying what is preserved of the adjoint relation when residuation, namely additivity, fails and it is replaced by the weaker notion of join-uniformity.

Uniformity has been introduced in [10] in the context of static program analysis for providing a lattice-theoretic characterisation of abstract domain compressors in abstract interpretation [3, 5]. In abstract interpretation a domain is uniquely determined by an upper closure operator. This models precisely the loss of precision in analysing undecidable program properties, such as the extensional semantic properties Π of programs [14]. If the objects of a complete lattice denote properties (viz., sets) of computed program states in \mathcal{S} , then extensively means that approximation produces larger sets (i.e., less tight properties), idempotence means that approximation is made all at once, and monotonicity specifies that approximation keeps the relative order of precision of properties. In this context we are able to associate with each program P and (possibly undecidable) property to verify Π , a (decidable) over approximation $\rho(P, \Pi)$ of Π (i.e., $\Pi \subseteq \rho(P, \Pi)$) [3]. $\rho(P, \Pi)$ can specify for instance the absence of specific run-time errors, such as run-time overflows of integer and floating point variables (see [6]). Of course when $P \in \rho(P, \Pi)$ nothing can be concluded about P and Π , but when $P \notin \rho(P, \Pi)$, surely we have $P \notin \Pi$. The function $\lambda X. \rho(P, X)$ is in this context an upper closure operator [5] on the lattice of all program properties, where the most precise approximation is the identity function, which is the bottom element in the lattice of all upper closures, and the less precise approximation is the function mapping any property to the whole set of all possible values \mathcal{S} . Refining approximations, i.e., upper closures, corresponds therefore precisely to act as a lower closure on abstract domains, i.e., any lower closure operator on the lattice of all upper closure operators is a refinement.

The notion of abstraction refinement is extensively studied in computer science, notably in automatic program analysis, system verification, and abstract model checking for removing false alarms and spurious behaviours when approximating undecidable properties of computer programs and systems [5, 13, 2]. In this context an abstraction refinement is *compressible* if it is join-uniform, namely if the least upper bound of all abstract domains having the same refinement, still has the same refinement. This unique domain is the least one that, by refinement, gives back a given domain. A function f defined on a complete lattice C is join-uniform whenever for any $Y \subseteq C$ such that f is constant on Y with value $f(y)$, then $f(\bigvee Y) = f(y)$. When f is join-uniform its inverse $f^-(x) = \bigvee \{ y \mid f(y) = f(x) \}$ is such that $f(f^-(x)) = f(x)$ and $f^-(x)$ is maximal for which this happens. Examples include f as the function mapping any subset S of a complete boolean algebra C into the corresponding sub-boolean algebra $f(S)$ of C and f^- as the reduction of

S into the sets of its join-irreducible elements [9]. This notion of reduction along a given partial order, e.g., file size, is shared by a number of algorithms for *lossless compression*, e.g., see the `ZIP` file format and the GNU tool `gzip`, as well as the `PNG` and `GIF` file formats. The lossless compression is perfectly captured by uniformity.

We show that uniformity plays a central role in weakening additivity and co-additivity in the existence of adjoint closures. In particular we prove residuation, which is additivity on complete lattices, can be weakened by join-uniformity yet keeping the possibility of having an adjoint on a modified partial order. We introduce the notion of pseudo-adjoint of a function which corresponds to the adjoint when the function is residuated. Then by weakening residuation the pseudo-adjoint still keeps some of the properties of the adjoint case. In particular, on the case of lower closures, as specified in domain refinement, join-uniformity is the only possible weakening of residuation yet keeping the adjoint relation on a modified partially ordered set. Finally we characterise the set of all join-uniform lower closure operators as the set of fix-points of a function defined on the set of all lower closures of a complete lattice.

2 Closures and residuated closures

Closures and notations. In the following, $\langle C, \leq, \top, \perp, \vee, \wedge \rangle$ is a generic complete lattice, denoted C_{\leq} for underlining the considered order on C . A monotone function f on a complete lattice C is denoted $f : C \xrightarrow{m} C$, the set of all monotone functions on C is denoted

$$\mathcal{F}_C^{\text{mono}} \stackrel{\text{def}}{=} \{ f : C \longrightarrow C \mid f \text{ monotone} \}^1$$

A reductive [extensive] function f on C is such that $\forall x \in C. f(x) \leq x$ [$f(x) \geq x$] and it is denoted $f : C \xrightarrow{\text{red}} C$ [$f : C \xrightarrow{\text{ext}} C$]. The set of all reductive [extensive] function is

$$\mathcal{F}_C^{\text{red}} \stackrel{\text{def}}{=} \{ f : C \longrightarrow C \mid f \text{ reductive} \} \quad [\mathcal{F}_C^{\text{ext}} \stackrel{\text{def}}{=} \{ f : C \longrightarrow C \mid f \text{ extensive} \}].$$

Finally, an idempotent function f on C is such that $\forall x \in C. f(f(x)) = f(x)$ and it is denoted $f : C \xrightarrow{\text{idem}} C$. The set of all idempotent functions is

$$\mathcal{F}_C^{\text{idem}} \stackrel{\text{def}}{=} \{ f : C \longrightarrow C \mid f \text{ idempotent} \}.$$

Let f, g two functions on a complete lattice C , we define the point-wise order $f \dot{\leq} g$ iff $\forall x \in C. f(x) \leq_C g(x)$.

Two monotone functions between the complete lattices C_{\leq_C} and A_{\leq_A} $\alpha : C \xrightarrow{m} A$ and $\gamma : A \xrightarrow{m} C$ form an *adjunction* if for any $x \in C$ and $y \in A$: $\alpha(x) \leq_A y \Leftrightarrow x \leq_C \gamma(y)$. In this case, α [resp. γ] is the *left [right]adjoint* of γ [α] and

¹In the following we omit the pedex C when clear from the context.

it is additive [co-additive], i.e. it preserves *lub*'s [*glb*] of all subsets of the domain (emptyset included). Let us define the following function transformations:

$$f^+ \stackrel{\text{def}}{=} \lambda x. \bigvee \{ y \mid f(y) \leq x \} \quad \text{and} \quad f^- \stackrel{\text{def}}{=} \lambda x. \bigwedge \{ y \mid x \leq f(y) \}$$

In the following, we call f^+ right *pseudo*-adjoint of f , and f^- left *pseudo*-adjoint of f . When f is an additive [co-additive] map then the right [left] adjoint exists and it is precisely f^+ [f^-] (see [1] for notation). In this case we say that $\langle f, f^+ \rangle$ and $\langle f^-, f \rangle$ are adjoint operators.

An upper [lower] closure operator $\rho : C \longrightarrow C$ is monotone, idempotent, and extensive [reductive], i.e. $f \in \mathcal{F}_C^{\text{mono}} \cap \mathcal{F}_C^{\text{idem}} \cap \mathcal{F}_C^{\text{ext}}$ [$f \in \mathcal{F}_C^{\text{mono}} \cap \mathcal{F}_C^{\text{idem}} \cap \mathcal{F}_C^{\text{red}}$]. The set of all upper [lower] closure operators on C is denoted by $\text{uco}(C)$ [$\text{lco}(C)$]. Recall that if C is a complete lattice, then $\langle \text{uco}(C), \sqsubseteq, \sqcup, \sqcap, \lambda x. \top, id \rangle$ is a complete lattice [15, 7, 12], where $id \stackrel{\text{def}}{=} \lambda x. x$ and for every $\rho, \eta \in \text{uco}(C)$, $\rho \sqsubseteq \eta$ iff $\forall y \in C. \rho(y) \leq \eta(y)$ iff $\eta(C) \subseteq \rho(C)$. Upper closures are uniquely determined by their fix-points $\rho(C)$. $X \subseteq C$ is the set of fix-points of an upper closure if it is a *Moore family*, i.e. $X = \mathcal{M}(X) \stackrel{\text{def}}{=} \{ \bigwedge S \mid S \subseteq X \}$. It is known that upper closures can be made additive. This transformation is called the *disjunctive completion* of δ and it is defined as: $\Upsilon(\delta) \stackrel{\text{def}}{=} \bigvee \{ \rho \in \text{uco}(C) \mid \rho \sqsubseteq \delta, \rho \text{ additive} \}$, which is an upper closure closed by concrete least upper bound, i.e., such that $\rho \circ \vee \circ \rho = \vee \circ \rho$ [8]².

Residuated closures. Janowitz, characterized the order theoretic structure and properties of residuated closure operators [1, 11]. Let us first recall the notion of quasi-residuated and residuated map.

Definition 2.1 [11] *Let $f : C \longrightarrow D$ be a monotone map. f is quasi-residuated if $\exists h : D \longrightarrow C. f \circ h \leq id$ iff $\forall x \in D. \{ y \in C \mid f(y) \leq x \} \neq \emptyset$. f is residuated if $\exists h : D \longrightarrow C. f \circ h \leq id \wedge h \circ f \geq id$.*

In the following, this function h is the right adjoint of f (called *residual* in [11]). The following theorem provides the Janovitz characterization of adjunction between residuated closure operators.

Theorem 2.2 [11, Th. 2.10]

Let $f : C \longrightarrow C$ be residuated map, i.e. $\langle f, f^+ \rangle$ ³ is a pair of adjoint operators on C , then

$$(1) \quad f \in \text{uco}(C) \Leftrightarrow f^+ \in \text{lco}(C) \Leftrightarrow f \circ f^+ = f^+ \Leftrightarrow f^+ \circ f = f$$

and

$$(2) \quad f \in \text{lco}(C) \Leftrightarrow f^+ \in \text{uco}(C) \Leftrightarrow f \circ f^+ = f \Leftrightarrow f^+ \circ f = f^+$$

²Note that this fact always hold for $\eta \in \text{lco}(C)$, hence for lower closures it does not corresponds to additivity.

³ f^+ is precisely the function defined in the previous section.

Dually, we have that if $f : C \longrightarrow C$ is a *dual-residuated*⁴ map and f^- is its left-adjoint (defined in the previous section), then

$$(3) \quad f \in uco(C) \Leftrightarrow f^- \in lco(C) \Leftrightarrow f \circ f^- = f \Leftrightarrow f^- \circ f = f^-$$

and

$$(4) \quad f \in lco(C) \Leftrightarrow f^- \in uco(C) \Leftrightarrow f^- \circ f = f \Leftrightarrow f \circ f^- = f^-$$

In the hypotheses of the theorem, f^+ is trivially dual-residuated, and f^- is residuated.

We observe that, on a complete lattice, a *residuated* function is a monotone function admitting right-adjoint [1, 11], namely a residuated function on a complete lattice is an additive function and viceversa. By this observation, whenever we consider complete lattices, we can use the notion of additivity and of residuated as synonymous. Dually, co-additivity on complete lattices corresponds to dual-residuation.

Pseudo-adjointing closures. Let $\eta \in lco(C)$ and $\delta \in uco(C)$, then we can define the following functions:

$$\begin{aligned} \eta_- &\stackrel{\text{def}}{=} \lambda x. \bigwedge \{ \eta(y) \mid \eta(y) \geq x \} & \eta_+ &\stackrel{\text{def}}{=} \lambda x. \bigvee \{ y \mid \eta(y) = \eta(x) \} \\ \delta_- &\stackrel{\text{def}}{=} \lambda x. \bigwedge \{ y \mid \delta(x) = \delta(y) \} & \delta_+ &\stackrel{\text{def}}{=} \lambda x. \bigvee \{ \delta(y) \mid \delta(y) \leq x \} \end{aligned}$$

The following results show the relation between the functions introduced above and the pseudo-adjoints of an upper or lower closure operator.

Proposition 2.3 *Let $\eta \in lco(C)$ and $\delta \in uco(C)$. Then*

$$(1) \quad \eta^- = \eta_- \quad \delta^+ = \delta_+ \tag{2.1}$$

$$(2) \quad \eta^+ = \eta_+ \quad \delta^- = \delta_- \tag{2.2}$$

if η is additive and δ is co-additive.

PROOF.

1. Let us prove first Eq. 2.1 on $\eta \in lco(C)$ (the proof for δ is dual). First of all note that $\eta^-(x) \leq \eta_-(x)$ since by idempotence of η $\{ \eta(y) \mid \eta(y) \geq x \} \subseteq \{ y \mid \eta(y) \geq x \}$. On the other hand, by reductivity of η we have

$$\eta_-(x) = \bigwedge \{ \eta(y) \mid \eta(y) \geq x \} \leq \bigwedge \{ y \mid \eta(y) \geq x \} = \eta^-(x)$$

Hence we have the equality.

⁴Defined by duality.

2. Consider $\eta \in \text{Ico}(C)$ additive, let us prove Eq. 2.2, for η (for δ is dual). First of all $\eta_+(x) \leq \eta^+(x)$ since, by reductivity of η we have $\{y \mid \eta(y) = \eta(x)\} \subseteq \{y \mid \eta(y) \leq x\}$. On the other hand, by Th. 2.2 we have that $\eta(x) = \eta(\eta^+(x))$, hence $\eta^+(x) \in \{y \mid \eta(y) = \eta(x)\}$, and therefore $\eta^+(x) \leq \eta_+(x)$, implying the equality.

□

These results tell us that the right pseudo-adjoint of an upper closure is always equal to δ_+ , while the right pseudo-adjoint of a lower closure is equal to η^+ if the closure is additive, namely when we are computing precisely its right adjoint.

3 Weakening residuation by join-uniformity

A monotone function on a complete lattice C , $f : C \xrightarrow{\text{m}} C$, is *join-uniform* if for all $Y \subseteq C$, $(\exists \bar{x} \in Y. \forall y \in Y. f(y) = f(\bar{x})) \Rightarrow (\exists \bar{x} \in Y. f(\bigvee Y) = f(\bar{x}))$ [10]. In this case we denote join-uniform functions as $f : C \xrightarrow{\text{ju}} C$. Meet-uniformity is dually defined, in particular we denote a meet-uniform function as $f : C \xrightarrow{\text{mu}} C$. It is obvious that join-uniformity is strictly weaker than additivity. It is known (cf. [10]) that a *lifted* partial order can be induced by join-uniform closures, and defined as follows for a join-uniform closure $\eta \in \text{Ico}(C)$:

$$x \leq_\eta y \Leftrightarrow \eta(x) \leq \eta(y) \wedge (\eta(y) \leq \eta(x) \Rightarrow x \leq y) \quad (3.1)$$

In general $\leq \subseteq \leq_\eta$ and, when C_\leq is a complete lattice, then also C_{\leq_η} is a complete lattice. Moreover η is always additive on C_{\leq_η} [10].

When adjoining an additive/co-additive closure we obtain two different results: (1) the (right/left) adjoint is still a closure operator and (2) they satisfy the relations in Theorem 2.2. In the following we will weaken additivity by join-uniformity and we study the preserved relation between f and f^+ both when $f \in \text{Ico}(C)$ and when $f \in \text{uco}(C)$.

3.1 Join uniformity of lower closures

Let us consider $\eta \in \text{Ico}(C)$. Next theorems analyze the results described in Th. 2.2 showing when residuation is sufficient, sufficient and necessary or not necessary. In particular, without any hypothesis on η we show that the pseudo-adjoint fails to be an uco but one equation between η and η^+ in Th 2.2 hold anyway, while the other corresponds to additivity.

Theorem 3.1 *Let C be a complete lattice. Let $\eta \in \text{Ico}(C)$, the following facts hold*

1. $\eta^+ \in \mathcal{F}^{\text{mono}} \cap \mathcal{F}^{\text{ext}}$ (it may lose idempotence);
2. $\eta^+ \circ \eta = \eta^+$ always holds.
3. $\eta \circ \eta^+ = \eta$ iff η additive;

PROOF.

1. Let us show that η^+ is monotone and extensive. Let $x \leq z$, then we have that $\{y \mid x \geq \eta(y)\} \subseteq \{y \mid z \geq \eta(y)\}$. But this implies that $\bigvee \{y \mid x \geq \eta(y)\} \leq \bigvee \{y \mid z \geq \eta(y)\}$. Extensivity holds by reductivity of η . In fact $x \geq \eta(x)$, hence $x \in \{y \mid x \geq \eta(y)\}$, which trivially implies that $x \leq \bigvee \{y \mid x \geq \eta(y)\} = \eta^+(x)$. In Fig. 1 we show an example where the lack of additivity implies a lack of idempotence of η^+ . On the left η , whose fix-points are represented by circled points, is not additive, while, on the right, η^+ (represented again by circled points) is not idempotent on x .

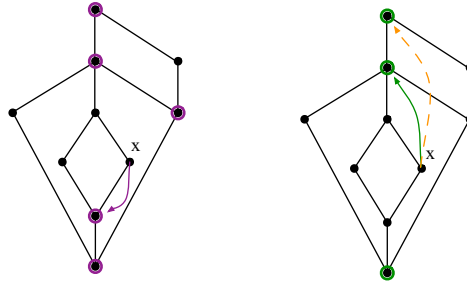


Figure 1: (left) η not additive, (right) η^+ , the dashed line is $\eta^+ \circ \eta^+(x)$.

2. If $\eta(z) \leq \eta(x)$ (for some $z \in C$), being η reductive we have $\eta(z) \leq x$. On the other hand, if $\eta(z) \leq x$ then by idempotence of η , $\eta(z) \leq \eta(x)$, hence we have that $\{z \mid \eta(z) \leq x\} = \{z \mid \eta(z) \leq \eta(x)\}$, namely $\eta^+(x) = \eta^+(\eta(x))$.
3. If η is additive then the thesis holds by Th. 2.2(2). Suppose that $\eta \circ \eta^+ = \eta$. We observe also that, being $\eta \in \text{Ico}(C)$ then it is closed by concrete least upper bound, hence, for any $X \subseteq C$ we have $\bigvee_{x \in X} \eta(x) = \eta(\bigvee_{x \in X} \eta(x))$. Let us prove that, for any X we have $\eta(\bigvee X) = \bigvee_{x \in X} \eta(x)$. Note that

$$\eta(\eta^+(\bigvee_{x \in X} \eta(x))) = \eta(\bigvee_{x \in X} \eta(x)) = \bigvee_{x \in X} \eta(x)$$

by hypothesis and because $\eta \in \text{Ico}(C)$. Observe that $\forall x \in X$ we have $\eta(x) \leq \bigvee_{x \in X} \eta(x)$, hence $X \subseteq \{y \mid \eta(y) \leq \bigvee_{x \in X} \eta(x)\}$ namely

$$\bigvee X \leq \bigvee \{y \mid \eta(y) \leq \bigvee_{x \in X} \eta(x)\} = \eta^+(\bigvee_{x \in X} \eta(x)).$$

By reductivity of η this implies that $\eta(\bigvee X) \leq \eta^+(\bigvee_{x \in X} \eta(x))$, then by idempotence of η , this implies that

$$\eta(\bigvee X) \leq \eta(\eta^+(\bigvee_{x \in X} \eta(x))) = \bigvee_{x \in X} \eta(x).$$

On the other hand, $x \leq \bigvee X$ implies by monotonicity that $\eta(x) \leq \eta(\bigvee X)$, hence $\bigvee_{x \in X} \eta(x) \leq \eta(\bigvee X)$, implying the equality and therefore additivity of η .

□

Consider $\eta \in \text{Ico}(C_{\leq})$ join-uniform, in [10], the authors observed that η^+ is not the right adjoint of η on C_{\leq} , while it is its right adjoint on \leq_{η} , denoting the order lifted by η as defined in Eq. 3.1. The following result shows that this happens iff η is join-uniform [10] and, in this case, η_+ is precisely its right adjoint.

Theorem 3.2 *Let $\eta \in \text{Ico}(C)$. Then the following facts are equivalent*

1. η join-uniform;
2. η is additive on \leq_{η} ;
3. $\langle \eta, \eta_+ \rangle$ are adjoints on \leq_{η} .

PROOF. Let's prove first that η join-uniform on \leq if and only if η is additive on \leq_{η} . When η is join-uniform on the standard order then η is additive on the lifted order by duality from [10, Th. 5.10]. Let us prove the other implication, and consider the definition of lifted least upper bound [10]

$$\bigvee_{\eta} Y = \begin{cases} \bigvee \{ y \in Y \mid \eta(y) = \eta(x) \} & \text{if } \exists x \in Y . \bigvee \eta(Y) = \eta(x) \\ \bigvee \eta(Y) & \text{otherwise} \end{cases}$$

Let $Y = \{ y \mid \eta(y) = \eta(x) \}$, then $\bigvee \eta(Y) = \bigvee \{ \eta(y) \mid \eta(y) = \eta(x) \} = \eta(x)$ and $x \in Y$, therefore

$$\begin{aligned} \bigvee_{\eta} \{ y \mid \eta(y) = \eta(x) \} &= \bigvee \{ y \in \{ z \mid \eta(z) = \eta(x) \} \mid \eta(y) = \eta(x) \} \\ &= \bigvee \{ y \mid \eta(y) = \eta(x) \} \quad (*) \end{aligned}$$

Hence we have

$$\begin{aligned} \eta(\bigvee \{ y \mid \eta(y) = \eta(x) \}) &= \eta(\bigvee_{\eta} \{ y \mid \eta(y) = \eta(x) \}) \\ &= \bigvee_{\eta} \{ \eta(y) \mid \eta(y) = \eta(x) \} = \eta(x) \end{aligned}$$

This is join-uniformity of η on the order \leq .

Finally, η join-uniform iff $\langle \eta, \eta_+ \rangle$ are adjoints on \leq_{η} is consequence of the previous result (*) and of Prop. 2.3. □

We have just shown that η_+ is the right adjoint of η on the lifted order \leq_{η} iff η is join-uniform. Next theorem shows how the relations in Th. 2.2 change for η_+ on \leq when η is only join-uniform.

Theorem 3.3 *Let C be a complete lattice. $\eta \in \text{Ico}(C)$*

1. *If η join-uniform then $\eta_+ \in \mathcal{F}^{idem} \cap \mathcal{F}^{ext}$ (it may lose monotonicity);*
2. *η join-uniform iff $\eta \circ \eta_+ = \eta$;*
3. *$\eta_+ \circ \eta = \eta_+$ always holds.*

PROOF.

1. Extensivity is trivial being $x \in \{y \mid \eta(y) = \eta(x)\}$. Idempotence comes directly from join-uniformity.

$$\begin{aligned}
 \eta_+(\eta_+(x)) &= \lambda x. \bigvee \left\{ y \mid \eta(y) = \eta(\eta_+(x)) \right\} \\
 &= \lambda x. \bigvee \left\{ y \mid \eta(y) = \eta\left(\bigvee \left\{ y \mid \eta(y) = \eta(x) \right\}\right) \right\} \\
 &= \lambda x. \bigvee \left\{ y \mid \eta(y) = \eta(x) \right\} \quad (\text{By join-uniformity}) \\
 &= \eta_+(x)
 \end{aligned}$$

In Figure 2 we show that in general η_+ is not monotone [10].

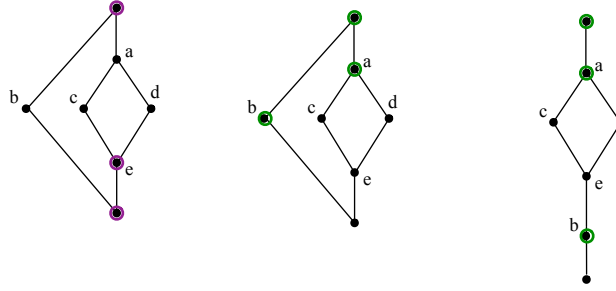


Figure 2: (left) η not additive, (center) η_+ on \leq , (right) η_+ on \leq_η .

2. η_+ is join-uniform iff $\eta \circ \eta_+ = \eta$ by definition of join-uniformity.
3. $\eta_+ \circ \eta = \eta_+$ trivially holds by idempotence of η .

□

3.2 Join-uniformity of upper closures

In the previous section we proved that join-uniformity of lower closures, when it holds, weakens additivity in the adjunction relation. In this section, we show that join-uniformity of upper closures, does not allow in general to weaken additivity of the adjoint relation. First of all let us recall that an uco is always join-uniform [8]. At this point, while it is trivial to show that $\delta^+ \circ \delta = \delta$ always holds for any $\delta \in \text{uco}(C)$, we prove that the key property which characterizes the right-adjoint of an upper closure, $\delta \circ \delta^+ = \delta^+$, cannot be weakened.

Theorem 3.4 *Let C a complete lattice. Let $\delta \in \text{uco}(C)$, the following facts hold*

1. $\delta^+ \in \text{lco}(C)$;
2. $\delta^+ \circ \delta = \delta$;
3. $\delta \circ \delta^+ = \delta^+$ iff δ additive.

PROOF.

1. Let us prove that δ^+ is monotone. Let $x, y \in C$ such that $x \leq y$, then $\{z \mid \delta(z) \leq x\} \subseteq \{z \mid \delta(z) \leq y\}$, which implies that $\bigvee \{z \mid \delta(z) \leq x\} \leq \bigvee \{z \mid \delta(z) \leq y\}$, namely $\delta^+(x) \leq \delta^+(y)$.
Note that $\{y \mid \delta(y) \leq x\} \subseteq \{y \mid y \leq x\}$ hence

$$\delta^+(x) = \bigvee \{y \mid \delta(y) \leq x\} \leq \bigvee \{y \mid y \leq x\} = x.$$

Therefore δ^+ is a lower operator. Finally, we have just proved that $\delta^+ \delta^+(x) \leq \delta^+(x)$. Let $w \in C$ such that $\delta(w) \leq x$ then $\delta(w) \in \{z \mid \delta(z) \leq x\}$. Hence, $\delta^+(x) = \bigvee \{z \mid \delta(z) \leq x\} \geq \delta(w)$, i.e., $w \in \{z \mid \delta(z) \leq \delta^+(x)\}$. We, therefore, proved that $\{z \mid \delta(z) \leq x\} \subseteq \{y \mid \delta(y) \leq \delta^+(x)\}$. But then

$$\delta^+(x) = \bigvee \{z \mid \delta(z) \leq x\} \leq \bigvee \{y \mid \delta(y) \leq \delta^+(x)\} = \delta^+ \delta^+(x).$$

2. $\delta^+(\delta(x)) = \delta_+(\delta(x)) = \bigvee \{\delta(z) \mid \delta(z) \leq \delta(x)\} = \delta(x)$.
3. First fo all, by Eq. 2.1 (Th. 2.3) $\delta^+(x) = \delta_+(x) = \bigvee \{\delta(y) \mid \delta(y) \leq x\}$.
If δ is additive then

$$\begin{aligned} \delta(\bigvee \{\delta(y) \mid \delta(y) \leq x\}) &= \bigvee \{\delta\delta(y) \mid \delta(y) \leq x\} \\ &\quad \text{by idempotence of } \delta \\ &= \bigvee \{\delta(y) \mid \delta(y) \leq x\} \end{aligned}$$

Suppose that the equality holds. Recall that, $\delta \in \text{uco}$ is additive iff for all $Z \subseteq \delta(C)$ we have $\delta(\bigvee Z) = \bigvee Z$. Note that $\bigvee Z \geq \bigvee \{\delta(y) \mid \delta(y) \leq \bigvee Z\}$ and that

$$\delta(y) \in Z \Rightarrow \delta(y) \leq \bigvee Z \Rightarrow \delta(y) \in \{\delta(y) \mid \delta(y) \leq \bigvee Z\}$$

Namely, $Z \subseteq \{\delta(y) \mid \delta(y) \leq \bigvee Z\}$ and $\bigvee Z \leq \bigvee \{\delta(y) \mid \delta(y) \leq \bigvee Z\}$. So we have the equality. Therefore we can conclude that

$$\delta(\bigvee Z) = \delta(\bigvee \{\delta(y) \mid \delta(y) \leq \bigvee Z\}) = \bigvee \{\delta(y) \mid \delta(y) \leq \bigvee Z\} = \bigvee Z$$

where the second equality hold by hypothesis and because $\delta^+ = \delta_+$.

□

Let $\text{Fix}(\delta) \stackrel{\text{def}}{=} \{ x \in C \mid \delta(x) = x \}$, we observe that $\text{Fix}(\delta) \subseteq \text{Fix}(\delta^+)$ if and only if $\delta^+ \circ \delta = \delta$. Analogously, $\text{Fix}(\delta) \supseteq \text{Fix}(\delta^+)$ iff $\delta \circ \delta^+ = \delta^+$. This implies that, if δ^+ is the right adjoint of δ , when δ is additive, then $\text{Fix}(\delta) = \text{Fix}(\delta^+)$. We proved in Th. 3.4 that, if δ is not additive, namely only join-uniform by construction, $\delta \circ \delta^+ = \delta^+$ does not hold, namely $\text{Fix}(\delta) \subsetneq \text{Fix}(\delta^+)$. We are interested in characterizing the points that are in $\text{Fix}(\delta^+) \setminus \text{Fix}(\delta)$, namely the points added by δ^+ . In particular we observe that these points are precisely those elements making δ not additive. This provides a further characterization of disjunctive completion as pseudo-adjoint of an upper closure operator.

Lemma 3.5 *Let $\delta \in \text{uco}(C)$, then $\delta^+ = (\Upsilon(\delta))^+$.*

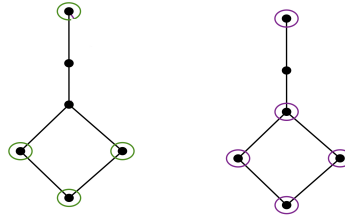
PROOF. We have to prove that $x \in \text{Fix}(\delta^+)$ iff $x \in \text{Fix}((\Upsilon(\delta))^+)$. Since Υ -closed upper closures are trivially additive, then $\Upsilon(\delta)$ admits right adjoint and by Th. 2.2-(1) we have that $\text{Fix}(\Upsilon(\delta)) = \text{Fix}((\Upsilon(\delta))^+)$. Moreover, we observed above that $\text{Fix}(\delta) \subseteq \text{Fix}(\delta^+)$, then

$$\begin{aligned} x \in \text{Fix}(\delta^+) & \text{ iff } x \in \text{Fix}(\delta) \vee \exists y. x = \bigvee \{ \delta(z) \mid \delta(z) \leq y \} \\ & \text{ iff } x \in \text{Fix}(\delta) \vee \exists Y \subseteq \text{Fix}(\delta). x = \bigvee Y \\ & \text{ iff } x \in \text{Fix}(\Upsilon(\delta)) = \text{Fix}((\Upsilon(\delta))^+) \end{aligned}$$

□

Proposition 3.6 *Let $\delta \in \text{uco}(C)$, let us define $\tilde{\delta} \in \text{uco}(C)$ by the set of fix-point $\text{Fix}(\tilde{\delta}) \stackrel{\text{def}}{=} \text{Fix}(\delta^+)$, then $\tilde{\delta} = \Upsilon(\delta)$.*

Example 3.7 *Consider the small lattice in the picture on the left. The circled points are the fix-points of an upper closure δ . On the right the circled points are the fix-points of $\delta^+ = \delta_+$ which, as we can observe, adds the point of disjunctive completion, making the closure additive.*



Let's collect all the results (and their duals) concerning pseudo-adjoints of closure operators in Table 1 and Table 2.

		JOIN-UNIFORM	ADDITIVE
$\delta \in uco(C)$		$\delta^+ \in lco(C) \quad \delta^+ \circ \delta = \delta$ $\delta^+ = (\bigvee(\delta))^+$	$\delta \circ \delta^+ = \delta^+$ $(\langle \delta, \delta^+ \rangle \text{ GC})$
$\eta \in lco(C)$	$\eta^+ \circ \eta = \eta^+$ $\eta_+ \circ \eta = \eta_+$	$\eta \circ \eta_+ = \eta$ $\eta_+ \in uco(C, \leq_\eta)$ $\langle \eta, \eta_+ \rangle \text{ GC on } \leq_\eta$	$\eta \circ \eta^+ = \eta$ $\eta^+ \in uco(C)$ $(\langle \eta, \eta^+ \rangle \text{ GC})$

Table 1: Properties of right pseudo-adjoints of closures

		MEET-UNIFORM	CO-ADDITIVE
$\delta \in uco(C)$	$\delta_- \circ \delta = \delta_-$ $\delta^- \circ \delta = \delta^-$	$\delta \circ \delta_- = \delta$ $\delta_- \in lco(C, \leq_\delta)$ $\langle \delta_-, \delta \rangle \text{ GC on } \leq_\delta$	$\delta \circ \delta^- = \delta$ $\delta^- \in lco(C)$ $(\langle \delta^-, \delta \rangle \text{ GC})$
$\eta \in lco(C)$		$\eta^- \in uco(C) \quad \eta^- \circ \eta = \eta$ $\eta^- = \mathcal{M}(\eta)^-$	$\eta \circ \eta^- = \eta^-$ $(\langle \eta^-, \eta \rangle \text{ GC})$

Table 2: Dual properties of left pseudo-adjoints of closures

4 Making (lower) closures uniform

We aim at characterizing the set of all uniform lower closures. We will follow [4] and we provide a characterization of uniform closures as images of suitable transformers on lower closure operators.

In [10, Theorem 4.2] the authors proved that the subdomain of $uco(C)$ of all the meet-uniform closures is a Moore family of $uco(C)$. This fact says that this domain forms itself a closure operator on $uco(C)$. Dually on lower closures, this means that, for any $\eta \in lco(C)$, the best join-uniform lower approximation

$$\bigsqcup \{ \delta \in lco(C) \mid \delta \sqsubseteq \eta, \delta \text{ is join-uniform} \}$$

exists. At this point, we precisely focus on this transformer mapping any lower closure operator η into the *greatest* join-uniform lower closure operator *smaller* than η . In the following if $\eta \in lco(C)$ we denote as $\eta(Z) = \text{const}$ the fact $\exists w \in \eta . \forall z \in Z . \eta(z) = w$ and as $Z \not\sqsubseteq x$ the fact $\forall z \in Z . z \not\sqsubseteq x$.

Theorem 4.1 *Let $\eta \in \text{Ico}(C)$. Then η is join-uniform if and only if*

$$\forall x \in \eta . \forall Z \subseteq C . \left((\eta(Z) = \text{const} \wedge Z \not\geq x) \Rightarrow \bigvee Z \not\geq x \right),$$

PROOF.

(\Rightarrow) We prove that if $\exists x \in \eta . \exists Z \subseteq C . (\eta(Z) = \text{const} \wedge Z \not\geq x \wedge \bigvee Z \geq x)$ then η is not join-uniform. We know that Z is such that $\exists w \in \eta . \forall z \in Z . \eta(z) = w$, this means that $\forall z \in Z . z \geq w$. Moreover the hypothesis $\bigvee Z \geq x$ implies, by monotonicity, that $\eta(\bigvee Z) \geq x$ and if $\eta(\bigvee Z) = w$ then we would have $w \geq x$ and this is absurd because otherwise we would have $\forall z \in Z . z \geq x$, which is avoided by the hypotheses on x and Z . Therefore $\eta(\bigvee Z) \neq w$, namely the closure η is not join-uniform.

(\Leftarrow) We prove that if η is not join-uniform then $\exists x \in \eta . \exists Z \subseteq C . (\eta(Z) = \text{const} \wedge Z \not\geq x \wedge \bigvee Z \geq x)$. Consider $w \in \eta$ and $Z = \{ z \in C \mid \eta(z) = w \}$, then $\forall z \in Z . z \geq w$ and this implies that $\bigvee Z \geq w$. By monotonicity this implies $\eta(\bigvee Z) \geq w$. We supposed that η was not join-uniform, this means that $\eta(\bigvee Z) > w$, i.e. there isn't the equality. Let $x = \eta(\bigvee Z)$, therefore $\bigvee Z \geq \eta(\bigvee Z) = x$. Moreover we have $\forall z \in Z . z \not\geq x$, otherwise if it exists $z \in Z$ such that $z \geq x$ we would have also $\eta(z) = w < x = \eta(x)$ by definition of x , which is absurd for the monotonicity of η . All these facts imply that Z is such that $\eta(Z) = \text{const}$, by construction, and $Z \not\geq x$ and $\bigvee Z \geq x$ for what we have just proved.

□

This theorem implies that we can isolate a set of fix-points in $\eta(C)$ which represent the closest join-uniform closure with respect to η .

$$(\eta)^\Delta \stackrel{\text{def}}{=} \{ x \in \eta \mid \forall Z \subseteq C . ((\eta(Z) = \text{const}, Z \not\geq x) \Rightarrow \bigvee Z \not\geq x) \}$$

Lemma 4.2 *If $\eta \in \text{Ico}(C)$ then $(\eta)^\Delta \in \text{Ico}(C)$ and*

$$(\eta)^\Delta = \bigsqcup \{ \delta \in \text{Ico}(C) \mid \delta \sqsubseteq \eta, \delta \text{ is join-uniform} \}.$$

PROOF. Consider a set Y of elements y such that $y \in (\eta)^\Delta(C)$, we have to prove that $\bigvee Y \in (\eta)^\Delta(C)$. The hypotheses imply that for each $y \in Y$ we have that $\forall Z \subseteq C . (\eta(Z) = w \wedge Z \not\geq y) \Rightarrow \bigvee Z \not\geq y$. Consider $Z \subseteq C$ such that $\eta(Z) = \text{const}$ then we prove that $Z \not\geq \bigvee Y$ implies $\bigvee Z \not\geq \bigvee Y$. Therefore suppose $Z \not\geq \bigvee Y$ and suppose $\bigvee Z \geq \bigvee Y$, this condition implies that $\forall y \in Y . \bigvee Z \geq y$. Consider now the condition $Z \not\geq \bigvee Y$. This means that $\forall z \in Z . z \not\geq \bigvee Y$, i.e. $\forall z \in Z . \exists y \in Y . z \not\geq y$. If we prove that, with these hypotheses, $\exists y \in Y . \forall z \in Z . z \not\geq y$, i.e. $Z \not\geq y$, then we would have an absurd because we have $\bigvee Z \geq y$ and that $Z \not\geq y$ when the hypothesis was that $y \in (\eta)^\Delta(C)$.

Suppose that $\forall y \in Y . \exists z . z \geq y$. Then we know that $\forall z \in Z . z \geq w$ and by monotonicity this implies that $w = \eta(z) \geq \eta(y) = y$. Therefore $\forall y \in Y . w \geq y$ and this implies that $\forall z \in Z . z \geq w \geq \bigvee Y$, by definition of \bigvee , that is absurd by the hypothesis made. This means that $\exists y \in Y . \forall z \in Z . z \not\geq y$, that for the absurd described above implies that $\bigvee Z \not\geq \bigvee Y$. Indeed if $Z \not\geq \bigvee Y$ then $\bigvee Z \not\geq \bigvee Y$, so $\bigvee Y \in (\eta)^\Delta(C)$.

Finally, we can prove that $(\eta)^\Delta$ is the greatest closure join-uniform smaller than η . Clearly, by construction and by Theorem 4.1, $(\eta)^\Delta$ is join-uniform. Suppose that there exists $\eta' \sqsubseteq \eta$ join-uniform and $(\eta)^\Delta \sqsubset \eta'$. This means that η' has more fix-points than $(\eta)^\Delta$, i.e., $\exists x \in \eta'(C) \subseteq \eta(C)$ such that $x' \notin (\eta)^\Delta$. Now, since η' is join-uniform, x is such that $\forall Z \subseteq C . ((\eta(Z) = \text{const}, Z \not\geq x) \Rightarrow \bigvee Z \not\geq x)$ by Theorem 4.1, but by construction all the elements in $\eta(C)$ satisfying this property are in $(\eta)^\Delta$, hence $x \in (\eta)^\Delta$, namely $\eta' = (\eta)^\Delta$. \square

Theorem 4.3 $\eta \in \text{Lco}(C)$ is join-uniform iff $(\eta)^\Delta = \eta$.

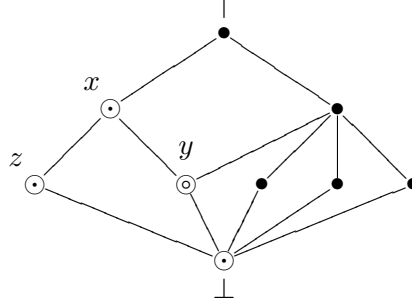
PROOF. Trivially we have that if η is join-uniform then it is image of itself. Indeed by Theorem 4.1 all the elements of η satisfy the condition imposed by $(\eta)^\Delta$. Analogously for each $\eta \in \text{Lco}(C)$ we have that $(\eta)^\Delta$ is join-uniform, again by Theorem 4.1. Moreover $(\eta)^\Delta$ is the most concrete join-uniform closure contained in η . Indeed if there exists another join-uniform closure operator η' contained in η such that $\eta' \sqsubset (\eta)^\Delta$, then there exists at least one element $x \in \eta'$ such that $x \notin (\eta)^\Delta$. By Theorem 4.1, this means that $x \in (\eta)^\Delta$ and consequently η' cannot be join-uniform. \square

Join-uniformity can be characterized on join-irreducible elements. Recall that x is join-irreducible if $x \neq \perp$ and $x = y \vee z$ implies that either $x = y$ or $x = z$.

Theorem 4.4 $(\eta \setminus (\eta)^\Delta) \cap \text{Jirr}(\eta) = \emptyset \Leftrightarrow \eta = (\eta)^\Delta$.

PROOF. Consider $\eta \neq (\eta)^\Delta$, namely $\exists x \in \eta$ such that $Z \not\geq x$ and $\bigvee Z \geq x$. Consider $Y \subseteq \text{Jirr}(\eta)$ such that $x = \bigvee Y$. We have to prove that there exists $y \in Y$ such that $Z \not\geq y$. Suppose that $\forall y \in Y . Z \geq y$, namely $\forall z \in Z . \forall y \in Y . z \geq y$. Let $w = \eta(Z)$, we can note that $z \geq y$ implies, by monotonicity of η , that $w = \eta(z) \geq \eta(y) = y$, and this holds for each $y \in Y$. We supposed that $x = \bigvee Y$, so by definition of \bigvee we have that $w \geq x$, but we know that for each z we have $z \geq w$, this would mean that $\forall z \in Z . z \geq x$, which is avoided by the hypotheses on Z and x . Therefore $\exists y \in Y . Z \not\geq y$. Finally, if we consider this y then we have that $\bigvee Z \geq x \geq y$, i.e. we have the thesis. On the other hand if $\eta = (\eta)^\Delta$ then $\eta \setminus (\eta)^\Delta = \emptyset$ and therefore the intersection is \emptyset . \square

Example 4.5 Consider the following lattice where the circled points, \odot and \ominus , are the points in the lower closure δ . η is clearly not join-uniform due to y . y is precisely the element to be removed to make η a join-uniform closure.



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