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Rapporto di ricerca
Research report
September 2010

RR 80/2010

An inference system for relationships between spatio-temporal granularities

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Abstract

Temporal, spatial, and spatio-temporal granularities allow one to qualify classical data locating them in time and space. In order to compare data qualified with different granularities and associate data to different granularities, it is necessary to know how the involved granularities are related. However, the explicit calculation of these relationships may be heavy from a computational point of view. Thus, in this paper, we propose an inference system for inferring definitely valid relationships starting from a set of already known valid relationships without to calculate them explicitly. We will prove the soundness and completeness of the system.

1 Introduction

Since the large growth of temporal and spatial datasets, it is becoming ever more important the storage, management, and querying of spatio-temporal databases.

A way to manage spatio-temporal data is to qualify and aggregate them by using temporal, spatial, and spatio-temporal granularities. Temporal granularities have been formalized by Bettini et al. [6, 11] for managing temporal data, while notions of spatial and spatio-temporal granularities have been proposed in [5]. Informally, a granularity represents a partition of a space in disjoint sets of points called granules. Granules can be used for qualifying and aggregating any information associated with them. Proposed frameworks for granularities include also some operations useful to create new granularities and some relationships describing how granules of two granularities are related [5, 11]. The same space can be partitioned by more granularities, representing it at different qualification levels (e.g., municipalities, provinces, regions).

The analysis and querying of spatio-temporally qualified data require in many cases to compare data associated to different granularities (e.g., car accidents happened in municipalities may be compared with provincial emergency services) or to convert data associated to a granularity into equivalent data associated to a different granularity (e.g., car accidents in regions can be calculated from accidents in municipalities). These analyses depend on the relationships between the involved granularities. For example, car accidents in regions can be obtained just summing car accidents in municipalities that partition regions. Knowledge about relationships between involved granularities allows one to know how data associated to different granularities can be compared.

Thus, it is important to know which relationships exist between granularities we are interested in. These relationships can be calculated by some algorithms that compare granules of granularities. However, especially in the spatial and spatio-temporal cases, these algorithms may compare many granules also by using geometrical functions that require a lot of resources. Thus, these algorithms can be very heavy from a computational point of view. As an alternative, in some cases it is possible to deduce some relationships between two granularities by observing other relationships already known valid between them or other related granularities. For example, by knowing that each granule of a granularity A is contained in a granule of B and that each granule of B is also a granule of C, we can deduce that each granule of A is contained in a granule of C. Inference rules similar to this one allow one to obtain some information about relationships between granularities without executing any algorithm, saving computational resources.

In this paper, we propose an inference system that, starting from a set of relationships between temporal, spatial, or spatio-temporal granularities, derives all other relationships definitely valid between the same granularities. The inference system is made up of a set of rules. Each rule concludes the validity of a relationship based on the validity of some premises. We also prove the soundness and completeness of the proposed inference system. The inference system includes also some rules allowing us to deduce the validity of some relationships between a granularity and the granularities used to define it. For example, knowing that A has been created by selecting some granules of B, the system infers that A is a subgranularity of B.

It is worth noting that, at the best of our knowledge, previous proposals focused only on inference rules on relationships between temporal or spatial regions, while no inference systems for granularities (i.e., sets of disjoint regions) have never been proposed in the past.

The rest of the paper is organised as follows. In the next section we briefly report main related work. In Section 3 some background notions about granularities are introduced, while in Section 4 we present our inference system and an application example. Finally, in Section 5, we conclude with some remarks and future work.

2 Related Work

In this section we discuss proposals in literature about inference systems on temporal, spatial, and spatio-temporal relationships.

The most important and well-formalized framework including temporal relationships is the Allen's interval algebra [1]. Allen proposed a notion of temporal interval and he defined a set of relationships between time intervals (e.g., before, during, equals). Relationships are managed and represented by a constraints network. The network is built and expanded by using a transitivity table. Given the relationship between two intervals A and B and the relationship between B and C, the transitivity table calculates the relationships definitely true between A and C. Inferred relationships are added to the already existing network, broaden information about the considered temporal intervals.

The Allen's approach and notions have been further developed and extended in several direction, e.g., fuzzy temporal intervals [2] and temporal semi-intervals [10].

Similar frameworks have been proposed also for spatial regions, especially based on Region Connection Calculus (RCC) [12]. In [12], Randell et al. present an algebra allowing one to express topological relationships (e.g., contains, overlap, disjoint) between spatial regions. Moreover, similarly to the Allen's proposal, they define a transitivity table in order to infer composition of two RCC relationships.

The RCC framework has been further extended to consider regions with a single hole in [15]. Following the original proposal about RCC, Vasardani and Egenhofer propose a set of relationships between single-holed spatial regions and a composition operator for them.

In [13], Systla and Yu presented a similar framework focused on pictorial databases. They propose nine relationships (e.g., left_of, above, inside) and study a deductive system able to infer new relationships from a given set. In particular they studied transitivity, symmetry, and other properties of the relationships.

Wiebrock et al. [17] propose a model to represent spatial relationships based on transformation matrices. Then, they define an inference system based on matrices manipulation.

Considering spatio-temporal locations and regions, in [8] Bittner proposed the notion of stratified spatio-temporal map spaces, a spatio-temporal extension of stratified map spaces [14]. Based on this notion, Bittner introduces a set of spatio-temporal relationships (e.g., different-time-same place, same-time-same-place, different-time-different-place) between two time-moving spatial regions. Finally, the author proposes a composition operator for these relationships, also considering different levels of map space (i.e., different temporal and spatial granularities).

However, all these proposals take in account just relationships between single regions, while our work introduce an inference system on relationships between granularities, i.e. sets of non-overlapping regions.

3 Background

In this section we briefly present the definitions of temporal, spatial, and spatio-temporal granularities already proposed in [5, 11].

3.1 Temporal Granularities

The notion of temporal granularity has been developed since the last years of 1990's. Based on several previous proposals, Bettini et al. developed in [6, 7, 16] the formalisation for temporal granularity now widely accepted by the temporal research community.

Informally, a temporal granularity represents a partition of a time domain. Each element of this partition (i.e., the granularity) is called *granule*. Describing a fact, we can use these granules to provide data with a temporal qualification at the suitable granularity. In other words, a temporal granularity represents a temporal unit of measure.

To give the definition of temporal granularity it is first of all necessary to define how we represent a time domain. A *time domain* is a pair $(T, \leq)$ where T is a non-empty set of time instants and $\leq$ is a total order over T. The domain represents the usual time line: it is the set of basic temporal entities used to interpret the other notions. Examples of discrete time domain are $(\mathbb{N}, \leq)$ and $(\mathbb{Z}, \leq)$.

Given a time domain T, a *temporal granularity* is a mapping G from an index set I to the power set of T such that:

1. if $i < j$ and $G(i)$ and $G(j)$ are non-empty, then each element of $G(i)$ is less than all elements of $G(j)$;

2. if $i < k < j$ and $G(i)$ and $G(j)$ are non-empty, then $G(k)$ is non-empty.

The first condition states that granules (i.e., the sets of instants corresponding to indexes) do not overlap each other and that the index order and the time domain order are the same. Instead, the second condition states that non-empty granules are contiguous. Usually, the index set is a subset of integers thus a granularity defines a countable set of granules, each one identified by its index. A granularity is *bounded* if there exist two indexes $k_1, k_2 \in I$ such that $G(i) = \emptyset$ for all $i < k_1$ and $i > k_2$. Often, granules are not referred by their indexes but using labels, i.e., textual representations. For this purpose it can be defined also a label mapping that associates to each label the corresponding granule.

Usual granularities are `Seconds`, `Minutes`, `Days`, `Years`. For example, if we consider the granularity `Years`, the granule `Years`(2008) corresponds to the time instants belonging to year 2008.

Thus, a granularity is made up of some non-empty granules. Granules represent sets of time instants perceived and used as indivisible entities. A granule can represent either a single instant, a time interval (i.e., a set of contiguous instants), or a set of non-contiguous instants.

Other notions are defined about a temporal granularity:

- the *origin* of a granularity G is a special granule designated as the initial granule, e.g., $G(0)$;

- the *image* of a granularity is the union of all granules in the granularity;

- the *extent* of a granularity is the smallest interval of the time domain that contains the image of the granularity. Formally, it is the set

$$\{t \in T | \exists a, b \in Im, a \leq t \leq b\}$$

where T is the time domain and Im is the image of the granularity.

Several relationships are defined between temporal granularities (assuming they have the same time domain). These relationships allow one to build hierarchies of granularities and address some issues related to the conversion and switching of information from a granularity to a related one. This ability is an important research theme about temporal information systems and temporal reasoning [7, 11].

The most important and used relationships among time granularities are (abbreviations for relationship names used hereinafter are enclosed in parenthesis):

- $\mathbf{GroupsInto}(G, H)$: (GI) G groups into H, denoted $G \trianglelefteq H$, if for each index j in the time domain of H there exists a (possibly infinite) subset S of the integers such that $H(j) = \bigcup_{i \in S} G(i)$.
- $\mathbf{FinerThan}(G, H)$: (FT) G is finer than H, denoted $G \preceq H$, if for each index i, there exists an index j such that $G(i) \subseteq H(j)$. If $G \preceq H$ we say that H is coarser than G ($H \succeq G$).
- $\mathbf{SubGranularity}(G, H)$: (SG) G is a subgranularity of H, denoted $G \sqsubseteq H$, if for each index i, there exists an index j such that $G(i) = H(j)$.
- $\mathbf{ShiftEquivalent}(G, H)$: (SE) G and H are shift equivalent, denoted $G \underline{\leftrightarrow} H$, if there exists an integer k such that $G(i) = H(i + k)$ for all i in the index set. Note that $G \underline{\leftrightarrow} H$ if and only if $G \sqsubseteq H$ and $H \sqsubseteq G$.
- $\mathbf{Partitions}(G, H)$: (P) G partitions H if $G \trianglelefteq H$ and $G \preceq H$.
- $\mathbf{GroupsPeriodicallyInto}(G, H)$: (GPI) G groups periodically into H if:
  1. $G \trianglelefteq H$;
  2. there exist $n, m \in Z^+$, where n is less than the number of non-empty granules of H, such that for all $i \in Z$, if $H(i) = \bigcup_{r=0}^{k} G(j_r)$ and $H(i + n) \neq \emptyset$ then $H(i + n) = \bigcup_{r=0}^{k} G(j_r + m)$.

Using these relationships it is possible to define two other useful notions: bottom granularity and calendar. Given a granularity relation $g\text{-}rel$ and a set of granularities over the same domain, a granularity G in the set is said to be a *bottom granularity* with respect to $g\text{-}rel$ if for each granularity H in the set, we have G $g\text{-}rel$ H. Moreover, we call *calendar* a set of granularities having the same domain and including a bottom granularity with respect to $\trianglelefteq$ ($\mathbf{GroupsInto}$). An usual example of calendar is the set $\{\texttt{Minutes}, \texttt{Hours}, \texttt{Days}, \texttt{Months}, \texttt{Years}\}$. Note that `Weeks` does not group into `Months` since a week can overlap two months.

Using these definitions and notations, Ning et al. [11] completed the framework for temporal granularity defining some operations useful to build new granularities from already existing ones. In particular they use an algebraic approach called *calendar algebra*.

The calendar algebra consists of operations allowing one to manage and build temporal granularities. These operations can be classified in two classes: grouping-oriented and granule-oriented operations. Operations in the first class combine the granules of a given granularity to form the granules of a new granularity, while operations in the second class construct a new granularity choosing some granules of the given parameter granularities.

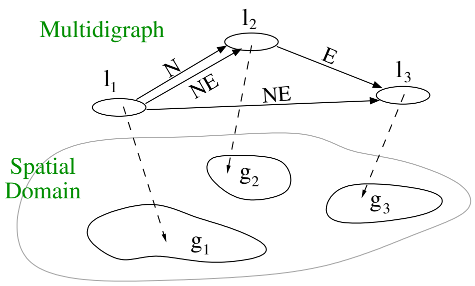

Figure 1: Structure of a vector-based spatial granularity

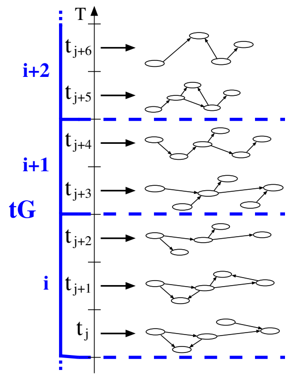

Figure 2: Structure of a vector-based spatio-temporal granularity

3.2 Spatial Granularities

A spatial granularity represents a partition of a space domain in regions, called *granules*. Each granule may have holes and may be composed by several disjoint areas. Each granule is an indivisible entity useful to spatially qualify classical information.

In [5] spatial granularities are defined by using a two-level model. The lower level represents the spatial domain, on which we can recognise geometrical information and in which data representing granules are defined. The higher level is an index structure used to access and manage granules. Conversely to other proposals in literature, in this approach spatial granularities do not represent only granules but also the relations between granules. Hence, for example, the granularity `Nations` may represent that Congo is west of Kenya. In order to represent in the same structure the granules and the relationships between them, multidigraphs are used as index sets. A multidigraph is a labelled directed graph with multiple labelled edges. This two-level structure of a spatial granularity is exemplified in Figure 1, where a spatial granularity with three granules and direction-based relationships is depicted.

We recall here the definitions of multidigraph and spatial granularity sketched in [4].

Definition 3.1 (Multidigraph)**.** *A labelled multidigraph is a labelled directed graph with multiple labelled edges defined as* $<V, MA, \Sigma_V, \Sigma_A, s, t, l_V, l_A>$ *where:*

- V *is the set of nodes;*
- $MA = (A, m)$ *is the multiset of edges. The multiset is composed of the set of edges* $A \subseteq V \times V$ *and the function* $m : A \to \mathbb{N}$ *that for each edge in* A *gives its multiplicity.*
- Σ_V *is the finite alphabet of node labels;*
- Σ_A *is the finite alphabet of edge labels;*
- $s : A \to V$ *is a function indicating the source node of an edge;*
- $t : A \to V$ *is a function indicating the target node of an edge;*
- $l_V : V \to \Sigma_V$ *is the labelling function for the nodes, it is a bijection;*

- *$l_A : A \to \mathcal{P}(\Sigma_A)$ is the labelling function for the edges. l_A must associate to each edge as many labels as its multiplicity, then we impose that, for each edge $e \in A$, $| l_A(e) |= m(e)$. Since an edge can have a multiplicity greater than one the labelling function can give a set of edge labels.*

The definition of spatial granularity includes the two levels we introduced and all the structures needed to link and manage them.

Definition 3.2 (Spatial granularity)**.** *A spatial granularity $\mathcal{G}$ is $<SD, MG, D_A, G>$ where:*

1. *SD is a spatial domain;*
2. *MG is a multidigraph;*
3. *D_A is a mapping that associates to each edge label the relation between granules that it represents in the granularity;*
4. *G is a mapping associating to each node of the multidigraph the non-empty spatial extent of the granule it represents. We impose that granules have pairwise disjoint interior.*

For any pair (v_1, v_2) of nodes of $MG.V$ and any label $l \in MG.\Sigma_A$, given $g_1 = G(v_1)$, $g_2 = G(v_2)$, and $R = D_A(l)$, in MG there must be an edge labelled with l between v_1 and v_2 iff the two granules g_1 and g_2 are related by the relation R.

In the multidigraph each node represents a spatial granule and it is mapped to its geometrical representation. On the other hand, edges represent relations between granules (e.g., direction- and distance-based relations). Each edge is labelled with the name of the relation it represents. The association of edge labels with the mathematical definition of relations they represent is maintained by the D_A mapping.

In the following, we distinguish spatio-temporal, temporal, and spatial granularities adding to the name of the granularity the prefixes st, t, and s, respectively.

The framework for spatial granularity may be completed defining relations and operations over spatial granularities. *Relations* between granularities allow one to compare granules belonging to different granularities. This feature is useful for aggregating data already associated to a granularity G by using a different granularity H. The following relationships between spatial granularities have been defined (abbreviations for relationship names used hereinafter are enclosed in parenthesis):

- $\mathbf{GroupsInto}(G, H)$: (GI) each granule of H is equal to the union of a set of granules of G, e.g., provinces group into regions;
- $\mathbf{FinerThan}(G, H)$: (FT) each granule of G is contained in one granule of H, e.g., each university campus is contained in a municipality;
- $\mathbf{SubGranularity}(G, H)$: (SG) for each granule of G, there exists a granule in H with the same spatial extent, e.g., European nations are a subgranularity of all nations;
- $\mathbf{Partition}(G, H)$: (P) G groups into and is finer than H, e.g., countries partition continents;
- $\mathbf{CoveredBy}(G, H)$: (CB) the image of G (i.e., the union of the spatial extent of its granules) is contained in the image of H, for example, national parks are covered by provinces, but they are not finer than, since a park can be shared by two provinces;

- $\mathbf{Disjoint}(G, H)$: (D) images of G and H are disjoint, e.g., national parks and municipal parks are disjoint;
- $\mathbf{Overlap}(G, H)$: (O) images of G and H overlap, e.g., national parks and lakes overlap each other.

Further, *operations* over spatial granularities have been proposed. They may be used to create new granularities from already defined ones. For example, if we have a granularity `Provinces` representing provinces, we may create automatically the granularity `Regions` representing regions by grouping appropriately granules of `Provinces`. In other cases, we may want to create a new granularity selecting only some granules of a given granularity.

Considering application-driven requests and users' requirements, the following operations over spatial granularities have been defined:

- $\mathbf{Grouping}(G, P)$ creates a new granularity grouping granules of G accordingly to one their partition P. For example, a new granularity partitioning a city in three granules representing rich, middle class, and poor quarters can be obtained grouping together quarters accordingly to a user-defined partition based on population wealth;
- $\mathbf{Combine}(G, H)$ groups together granules of G included in one granule of H, e.g., European parks can be grouped together with respect the nation they belong to;
- $\mathbf{Subset}(G, S)$ selects only granules of G belonging to the set S, e.g., from the European nations only those that do not touch any sea can be selected;
- $\mathbf{SelectInside}(G, H)$ selects only granules of G that are contained in a granule of H, e.g., we can select only those national parks belonging to a single province;
- $\mathbf{SelectContain}(G, H)$ selects only granules of G that contain at least one granule of H, e.g., given the granularity representing nuclear plants, we can select only those provinces having at least one plant;
- $\mathbf{SelectIntersect}(G, H)$ selects only granules of G that intersect at least one granule of H, e.g., from provinces we can obtain only those overlapping national parks;
- $\mathbf{Union}(G, H)$ creates a new granularity containing all granules of G and H (eventually deleting from the granules of H the extents already contained in G), e.g., we can obtain all parks by joining national, regional, and municipal parks;
- $\mathbf{Intersect}(G, H)$ creates a new granularity containing only granules representing intersections of one granule of G with one of H, e.g., we can refine national parks dividing them with respect to provinces;
- $\mathbf{Difference}(G, H)$ creates a new granularity obtained from G by deleting those areas covered also by some granules of H, e.g., we can obtain only terrestrial extent of provinces by deleting from them the lakes.

3.3 Spatio-temporal Granularities

A spatio-temporal granularity represents the evolution over time of a spatial granularity (see Figure 2). A spatio-temporal granularity has two components. The former is a temporal granularity, tG, that aggregates time points, while the latter is a mapping (called *spatial evolution*) that associates to each time point t the spatial granularity valid on it.

Definition 3.3 (Spatial evolution)**.** *Let T be a (possibly infinite) temporal domain and $GF = \{sG_k\}_k$ a set of spatial granularities with the same edge label set and representing the same spatial relations. A* spatial evolution *E is a mapping from T to GF such that:*

$$\exists t_1, t_2 \in T, t_1 \leq t_2 : \forall t \in [t_1, t_2] : \exists sG_k \in GF : E(t) = sG_k,$$

i.e., given a temporal point t between a lower t_1 and upper bound t_2, $E(t)$ provides the spatial granularity valid on t. Given a spatial node label j, $E(t)(j)$ represents the spatial granule j valid at point t.

Lower and upper bounds in the evolution definition allow one to represent finite evolutions. However, it is possible to represent infinite evolutions by using infinite bounds.

Definition 3.4 (Spatio-temporal granularity)**.** *Let tG be a temporal granularity and E a spatial evolution both over domain T. A* spatio-temporal granularity *stG is a pair $\langle tG, E\rangle$. Moreover, given a temporal granule index i and a spatial node label j, $stG(i, j) = \{E(t)(j)\}_{t \in tG(i)}$ is the* spatio-temporal granule *representing the evolution of spatial granule j during the temporal granule i.*

Also for spatio-temporal granularities some relations and operations have been defined. In both cases, the spatial definitions have been extended to the spatio-temporal context by adding a temporal dimension.

Formally, the following Boolean functions summarise the type of properties a user can be interested to check over spatio-temporal granularities, considering in particular their spatial part (the temporal part will be considered later by introducing temporal quantifiers):

- $\mathbf{GranuleRel}(stG, S_1, S_2, R)$ checks whether a given spatial relation R (e.g., North) exists between two given spatial granules S_1 and S_2 of stG;
- $\mathbf{Belongs}(stG, S)$ checks whether a given spatial granule S belongs to spatial granularities in stG;
- $R(stG, stH)$ checks whether the spatial granularities stored in spatio-temporal granularities stG and stH are related by the relationship R (e.g., $\mathbf{GroupsInto}$).

All these functions check spatial relationships. Hence, now these spatial relations have to be combined with time. To do that the "always" and "sometimes" temporal quantifiers are used. These quantifiers, together with spatial reasoning, allow to represent concepts similar to "spatial relation R is always valid" or "exists a time in which spatial relation S is valid". Considering time, the framework represent two levels: point level and granule level. At both levels always ($\forall$) and exists ($\exists$) operators can be applied. Since the granule level is always considered before the point level, we obtain four quantifiers: "for each granule–for each instant" ($\forall\forall$), "for each granule–exists an instant" ($\forall\exists$), "exists a granule–for each instant" ($\exists\forall$), "exists a granule–exists an instant" ($\exists\exists$). Combining these quantifiers with spatial relationships we obtain spatio-temporal relationships as, for example, $\forall\exists\mathbf{GranuleRel}$ (i.e., for each granule exists an instant in which $\mathbf{GranuleRel}$ holds).

To better understand, let us consider an example. Epidemiologists may want to know whether every year there exists time points during which spatial granularity representing bird migration areas intersects the granularity representing administrative regions where Psittacosis (an infectious disease that is spread by birds) [9] cases have been surveyed. This issue can be addressed by the relation $\forall\exists\mathbf{Overlap}(A, B)$ where $B = \texttt{Psittacosis}_{\texttt{Region-Year}}$ is the spatio-temporal granularity based on years representing regions where Psittacosis cases have been surveyed and A is the spatio-temporal granularity $\texttt{BirdMigr}_{\texttt{Region-Year}}$ based on years representing bird migration regions.

A similar extension has been defined for spatio-temporal operations. Spatio-temporal operations apply the original spatial operations to each spatial granularity recorded in a given spatio-temporal granularity. Spatio-temporal operations allow one to compute, for example, grouping, union, and selection by containment over spatio-temporal granularities. For example, we can calculate the granularity representing, instant by instant, African countries intersecting areas where cholera cases have been surveyed.

For example, $stSubset(stG, S)$, where S is a set of spatial node labels, returns a spatio-temporal granularity where, at every instant t, the associated spatial granularity is obtained from the one valid at time t in stG selecting only granules whose label belongs to S.

Formally, let $\langle Oper\rangle$ be an unary operation over spatial granularities requiring a set of (possibly empty) parameters Par. The corresponding operation over spatio-temporal granularities, $stG' = st\langle Oper\rangle(stG, Par)$, is defined such that, for each instant t, $stG'.E(t) = \langle Oper\rangle(stG.E(t), Par)$. In other words, $st\langle Oper\rangle$ applies the original spatial operation $\langle Oper\rangle$ to each spatial granularity recorded in the spatio-temporal one.

A similar extension can be defined for binary operations.

4 The Inference System

In this section we present our inference system for relationships between spatial, temporal, and spatio-temporal granularities.

In the next subsections, we will discuss the system for spatial granularities. We will present the semantics of our system, the inference rules, and we will prove that the system is sound and complete.

The ideas we used for studying the inference systems are the same for the three kinds of granularities (i.e., temporal, spatial, and spatio-temporal granularities). Thus, we will discuss only the system for spatial granularities. Temporal and spatio-temporal systems are briefly discussed in Section 4.6, in which we present features distinguishing them from the spatial one.

4.1 Basic Ideas

Given a set $\mathcal{R}$ of relationships between spatial granularities over a set $\mathcal{G}$ of granularities, the inference system automatically infers all other relationships definitely valid over $\mathcal{G}$. In other words, it propagates the given constraints.

The inference system does not know the actual definition of the granularities over which it will operate (i.e., their graphs, granules, and granules extent), it knows only some relationships between them. In other words, the system works on an abstraction of the real granularities contained in the database. This abstraction does not consider the low level representation of the granularities, i.e., granules, extents, and graphs, but only the properties, the relationships of the granularities. Thus, the system is not able to compute a truth value for each possible relationship between two given granularities, but only for someones. In some cases, nothing can be decided about some relationships.

We studied several kinds of rules for answering to the following questions.

- Are relationships reflexive?
- Knowing that $R(G, H)$ is valid, what other relationships must be valid between G and H?
- Knowing that $R(G, H)$ is valid, what other relationships must be valid between H and G?
- What other relationships can be inferred between G and H knowing that R_1 and R_2 are both valid between them?
- Is it possible to concatenate the relationships? In other words, knowing that $R_1(G_1, G_2)$ and $R_2(G_2, G_3)$ are valid, what can we infer about G_1 and G_3?

  Note that, assuming R_1 and R_2 are the same relationship, we study also the transitivity of the relationships.

Rules in proposed inference system allow only premises composed either by one relationship or by a conjunction of relationships. While the conclusion of rules must be just one relationship. Disjunctions, in premises and conclusions, are not permitted.

The proposed inference system does not infer only relationships that definitely hold on the considered set of granularities, it returns also the set of relationships that surely do not hold (assuming that considered granularities are not equivalent). We denote that a relationship $R(G, H)$ does not hold with $\neg R(G, H)$.

Besides relationships presented in Section 3, we consider also the equivalence relationship. We say that two granularities are equivalent, $G \approx H$, if both contain exactly the same granules with the same extents, without regard to their labels, i.e. for each granule of G, there exists a granule in H with the same spatial extent and vice versa. Equivalence is important because if two granularities are equivalent all relationships hold between them. In other words, the equivalence relationship implies any other relationships. Thus, it is important to note that in our inference system all rules concluding that a relationship cannot be valid assume that considered granularities are not equivalent.

Thus, for example, if $GroupsInto(G, H)$ is in the given relationships set, the system infers that also $CoveredBy(H, G)$ must be valid. Moreover, if also $\neg G \approx H$ is in the set, the system infers that $SubGranularity(G, H)$ cannot be true, i.e., $\neg SubGranularity(G, H)$.

The inference system is based on the application of the set of rules on a starting set of relationships. This set can be obtained also by analysing how the granularities we are interested in have been created. Knowing which operation has been used to define a granularity (see Section 3), we can infer some relationships between it and the operand granularities. However, these rules are not really part of the inference system because they do not infer relationships from other relationships. These rules can be considered as the first and starting point for the application of the inference system. For example, if $G' = SelectInside(G_1, G_2)$ we can deduce that G_1 groups into G' and that G' is finer than both G_1 and G_2.

Inference rules from operations used to define spatial granularities are depicted in Tables 6 and 7.

4.2 Semantics of Relationships Between Granularities

As we said in the previous section, the inference system is theoretically based on an abstraction of our frameworks for granularities. In this model only relationships are important, while it does not regard how granularities are defined. In this section we present this abstract model and its semantics. Moreover we show that this abstraction in consistent with our theoretical framework.

Definition 4.1. *A spatial granularity model is a pair* $(\mathcal{W}, \mathcal{R})$*, where* $\mathcal{W}$ *is a non-empty set of worlds and* $\mathcal{R}$ *is a set of binary relationships over* $\mathcal{W}$*.*

Given a model $\mathcal{M}$ and an interpretation λ on it that maps each label representing a granularity to a world in $\mathcal{W}$, validity of relationships between spatial granularities is represented by the smallest relation, $\models^{\mathcal{M},\lambda}$, satisfying the following constraints (where Γ represents a set of relationships) that represent the general system behaviour:

- $\models^{\mathcal{M},\lambda} R(G_1, G_2)$ iff $R(\lambda(G_1), \lambda(G_2)) \in \mathcal{R}$
- $\models^{\mathcal{M},\lambda} \neg R(G_1, G_2)$ iff $R(\lambda(G_1), \lambda(G_2)) \notin \mathcal{R}$
- $\models^{\mathcal{M},\lambda} R_1(G_1, G_2) \wedge \Gamma$ iff $\models^{\mathcal{M},\lambda} R(G_1, G_2)$ and $\models^{\mathcal{M},\lambda} \Gamma$
- $\Gamma \models^{\mathcal{M},\lambda} R(G_1, G_2)$ iff $\models^{\mathcal{M},\lambda} \Gamma$ implies $\models^{\mathcal{M},\lambda} R(G_1, G_2)$

and the constraints in Figures 3 and 4 that represent the behaviour with respect to relationships between granularities, which semantics is the one informally presented in Section 3. For example, the (DR) constraint imposes that if in the model $D(G_1, G_2)$ is valid, then in the model also $\neg R(G_1, G_2)$ must be valid for each spatial relationship R. On the other hand, (EGI) says that when $GI(G_1, G_2)$ is valid in the model then also $CB(G_2, G_1)$ is valid.

Now, we show that this model is sound, that is, that each assertion on it is true also in the theoretical model for spatial granularities. We remark that this is needed in order to show that the abstraction over which the inference system is based is a "good" abstraction, i.e., it respects the proposed framework.

Proposition 4.1. The proposed model, and its semantics, for the inference system over spatial granularities is sound.

Proof. We have to show that each property in Figures 3 and 4 is valid also in the theoretical framework for spatial granularities. We present just some cases while the other ones are analogous.

(trans) We need to prove that $GroupsInto$, $FinerThan$, $SubGranularity$, $Partition$, and $CoveredBy$ are transitive also in our framework. Let us consider $GroupsInto$ relationship and let G_1, G_2, and G_3 such that G_1 groups into G_2 and G_2 groups into G_3. Thus, by the definition of $GroupsInto$, each granule of G_2 is equal to the union of a set of granules of G_1, and each granule of G_3 is equal to the union of a set of granules of G_2. Since, each granule of G_3 is the union of granules in G_2 and each of these is in turn equal to the union of some granules of G_1, then we can conclude that each granule of G_3 is equal to the union of some granules in G_1. Thus, G_1 groups into G_3.

The treatment of $FinerThan$, $SubGranularity$, $Partition$, and $CoveredBy$ is very similar.

(refl) We need to show that $GroupsInto$, $FinerThan$, $SubGranularity$, $Partition$, and $CoveredBy$ are reflexive. Let us consider $GroupsInto$

relationship and let G a spatial granularity. Obviously, each granule of G is equal to union of the singleton made up of just itself, thus G groups into itself, i.e., $GroupsInto(G, G)$.

The treatment of $FinerThan$, $SubGranularity$, $Partition$, and $CoveredBy$ is very similar.

(antirefl) We need to show that $Disjoint$, $Overlap$ are antireflexive. Let us consider $Disjoint$ and let G a spatial granularity. Obviously the image of G cannot be disjoint by itself, thus G cannot be disjoint by itself, i.e., $\neg Disjoint(G, G)$.

The treatment of $Overlap$ is very similar.

(antisymm) We need to show that $GroupsInto$, $FinerThan$, $SubGranularity$, and $Partition$ are antisymmetric. Let us consider $GroupsInto$ relationship and let G_1 and G_2 two spatial granularities such that G_1 groups into G_2 and G_2 groups into G_1. Each granule g_1 of G_1 is equal to the union of a set S_2 of granules of G_2, and, in turn, each granule g_2 in this set is equal to the union of a set S_1 of granules in G_1. But, S_2 and S_1 must be the singletons containing just g_2 and g_1 respectively, otherwise g_1 would be equal to the union of other granules in G_1, that it is not possible since granules in a granularity cannot intersect each other. Thus, each granule g_1 in G_1 must be equal to a granule g_2 in G_2. Repeating the same argument starting from granules in G_2, we obtain that G_1 and G_2 must contain the same granules, i.e., they are equivalent.

The treatment of $FinerThan$, $SubGranularity$, and $Partition$ is very similar.

□

4.3 Inference System

Tables from 1 to 7 present, in tabular form, the rules of the inference system for spatial granularities. These rules answer to questions reported in Section 4.1. We remind that all rules concluding the non validity of a relationship have also the premise that considered granularities are not equivalent, otherwise all relationships hold obviously between them.

In these tables, the ✓symbol means that the corresponding relationship can be inferred, the × symbol means that the relationship does not hold, and, finally, the - symbol means that nothing can be decided about that relationship.

In particular, Table 1 tell us which other relationships between G and H can be inferred from a relationship $R(G, H)$. Rules in Table 2 infer relationships between H and G starting from $R(G, H)$. Table 3 contains rules inferring from a pair of relationships $R_1(G, H)$ and $R_2(G, H)$. Finally, tables 4 and 5 represent the composition rules for relationships between spatial granularities.

The antisymmetry of relationships is stated, as a special case, by Table 2, while transitivity is a special case of the relationship concatenation studied in tables 4 and 5.

Besides rules in these tables the inference system includes also rules stating the reflexivity

$$\bigwedge_{R\in\{GI,FT,SG,P,CB\}} . \vdash R(G, G)$$

and antireflexivity

$$\bigwedge_{R\in\{D,O\}} . \nvdash R(G,G)$$

Moreover, the system needs also some rules for describing the behaviour of equivalence (where the meaning of R is not specified it may be any relationship we introduced):

- $\vdash G \approx G$
- $G \approx H \vdash H \approx G$
- $G_1 \approx G_2 \wedge G_2 \approx G_3 \vdash G_1 \approx G_3$
- $\bigwedge_{R\in\{GI,FT,SG,P,CB\}} .G \approx H \vdash R(G,H)$
- $R(G_1,G_2) \wedge G_1 \approx G_3 \vdash R(G_3,G_2)$
- $R(G_1,G_2) \wedge G_2 \approx G_3 \vdash R(G_1,G_3)$

The first three rules state reflexivity, symmetry, and transitivity of equivalence, respectively. The fourth rule states that equivalence implies any other relationship, while last two rules state the monotonicity of equivalence with respect to all other relationships.

When the set of relationships the system is considering contains both a relationship and its negation, the set is said to be inconsistent and the system infers the special symbol $\bot$, i.e., $R(G,H) \wedge \neg R(G,H) \vdash \bot$.

Finally, we introduce the RAA rule representing the *reductio ad absurdum*.

$$\frac{\begin{array}{c}[\neg R(G,H)]\\ \vdots \\ \bot\end{array}}{R(G,H)}$$

When a derivation that includes $\neg R(G,H)$ reaches a contradiction, the system can infer that $R(G,H)$ must hold.

This rule is suitable for our inference system because all relationships are Boolean and each relationship has for sure a truth value, i.e. either $R(G,H)$ or $\neg R(G,H)$ must be true. This rule is based on the notion of discharged assumption that is standard in Natural Deduction [3] proof systems. The relationship $\neg R(G,H)$ is discharged during the rule application.

Note that by using RAA we can obtain contrapositives of the other rules. For example, from a rule of the form $R(G_1,H_1) \vdash R(G_2,H_2)$ we can obtain the equivalent rule $\neg R(G_2,H_2) \vdash \neg R(G_1,H_1)$ with the following proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{[\neg\neg R(G_1,H_1)]^2 \quad [\neg R(G_1,H_1)]^1}{\bot}}{R(G_1,H_1)}\,1}{R(G_2,H_2)} \qquad \neg R(G_2,H_2)}{\cfrac{\bot}{\neg R(G_1,H_1)}\,2}$$

Contraposition allows us, for example, to obtain from rules in the inference system that if G is not covered by H, then G is not finer than H.

With very similar proofs, from rules like $R_1(G_1, H_1) \wedge R_2(G_2, H_2) \vdash R_3(G_3, H_3)$ we can obtain equivalent rules $R_1(G_1, H_1) \wedge \neg R_3(G_3, H_3) \vdash \neg R_2(G_2, H_2)$ and $R_2(G_2, H_2) \wedge \neg R_3(G_3, H_3) \vdash \neg R_1(G_1, H_1)$. In this cases contraposition allows us to obtain for example from the antisymmetry rule $\neg G \approx H \wedge GI(G, H) \vdash \neg GI(H, G)$ the rule $GI(G, H) \wedge GI(H, G) \vdash G \approx H$, i.e., if a granularity G groups into H and vice versa then G and H are equivalent.

Finally, also the contrapositive of RAA can be obtained in the same way.

4.4 Soundness and Completeness of Inference System

In this section we study soundness and completeness of proposed inference system for spatial granularities.

Theorem 4.1 (Soundness)**.** The proposed inference system for spatial granularities is sound, i.e. $\mathcal{R} \vdash R(G, H)$ implies $\mathcal{R} \models^{\mathcal{M},\lambda} R(G, H)$ for every model $\mathcal{M}$ and every interpretation λ.

Proof. The proof is by induction on the structure of the derivation of $R(G, H)$. The base case is when $R(G, H) \in \mathcal{R}$ and is trivial. Due to the similarity of inference rules and semantics, the proof that all rules are sound is trivial. The only interesting case is the application of RAA.

$$\frac{\begin{array}{cc} \mathcal{R} & [\neg R(G,H)] \\ & \vdots \\ & \bot \end{array}}{R(G,H)}$$

Let $\mathcal{R}_1$ be $\mathcal{R} \cup \{\neg R(G, H)\}$. By the induction hypothesis, $\mathcal{R}_1 \models^{\mathcal{M},\lambda} \bot$ for every model $\mathcal{M}$ and every interpretation λ. Now, consider an arbitrary model $\mathcal{M}$ and an arbitrary interpretation λ, we assume $\models^{\mathcal{M},\lambda} \mathcal{R}$ and prove $\models^{\mathcal{M},\lambda} R(G, H)$. Since $\not\models^{\mathcal{M},\lambda} \bot$, by the induction hypothesis we obtain $\not\models^{\mathcal{M},\lambda} \mathcal{R}_1$, that, given the assumption $\models^{\mathcal{M},\lambda} \mathcal{R}$ leads to $\not\models^{\mathcal{M},\lambda} \neg R(G, H)$. Thus, since a model associates a truth value to each relationship, $\models^{\mathcal{M},\lambda} R(G, H)$. □

Now, we prove the completeness of the proposed inference system for spatial granularities.

Definition 4.2 (Consistency)**.** *A set $\mathcal{R}$ of relationships between granularities is said to be* consistent *if $\mathcal{R} \nvdash \bot$. It is said* inconsistent *otherwise.*

Proposition 4.2. Let $\mathcal{R}$ be a consistent set of relationships between granularities. For each relationship $R(G_1, G_2)$, either $\mathcal{R} \cup \{R(G_1, G_2)\}$ is consistent or $\mathcal{R} \cup \{\neg R(G_1, G_2)\}$ is consistent.

Proof. Let us suppose that $\mathcal{R} \cup \{R(G_1, G_2)\}$ and $\mathcal{R} \cup \{\neg R(G_1, G_2)\}$ are both inconsistent. Thus, $\mathcal{R} \cup \{R(G_1, G_2)\} \vdash \bot$ and $\mathcal{R} \cup \{\neg R(G_1, G_2)\} \vdash \bot$. By using RAA, we obtain that $\mathcal{R} \vdash \neg R(G_1, G_2)$ and $\mathcal{R} \vdash R(G_1, G_2)$ then $\mathcal{R}$ is inconsistent (contradiction). □

Let $\mathcal{R}$ be a maximally consistent set of relationships between granularities. With $G^{\mathcal{R}}$ we denote the set of constants, representing granularities, occurring in $\mathcal{R}$.

Definition 4.3 (Maximal consistency)**.** *A set $\mathcal{R}$ of relationships between granularities is maximally consistent with respect to a set of granularities $\mathcal{G}$ iff the following two conditions hold:*

1. *$\mathcal{R}$ is consistent;*

2. *for each relationship $R(G_1, G_2)$ with $G_1, G_2 \in \mathcal{G}$, either $R(G_1, G_2) \in \mathcal{R}$ or $\neg R(G_1, G_2) \in \mathcal{R}$.*

Lemma 4.1. Each set $\mathcal{R}$ of relationships between granularities can be extended to $\mathcal{R}^*$, a maximally consistent set with respect to $G^{\mathcal{R}}$.

Proof. Let $r_1, r_2, \ldots$ be an enumeration of all possible relationships, and their negation, over $G^{\mathcal{R}}$.

We iteratively build a sequence of consistent sets of relationships by defining $\mathcal{R}_0 = \mathcal{R}$ and

$$\mathcal{R}_{i+1} = \begin{cases} \mathcal{R}_i & \text{if } \mathcal{R}_i \cup \{r_{i+1}\} \text{ is inconsistent} \\ \mathcal{R}_i \cup \{r_{i+1}\} & \text{if } \mathcal{R}_i \cup \{r_{i+1}\} \text{ is consistent} \end{cases}$$

We define $\mathcal{R}^* = \bigcup_{i \geq 0} \mathcal{R}_i$. Now we prove that $\mathcal{R}^*$ is maximally consistent.

1. First we prove consistency. Suppose that $\mathcal{R}^*$ is inconsistent, then it exists i such that R_{i-1} is consistent while R_i is inconsistent. Of course, it is not possible that R_i is inconsistent since it has been built from R_{i-1} adding r_i only if it remain consistent. Thus, $\mathcal{R}^*$ is consistent.

2. We prove now the maximality. Suppose $\mathcal{R}^*$ is not maximal, thus there exists $R(G_1, G_2)$ such that $R(G_1, G_2) \notin \mathcal{R}^*$ and $\neg R(G_1, G_2) \notin \mathcal{R}^*$. There exist i and j such that $R(G_1, G_2) = r_i$ and $\neg R(G_1, G_2) = r_j$. Let us suppose that $i < j$ (the other case is symmetric). Since $R(G_1, G_2) \notin \mathcal{R}^*$ we know that $\mathcal{R}_{i-1} \cup \{r_i\}$ is inconsistent, thus also $\mathcal{R}_{j-1} \cup \{r_i\}$ it is (since $i < j$, $\mathcal{R}_{j-1}$ includes at least $\mathcal{R}_{i-1}$). By the Proposition 4.2 we can conclude that $\mathcal{R}_{j-1} \cup \{r_j\} = \mathcal{R}_j$ must be consistent and thus $r_j = \neg R(G_1, G_2) \in \mathcal{R}^*$ (contradiction).

□

Definition 4.4. *Let $\mathcal{R}$ be a maximally consistent set of relationships between granularities. Let $G^{\mathcal{R}}$ be the set of constants, representing granularities, occurring in $\mathcal{R}$. We define the binary relation $\equiv^{\mathcal{R}}$ over $G^{\mathcal{R}}$ such that for each $G_1, G_2 \in G^{\mathcal{R}}$, $G_1 \equiv^{\mathcal{R}} G_2$ iff $G_1 \approx G_2 \in \mathcal{R}$.*

Proposition 4.3. Given $\mathcal{R}$ a maximally consistent set of relationships between granularities, $\equiv^{\mathcal{R}}$ is an equivalence relation.

Proof. It is trivial by the rules stating reflexivity, symmetry, transitivity, and monotonicity of equivalence relationship. □

In the following we will use the notation $[G]^{\mathcal{R}}$ to indicate the equivalence class containing the constant G, i.e.

$$[G]^{\mathcal{R}} = \{H \mid G \equiv^{\mathcal{R}} H\}$$

Definition 4.5. *Let $\mathcal{R}$ be a maximally consistent set of relationships between granularities. We define the* canonical model *$\mathcal{M} = (\mathcal{W}, \Sigma)$ as follows:*

- *$\mathcal{W} = \{[G]^{\mathcal{R}} \mid G \in G^{\mathcal{R}}\}$;*
- *$\Sigma = \{R(G_1, G_2) \in \mathcal{R}\}$.*

We define the canonical interpretation $\lambda : G^{\mathcal{R}} \rightarrow \mathcal{W}$ *such that* $\lambda(G) = [G]^{\mathcal{R}}$ *for each* $G \in G^{\mathcal{R}}$.

Proposition 4.4. Given $\mathcal{R}$ a maximally consistent set of relationships between granularities, its canonical model $\mathcal{M}$ is a Kripke model for our inference system.

Proof. We have to prove that the model respects properties stated in the semantics of the relationships between spatial granularities. We will show only some cases, the other ones are similar.

(Trans) Suppose there exist three worlds $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in \mathcal{W}$ such that $R(\mathcal{W}_1, \mathcal{W}_2) \in \Sigma$ and $R(\mathcal{W}_2, \mathcal{W}_3) \in \Sigma$, but $R(\mathcal{W}_1, \mathcal{W}_3) \notin \Sigma$ (with $R \in \{GI, FT, SG, P, CB\}$). Thus, there exist three labels $G_1, G_2, G_3 \in G^{\mathcal{R}}$ such that $\lambda(G_1) = \mathcal{W}_1$, $\lambda(G_2) = \mathcal{W}_2$, $\lambda(G_3) = \mathcal{W}_3$, $R(G_1, G_2) \in \mathcal{R}$, and $R(G_2, G_3) \in \mathcal{R}$, but $R(G_1, G_3) \notin \mathcal{R}$. Thus, by the maximality of $\mathcal{R}$, $\neg R(G_1, G_3) \in \mathcal{R}$. But this leads to the inconsistency of $\mathcal{R}$, given that from $R(G_1, G_2)$ and $R(G_2, G_3)$ we can derive $R(G_1, G_3)$ in $\mathcal{R}$ and thus $\perp$.

(Refl) Suppose there exists a world $\mathcal{W}_1 \in \mathcal{W}$ such that $R(\mathcal{W}_1, \mathcal{W}_1) \notin \Sigma$ (with $R \in \{GI, FT, SG, P, CB\}$). Thus, there exists a label $G_1 \in G^{\mathcal{R}}$ such that $\lambda(G_1) = \mathcal{W}_1$ and $R(G_1, G_1) \notin \mathcal{R}$ By the maximality of $\mathcal{R}$, we know that $\neg R(G_1, G_1) \in \mathcal{R}$. This leads to the inconsistency of $\mathcal{R}$, given that in $\mathcal{R}$ we can derive $R(G_1, G_1)$ and then $\perp$.

(Antirefl) Suppose there exists a world $\mathcal{W}_1 \in \mathcal{W}$ such that $D(\mathcal{W}_1, \mathcal{W}_1) \in \Sigma$ or $O(\mathcal{W}_1, \mathcal{W}_1) \in \Sigma$. Thus, there exists a label $G_1 \in G^{\mathcal{R}}$ such that $\lambda(G_1) = \mathcal{W}_1$ and $D(G_1, G_1) \in \mathcal{R}$ or $O(G_1, G_1) \in \mathcal{R}$. This leads to the inconsistency of $\mathcal{R}$, given that in $\mathcal{R}$ we can derive $\neg D(G_1, G_1)$ and $\neg O(G_1, G_1)$ and then $\perp$.

(Antisymm) Suppose there exist two worlds $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{W}$ such that $R(\mathcal{W}_1, \mathcal{W}_2) \in \Sigma$ and $R(\mathcal{W}_2, \mathcal{W}_1) \in \Sigma$, but $\mathcal{W}_1$ and $\mathcal{W}_2$ are different (with $R \in \{GI, FT, SG, P\}$). Thus, there exist two labels $G_1, G_2 \in G^{\mathcal{R}}$ such that $\lambda(G_1) = \mathcal{W}_1$, $\lambda(G_2) = \mathcal{W}_2$, $R(G_1, G_2) \in \mathcal{R}$, and $R(G_2, G_1) \in \mathcal{R}$, but $G_1 \approx G_2 \notin \mathcal{R}$. By the maximality of $\mathcal{R}$, $\neg G_1 \approx G_2 \in \mathcal{R}$. This leads to the inconsistency of $\mathcal{R}$ applying the rule $R(G_1, G_2) \wedge \neg G_1 \approx G_2 \vdash \neg R(G_2, G_1)$.

□

Lemma 4.2. Let $\mathcal{R}$ be a maximally consistent set of relationships between granularities, $\mathcal{R} \vdash R(G_1, G_2)$ iff $R(G_1, G_2) \in \mathcal{R}$.

Proof.

($\Leftarrow$) If $R(G_1, G_2) \in \mathcal{R}$ then trivially $\mathcal{R} \vdash R(G_1, G_2)$.

($\Rightarrow$) Suppose $R(G_1, G_2) \notin \mathcal{R}$, thus, by the maximality of $\mathcal{R}$, $\neg R(G_1, G_2) \in \mathcal{R}$ and $\mathcal{R} \vdash \neg R(G_1, G_2)$. This leads to the inconsistency of $\mathcal{R}$ since, by hypothesis, $\mathcal{R} \vdash R(G_1, G_2)$. Contradiction.

□

Lemma 4.3. Let $\mathcal{R}$ be a maximally consistent set of relationships between granularities, $\mathcal{M}$ its canonical model, and λ the canonical interpretation. Then, $R(G_1, G_2) \in \mathcal{R}$ iff $\mathcal{R} \models^{\mathcal{M},\lambda} R(G_1, G_2)$.

Proof.

($\Rightarrow$) if $R(G_1, G_2) \in \mathcal{R}$ and $\models^{\mathcal{M},\lambda} \mathcal{R}$ then trivially $\models^{\mathcal{M},\lambda} R(G_1, G_2)$.

($\Leftarrow$) By hypothesis $\mathcal{R} \models^{\mathcal{M},\lambda} R(G_1, G_2)$. Suppose $R(G_1, G_2) \notin \mathcal{R}$. By the maximality of $\mathcal{R}$, $\neg R(G_1, G_2) \in \mathcal{R}$, hence $\mathcal{R} \models^{\mathcal{M},\lambda} \neg R(G_1, G_2)$, and then $\mathcal{R} \models^{\mathcal{M},\lambda} \bot$ (contradiction).

□

Theorem 4.2 (Completeness)**.** The inference system for relationships between spatial granularities is complete, i.e., if $\mathcal{R} \nvdash R(G_1, G_2)$ then there exist a model $\mathcal{M}$ and an interpretation λ such that $\mathcal{R} \not\models^{\mathcal{M},\lambda} R(G_1, G_2)$.

Proof. If $\mathcal{R} \nvdash R(G_1, G_2)$ then $\mathcal{R} \cup \{\neg R(G_1, G_2)\}$ is consistent, otherwise $\mathcal{R} \cup \{\neg R(G_1, G_2)\} \vdash \bot$ and hence (by RAA) $\mathcal{R} \vdash R(G_1, G_2)$. $\mathcal{R} \cup \{\neg R(G_1, G_2)\}$ can be extended to a maximally consistent set $\mathcal{R}^*$. Let $\mathcal{M}$ be its canonical model and λ its canonical interpretation. $\mathcal{R}^* \models^{\mathcal{M},\lambda} \neg R(G_1, G_2)$ thus $\mathcal{R}^* \not\models^{\mathcal{M},\lambda} R(G_1, G_2)$. Then we can conclude that $\mathcal{R} \not\models^{\mathcal{M},\lambda} R(G_1, G_2)$. □

4.5 Example

To better understand the proposed system, let us consider an example about spatial granularities. Let A, B, and C three spatial granularities. About them we know only that A is a subgranularity of B and groups into C, and that B is finer than C. By just using these three information the inference system deduce, by applying the concatenation rule to $SubGranularity(A, B)$ and $FinerThan(B, C)$ that also A is finer than C. Thus, since A groups into and is finer than C, A partitions C. Moreover, by applying rules reported in Tables 1 and 2 the inference system deduces also that the granularities are all covered by each other, i.e., they have the same image. Finally, applying again rules in Table 2 to relationships $SubGranularity(A, B)$ and $CoveredBy(B, A)$, the inference system discovers that A and B are equivalent, i.e., they have the same granules eventually with different labels. Thus, of course, also B partitions C.

4.6 The temporal and spatio-temporal inference systems

In the previous section we presented the inference system for spatial granularities and we proved its soundness and completeness. Notions and proofs about the inference system for temporal granularities are very similar to the spatial ones and can be obtained from them just knowing that in the temporal case:

- $CoveredBy$, $Disjoint$, and $Overlap$ relationships are not considered;
- the $GroupsPeriodicallyInto$ relationship is added, but it is just a special case of the $GroupsInto$;
- the equivalence relationship is replaced by $ShiftEquivalent$.

Table 8 describes which relationships between G and H can be inferred from a relationship $R(G, H)$. Rules in Table 9 infer relationships between H and G starting from $R(G, H)$. Table 10 contains rules inferring from a pair of relationships $R_1(G, H)$ and $R_2(G, H)$. Finally, Table 11 represents the composition rules for relationships between temporal granularities. Finally, tables 12 and 13 infer relationships knowing which operations have been used to define temporal granularities.

On the other hand, the spatio-temporal case is slightly different thus we briefly detail this case.

A spatio-temporal granularity represents the evolution over time points (aggregated by a temporal granularity) of spatial granularities. Thus, the inference system for spatio-temporal granularities has to manage both temporal and spatial granularities and adds some new ad-hoc rules over spatio-temporal granularities. For the same reason, a model for spatio-temporal granularities contains spatial, temporal, and spatio-temporal granularities; thus, its semantics extends the semantics of spatial and temporal granularities models and add the new constraints depicted in figures 6 and 7.

We remind that a spatio-temporal relationship is made up of four parts:

- a temporal quantifier Q composed by a quantification on time granules and one on time points;
- the spatial relationship R that is temporally quantified by Q;
- two spatio-temporal granularities compared by the relationship.

Rules we studied answer to the same questions used for spatial and temporal granularities (see Section 4.1). As for the other inference systems, also in this case each rule has a premise that can be a conjunction of spatial, temporal, and spatio-temporal relationships, and one conclusion that is a spatio-temporal relationship.

These rules specify which quantifier can be inferred in the conclusion relationship while the spatial relationship is obtained applying the inference system for the spatial granularities.

Note that, premises of spatio-temporal rules include also the temporal relationship between the temporal granularities of the involved spatio-temporal granularities, because the quantifier in the conclusion relationship depends also by it.

Tables 14 and 15 contain rules inferring relationships between stG and stH starting from a relationship $QR(stG, stH)$ and the relationships between the temporal granularities involved in stG and stH. Rules in Table 16 infer relationships between stH and stG starting from a relationship $QR(stG, stH)$. Tables from 18 to 26 represent concatenation table for spatio-temporal relationships.

For example, from Table 16 we can conclude that from $\forall\exists R(stG, stH)$ we may infer $\exists\exists R'(stH, stG)$ if the temporal granularity of stG is a subgranularity of the one of stH, where R' is such that in the spatial inference system $R(G, H) \vdash R'(H, G)$.

As in spatial and temporal inference systems, the spatio-temporal system needs some auxiliary rules. Table 27 describes when it is possible to infer $\perp$, i.e., a contradiction. Temporal quantification introduces uncertainty (e.g., in $\exists\exists GI(G, H)$ we do not know the exact time point in which $GroupsInto$ relationship hold) and this uncertainty influences inference rules. For example, from $\exists\forall R(G, H)$ and $\exists\forall\neg R(G, H)$ we cannot infer $\perp$ since the two considered temporal granules may be different. But we can do that from $\forall\forall R(G, H)$ and $\exists\exists\neg R(G, H)$. Table 27 shows the quantifiers combinations that allow the system to infer a contradiction.

Considering the RAA rule for spatio-temporal granularities, its trivial formulation would be

$$\frac{\begin{matrix} QR(stG, stH) \\ \vdots \\ \bot \end{matrix}}{\neg QR(stG, stH)}$$

However, since our framework for spatio-temporal granularities does not allow to put a negation before the quantifiers, but only to quantify a negation of a relationship, we always rewrite $\neg Q$ by using the usual translations for which $\forall \equiv \neg\exists\neg$ and $\exists \equiv \neg\forall\neg$. Table 28 summarises the RAA rule for spatio-temporal granularities.

Since spatio-temporal relationships are a temporally quantified version of spatial relationships, we remind that in any case the inference system from any relationship $QR(stG, stH)$ can infer $QR'(stG, stH)$ where R' is such that $R(G, H) \vdash R'(G, H)$ in the spatial inference system.

We say that two spatio-temporal granularities are equivalent, $stG \approx stH$ iff they are based on the same temporal granularity and $\forall\forall \approx (stG, stH))$. Note that equivalence of spatial granularities can be treated like all other spatial relationships, thus reflexivity, symmetry, and transitivity in the spatio-temporal case follow from spatial and spatio-temporal rules. Thus, we have just to specify the monotonicity rules:

- $QR(stG_1, stG_2) \wedge stG_1 \approx stG_3 \vdash QR(stG_3, stG_2)$
- $QR(stG_1, stG_2) \wedge stG_2 \approx stG_3 \vdash QR(stG_1, stG_3)$

Considering, changes and considerations described in this section, soundness and completeness proofs for spatio-temporal inference system is similar to those for spatial and temporal inference systems.

5 Conclusions and Future Work

In this paper we proposed an inference system that allows one to obtain the set of relationships valid surely between a set of temporal, spatial, and spatio-temporal granularities starting from a set of already assumed valid relationships. The system has been proved to be sound (all the inferred relationships are surely valid) and complete (the system infers all the relationships definitely valid). Moreover, we proved also that the model on which the system is based is consistent with the frameworks for granularities proposed in previous papers [5, 11]. The system allows also to deduce which relationships are valid between a granularity and the granularities used to create it through the operations defined in the frameworks. These rules allow to build a starting set of relationships, between the granularities we are interested in, on which the inference system can be applied.

As for future work we want to implement the inference system (along the procedure for deciding the validity of relationships), extending our proposal for a database able to manage granularities with inference capabilities.

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A Semantics of Temporal and Spatio-temporal Granularities

$$\bigwedge_{R\in\{GI,FT,SG,P,CB\}} .R(G_1,G_2)\wedge R(G_2,G_3)\vDash^{\mathcal{M},\lambda} R(G_1,G_3) \tag{trans}$$

$$\bigwedge_{R\in\{GI,FT,SG,P,CB\}} .\vDash^{\mathcal{M},\lambda} R(G,G) \tag{refl}$$

$$\bigwedge_{R\in\{D,O\}} .\vDash^{\mathcal{M},\lambda} \neg R(G,G) \tag{antirefl}$$

$$\bigwedge_{R\in\{D,O\}} .R(G_1,G_2)\vDash^{\mathcal{M},\lambda} R(G_2,G_1) \tag{symm O/D}$$

$$\bigwedge_{R\in\{GI,FT,SG,P\}} .R(G_1,G_2)\wedge R(G_2,G_1)\vDash^{\mathcal{M},\lambda} G_1\approx G_2 \tag{antisymm}$$

$$P(G_1,G_2)\vDash^{\mathcal{M},\lambda} GI(G_1,G_2) \tag{EP1}$$

$$P(G_1,G_2)\vDash^{\mathcal{M},\lambda} FT(G_1,G_2) \tag{EP2}$$

$$GI(G_1,G_2)\wedge FT(G_1,G_2)\vDash^{\mathcal{M},\lambda} P(G_1,G_2) \tag{IP}$$

$$\bigwedge_{R\in\{GI,FT,SG,P,CB,O\}} .D(G_1,G_2)\vDash^{\mathcal{M},\lambda} \neg R(G_1,G_2) \tag{DR}$$

$$\bigwedge_{R\in\{GI,FT,SG,P,CB,D\}} .O(G_1,G_2)\vDash^{\mathcal{M},\lambda} \neg R(G_1,G_2) \tag{OR}$$

$$FT(G_1,G_2)\vDash^{\mathcal{M},\lambda} CB(G_1,G_2) \tag{EFT}$$

$$SG(G_1,G_2)\vDash^{\mathcal{M},\lambda} FT(G_1,G_2) \tag{ESG1}$$

$$GI(G_1,G_2)\vDash^{\mathcal{M},\lambda} CB(G_2,G_1) \tag{EGI}$$

$$SG(G_1,G_2)\vDash^{\mathcal{M},\lambda} GI(G_2,G_1) \tag{ESG2}$$

$$GI(G_1,G_2)\wedge CB(G_1,G_2)\vDash^{\mathcal{M},\lambda} FT(G_1,G_2) \tag{IFT}$$

$$CB(G_1,G_2)\wedge CB(G_2,G_3)\vDash^{\mathcal{M},\lambda} CB(G_1,G_3) \tag{conc1}$$

$$CB(G_1,G_2)\wedge D(G_2,G_3)\vDash^{\mathcal{M},\lambda} D(G_1,G_3) \tag{conc2}$$

$$GI(G_1,G_2)\wedge D(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg CB(G_1,G_3) \tag{conc3}$$

$$D(G_1,G_2)\wedge CB(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg GI(G_1,G_3) \tag{conc4}$$

$$GI(G_1,G_2)\wedge O(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg FT(G_1,G_3) \tag{conc5}$$

$$GI(G_1,G_2)\wedge O(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg D(G_1,G_3) \tag{conc6}$$

$$P(G_1,G_2)\wedge O(G_2,G_3)\vDash^{\mathcal{M},\lambda} O(G_1,G_3) \tag{conc7}$$

$$D(G_1,G_2)\wedge O(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg GI(G_1,G_3) \tag{conc8}$$

$$O(G_1,G_2)\wedge GI(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg FT(G_1,G_3) \tag{conc9}$$

$$O(G_1,G_2)\wedge P(G_2,G_3)\vDash^{\mathcal{M},\lambda} O(G_1,G_3) \tag{conc10}$$

$$O(G_1,G_2)\wedge D(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg FT(G_1,G_3) \tag{conc11}$$

$$O(G_1,G_2)\wedge CB(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg GI(G_1,G_3) \tag{conc12}$$

$$O(G_1,G_2)\wedge CB(G_2,G_3)\vDash^{\mathcal{M},\lambda} \neg D(G_1,G_3) \tag{conc13}$$

Figure 3: Semantics of spatial relationships. First part.

$$GI(G_1,G_2) \wedge SG(G_1,G_2) \models^{\mathcal{M},\lambda} G_1 \approx G_2 \quad \text{(I}\approx\text{1)}$$
$$SG(G_1,G_2) \wedge CB(G_2,G_1) \models^{\mathcal{M},\lambda} G_1 \approx G_2 \quad \text{(I}\approx\text{2)}$$
$$G_1 \approx G_2 \wedge G_2 \approx G_3 \models^{\mathcal{M},\lambda} G_1 \approx G_3 \quad \text{(trans}\approx\text{)}$$
$$\models^{\mathcal{M},\lambda} G \approx G \quad \text{(rifl}\approx\text{)}$$
$$G_1 \approx G_2 \models^{\mathcal{M},\lambda} G_2 \approx G_1 \quad \text{(simm}\approx\text{)}$$
$$\bigwedge_{R\in\{GI,FT,SG,P,CB\}} .G_1 \approx G_2 \models^{\mathcal{M},\lambda} R(G_1,G_2) \quad \text{(E}\approx\text{)}$$
$$R(G_1,G_2) \wedge G_1 \approx G_3 \models^{\mathcal{M},\lambda} R(G_3,G_2) \quad \text{(mon1)}$$
$$R(G_1,G_2) \wedge G_2 \approx G_3 \models^{\mathcal{M},\lambda} R(G_1,G_3) \quad \text{(mon2)}$$

Figure 4: Semantics of spatial relationships. Second part.

$$\bigwedge_{R\in\{GI,FT,SG,P,GPI,SE\}} .R(G_1,G_2) \wedge R(G_2,G_3) \models^{\mathcal{M},\lambda} R(G_1,G_3) \quad \text{(trans)}$$
$$\bigwedge_{R\in\{GI,FT,SG,P,GPI,SE\}} . \models^{\mathcal{M},\lambda} R(G,G) \quad \text{(refl)}$$
$$SE(G_1,G_2) \models^{\mathcal{M},\lambda} SE(G_2,G_1) \quad \text{(symmSE)}$$
$$\bigwedge_{R\in\{GI,FT,SG,P,GPI\}} .R(G_1,G_2) \wedge R(G_2,G_1) \models^{\mathcal{M},\lambda} G_1 \approx G_2 \quad \text{(antisymm)}$$
$$P(G_1,G_2) \models^{\mathcal{M},\lambda} GI(G_1,G_2) \quad \text{(EP1)}$$
$$P(G_1,G_2) \models^{\mathcal{M},\lambda} FT(G_1,G_2) \quad \text{(EP2)}$$
$$GI(G_1,G_2) \wedge FT(G_1,G_2) \models^{\mathcal{M},\lambda} P(G_1,G_2) \quad \text{(IP)}$$
$$GPI(G_1,G_2) \models^{\mathcal{M},\lambda} GP(G_1,G_2) \quad \text{(IGP)}$$
$$SG(G_1,G_2) \models^{\mathcal{M},\lambda} FT(G_1,G_2) \quad \text{(ESG1)}$$
$$SG(G_1,G_2) \models^{\mathcal{M},\lambda} GI(G_2,G_1) \quad \text{(ESG2)}$$
$$GI(G_1,G_2) \wedge SG(G_1,G_2) \models^{\mathcal{M},\lambda} SE(G_1,G_2) \quad \text{(ISE)}$$
$$G_1 \approx G_2 \wedge G_2 \approx G_3 \models^{\mathcal{M},\lambda} G_1 \approx G_3 \quad \text{(trans}\approx\text{)}$$
$$\models^{\mathcal{M},\lambda} G \approx G \quad \text{(rifl}\approx\text{)}$$
$$G_1 \approx G_2 \models^{\mathcal{M},\lambda} G_2 \approx G_1 \quad \text{(simm}\approx\text{)}$$
$$\bigwedge_{R\in\{GI,FT,SG,P,CB\}} .SE(G_1,G_2) \models^{\mathcal{M},\lambda} R(G_1,G_2) \quad \text{(ESE)}$$
$$G_1 \approx G_2 \models^{\mathcal{M},\lambda} SE(G_1,G_2) \quad \text{(E}\approx\text{)}$$
$$R(G_1,G_2) \wedge SE(G_1,G_3) \models^{\mathcal{M},\lambda} R(G_3,G_2) \quad \text{(mon1)}$$
$$R(G_1,G_2) \wedge SE(G_2,G_3) \models^{\mathcal{M},\lambda} R(G_1,G_3) \quad \text{(mon2)}$$

Figure 5: Semantics of temporal relationships

$$QR(stG, stH) \vDash QR'(stG, stH) \qquad \text{where } R(G, H) \vDash R'(G, H) \tag{space}$$

$$\bigwedge_{Q \in \{\forall\forall, \forall\exists, \exists\forall, \exists\exists\}} . \forall\forall R(stG, stH) \vDash QR(stG, stH) \tag{allall}$$

$$\bigwedge_{Q \in \{\forall\exists, \exists\exists\}} . \forall\exists R(stG, stH) \vDash QR(stG, stH) \tag{allexists}$$

$$\bigwedge_{Q \in \{\exists\forall, \exists\exists\}} . \exists\forall R(stG, stH) \vDash QR(stG, stH) \tag{existsall}$$

$$\forall\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(stG.tG, tG') \vDash \forall\forall R(stG', stH) \tag{rel1}$$

$$\forall\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge SG(stG.tG, tG') \vDash \exists\forall R(stG', stH) \tag{rel2}$$

$$\forall\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge FT(stG.tG, tG') \vDash \exists\exists R(stG', stH) \tag{rel3}$$

$$\forall\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(stG.tG, tG') \vDash \forall\exists R(stG', stH) \tag{rel4}$$

$$\forall\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge FT(stG.tG, tG') \vDash \exists\exists R(stG', stH) \tag{rel5}$$

$$\exists\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge FT(stG.tG, tG') \vDash \exists\exists R(stG', stH) \tag{rel6}$$

$$\exists\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge SG(stG.tG, tG') \vDash \exists\forall R(stG', stH) \tag{rel7}$$

$$\exists\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge FT(stG.tG, tG') \vDash \exists\exists R(stG', stH) \tag{rel8}$$

$$\forall\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(tG', stG.tG) \vDash \exists\forall R(stG', stH) \tag{rel9}$$

$$\forall\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge FT(tG', stG.tG) \vDash \forall\forall R(stG', stH) \tag{rel10}$$

$$\forall\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(tG', stG.tG) \vDash \exists\exists R(stG', stH) \tag{rel11}$$

$$\forall\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge SG(tG', stG.tG) \vDash \forall\exists R(stG', stH) \tag{rel12}$$

$$\exists\forall R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(tG', stG.tG) \vDash \exists\forall R(stG', stH) \tag{rel13}$$

$$\exists\exists R(stG, stH) \wedge stG' = \langle tG', stG.E \rangle \wedge GI(tG', stG.tG) \vDash \exists\exists R(stG', stH) \tag{rel14}$$

Figure 6: Semantics of spatio-temporal relationships. First part.

$$\forall\forall R(stG, stH) \wedge GI(stG.tG, stH.tG) \vDash \forall\forall R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv1}$$

$$\forall\forall R(stG, stH) \wedge FT(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv2}$$

$$\forall\forall R(stG, stH) \wedge SG(stG.tG, stH.tG) \vDash \exists\forall R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv3}$$

$$\forall\exists R(stG, stH) \wedge GI(stG.tG, stH.tG) \vDash \forall\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv4}$$

$$\forall\exists R(stG, stH) \wedge FT(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv5}$$

$$\exists\forall R(stG, stH) \wedge GI(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv6}$$

$$\exists\forall R(stG, stH) \wedge FT(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv7}$$

$$\exists\forall R(stG, stH) \wedge SG(stG.tG, stH.tG) \vDash \exists\forall R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv8}$$

$$\exists\exists R(stG, stH) \wedge GI(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv9}$$

$$\exists\exists R(stG, stH) \wedge FT(stG.tG, stH.tG) \vDash \exists\exists R'(stH, stG) \quad \text{where } R(G,H) \vDash R'(H,G) \tag{inv10}$$

$$\forall\forall R(stG, stH) \wedge \forall\forall R'(stG, stH) \vDash \forall\forall R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair1}$$

$$\forall\forall R(stG, stH) \wedge \forall\exists R'(stG, stH) \vDash \forall\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair2}$$

$$\forall\forall R(stG, stH) \wedge \exists\forall R'(stG, stH) \vDash \exists\forall R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair3}$$

$$\forall\forall R(stG, stH) \wedge \exists\exists R'(stG, stH) \vDash \exists\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair4}$$

$$\forall\exists R(stG, stH) \wedge \forall\forall R'(stG, stH) \vDash \forall\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair5}$$

$$\forall\exists R(stG, stH) \wedge \exists\forall R'(stG, stH) \vDash \exists\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair6}$$

$$\exists\forall R(stG, stH) \wedge \forall\forall R'(stG, stH) \vDash \exists\forall R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair7}$$

$$\exists\forall R(stG, stH) \wedge \forall\exists R'(stG, stH) \vDash \exists\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair8}$$

$$\exists\exists R(stG, stH) \wedge \forall\forall R'(stG, stH) \vDash \exists\exists R''(stG, stH) \quad \text{where } R(G,H) \wedge R'(G,H) \vDash R''(G,H) \tag{pair9}$$

Figure 7: Semantics of spatio-temporal relationships. Second part.

$$\forall\forall R(stG, stH) \wedge \forall\forall R'(stH, stI) \vDash \forall\forall R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc1}$$

$$\forall\forall R(stG, stH) \wedge \forall\exists R'(stH, stI) \wedge GI(stG.tG, stH.tG) \vDash \exists\exists R''(stG, stI) \text{ where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc2}$$

$$\forall\forall R(stG, stH) \wedge \forall\exists R'(stH, stI) \wedge SG(stG.tG, stH.tG) \vDash \forall\exists R''(stG, stI) \text{ where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc3}$$

$$\forall\forall R(stG, stH) \wedge \exists\forall R'(stH, stI) \wedge GI(stG.tG, stH.tG) \vDash \exists\forall R''(stG, stI) \text{ where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc4}$$

$$\forall\forall R(stG, stH) \wedge \exists\exists R'(stH, stI) \wedge GI(stG.tG, stH.tG) \vDash \exists\exists R''(stG, stI) \text{ where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc5}$$

$$\forall\exists R(stG, stH) \wedge \forall\forall R'(stH, stI) \vDash \forall\exists R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc6}$$

$$\forall\exists R(stG, stH) \wedge \exists\forall R'(stH, stI) \wedge GI(stG.tG, stH.tG) \vDash \exists\exists R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc7}$$

$$\exists\forall R(stG, stH) \wedge \forall\forall R'(stH, stI) \vDash \exists\forall R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc8}$$

$$\exists\forall R(stG, stH) \wedge \forall\exists R'(stH, stI) \wedge SG(stG.tG, stH.tG) \vDash \exists\exists R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc9}$$

$$\exists\exists R(stG, stH) \wedge \forall\forall R'(stH, stI) \vDash \exists\exists R''(stG, stI) \quad \text{where } R(G,H) \wedge R'(H,I) \vDash R''(G,I) \tag{conc10}$$

Figure 8: Semantics of spatio-temporal relationships. Third part.

B Inference Rules for Spatial Relationships

$R(G_1, G_2)$	$?(G_1, G_2)$	
GroupsInto	GroupsInto	✓
	FinerThan	-
	Subgranularity	×
	Partition	-
	CoveredBy	-
	Disjoint	×
	Overlap	×
FinerThan	GroupsInto	-
	FinerThan	✓
	Subgranularity	-
	Partition	-
	CoveredBy	✓
	Disjoint	×
	Overlap	×
Subgranularity	GroupsInto	×
	FinerThan	✓
	Subgranularity	✓
	Partition	×
	CoveredBy	✓
	Disjoint	×
	Overlap	×
Partition	GroupsInto	✓
	FinerThan	✓
	Subgranularity	×
	Partition	✓
	CoveredBy	✓
	Disjoint	×
	Overlap	×

$R(G_1, G_2)$	$?(G_1, G_2)$	
CoveredBy	GroupsInto	-
	FinerThan	-
	Subgranularity	-
	Partition	-
	CoveredBy	✓
	Disjoint	×
	Overlap	×
Disjoint	GroupsInto	×
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	×
	Disjoint	✓
	Overlap	×
Overlap	GroupsInto	×
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	×
	Disjoint	×
	Overlap	✓

Table 1: $\neg G_1 \approx G_2 \wedge R(G_1, G_2) \vdash ?(G_1, G_2)$

$R(G_1, G_2)$	$?(G_2, G_1)$	
GroupsInto	GroupsInto	×
	FinerThan	-
	Subgranularity	-
	Partition	×
	CoveredBy	✓
	Disjoint	×
	Overlap	×
FinerThan	GroupsInto	-
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	-
	Disjoint	×
	Overlap	×
Subgranularity	GroupsInto	✓
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	×
	Disjoint	×
	Overlap	×
Partition	GroupsInto	×
	FinerThan	-
	Subgranularity	×
	Partition	×
	CoveredBy	✓
	Disjoint	×
	Overlap	×

$R(G_1, G_2)$	$?(G_2, G_1)$	
CoveredBy	GroupsInto	-
	FinerThan	-
	Subgranularity	
	Partition	-
	CoveredBy	-
	Disjoint	×
	Overlap	×
Disjoint	GroupsInto	×
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	×
	Disjoint	✓
	Overlap	×
Overlap	GroupsInto	×
	FinerThan	×
	Subgranularity	×
	Partition	×
	CoveredBy	×
	Disjoint	×
	Overlap	✓

Table 2: $\neg G_1 \approx G_2 \wedge R(G_1, G_2) \vdash ?(G_2, G_1)$

$R_1(G_1, G_2)$	$R_2(G_1, G_2)$	$?(G_1, G_2)$
GroupsInto	**FinerThan**	See Partition in Table 1
	Subgranularity	$G_1 = G_2$
	Partition	See Partition in Table 1
	CoveredBy	See GroupsInto+CoveredBy in Table 1
	Disjoint	Impossible
	Overlap	Impossible
FinerThan	**GroupsInto**	See Partition in Table 1
	Subgranularity	See Subgranularity in Table 1
	Partition	See Partition in Table 1
	CoveredBy	See FinerThan in Table 1
	Disjoint	Impossible
	Overlap	Impossible
Subgranularity	**GroupsInto**	$G_1 = G_2$
	FinerThan	See Subgranularity in Table 1
	Partition	$G_1 = G_2$
	CoveredBy	See Subgranularity in Table 1
	Disjoint	Impossible
	Overlap	Impossible
Partition	**GroupsInto**	See Partition in Table 1
	FinerThan	See Partition in Table 1
	Subgranularity	$G_1 = G_2$
	CoveredBy	See Partition in Table 1
	Disjoint	Impossible
	Overlap	Impossible
CoveredBy	**GroupsInto**	See GroupsInto+CoveredBy in Table 1
	FinerThan	See FinerThan in Table 1
	Subgranularity	See Subgranularity in Table 1
	Partition	See Partition in Table 1
	Disjoint	Impossible
	Overlap	Impossible
Disjoint	**GroupsInto**	Impossible
	FinerThan	Impossible
	Subgranularity	Impossible
	Partition	Impossible
	CoveredBy	Impossible
	Overlap	Impossible
Overlap	**GroupsInto**	Impossible
	FinerThan	Impossible
	Subgranularity	Impossible
	Partition	Impossible
	CoveredBy	Impossible
	Disjoint	Impossible

Table 3: $\neg G_1 \approx G_2 \wedge R_1(G_1, G_2) \wedge R_2(G_1, G_2) \vdash ?(G_1, G_2)$

$R(G_1, G_2)$	$S(G_2, G_3)$													
	GroupsInto		**FinerThan**		**SubGranul.**		**Partition**		**CoveredBy**		**Disjoint**		**Overlap**	
GroupsInto	GroupsInto	$\checkmark$	GroupsInto	-	GroupsInto	-	GroupsInto	$\checkmark$	GroupsInto	-	GroupsInto	-	GroupsInto	-
	FinerThan	-[a]	FinerThan	-[b]	FinerThan	-	FinerThan	-[b]	FinerThan	-	FinerThan	$\times$	FinerThan	$\times$
	SubGranul.	$\times$	SubGranul.	-	SubGranul.	-	SubGranul.	$\times$	SubGranul.	-	SubGranul.	$\times$	SubGranul.	$\times$
	Partition	-[a]	Partition	-	Partition	-	Partition	-[b]	Partition	-	Partition	$\times$	Partition	$\times$
	CoveredBy	-	CoveredBy	-	CoveredBy	-	CoveredBy	-[b]	CoveredBy	-	CoveredBy	$\times$	CoveredBy	$\times$
	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	-	Disjoint	$\times$
	Overlap	$\times$	Overlap	-	Overlap	-	Overlap	$\times$	Overlap	-	Overlap	-	Overlap	-
FinerThan	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	$\times$	GroupsInto	-
	FinerThan	-	FinerThan	$\checkmark$	FinerThan	$\checkmark$	FinerThan	$\checkmark$	FinerThan	-	FinerThan	$\times$	FinerThan	-
	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	$\times$	SubGranul.	-
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-	Partition	$\times$	Partition	-
	CoveredBy	-	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\times$	CoveredBy	-
	Disjoint	-	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\checkmark$	Disjoint	-
	Overlap	-	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	-
SubGranul.	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	$\times$	GroupsInto	-
	FinerThan	-	FinerThan	$\checkmark$	FinerThan	$\checkmark$	FinerThan	$\checkmark$	FinerThan	-	FinerThan	$\times$	FinerThan	-
	SubGranul.	-	SubGranul.	-	SubGranul.	$\checkmark$	SubGranul.	-	SubGranul.	-	SubGranul.	$\times$	SubGranul.	-
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-	Partition	$\times$	Partition	-
	CoveredBy	-	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\checkmark$	CoveredBy	$\times$	CoveredBy	-
	Disjoint	-	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\times$	Disjoint	$\checkmark$	Disjoint	-
	Overlap	-	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	$\times$	Overlap	-

[a] $\checkmark$ if $CoveredBy(G_1, G_2)$ and $CoveredBy(G_2, G_3)$
[b] $\checkmark$ if $CoveredBy(G_1, G_2)$

Table 4: $\neg G_1 \approx G_3 \wedge R(G_1, G_2) \wedge S(G_2, G_3) \vdash ?(G_1, G_3)$. First part.

$R(G_1, G_2)$	$S(G_2, G_3)$													
	GroupsInto		**FinerThan**		**SubGranul.**		**Partition**		**CoveredBy**		**Disjoint**		**Overlap**	
Partition	GroupsInto	✓	GroupsInto	-	GroupsInto	-	GroupsInto	✓	GroupsInto	-	GroupsInto	×	GroupsInto	×
	FinerThan	-	FinerThan	✓	FinerThan	✓	FinerThan	✓	FinerThan	-	FinerThan	×	FinerThan	×
	SubGranul.	×	SubGranul.	-	SubGranul.	-	SubGranul.	×	SubGranul.	-	SubGranul.	×	SubGranul.	×
	Partition	-	Partition	-	Partition	-	Partition	✓	Partition	-	Partition	×	Partition	×
	CoveredBy	-	CoveredBy	✓	CoveredBy	✓	CoveredBy	✓	CoveredBy	✓	CoveredBy	×	CoveredBy	×
	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	✓	Disjoint	×
	Overlap	×	Overlap	×	Overlap	×	Overlap	×	Overlap	×	Overlap	×	Overlap	✓
CoveredBy	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	×	GroupsInto	-
	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	×	FinerThan	-
	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	-	SubGranul.	×	SubGranul.	-
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-	Partition	×	Partition	-
	CoveredBy	-	CoveredBy	✓	CoveredBy	✓	CoveredBy	✓	CoveredBy	✓	CoveredBy	×	CoveredBy	-
	Disjoint	-	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	✓	Disjoint	-
	Overlap	-	Overlap	×	Overlap	×	Overlap	×	Overlap	×	Overlap	×	Overlap	-
Disjoint	GroupsInto	×	GroupsInto	×	GroupsInto	×	GroupsInto	×	GroupsInto	×	GroupsInto	-	GroupsInto	×
	FinerThan	×	FinerThan	-	FinerThan	-	FinerThan	×	FinerThan	-	FinerThan	-	FinerThan	-
	SubGranul.	×	SubGranul.	-	SubGranul.	-	SubGranul.	×	SubGranul.	-	SubGranul.	-	SubGranul.	-
	Partition	×	Partition	×	Partition	×	Partition	×	Partition	×	Partition	-	Partition	×
	CoveredBy	×	CoveredBy	-	CoveredBy	-	CoveredBy	×	CoveredBy	-	CoveredBy	-	CoveredBy	-
	Disjoint	✓	Disjoint	-	Disjoint	-	Disjoint	✓	Disjoint	-	Disjoint	-	Disjoint	-
	Overlap	×	Overlap	-	Overlap	-	Overlap	×	Overlap	-	Overlap	-	Overlap	-
Overlap	GroupsInto	-	GroupsInto	×	GroupsInto	×	GroupsInto	×	GroupsInto	×	GroupsInto	-	GroupsInto	-
	FinerThan	×	FinerThan	-	FinerThan	-	FinerThan	×	FinerThan	-	FinerThan	×	FinerThan	-
	SubGranul.	×	SubGranul.	-	SubGranul.	-	SubGranul.	×	SubGranul.	-	SubGranul.	×	SubGranul.	-
	Partition	×	Partition	×	Partition	×	Partition	×	Partition	×	Partition	×	Partition	-
	CoveredBy	×	CoveredBy	-	CoveredBy	-	CoveredBy	×	CoveredBy	-	CoveredBy	×	CoveredBy	-
	Disjoint	-	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	×	Disjoint	-	Disjoint	-
	Overlap	-	Overlap	-	Overlap	-	Overlap	✓	Overlap	-	Overlap	-	Overlap	-

Table 5: $\neg G_1 \approx G_3 \wedge R(G_1, G_2) \wedge S(G_2, G_3) \vdash ?(G_1, G_3)$. Second part.

	GroupsInto		FinerThan		Subgranularity		Partition		CoveredBy		Disjoint		Overlap	
$G' = Grouping(G, P, L)$	$G' \to G$	$\times$	$G' \to G$	$\times$	$G' \to G$	$\times$	$G' \to G$	$\times$	$G' \to G$	$\checkmark$	$G' \to G$	$\times$	$G' \to G$	$\times$
	$G \to G'$	$\checkmark$	$G \to G'$	$\checkmark$	$G \to G'$	$\times$	$G \to G'$	$\checkmark$	$G \to G'$	$\checkmark$	$G \to G'$	$\times$	$G \to G'$	$\times$
$G' = Combine(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	$\times$	$G' \to G_1$	$\times$
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	$\checkmark$	$G' \to G_2$	$\times$	$G' \to G_2$	$\times$
	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	$\times$	$G_1 \to G'$	$\times$
	$G_2 \to G'$	$\checkmark$	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	$\times$	$G_2 \to G'$	$\times$
$G' = Subset(G, S)$	$G' \to G$	$\times$	$G' \to G$	$\checkmark$	$G' \to G$	$\checkmark$	$G' \to G$	$\times$	$G' \to G$	$\checkmark$	$G' \to G$	$\times$	$G' \to G$	$\times$
	$G \to G'$	$\checkmark$	$G \to G'$	$\times$	$G \to G'$	$\times$	$G \to G'$	$\times$	$G \to G'$	$\times$	$G \to G'$	$\times$	$G \to G'$	$\times$
$G' = Subset(g, G, R)$	$G' \to G$	-	$G' \to G$	$\checkmark$	$G' \to G$	$\checkmark$	$G' \to G$	-	$G' \to G$	$\checkmark$	$G' \to G$	$\times$	$G' \to G$	$\times$
	$G \to G'$	$\checkmark$	$G \to G'$	-	$G \to G'$	-	$G \to G'$	-	$G \to G'$	-	$G \to G'$	$\times$	$G \to G'$	$\times$
$G' = SelectContain(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	$\checkmark$	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	$\times$	$G' \to G_1$	$\times$
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	$\times$	$G' \to G_2$	-
	$G_1 \to G'$	$\checkmark$	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	$\times$	$G_1 \to G'$	$\times$
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	$\times$	$G_2 \to G'$	-
$G' = SelectInside(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	$\checkmark$	$G' \to G_1$	-	$G' \to G_1$	$\checkmark$	$G' \to G_1$	$\times$	$G' \to G_1$	$\times$
	$G' \to G_2$	-	$G' \to G_2$	$\checkmark$	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	$\checkmark$	$G' \to G_2$	$\times$	$G' \to G_2$	$\times$
	$G_1 \to G'$	$\checkmark$	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	$\times$	$G_1 \to G'$	$\times$
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	$\times$	$G_2 \to G'$	$\times$

Table 6: Rules to infer relationships between operands and result of an operation. First part.

	GroupsInto		FinerThan		Subgranularity		Partition		CoveredBy		Disjoint		Overlap	
$G' = SelectIntersect(G_1, G_2)$	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	×	$G' \rightarrow G_1$	×
	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	✓
	$G_1 \rightarrow G'$	✓	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	×	$G_1 \rightarrow G'$	×
	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	✓
$G' = Union(G_1, G_2)$	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	×	$G' \rightarrow G_1$	×
	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×
	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	✓	$G_1 \rightarrow G'$	✓	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	✓	$G_1 \rightarrow G'$	×	$G_1 \rightarrow G'$	×
	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	✓	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×
$G' = Intersect(G_1, G_2)$	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	×	$G' \rightarrow G_1$	×
	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	✓	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	-	$G' \rightarrow G_2$	✓	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×
	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	×	$G_1 \rightarrow G'$	×
	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	-	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×
$G' = Difference(G_1, G_2)$	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	-	$G' \rightarrow G_1$	✓	$G' \rightarrow G_1$	×	$G' \rightarrow G_1$	×
	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	×	$G' \rightarrow G_2$	✓	$G' \rightarrow G_2$	×
	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	-	$G_1 \rightarrow G'$	×	$G_1 \rightarrow G'$	×
	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	×	$G_2 \rightarrow G'$	✓	$G_2 \rightarrow G'$	×

Table 7: Rules to infer relationships between operands and result of an operation. Second part.

C Inference Rules for Temporal Relationships

$R(G_1, G_2)$	$?(G_1, G_2)$	
GroupsInto	GroupsInto	✓
	FinerThan	-
	Subgranularity	×
	Partition	-
	GroupsPeriodInto	-
FinerThan	GroupsInto	-
	FinerThan	✓
	Subgranularity	-
	Partition	-
	GroupsPeriodInto	-
Subgranularity	GroupsInto	×
	FinerThan	✓
	Subgranularity	✓
	Partition	×
	GroupsPeriodInto	×
Partition	GroupsInto	✓
	FinerThan	✓
	Subgranularity	×
	Partition	✓
	GroupsPeriodInto	-
GroupsPeriodInto	GroupsInto	✓
	FinerThan	-
	Subgranularity	×
	Partition	-
	GroupsPeriodInto	✓

Table 8: $\neg G_1 \approx G_2 \wedge R(G_1, G_2) \vdash ?(G_1, G_2)$

$R(G_1, G_2)$	$?(G_2, G_1)$	
GroupsInto	GroupsInto	×
	FinerThan	-
	Subgranularity	-
	Partition	×
	GroupsPeriodInto	×
FinerThan	GroupsInto	-
	FinerThan	×
	Subgranularity	×
	Partition	×
	GroupsPeriodInto	-
Subgranularity	GroupsInto	✓
	FinerThan	×
	Subgranularity	×
	Partition	×
	GroupsPeriodInto	-
Partition	GroupsInto	×
	FinerThan	×
	Subgranularity	×
	Partition	×
	GroupsPeriodInto	×
GroupsPeriodInto	GroupsInto	×
	FinerThan	-
	Subgranularity	-
	Partition	×
	GroupsPeriodInto	×

Table 9: $\neg G_1 \approx G_2 \wedge R(G_1, G_2) \vdash ?(G_2, G_1)$

$R_1(G_1, G_2)$	$R_2(G_1, G_2)$	$?(G_1, G_2)$
GroupsInto	**FinerThan**	See Partition in Table 8
	Subgranularity	$G_1 = G_2$
	Partition	See Partition in Table 8
	GroupsPeriodInto	See GroupsPeriodInto in Table 8
FinerThan	**GroupsInto**	See Partition in Table 8
	Subgranularity	See SubGranularity in Table 8
	Partition	See Partition in Table 8
	GroupsPeriodInto	See GroupsPeriodInto+Partition in Table 8
Subgranularity	**GroupsInto**	$G_1 = G_2$
	FinerThan	See SubGranularity in Table 8
	Partition	$G_1 = G_2$
	GroupsPeriodInto	$G_1 = G_2$
Partition	**GroupsInto**	See Partition in Table 8
	FinerThan	See Partition in Table 8
	Subgranularity	$G_1 = G_2$
	GroupsPeriodInto	See GroupsPeriodInto+Partition in Table 8
GroupsPeriodInto	**GroupsInto**	See GroupsPeriodInto in Table 8
	FinerThan	See GroupsPeriodInto+Partition in Table 8
	Subgranularity	$G_1 = G_2$
	Partition	See GroupsPeriodInto+Partition in Table 8

Table 10: $\neg G_1 \approx G_2 \wedge R_1(G_1, G_2) \wedge R_2(G_1, G_2) \vdash ?(G_1, G_2)$

$R(G_1, G_2) \setminus S(G_2, G_3)$	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
GroupsInto	GroupsInto	✓	GroupsInto	-	GroupsInto	-	GroupsInto	✓	GroupsInto	✓
	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-
	Subgranularity	×	Subgranularity	-	Subgranularity	-	Subgranularity	×	Subgranularity	×
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-
	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-
FinerThan	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-
	FinerThan	-	FinerThan	✓	FinerThan	✓	FinerThan	✓	FinerThan	-
	Subgranularity	-	Subgranularity	-	Subgranularity	-	Subgranularity	-	Subgranularity	-
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-
	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-
Subgranularity	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-	GroupsInto	-
	FinerThan	-	FinerThan	✓	FinerThan	✓	FinerThan	✓	FinerThan	-
	Subgranularity	-	Subgranularity	-	Subgranularity	✓	Subgranularity	-	Subgranularity	-
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-
	GroupsPeriodInto	-	GroupsPeriodInto	×	GroupsPeriodInto	×	GroupsPeriodInto	×	GroupsPeriodInto	-
Partition	GroupsInto	✓	GroupsInto	-	GroupsInto	-	GroupsInto	✓	GroupsInto	✓
	FinerThan	-	FinerThan	✓	FinerThan	✓	FinerThan	✓	FinerThan	-
	Subgranularity	×	Subgranularity	-	Subgranularity	-	Subgranularity	×	Subgranularity	×
	Partition	-	Partition	-	Partition	-	Partition	✓	Partition	-
	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-
GroupsPeriodInto	GroupsInto	✓	GroupsInto	-	GroupsInto	-	GroupsInto	✓	GroupsInto	✓
	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-	FinerThan	-
	Subgranularity	×	Subgranularity	-	Subgranularity	-	Subgranularity	×	Subgranularity	×
	Partition	-	Partition	-	Partition	-	Partition	-	Partition	-
	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	-	GroupsPeriodInto	✓

Table 11: $\neg G_1 \approx G_3 \wedge R(G_1, G_2) \wedge S(G_2, G_3) \vdash ?(G_1, G_3)$

	GroupsInto		FinerThan		Subgranularity		ShiftEquiv		Partition		GroupsPeriodInto	
$G' = Group_m(G)$	$G' \to G$	×	$G' \to G$	×	$G' \to G$	×	$G' \to G$	×	$G' \to G$	×	$G' \to G$	×
	$G \to G'$	✓	$G \to G'$	✓	$G \to G'$	×	$G \to G'$	×	$G \to G'$	✓	$G \to G'$	✓ (1, m)
$G' = AlteringTick^m_{l,k}(G_1, G_2)$	$G' \to G_1$	×	$G' \to G_1$	×	$G' \to G_1$	×	$G' \to G_1$	×	$G' \to G_1$	×	$G' \to G_1$	×
	$G' \to G_2$	×	$G' \to G_2$	×	$G' \to G_2$	×	$G' \to G_2$	×	$G' \to G_2$	×	$G' \to G_2$	×
	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×
	$G_2 \to G'$	✓	$G_2 \to G'$	✓	$G_2 \to G'$	×	$G_2 \to G'$	×	$G_2 \to G'$	✓	$G_2 \to G'$	✓
$G' = Shift_m(G)$	$G' \to G$	✓	$G' \to G$	✓	$G' \to G$	✓	$G' \to G$	✓	$G' \to G$	✓	$G' \to G$	✓ (1,1)
	$G \to G'$	✓	$G \to G'$	✓	$G \to G'$	✓	$G \to G'$	✓	$G \to G'$	✓	$G \to G'$	✓ (1,1)
$G' = Combine(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-
	$G_2 \to G'$	✓	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-
$G' = AnchoredGrouping(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	✓	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	✓	$G_1 \to G'$	-
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-
$G' = Subset^n_m(G)$	$G' \to G$	×	$G' \to G$	✓	$G' \to G$	✓	$G' \to G$	×	$G' \to G$	×	$G' \to G$	×
	$G \to G'$	✓	$G \to G'$	×	$G \to G'$	×	$G \to G'$	×	$G \to G'$	×	$G \to G'$	✓
$G' = SelectDown^l_k(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	✓	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	✓	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-

Table 12: Rules to infer relationships between operands and result of an operation. First part.

	GroupsInto		FinerThan		Subgranularity		ShiftEquiv		Partition		GroupsPeriodInto	
$G' = SelectUp(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	✓	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-
$G' = SelectByIntersect_k^l(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	✓	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-
$G' = Union(G_1, G_2)$	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-[a]
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	-	$G_1 \to G'$	✓	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-
$G' = Intersection(G_1, G_2)$	$G' \to G_1$	×	$G' \to G_1$	✓	$G' \to G_1$	✓	$G' \to G_1$	×	$G' \to G_1$	×	$G' \to G_1$	×
	$G' \to G_2$	×	$G' \to G_2$	✓	$G' \to G_2$	✓	$G' \to G_2$	×	$G' \to G_2$	×	$G' \to G_2$	×
	$G_1 \to G'$	✓	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	×	$G_1 \to G'$	-[a]
	$G_2 \to G'$	✓	$G_2 \to G'$	×	$G_2 \to G'$	×	$G_2 \to G'$	×	$G_2 \to G'$	×	$G_2 \to G'$	-[a]
$G' = Difference(G_1, G_2)$	$G' \to G_1$	-	$G' \to G_1$	✓	$G' \to G_1$	✓	$G' \to G_1$	-	$G' \to G_1$	-	$G' \to G_1$	-
	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-	$G' \to G_2$	-
	$G_1 \to G'$	✓	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-	$G_1 \to G'$	-[a]
	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-	$G_2 \to G'$	-

[a] ✓ if G_1, G_2 are periodic

Table 13: Rules to infer relationships between operands and result of an operation. Second part.

D Inference Rules for Spatio-temporal Relationships

Q	$S(tG, tG')$											
	$\approx$		**GroupsInto**		**FinerThan**		**SubGranul.**		**Partition**		**GroupsPeriodInto**	
∀∀	∀∀	✓	∀∀	✓	∀∀	-	∀∀	-	∀∀	✓	∀∀	✓
	∀∃	✓	∀∃	✓	∀∃	-	∀∃	-	∀∃	✓	∀∃	✓
	∃∀	✓	∃∀	✓	∃∀	-	∃∀	✓	∃∀	✓	∃∀	✓
	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
∀∃	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	✓	∀∃	✓	∀∃	-	∀∃	-	∀∃	✓	∀∃	✓
	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
∃∀	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
	∃∀	✓	∃∀	-	∃∀	-	∃∀	✓	∃∀	-	∃∀	-
	∃∃	✓	∃∃	-	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	-
∃∃	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
	∃∃	✓	∃∃	-	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	-

Table 14: $QR(stG, stH) \wedge S(tG, tG') \vdash ?R(stG', stH)$

Q	$S(tG', tG)$											
	$\approx$		**GroupsInto**		**FinerThan**		**SubGranul.**		**Partition**		**GroupsPeriodInto**	
∀∀	∀∀	✓	∀∀	-	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	-
	∀∃	✓	∀∃	-	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	-
	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
∀∃	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	✓	∀∃	-	∀∃	-	∀∃	✓	∀∃	-	∀∃	-
	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
	∃∃	✓	∃∃	✓	∃∃	-	∃∃	✓	∃∃	✓	∃∃	✓
∃∀	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
	∃∀	✓	∃∀	✓	∃∀	-	∃∀	-	∃∀	✓	∃∀	✓
	∃∃	✓	∃∃	✓	∃∃	-	∃∃	-	∃∃	✓	∃∃	✓
∃∃	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
	∃∃	✓	∃∃	✓	∃∃	-	∃∃	-	∃∃	✓	∃∃	✓

Table 15: $QR(stG, stH) \wedge S(tG', tG) \vdash ?R(stG', stH)$

Q	$S(tG', tG)$											
	$\approx$		**GroupsInto**		**FinerThan**		**SubGranul.**		**Partition**		**GroupsPeriodInto**	
$\forall\forall$	$\forall\forall$	$\checkmark$	$\forall\forall$	$\checkmark$	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	$\checkmark$	$\forall\forall$	$\checkmark$
	$\forall\exists$	$\checkmark$	$\forall\exists$	$\checkmark$	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	$\checkmark$	$\forall\exists$	$\checkmark$
	$\exists\forall$	$\checkmark$	$\exists\forall$	$\checkmark$	$\exists\forall$	-	$\exists\forall$	$\checkmark$	$\exists\forall$	$\checkmark$	$\exists\forall$	$\checkmark$
	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$
$\forall\exists$	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-
	$\forall\exists$	$\checkmark$	$\forall\exists$	$\checkmark$	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	$\checkmark$	$\forall\exists$	$\checkmark$
	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-
	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$
$\exists\forall$	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-
	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-
	$\exists\forall$	$\checkmark$	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	$\checkmark$	$\exists\forall$	-	$\exists\forall$	-
	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$
$\exists\exists$	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-	$\forall\forall$	-
	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-	$\forall\exists$	-
	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-	$\exists\forall$	-
	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$	$\exists\exists$	$\checkmark$

Table 16: $QR(stG, stH) \wedge S(tG, tH) \vdash ?R'(stH, stG)$. R' is such that $R(sG, sH) \vdash R'(sH, sG)$

Q	Q'	?
$\forall\forall$	$\forall\forall$	$\forall\forall R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\forall\exists$	$\forall\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\exists\forall$	$\exists\forall R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\exists\exists$	$\exists\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
$\forall\exists$	$\forall\forall$	$\forall\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\forall\exists$	-
	$\exists\forall$	$\exists\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\exists\exists$	-
$\exists\forall$	$\forall\forall$	$\exists\forall R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\forall\exists$	$\exists\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\exists\forall$	-
	$\exists\exists$	-
$\exists\exists$	$\forall\forall$	$\exists\exists R''$, where $R(G,H) \wedge R'(G,H) \vdash R''(G,H)$
	$\forall\exists$	-
	$\exists\forall$	-
	$\exists\exists$	-

Table 17: $QR(stG, stH) \wedge Q'R'(stG, stH) \vdash ?R''(stG, stH)$. R'' is such that $R(sG, sH) \wedge R'(sG, sH) \vdash R''(sG, sH)$

		∀∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∀	**GroupsInto**	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Subgranularity	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓	∀∀	✓
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 18: $Q_1 R_1(stG, stH) \wedge Q_2 R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3 R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∀∃									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∀	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 19: $Q_1R_1(stG, stH) \wedge Q_2R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∃∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∀	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 20: $Q_1R_1(stG, stH) \wedge Q_2R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∃∃									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∀	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 21: $Q_1R_1(stG, stH) \wedge Q_2R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∀∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∃	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓	∀∃	✓
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 22: $Q_1 R_1(stG, stH) \wedge Q_2 R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3 R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∃∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∀∃	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 23: $Q_1R_1(stG, stH) \wedge Q_2R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∀∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∃∀	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓	∃∀	✓
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 24: $Q_1R_1(stG, stH) \wedge Q_2R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∀∃									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∃∀	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	-	∃∃	-	∃∃	-	∃∃	-	∃∃	-

Table 25: $Q_1 R_1(stG, stH) \wedge Q_2 R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3 R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

		∀∀									
		T_2									
	T_1	**GroupsInto**		**FinerThan**		**Subgranularity**		**Partition**		**GroupsPeriodInto**	
∃∃	**GroupsInto**	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	FinerThan	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Subgranularity	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	Partition	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓
	GroupsPeriodInto	∀∀	-	∀∀	-	∀∀	-	∀∀	-	∀∀	-
		∀∃	-	∀∃	-	∀∃	-	∀∃	-	∀∃	-
		∃∀	-	∃∀	-	∃∀	-	∃∀	-	∃∀	-
		∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓	∃∃	✓

Table 26: $Q_1 R_1(stG, stH) \wedge Q_2 R_2(stH, stI) \wedge T_1(tG, tH) \wedge T_2(tH, tI) \Rightarrow Q_3 R_3(stG, stI)$. R_3 is such that $R_1(stG.E(t), stH.E(t)) \wedge R_2(stH.E(t), stI.E(t)) \vdash R_3(stG.E(t), stI.E(t))$, applying the concatenation rules for relationships between spatial granularities.

Q	Q'			
	$\forall\forall$	$\forall\exists$	$\exists\forall$	$\exists\exists$
$\forall\forall$	✓	✓	✓	✓
$\forall\exists$	✓	-	✓	-
$\exists\forall$	✓	✓	-	-
$\exists\exists$	✓	-	-	-

Table 27: $QR(stG, stH) \wedge Q'\neg R(stG, stH) \vdash \bot$

Q	**Q'**
$\forall\forall$	$\exists\exists$
$\forall\exists$	$\exists\forall$
$\exists\forall$	$\forall\exists$
$\exists\exists$	$\forall\forall$

Table 28: $QR(stG, stH) \vdash \ldots \vdash \bot \vdash Q'\neg R(stG, stH)$

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