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Investigation of the Tradeoff between Expressiveness and Complexity in Description Logics with Spatial Operators

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alla mia famiglia
per avermi permesso
di assentarmi dal mondo

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Contents

List of Tables

List of Figures

1

Introduction

1.1 Motivation

In the past decades, several scholars have posed their attention upon different aspects of expressiveness in Terminological Reasoning. Mostly, those investigations focused on the reduction of the usage scope for logical operators, as in the case of atomic negation in $\mathcal{AL}$. There exist numerous and deep analyses in the current literature of Description Logic that study the effects of extending them to include aspects that are not explicit, such as time, relational algebra aspects, or spatial reasoning. In particular the combination of Terminological Reasoning and Spatial Knowledge Representation techniques has received some attention only in the past ten years. The purpose of these approaches is to provide formal tools that allow Description Logics users to model concepts restricted by means of topological relations. We can consider the definition of the concept of "Italian town" that can be obtained by the role "part of". We are specifically interested in guaranteeing that, since Italy is part of Europe, then any "Italian town" is also a "European town", being such inference provided by means of the algebraic properties of the relation "part of". We henceforth denominate a terminological language that includes topological operators *Description Logic with Spatial Operators*, shortly DLSO.

An application of DLSOs is the implementation of spatial reasoners supporting interoperability of Geographic Information Systems (GISs). A set of databases is said to be interoperable if data manipulation operations performed on those databases return coherent results independently of the schemas, the architecture and the formats of the single databases. Making databases interoperable is quite a demanding task to be achieved in current database architectural research, and numerous approaches to such problems have been designed that make direct use of the notion of Formal Ontology.

Formal Ontologies are engineering objects, generally designed in some decidable logic language, that constrain the behavior of a finite set of terms that common sense suggests to be relevant in a specific domain of knowledge. Formal Ontologies are often defined in Description Logic, for several different reasons, and mostly because those languages are not only decidable but also scalable, in the sense that the reduction of usage scope of some operators makes the language itself tractable, and moreover because there exist effective and efficient implementations of those languages. In the case of interoperability issues for GISs, the implementation of a standard ontological layer shared by a set of GISs, used as information integration tool for geographic entities stored in the single

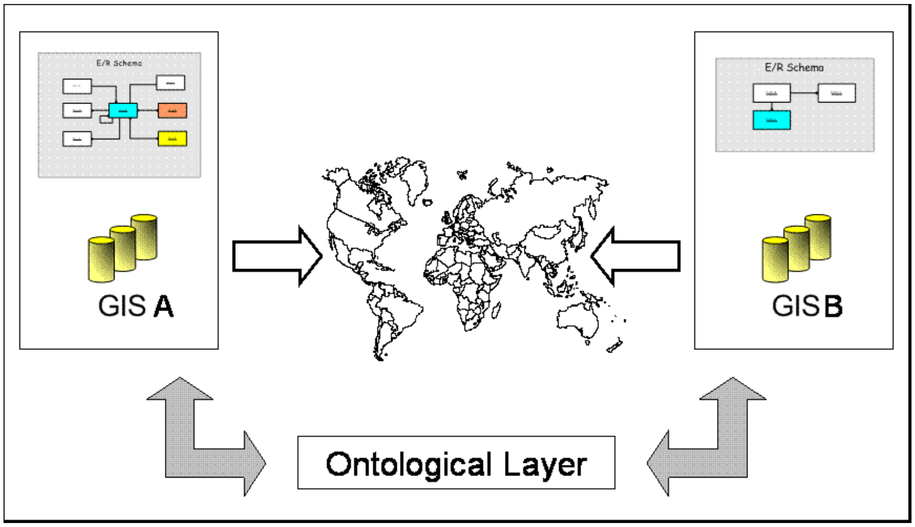

Fig. 1.1. A general ontological layer is a good candidate as *standard user interface* in order to achieve the interoperability between GISs.

GISs (see Figure 1.1) can be viewed as a problem of tradeoff between expressiveness and complexity for DLSOs. Indeed, the named ontological layer can be developed with actual Description Logics with Spatial Operators that are in fact a family of expressive formalisms that present not tractable worst-case complexity or even undecidability.

However there are domains of application for which such formalisms are "too" expressive. For this reason, it is important to conduct an accurate investigation to understand the behavior of the complexity of these formalisms with respect to the expressiveness. In particular we found that this area of research lacks of an exhaustive investigation on complexity required by each spatial reasoning task and by each spatial operator: a sort of taxonomy of complexity and expressiveness of spatial operators integrated within description logics.

1.2 The problem of investigating DLSOs

The term Description Logics, henceforth a DL, describes a family of Knowledge Representation formalisms. These formalisms aim at representing a knowledge domain, first defining the main concepts involved in it, and then using this structure to make assertions on the individuals that populate the domain. For instance we can define "formally" the concept of *woman* as follows:

woman **is defined as** *female* and *human*

It is possible to use the notion of *woman* in defining other complex notions (as *mother*) and make assertions on specific individuals as follows:

mother **is defined as** *woman* and *having at least one child*

Linda **is a** *mother*

DLs are provided with formal, logic-based semantics and are used in knowledge-based systems to represent and reason about conceptual and terminological knowledge of a problem domain (see Chapter 3 for formal definitions). DLs are based on the notions of concepts (classes, unary predicates) and roles (binary relations) and are characterized mainly by set operators that allow complex concepts and roles to be built from atomic ones. A specific DL is mainly characterized by the set of constructors it provides to build more complex concepts and roles out of atomic ones. A DL system allows concept descriptions to be interrelated and implicit knowledge can be derived from the explicitly represented knowledge using inference services. In particular a DL system must infer knowledge about classification of concepts and individuals. The main reasoning task for terminological languages is to check the consistency of concept definitions. Usually reasoning tasks exhibit high computational costs; particularly the problem of checking consistency of spatial concept descriptions is often undecidable.

The problem of representing spatial knowledge has been largely investigated by the AI community and deals with the formalization of relevant spatial information. One of the most important aspects of this investigation is the definition of *relevant spatial information*: the first question is if we are interested in qualitative rather then quantitative information. In my thesis I follow the most important research field of spatial reasoning that considers a qualitative approach to the problem of representing and reasoning about spatial knowledge. For this reason I define spatial extensions of DL based on the most referenced Qualitative Spatial Reasoning frameworks introduced formally in Chapter 2.

The aim of my thesis is to understand the expressive power of DLs extended with spatial operators and consider the computational drawbacks of these extensions. In particular I consider two main hybridization techniques in order to improve the expressive power of basic terminological languages. The first technique is based on the definition of specific set of axioms (called *role axioms*) to state explicitly the formal properties of spatial relations (for instance the transitivity of a *part of* relation). This axiomatic formalization of spatial relations ensure the soundness w.r.t. the corresponding spatial formalism. The second hybridization technique relies on the idea of *external concrete domain*: the logic is extended by means of an external well structured domain equipped with specific predicates. The challenge is the definition of spatial concrete domains in order to improve the expressiveness of the extended DL w.r.t. formal spatial representation formalisms that turn out to be of some interest for practical applications.

The results which have been obtained during the investigation carried out in my doctoral program are marked up in the text henceforth by a star ($*$). For all the results that can be found in the current literature, I used two different conventions, depending upon the context. In those contexts in which, for the purposes of argumentation it is necessary to include results from many authors I cite the scholar within the result itself, providing a reference to the relevant publication; when a single stream of results of the same provenance have been used through a section, I cite the relevant publication in front of that stream and avoid repetition on each separate occasion.

1.3 Structure of the Thesis

- In **Chapter 2** we provide a very general overview of the problem of representing spatial information and we introduce the basics of Qualitative Spatial Reasoning. We consider in particular two aspects of space, topology and direction, and provide the formal definition of the related most important theoretical frameworks.

- In **Chapter 3** we introduce the basics of description logics: their syntax, semantics, and standard reasoning problems. We also provide an introduction to family of description logics equipped with concrete domains.
- The **Chapter 4** is about the hybridization technique with fixed role box.
  - In Section 4.1 we present Wessel's idea of a spatial logic extended with a composition-based role box defined for RCC relations [140]. Given the undecidability of the logics $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ (proved in Section 4.2), we provide a tableau procedure to gain under some conditions a para-decidability of the two languages.
  - In Section 4.2 we present a correspondence theory between $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ and some undecidable modal logics defined by Lutz and Wolter, proving the undecidability of the two description logics
  - In Section 4.3 we generalize Wessel's idea of RCC composition-based role box to general constraint languages based on finite relation algebras. We provide the definition of new hybrid languages based on Ligozat's Cardinal Direction Calculus (decidable $\mathcal{DL}$), the Rectangle Algebra (undecidable $\mathcal{DL}$) and on a combined topological and directional framework presented by Li (undecidable $\mathcal{DL}$).
- In **Chapter 5** we investigate spatial concrete domains. Each section refers to a different QSRR formalism and investigate properties of both the corresponding general concrete domain and constraint system, when definable.
- In **Chapter 6** we provide a systematic analysis of expressivity and computational properties of the considered spatial description logics.
  - Section 6.1 provides a brief introduction to the tradeoff analysis.
  - In Section 6.2 we provide examples of the expressivity of the spatial languages and an analysis of their computational properties. We also defined the generalization of paraconsistency for a generic $\mathcal{ALCI}_{\mathsf{C}}$ logic with strong EQ semantics.
  - In Section 6.3 we analyze spatial hybridizations based on the concrete domains and constraint systems investigated in Chapter 5. We present the pros and cons of all the different approaches and consider the computational results.
- In **Chapter 7** we give a summary of our results and discuss possible future extensions of our work.

2

Qualitative Spatial Representation and Reasoning

2.1 Introduction

The research field of Artificial Intelligence deals with the problems of representing and storing information and reasoning on the stored knowledge. The study of the human process of information management and reasoning represent an important research field in AI, particularly the study of knowledge management. The books written by Johnson-Laird in 1983 [80] and 1998 [81] can be consider seminal works for more recent papers on spatial mental models like [82], [78] and [111]. As a matter of fact the human-being is the paradigm of non-formal knowledge and uncertainty management systems. The main difference between a person and a computing system is the use of a natural language versus a formal language and the capability of irrational and intuitive reasoning versus a strictly regulated computation. One of the main objectives of AI research is the formalization of knowledge representation and of reasoning processes for artificial, non-human systems; nevertheless the choice of knowledge frameworks that worth to be investigated is heavily influenced by human information management processes. For this reason research in AI aims to formalize knowledge focusing in particular on formal representation and reasoning systems that handle "commonsense" information. AI research is focused on "vertical" investigation of knowledge representation considering the partition into different conceptual domains. This is given principally by the high computational cost both in time and space of the reasoning process. A "vertical" analysis focused on a single conceptual domain allows the definition of optimization techniques for both the storing and the reasoning phases.

The field of Spatial Reasoning is an example of *vertical* analysis and *human-tuned* research: the core research activity is focused on qualitative aspects of space rather that quantitative. In the last twenty years Artificial Intelligence researchers showed a big interest for the representation of spatial knowledge and in particular in qualitative spatial knowledge. There are different approaches to this problem and many possible applications: from Geographical Information Systems (henceforth GISs) to robotics, from navigation to biology.

Qualitative spatial knowledge representation and reasoning is an important field of research in artificial intelligence. As pointed out by Renz and Nebel in [117]:

> [...]in qualitative spatial reasoning it is common to consider a particular aspect of space such as topology, direction, or distance and to develop a system of

qualitative relationships between spatial entities which cover this aspect of space to some degree and which appear to be useful from an applicational or from a cognitive perspective.

One of the most important field of application for QSR techniques is represented by the GISs that, as pointed out by Cohn and Hazarika in [36], do not sufficiently support intuitive human-computer interaction. "User may wish to abstract from numerical data and specify a query in a way which is essentially qualitative" [36]. For this reason my work aims to embed qualitative spatial knowledge in knowledge representation systems that might be used in the future as an intelligent interface between the user and the program.

In my thesis I consider two aspects of space, topology and direction, and the related most important theoretical frameworks. In particular as topological frameworks I consider the Region Connection Calculus of Randell, Cui and Cohn [113], the Boolean Region Connection Calculus [143], the set of spatial relations defined by Clementini et al. in [33]. With respect to direction information I consider the following theoretical frameworks: the Cardinal Direction Calculus defined by Ligozat in [89], the Rectangle Algebra as studied by Balbiani et al. in [16], the directional calculus investigated by Skiadopoulos et al. in [127] and the combined set of topological and directional relations investigated by Li in [88].

2.2 Commonsense Spatial Knowledge

Spatial information deals with a variety of aspects of "spatial objects" called spatial features. Researchers related to the field of spatial cognition and cognitive science consider many basic features that differ from the basis of QSRR research. In a work by Amitabha Mukerjee and Mausoom Sarkar in [105] basic spatial features are for instance simple shaped objects like "square", "circle" and simple measures like "small"or "big". This research field aims to investigate human spatial competence pointing out how spatial information are represented, as well as the mechanisms that are available for manipulating those representations. This is a fascinating research based in many cases on evidence from neuropsychological investigations of human spatial ability (see for instance [68]). Nevertheless the purpose of automated spatial reasoning capable to aswer to complex "commonsense" queries requires a different approach based on different basic spatial features. The main difference between Spatial Cognition and Qualitative Spatial Reasoning may be found in the effort of QSR researchers to optimize both sides of the problem: the choice of a knowledge representation system proper for a computer and the choice of qualitative features expressive for a human user.

The purpose of investigating all possible spatial frameworks and formal representation systems is very ambitious. For this reason during the research for my thesis I needed to narrow the field of investigation to 2D space. Then I asked myself what a person would like to say speaking about spatial and geographic knowledge. What are the building bricks of spatial reasoning and how can we efficiently organize and manage spatial information? We can consider a geographical framework as a expressive example of spatial knowledge. It is well known that knowledge can be represented with hierarchical structures based on many different relations (containment, kinship and many others). The literature of ontological research activity is rich of such examples. Commonsense concepts of spatial knowledge are usually structured into hierarchical trees. An example is given by the entities connected with the territory: so *region*, *district*, *town*, *natural park*, *regional natural park* are usually considered as connected by order relationships as containment or subsumption. The problem of organizing knowledge is very important in order to implement a reasoner, nevertheless a propaedeutic question is crucial for knowledge representation. Which are the primitives to represent spatial knowledge? Following the approach of cognitive science we can consider as primitives basic polytopes (as cycle, square, triangle, ...), coarse measures (as little, big, medium) and even colors ("the little blue square"). Nevertheless according to the goal of generalization we can consider more general spatial entities as *region* or *point*. The question is "which spatial entities are necessary and sufficient to represent spatial knowledge?". In the following paragraphs we try to give an answer to this problem.

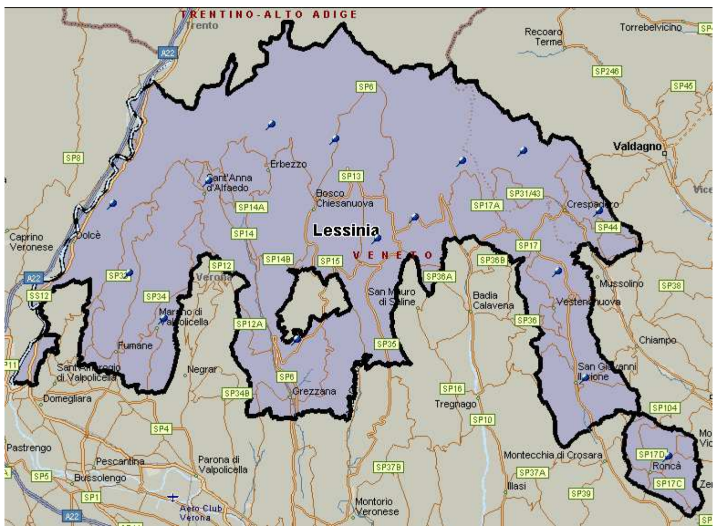

Fig. 2.1. The "Lessinia" regional natural park in the North of Italy.

2.2.1 A commonsense example of spatial knowledge

Let us consider the "Lessinia" regional natural park (see figure 2.1) divided between the district of Verona and the district of Vicenza in the North of Italy. This natural park is given by an aggregation of neighboring municipalities except for the town of Cerro Veronese which represents a hole in the area of the park.

If we think to the simplest mental representation of the park for a person it seems reasonable to consider the generic concepts of *region*, *boundary* and *hole*. Given the idea of boundary it is quite obvious to consider even the concept of *line* as a primitive of human representation of spatial information. In fact many elements in abstract representations of space are sketched by means of lines: roads are a common example.

Going a little bit further we can consider the fact that a person could be interested to a hotel or a refuge that are usually represented as *points* in a map (see figure 2.2). Another

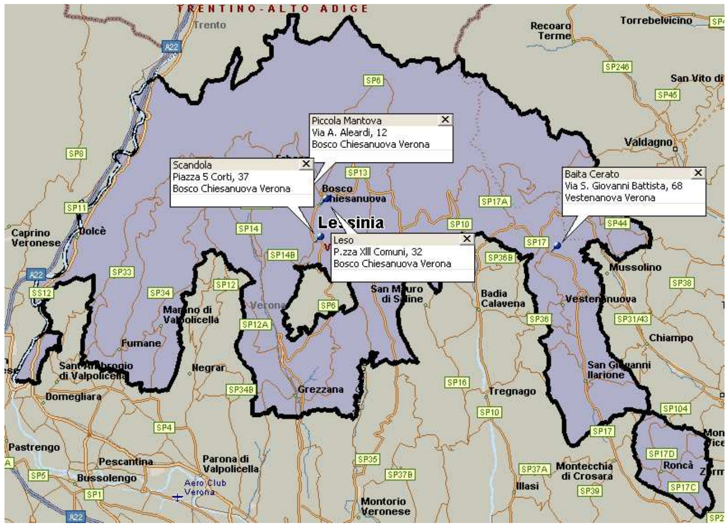

Fig. 2.2. Punctual references to some hotels and refuges in the area of the "Lessinia" natural park.

important brick of spatial knowledge structures considering commonsense knowledge is the concept of *aggregation*. The example of the park is still relevant because it is a consortium of municipalities, but a person will consider the area of the park as a single unit even if it is the union of the areas of different towns (see figure 2.3).

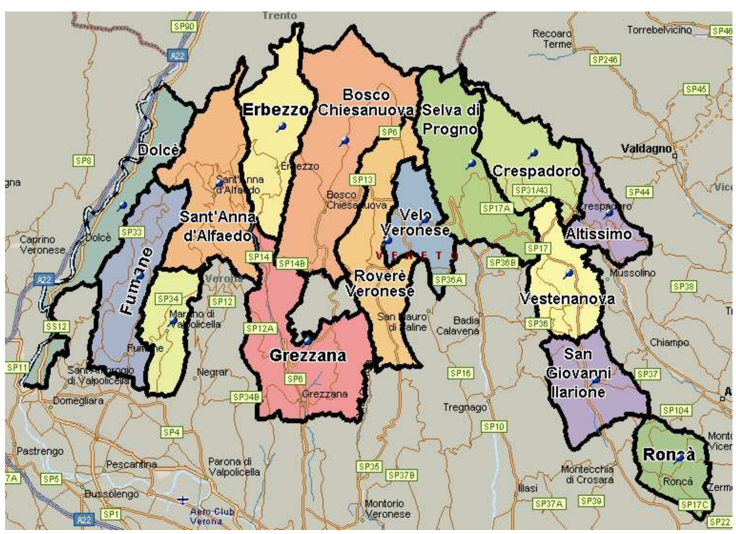

Fig. 2.3. The consortium of Lessinian villages corresponds to the natural park.

One last aspect of commonsense spatial knowledge that must be considered deals with the case of overlapping areas. As introduced before, the "Lessinia" natural park overlaps two different districts: Verona and Vicenza (see figure 2.4).

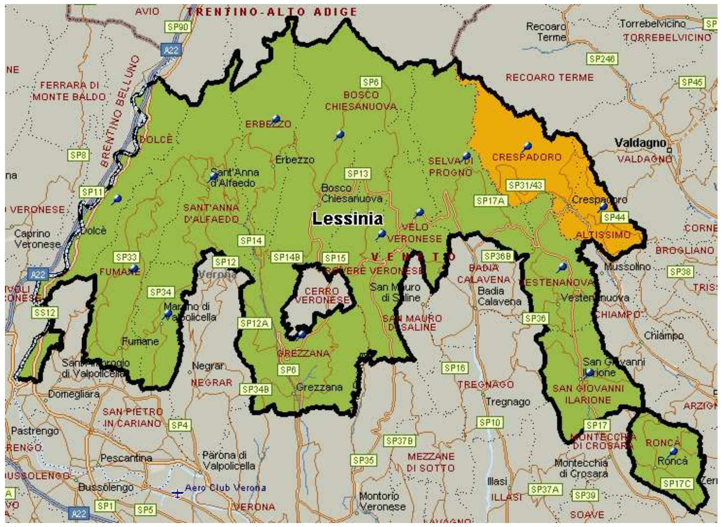

Fig. 2.4. The natural park is divided into two parts belonging one to the district of Verona (green) and one to the district of Vicenza (orange).

The problem of the overlapping of regions in commonsense reasoning is overcome considering the *intersection*. A person interested to the part of the natural park in the district of Verona will obviously consider the intersection between the area of the park and the area of the district of Verona.

2.2.2 Bricks for commonsense spatial reasoning

The problem of qualitative spatial representation and reasoning has been studied with many different approaches that gave birth to many theoretical frameworks and formal knowledge representation systems. As pointed out by Cohn and Renz in [37]:

> There are many different aspects to space and therefore to its representation. Not only do we have to decide on what kind of spatial entity we will admit [. . .], but also we can consider developing different kinds of ways of describing the relationship between these kinds of spatial entities; for example, we may consider just their topology, or their sizes or the distance between them, their relative orientation or their shape.

The example of commonsense spatial knowledge brought us to consider primitives as points, lines and regions that are considered by QSR researchers purely spatial entities. In the previous paragraph we presented these concepts without any mathematical consideration and omitted any detail on the punctual nature of regions considered as union of points. Nevertheless we now must consider which are the most relevant theoretical framework for spatial reasoning; in other words which are the most important relations among spatial objects? The literature on “spatial matters” (see for instance the Chapter on Qualitative Spatial Reasoning [37] in the Handbook of Knowledge Representation and the Handbook of Spatial Logics [1]) reveals the great importance of topological, mereological, mereotopological and directional information compared to morphological or size information.

The combination of spatial primitives with sets of spatial relationships characterizes families of formal knowledge representation systems (see figure 2.23). This can be an intuitive method to make a first comparison of the expressivity of knowledge representation systems. We refer the reader for further details to the section 2.5 of this chapter.

As formalized in [37] by Cohn and Renz the most studied spatial frameworks are topology, mereology and mereotopology that in some sense is the union of the previous two theories.

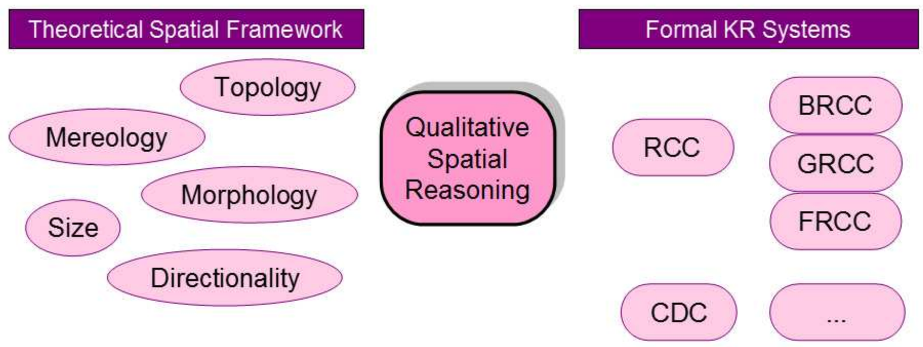

Fig. 2.5. The research activity in the field of QSR aims to the definition of both theoretical frameworks (main examples are topology, mereology, directionality and some others) and of knowledge representation systems (as the Region Connection Calculus (RCC) and its derived systems like Boolean RCC, Generalized RCC, Fuzzy RCC and the Cardinal Direction Calculus (CDC)).

2.2.3 Mereology and Mereotopology

Mereology is based on the concept of *part* and considers those relations between spatial objects that are connected with that notion. As pointed out by Cohn and Renz in [37], the most widely used mereological theory is the *minimal extensional mereology* defined by Simons in [123]. This theory takes the *proper part* relation (PP) as primitive and presents the following logical schema:

(a) Any axiom set sufficient for first-order predicate calculus with identity.
(b) $\forall x, y[\mathsf{PP}(x, y) \rightarrow \neg\mathsf{PP}(y, x)]$.
(c) $\forall x, y, z[[\mathsf{PP}(x, y) \wedge \mathsf{PP}(y, z)] \rightarrow \mathsf{PP}(x, z)]$.
(d) $\forall x, y[\mathsf{PP}(x, y) \rightarrow \exists z[\mathsf{PP}(z, y) \wedge \neg\mathsf{O}(z, x)]]$.
(e) $\forall x, y[\mathsf{O}(x, y) \rightarrow \exists z \forall w[\mathsf{P}(w, z) \equiv \mathsf{P}(w, x) \wedge \mathsf{P}(w, y)]]$.

The axioms (b) and (c) state that the proper part relation is a strict partial order (i.e. irreflexive, transitive and antisymmetric). Adding the definition of the *overlapping* relation O in the sense of “having a common part” it is possible to express the axiom (d) requiring that an individual cannot have a single proper part. The axiom (e) considers the *part* P relation which is a partial order (i.e. reflexive, transitive and antisymmetric) and states the existence of a unique product for overlapping individuals. For further details we refer the reader to the book of Simons [123], the book of Casati and Varzi [30] and Varzi's chapter in the Handbook of Spatial Logics [133].

As pointed out by Kontchakov et al. in [84] the intended models of standard spatial logics (and therefore of spatial reasoning theoretical frameworks) are based on "mathematical spaces" such as topological spaces and their relational or algebraic representation. According to the definition of Kontchakov et al. a *topological space* is a pair $(U, (i))$ where U is a nonempty set called the *universe* of the space, and i is the *interior operator* on U satisfying the *Kouratowski axioms* stating that for all $X, Y \subseteq U$:

(a) $\mathsf{i}(X \cap Y) = \mathsf{i}X \cap \mathsf{i}Y$,
(b) $\mathsf{i}X = \mathsf{ii}X$,
(c) $\mathsf{i}X \subseteq X$,
(d) $\mathsf{i}U = U$.

The *interior operator* (with its dual: the *closure operator* c) allows the distinction between *open* and *closed* sets intended as sets without and with boundary respectively. Starting from a topological operator it is possible to define the *connection* relation among elements of a topology. There exist three different notions of *connection*:

- $\mathsf{C}_1(X,Y) = \mathsf{i}(X) \cap \mathsf{i}(Y) \neq \emptyset$
- $\mathsf{C}_2(X,Y) = \mathsf{i}(X) \cap \mathsf{c}(Y) \neq \emptyset$ or $\mathsf{c}(X) \cap \mathsf{i}(Y) \neq \emptyset$
- $\mathsf{C}_3(X,Y) = \mathsf{c}(X) \cap \mathsf{c}(Y) \neq \emptyset$

The ordinary point-set topology (see [83] for further details) appears to conflict with common sense particularly for the distinction between *open* and *closed* sets as summarized by Varzi in [133]. Cohn and Renz in [37] say that:

> Although topology has been studied extensively within the mathematical literature, much of it is too abstract to be of relevance to those attempting to formalise common sense spatial reasoning.

To overcome this conflict the most studied theoretical framework studied in QSR is mereotopology. *Mereotopology* is a theoretical framework integrating topology and mereology. There are three main strategies to pursue the integration: a generalization of a mereology adding topological primitives (see the work of Borgo et al. [27]), the introduction of the topology as a specific subtheory of mereology (see [46]), the definition of mereological *part* with the topological *connection* primitive (see for instance [9]).

The best known case of mereotopology defined with the C_3 connection primitive is the Region Connection Calculus (henceforth RCC) presented in [113] and then studied in [35], [34] and many other papers. RCC and some variations of the original theory are presented in Section 2.3.

2.2.4 Orientation

Orientation is, like topology, a well suited framework for the qualitative approach as explained by Frank in [49]:

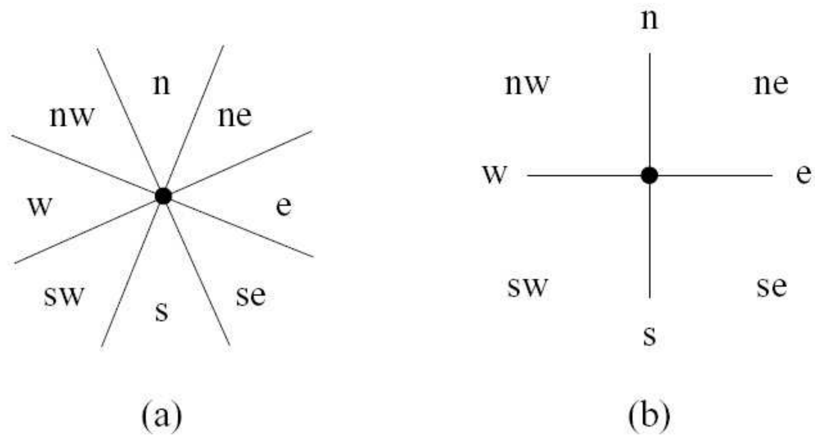

Fig. 2.6. Orientation relation between points: (a) cone-based and (b) projection-based.

> Verbal information about locations of places can leave certain aspects imprecise and humans deduce information from such descriptions. [...] It is clear that the qualitative approach loses some precision, but simplifies reasoning and allows deductions when precise information are not available.

An important feature of orientation information is the ternary nature of this family of spatial relations that depends on the *located object*, the *reference object*, and the *frame of reference* which can be specified either by a third object or by a given direction. The literature presents three different kinds of frames of reference:

extrinsic - when external factors impose an orientation on the reference object;
intrinsic - the orientation is given by some inherent property of the reference object;
deictic - the orientation is imposed by the point of view from which the reference object is seen.

If the frame of reference is given, orientation can be expressed in terms of binary relationships with respect to the given frame of reference. We refer the reader to the "Orientation" section in the Renz and Nebel work [117] for further details.

Many approaches to qualitative orientation information are based on points as primitives of spatial entities and consider only two-dimensional space. Frank in [48] proposed two main methods to describe the cardinal direction of a point with respect to a reference point in a geographical space: *cone-based* and *projection-based* (see Figure 2.6). Frank in 1996 proposed an investigation of deduction rules for reasoning on directional information. In order to formalize a reasoning technique Frank defined an algebra built on a set of symbols like {N, E, S, W}, a set of operations and axioms that define the outcome of the operations. The properties of cardinal directions can be expressed by means of an algebra with two operations on direction symbols: the *inverse*, to reverse the direction of a travel, and the *composition* of direction symbol of two consecutive segment of a path. The possibility to express a higher granularity is given by a wider set of symbols {N, NE, E, SE, S, SW, W, NW}. There exist other important formalizations of point-based cardinal information such as the *Star Calculus* proposed by Renz and Mitra in [115], which is a generalization of the *cardinal algebra* of Ligozat, or such as the *double-cross calculus* of Freksa [50] further investigated by Scivos and Nebel [122] which defines the direction of a located point to a reference point with respect to a perspective point. The projection-based approach proposed by Frank allows us to represent orientation relations in terms of the point algebra and for the purpose of this thesis it is the formalism best suited to be embed in a reasoning theory. The formal semantics of the projection-based approach and its computational properties were studied by Ligozat in [89], see section 2.4.1 for a wider introduction to this formalism.

An important challenge for qualitative orientation reasoning is to consider extended spatial entities as primitives. The problem is that extended objects can have an intrinsic direction or a complex shape such that even a natural language expression could hardly describe orientation relations between such objects. Many approaches to orientation relation among extended objects are based on approximation of regions or on specific "families" of regions. A representation strategy to describe orientation information with regions is to restrict regions to be rectangles with sides parallel to the axes determined by the frame of reference ([67], [108], [15]). The section 2.4 will present the main approaches to orientation information representation.

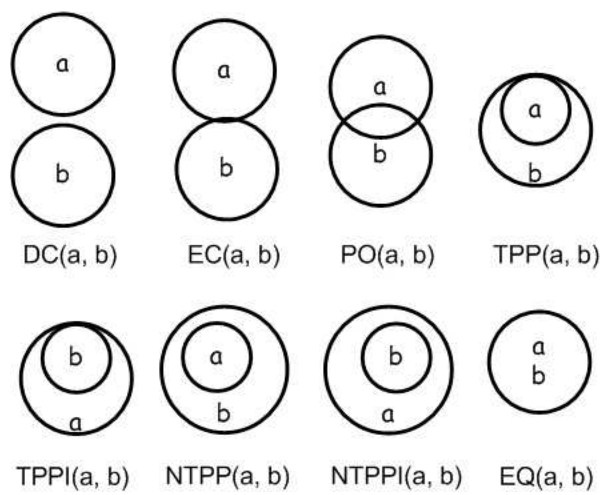

Fig. 2.7. The RCC8 JEPD spatial relations

2.3 KR Systems for Mereological and Mereotopological Relations

2.3.1 The Region Connection Calculus

The most referenced formalism for Spatial Reasoning is the Region Connection Calculus (RCC) first described by Randell, Cui and Cohn in [40], [112] and [113]. RCC is a first order theory based on the notion of *connection*, a primitive binary symmetric relation. This notion of connectedness is used to define a set of binary relations between spatial entities. In particular the eight relations illustrated in Fig. 2.7 are of great importance, in fact they are a *Jointly Exhaustive and Pairwise Disjoint* (**JEPD**) set: any two spatial entities (regions) stand to each other in exactly one these relations. This set of relations is known in literature as RCC-8 (see [34] for an extensive survey). The same set of relations has been identified as significant in the context of Geographical Information Systems [44], [45].

RCC takes regions rather than points as a fundamental notion. This region-based approach to spatial reasoning closely mirrors Allen's interval-based approach to temporal reasoning [3]. In fact they both take extended entities, rather than points, as primitives. As a matter of fact, the construction of the RCC theory of spatial regions was greatly influenced by the works of Allen and Hayes [2], [73], [75], [74] and consequently its development followed a similar pattern: a first order theory was introduced and investigated and then useful constraint languages were identified to provide a reasoning mechanism.

The language of RCC-8 contains individual variables $X_1, X_2, \ldots$ called *region variables*, eight binary predicates EC, DC, PO, EQ, TPP, TPPI, NTPP, NTPPI, and the Boolean connectives $\wedge, \vee, \rightarrow, \neg$. The well-formed formulas of this language will be called *spatial formulas*. Spatial formulas are interpreted in topological spaces $\mathfrak{T} = \langle U, \mathsf{i} \rangle$ where i is an *interior operator* on a set U satisfying the standard Kuratowsky's axioms. The region variables are assume to range over *regular standard closed sets* of $\mathfrak{T}$. The meaning of the eight basic RCC-8 predicates is defined as follows:

$$\mathsf{DC}(X_1, X_2) \Leftrightarrow \neg\exists x, x \in X_1 \cap X_2,$$

$$\mathsf{EC}(X_1, X_2) \Leftrightarrow (\exists x, x \in X_1 \cap X_2) \wedge (\neg\exists x, x \in \mathsf{i}X_1 \cap \mathsf{i}X_2),$$

$$\mathsf{PO}(X_1, X_2) \Leftrightarrow \exists x, x \in \mathsf{i}X_1 \cap \mathsf{i}X_2 \wedge \exists x, x \in \mathsf{i}X_1 \cap \neg X_2 \wedge \exists x, x \in \neg X_1 \cap \mathsf{i}X_2,$$

$$\mathsf{EQ}(X_1, X_2) \Leftrightarrow \forall x, (x \in X_1 \leftrightarrow x \in X_2),$$

$$\mathsf{TPP}(X_1, X_2) \Leftrightarrow \forall x, x \in X_1 \cup X_2 \wedge \exists x, x \in X_1 \cap \mathsf{i}\neg X_2 \wedge \exists x, x \in \neg X_1 \cap X_2,$$

$$\mathsf{NTPP}(X_1, X_2) \Leftrightarrow \forall x, x \in \neg X_1 \cup \mathsf{i}X_2 \wedge \exists x, x \in \neg X_1 \cap X_2,$$

$$\mathsf{TPPI}(X_1, X_2) \Leftrightarrow \mathsf{TPP}(X_2, X_1),$$

$$\mathsf{NTPPI}(X_1, X_2) \Leftrightarrow \mathsf{NTPP}(X_2, X_1).$$

An *assignment* in $\mathfrak{T}$ is a map $\mathfrak{a}$ associating every variable X to a set $\mathfrak{a}(X) \subseteq U$ such that $\mathfrak{a}(X) = \mathsf{ci}\mathfrak{a}(X)$, where c is the closure operator on U dual to i. A spatial formula φ is said to be *satisfiable* of there exists a topological space $\mathfrak{T}$ and an assignment $\mathfrak{a}$ in it under which φ is true in $\mathfrak{T}$ (in symbols $\mathfrak{T} \models^{\mathfrak{a}} \varphi$)

A central role in qualitative reasoning (as in temporal reasoning) is played by the *composition table* recalled for all RCC formalisms in Tables 2.1, 2.2, 2.3, 2.4 and 2.5. This table is used to solve one basic inference problem: given the relationship between two objects a and b e.g. $S(a, b)$ and a second relationship between b and another object c e.g. $T(b, c)$, then what are the possible relationship between a and c? The answer is in the composition table that lists the possible relationships for $S \circ T$. For example if we consider $contains \circ contains$ we should get again $contains$ since this relation is transitive.

Researchers defined also coarser versions of RCC-8 derived from this formalism collapsing some relationships. Here we briefly present several of these versions (see figure 2.8).

∘	**SR(a, b)**
SR(b, c)	*

Table 2.1. The RCC-1 composition table where * is the set of all base relations (in this case only SR (Spatially Related)).

∘	**DR(a, b)**	**O(a, b)**
DR(b, c)	*	*
O(b, c)	*	*

Table 2.2. The RCC-2 composition table where * is the set of all base relations (in this case DR (Discrete) and O (Overlapping)).

∘	**DR(a, b)**	**ONE(a, b)**	**EQ(a, b)**
DR(b, c)	*	DR, ONE	DR
ONE(b, c)	DR, ONE	*	ONE
EQ(b, c)	DR	ONE	EQ

Table 2.3. The RCC-3 composition table where * is the set of all base relations (in this case DR (Discrete), ONE (Overlapping Non Equal) and EQ (Equal)).

∘	**DR(a, b)**	**PO(a, b)**	**EQ(a, b)**	**PPI(a, b)**	**PP(a, b)**
DR(b, c)	*	DR, PO, PPI	DR	DR, PO, PPI	DR
PO(b, c)	DR, PO, PP	*	PO	PO, PPI	DR, PO, PP
EQ(b, c)	DR	PO	EQ	PPI	PP
PP(b, c)	DR, PO, PP	PO, PP	PP	PO, EQ, PP, PPI	PP
PPI(b, c)	DR	DR, PO, PPI	PPI	PPI	*

Table 2.4. The RCC-5 composition table where * is the set of all base relations (in this case DR (Discrete), PO (Partially Overlapping), EQ (Equal), PPI (Proper Part Inverse) and PP (Proper Part)).

- ***RCC5*:** {DR, PO, EQ, PP, PPI} where
  $PP = \{TPP, NTPP\}$
  $PPI = \{TPPI, NTPPI\}$
  $DR = \{EC, DC\}$
- ***RCC3*:** {DR, ONE, EQ} where
  $ONE = \{PP, PPI, PO\}$

In the framework of spatial and temporal reasoning representation, a fundamental result on the limit of relationships between logic and topology was given in 1951 by Grzegorczyk [66]. Grzegorczyk stated that a complete axiomatic theory for a topology is not decidable. In other words given a logical system L with a relation r and given an interpretation I of L, if the domain of the interpretation is a topological space with respect to r – r is a relation of connection – then L is not decidable. This important limit is the base of the work of Bennett, that in [24] presented a topological interpretation of both classical and intuitionistic propositional logic. A problem with RCC is computing inferences, in fact one can use any 1st-order theorem prover, but the complexity of the theory means that for many significant problems this approach is impractical. In [24] Bennett proposed a different approach to provide efficient reasoning about a large class of spatial

∘	**DC**(a, b)	**EC**(a, b)	**PO**(a, b)	**TPP**(a, b)	**NTPP**(a, b)	**TPPI**(a, b)	**NTPPI**(a, b)	**EQ**(a, b)
DC(b, c)	*	DC EC PO TPPI NTPPI	DC EC PO TPPI NTPPI	DC	DC	DC EC PO TPPI NTPPI	DC EC TPPI NTPPI	DC
EC(b, c)	DC EC PO TPP NTPP	DC EC PO TPP TPPI EQ	DC EC PO TPPI NTPPI	DC EC	DC	EC PO TPPI NTPPI	PO TPPI NTPPI	EC
PO(b, c)	DC EC PO TPP NTPP	DC EC PO TPP NTPP	*	DC EC PO TPP NTPP	DC EC PO TPP NTPP	PO TPPI NTPPI	PO TPPI NTPPI	PO
TPP(b, c)	DC EC PO TPP NTPP	EC PO TPP NTPP	PO TPP NTPP	TPP NTPP	NTPP	PO EQ TPP TPPI	PO TPPI NTPPI	TPP
NTPP(b, c)	DC EC PO TPP NTPP	PO TPP NTPP	PO TPP NTPP	NTPP	NTPP	PO TPP NTPP	PO TPPI NTPPI TPP NTPP EQ	NTPP
TPPI(b, c)	DC	DC EC	DC EC PO TPPI NTPPI	DC EC PO TPP TPPI EQ	DC EC PO TPP NTPP	TPPI NTPPI	NTPPI	TPPI
NTPPI(b, c)	DC	DC	DC EC PO TPPI NTPPI	DC EC PO TPPI NTPPI	*	NTPPI	NTPPI	NTPPI
EQ(b, c)	DC	EC	PO	TPP	NTPP	TPPI	NTPPI	EQ

Table 2.5. The RCC-8 composition table where * is the set of all base relations: DC (Disconnected), EC (Externally Connected), PO (Partially Overlapping), EQ (Equal), TPPI (Tangential Proper Part Inverse), TPP (Tangential Proper Part), NTPPI (Non-Tangential Proper Part Inverse) and NTPP (Non-Tangential Proper Part).

relations: given a finite set of spatial relationships he provides for them a propositional representation. Since the propositional calculus is totally computable, there is an effective procedure for deciding whether this set describes a possible situation. The interpretations allowed for the regions assigned to every variable in a constraint satisfaction problem defined against RCC is a generic regular region of any topological space, unlike Egenhofer's 9-intersection and 4-intersection models that require simply connected regions. Based upon this general assumption, Bennett [24] proved that the basic relations of RCC can be decided polynomially.

In 2003 S. Li and M. Ying obtained an important result about a conjecture raised by Bennett in [23], [25] about the extensional interpretation of RCC-8 . Li and Ying, after an exhaustive analysis, showed that no RCC model can be interpreted extensionally and posed an important limit to this formalism. In [116] Renz and Nebel proved the following theorem:

Theorem 2.1. *RCC-8 and RCC-5 are NP-hard.*

They also showed that using a modal encoding RCC-8 can be reduced to the satisfiability problem of well formed formulae of the first order calculus. In spite of the above mentioned complexity issues, RCC is still the most referenced formalism for qualitative spatial reasoning and, as we will see in the following chapters, a big effort has been spent to merge the RCC with reasoning tools like description logics. Nevertheless, as pointed out by Wolter and Zakharyaschev in [144], the expressive power of RCC-8 is rather limited. It only operates with simple regions and does not distinguish between connected and disconnected ones, regions with and without holes, etc. (Egenhofer and Herring [43]). For this reason it is important to evaluate other formal systems for qualitative spatial representation and reasoning.

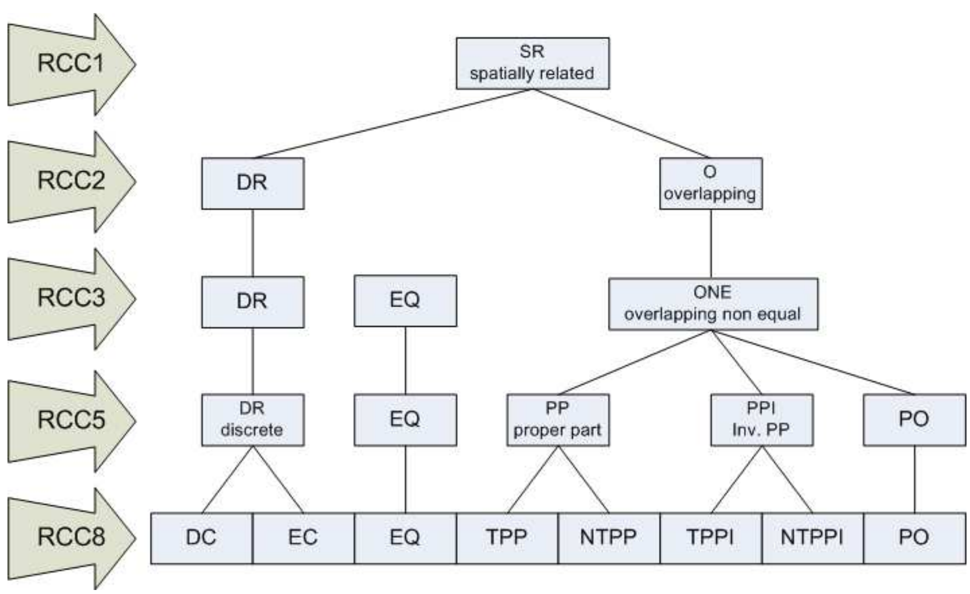

Fig. 2.8. The RCC family

2.3.2 The Boolean RCC

The language **Boolean Region Connection Calculus** (BRCC-8) was first studied by Wolter and Zakharyaschev in [143], [144] and [84] and then further investigated by Gabelaia et al. in [52], [53]. The BRCC-8 (see also the work of Balbiani, Tinchev and Vakarelov [17]) extends RCC-8 by allowing more complex region terms. It uses the same eight binary predicates as RCC-8 and allows not only atomic regions (interpreted as regular closed sets) but also their intersections, unions and complements. The boolean combinations of region variables are called *region terms* and are interpreted as follows. Given a topological space $\mathfrak{T} = \langle U, \mathsf{i} \rangle$, an assignment $\mathfrak{a}$ on it and two region terms t, t' the semantical meaning of combinations is the following:

- $\mathfrak{a}(t \vee t') = \mathsf{ci}(\mathfrak{a}(t) \cup \mathfrak{a}(t')) = \mathfrak{a}(t) \cup \mathfrak{a}(t')$,
- $\mathfrak{a}(t \wedge t') = \mathsf{ci}(\mathfrak{a}(t) \cap \mathfrak{a}(t'))$,
- $\mathfrak{a}(\neg t) = \mathsf{ci}(U - \mathfrak{a}(t))$

Every region term is interpreted as a regular closed set of the topological space. As noted in by Gabelaia et al. in [53], in BRCC-8 we can express the fact that a region is the intersection of two other regions, proposing the following example: consider the Swiss Alps which are the intersection of Switzerland and the Alps with the following syntax.

$$\mathsf{EQ}(SwissAlps, Switzerland \sqcap Alps)$$

It is of interest to note that Boolean region terms do not increase the complexity of reasoning in arbitrary topological models: the satisfiability problem for BRCC-8 formulas is still NP-complete. However, it becomes PSPACE-complete if all intended models are based on connected spaces as proved by Wolter and Zakharyaschev in [143].

2.3.3 Dimension Extended Method and Object-Calculus

One of the most important approaches to the problem of qualitative spatial representation raised in the context of Geographic Information Systems, is the Dimension Extended Method (DEM) presented by Clementini et al. in [33] based on the Object Calculus by Clementini and Di Felice introduced in [32]. This approach is the formal theory under the GeoUML language [22] developed in the IntesaGIS project [4] in recent years with the aim of standardizing GISs. The DEM approach takes into account the dimension of the result of the intersection between spatial objects considering a small number of topological relationships mutually exclusive and pairwise disjoint. The authors prove either that the combination of the terms with a boundary operator for line and area features is expressive enough to represent all possible cases in the DEM. The set of formal definitions of geometric features (objects) and relations are based on the point-set approach whose primitives are points of a topological space $\mathbb{R}^2$. The theory is based on the concepts of point (as primitive), line and region (both considered as sets of points) defined as closed sets for which hold the following rules:

- area features (regions) are connected areas without holes;
- line features are lines without self intersections, either circular or with two distinct end-points;

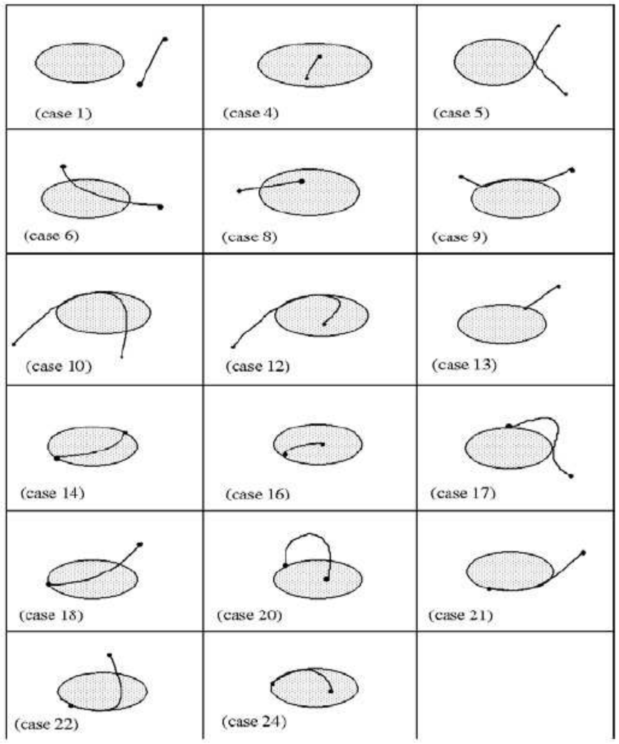

Fig. 2.9. The 17 different line/area cases in the dimension extended method. Taken from [33].

- point features may contain only one point.

The function dim returns the dimension of a point-set S as follows:

$$dim(S) = \begin{cases} - & \text{if } S = \emptyset \\ 0 & \text{if } S \text{ contains at least a point (no line, no area)} \\ 1 & \text{if } S \text{ contains at least a line (no area)} \\ 2 & \text{if } S \text{ contains at least an area} \end{cases}$$

One main aspect of this formalism concerns with the definitions of *boundary* and *interior* for the three basic types of spatial features.

Definition 2.2 (Boundary). *Given a spatial feature λ which can be either a point, a line or an area, its boundary denoted by $\partial\lambda$ is defined as follows:*

- $\partial\lambda = \emptyset$ *if λ is a point;*
- $\partial\lambda = \emptyset$ *in the case of a circular line or* $\partial\lambda = (P_1, P_2)$ *if the feature is a line with two distinct end-points P_1, P_2;*
- $\partial\lambda = L$ *where L is the circular line that bounds the area λ.*

Definition 2.3 (Interior). *The interior operator of a feature λ is denoted by λ° and defined as $\lambda^\circ = \lambda - \partial\lambda$. In particular the interior of a point and of a circular line is equal to the feature itself.*

The combination of the concepts defined above allows the definition of a large number of relations among spatial objects. This might be seen as a drawback of this approach, but the authors provided a user-friendly method for managing the expressive power of the formalism. We introduce now briefly the notation of the Object-Calculus presented in [32]. A *fact* is a triple $\langle \lambda_1, r, \lambda_2 \rangle$ means that a spatial feature λ_1 which can be either a point, a line or an area in in relation r with the feature λ_2. Facts can be combined by the use of *and* $(\wedge)$, *or* $(\vee)$ boolean operators.
In the DEM approach Clementini et al. make available to the users only boundary operators for area and line features and five topological relations: *touch, in, cross, overlap, disjoint*. As pointed out by authors in [33]:

> The set of topological relationships is close to the normal human use of these concepts and still powerful enough to represent a wide variety of cases.

We recall here the formal definitions of topological relations given by Clementini et al. in [33]. As pointed out by authors the advantage of this approach is to provide relationship names that have reasonably intuitive meaning for users of spatial applications.

Definition 2.4 (Touch). *The* touch *relation applies not to the point/point situation and it is defined as follows:*

$$\langle \lambda_1, touch, \lambda_2 \rangle \Leftrightarrow (\lambda_1^\circ \cap \lambda_2^\circ = \emptyset) \wedge (\lambda_1 \cap \lambda_2 \neq \emptyset)$$

Intuitively the *touch* relation applies when the intersection of two spatial objects is contained in the union of their boundaries (see Figure 2.10 taken from [33]).

Definition 2.5 (In). *The* in *relation applies to every possible situation and it is defined as follows:*

$$\langle \lambda_1, in, \lambda_2 \rangle \Leftrightarrow (\lambda_1 \cap \lambda_2 = \lambda_1) \wedge (\lambda_1^\circ \cap \lambda_2^\circ \neq \emptyset)$$

The *in* relation applies when a spatial object is completely contained in an other (see Figure 2.11 taken from [33]).

Definition 2.6 (Cross). *The* cross *relation applies to line/line and line/area situations and it is defined as follows:*

$$\begin{aligned}\langle \lambda_1, cross, \lambda_2 \rangle \Leftrightarrow & dim(\lambda_1^\circ \cap \lambda_2^\circ) = (max(dim(\lambda_1^\circ), dim(\lambda_2^\circ)) - 1) \wedge \\ & (\lambda_1 \cap \lambda_2 \neq \lambda_1) \wedge (\lambda_1 \cap \lambda_2 \neq \lambda_2)\end{aligned}$$

The informal explanation of this relation is that two spatial objects are in the *cross* relation when they meet on an internal point, so it applies only to spatial objects whose interior is different from the empty set (see Figure 2.12 taken from [33]).

Definition 2.7 (Overlap). *The* overlap *relation applies to area/area and line/line situations and it is defined as follows:*

$$\begin{aligned}\langle \lambda_1, overlap, \lambda_2 \rangle \Leftrightarrow & (dim(\lambda_1^\circ = dim(\lambda_2^\circ)) = dim(\lambda_1^\circ \cap \lambda_2^\circ)) \wedge \\ & (\lambda_1 \cap \lambda_2 \neq \lambda_1) \wedge (\lambda_1 \cap \lambda_2 \neq \lambda_2)\end{aligned}$$

Unlike the case of the *cross* relation, the case of *overlapping* is related to homogeneous objects, because the result of the intersection must be a third spatial object of the same dimension: two lines overlap if the intersection is still a line and two areas overlap if their intersection is still an area (see Figure 2.13 taken from [33]).

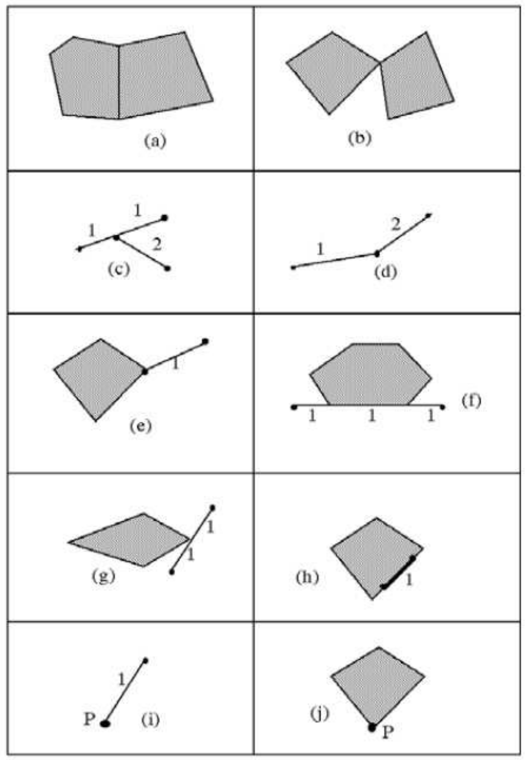

Fig. 2.10. The *touch* relation between two areas (a, b), two lines (c, d), a line and an area (e - h), a point and a line (i), a point and an area (j). Taken from [33].

Definition 2.8 (Disjoint). *The* disjoint *relation applies to every situation and it is defined as follows:*

$$\langle \lambda_1, disjoint, \lambda_2 \rangle \Leftrightarrow \lambda_1 \cap \lambda_2 = \emptyset$$

The *disjoint* relation refers to all couples of spatial objects whose intersection is the empty set. From the definitions it comes that all relations are symmetric (in the sense that $\langle \lambda_1, r, \lambda_2 \rangle \Leftrightarrow \langle \lambda_2, r, \lambda_1 \rangle$), except for the *in* relation which is transitive (in the sense that $\langle \lambda_1, r, \lambda_2 \rangle \wedge \langle \lambda_2, r, \lambda_3 \rangle \Rightarrow \langle \lambda_1, r, \lambda_3 \rangle$). Two more definitions are given in order to extract boundaries from areas and lines according to the notion of boundary recalled in definition 2.2.

Definition 2.9 (Boundary Operators). *The* boundary operators *are defined w.r.t. the considered spatial entity:*

- *given an area A and a boundary operator b, the pair (A, b) returns the circular line ∂A;*
- *given a non-circular line L and a couple of boundary operators f (from)and t (to), the pair (L, f) returns the initial point and the pair (L, t) returns the end point, where both points belong to the set ∂A.*

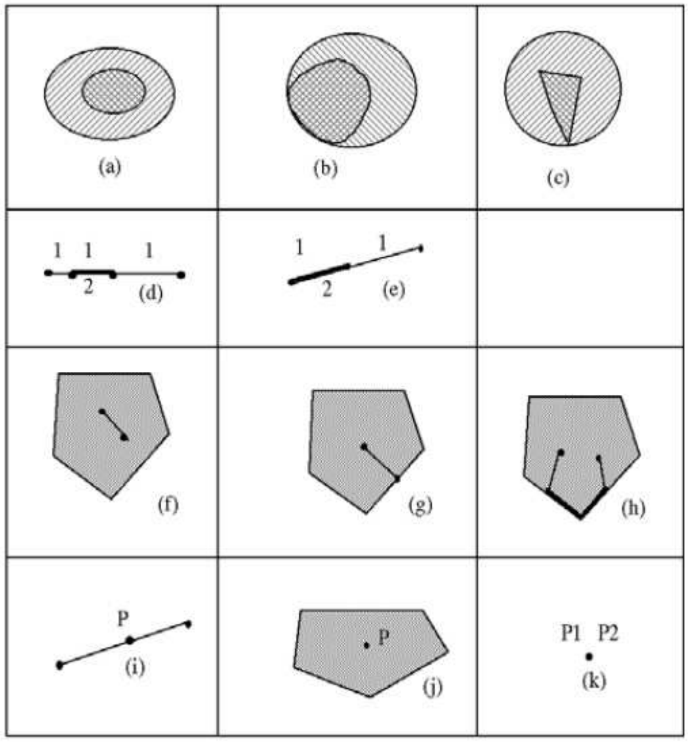

Fig. 2.11. The *in* relation between two areas (a, c), two lines (d, e), a line and an area (f - h), a point and a line (i), a point and an area (j), two points (k). Taken from [33].

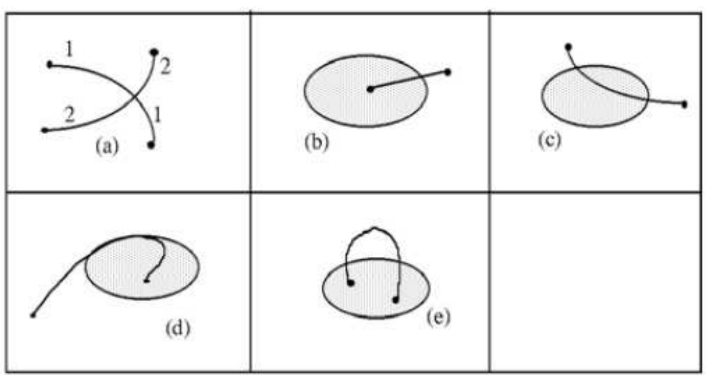

Fig. 2.12. The *cross* relation between two lines (a), a line and an area (b - e). Taken from [33].

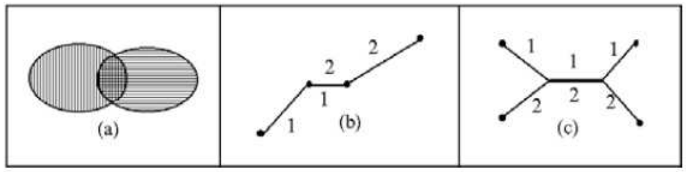

Fig. 2.13. The *overlap* relation between two areas (a), two lines (b, c). Taken from [33].

The most important feature of this set of spatial relations is that they are mutually exclusive (it cannot be the case that two different relationships hold between two spatial objects) and that they constitute a covering of all possible topological pairwise configurations.

2.4 Cardinal Directions

2.4.1 The Cardinal Direction Calculus

In 1998 Ligozat presented in [89] a practical approach to reason on orientation information. This work was heavily influenced by the work of Frank [48], [49] on qualitative spatial reasoning with cardinal directions.

In his work Ligozat considers the projection-based approach proposed by Frank which consists in choosing a system of axes centered at the reference point: an horizontal axis points from West to East and a vertical axis from South to North. The nine zones (N, NE, E, SE, S, SW, W, NW, EQ) are entirely determined by their projections. The calculus is basically one-dimensional, because as pointed out by Ligozat the inferences involved are merely consequences of the one dimensional case, which is the *time point calculus* (see [136]), a qualitative reasoning about points on a time line with $<$, $>$ and $=$ as basic relations. With this approach each object (a point) partitions the space into nine regions:

- its location, which corresponds to a point;
- four regions, which are half lines (semiaxes) corresponding to full North, full South, full East and full West;
- the four intermediate two-dimensional regions Northwest, Northeast, Southwest, Southeast.

Each region corresponds to a basic cardinal relation *(n, ne, e, se, s, sw, w, nw, eq)* and each cardinal relation is characterized by the two projections on the axes. In fact, if a point B is chosen as a reference point, the relation of any other point A with respect to B is represented as a pair of relations between the projection-points on a line. The point A is unambiguously identified by the projection on the axis pointing East, A_1, and on the axis pointing North, A_2. For instance the fact that A is North of B is characterized by

$$\begin{cases} A_1 = B_1 \\ A_2 > B_2 \end{cases}$$

or more concisely by the pair $(=, >)$, see table 2.6 for the full list of pair of relations. Two natural operations can be defined on the set of basic relations: *conversion* and *composition*. The first operation corresponds to switching roles and permutes n and s, e and w and leaves eq fixed. Composing knowledge can be expressed in terms of composition between the nine basic relations, where the result can be a disjunction of possibilities.

n	**e**	**s**	**w**	**ne**	**nw**	**se**	**sw**	**eq**
$(=,>)$	$(>,=)$	$(=,<)$	$(<,=)$	$(>,>)$	$(<,>)$	$(>,<)$	$(<,<)$	$(=,=)$

Table 2.6. Two notations for cardinal direction relations

Ligozat in his work considered binary constraint networks labeled with disjunctions of cardinal relations and presented the following results:

(a) Solving the consistency problem for general networks labeled with arbitrary disjunctions is NP-complete.
(b) There exists a subset of disjunctive relations, the subset of *pre-convex relations* such that the problem, when restricted to this subset, is polynomial and path-consistency implies consistency.
(c) The class of pre-convex relations is the maximal tractable subclass of disjunctive relations containing all basic relations.

The composition can be computed componentwise: on each axis, what we get is exactly what is called the time point calculus in temporal reasoning. For this reason the composition can be deduced from the simple composition table of the time point calculus, where the "=" relation in the neutral element of the operation. The element $eq := (=,=)$ is a neutral element for composition, because $=$ is a neutral element on each component. The set of disjunctive relations is a relations algebra with $2^9 = 512$ elements with the conversion and composition.

$\circ$	$n(=,>)$	$<(a,b)$	$>(a,b)$
$=(b,c)$	$=$	$<$	$>$
$<(b,c)$	$<$	$<$	$<,=,>$
$>(b,c)$	$>$	$<,=,>$	$>$

Table 2.7. The composition table of the time point calculus. The composition of projection-based cardinal relations can be computed componentwise according to this composition table.

For the aim of this work it is important to recall the formal definitions of convexity and preconvexity.

Definition 2.10 (Dimension). *Given a point B, the nine basic relations are the regions determined by B and the canonical directions. The dimension of a relation is the dimension of the associated region.* eq *correspond to a region of dimension 0,* n, e, s, w *are half-lines of dimension 1 and* ne, nw, se, sw *are regions of dimension 2.*

Definition 2.11 (Closure). *Given a relation r, associated to a region R, the topological closure of r corresponds to the closure of the region R.*

$\circ$	**n** $(=,>)$	**e** $(>,=)$	**s** $(=,<)$	**w** $(<,=)$	**ne** $(>,>)$	**nw** $(<,>)$	**se** $(>,<)$	**sw** $(<,<)$	**eq** $(=,=)$
n $(=,>)$	n $(=,>)$	ne $(>,>)$	n, eq, s $(=,\{*\})$	nw $(<,>)$	ne $(>,>)$	nw $(<,>)$	ne, e, se $(>,\{*\})$	nw, w, sw $(<,\{*\})$	n $(=,>)$
e $(>,=)$	ne $(>,>)$	e $(>,=)$	se $(>,<)$	e, eq, w $(\{*\},=)$	ne $(>,>)$	ne, n, nw $(\{*\},>)$	se $(>,<)$	se, s, sw $(\{*\},<)$	e $(>,=)$
s $(=,<)$	n, eq, s $(=,\{*\})$	se $(>,<)$	s $(=,<)$	sw $(<,<)$	ne, e, se $(>,\{*\})$	nw, w, sw $(<,\{*\})$	se $(>,<)$	sw $(<,<)$	s $(=,<)$
w $(<,=)$	nw $(<,>)$	e, eq, w $(\{*\},=)$	sw $(<,<)$	w $(<,=)$	ne, n, nw $(\{*\},>)$	nw $(<,>)$	se, s, sw $(\{*\},<)$	sw $(<,<)$	w $(<,=)$
ne $(>,>)$	ne $(>,>)$	ne $(>,>)$	ne, e, se $(>,\{*\})$	ne, n, nw $(\{*\},>)$	ne $(>,>)$	ne, n, nw $(\{*\},>)$	ne, e, se $(>,\{*\})$	all $(\{*\},\{*\})$	ne $(>,>)$
nw $(<,>)$	nw $(<,>)$	ne, n, nw $(\{*\},>)$	nw, w, sw $(<,\{*\})$	nw $(<,>)$	ne, n, nw $(\{*\},>)$	nw $(<,>)$	all $(\{*\},\{*\})$	nw, w, sw $(<,\{*\})$	nw $(<,>)$
se $(>,<)$	ne, e, se $(>,\{*\})$	se $(>,<)$	se $(>,<)$	se, s, sw $(\{*\},<)$	ne, e, se $(>,\{*\})$	all $(\{*\},\{*\})$	se $(>,<)$	se, s, sw $(\{*\},<)$	se $(>,<)$
sw $(<,<)$	nw, w, sw $(<,\{*\})$	se, s, sw $(\{*\},<)$	sw $(<,<)$	sw $(<,<)$	all $(\{*\},\{*\})$	nw, w, sw $(<,\{*\})$	se, s, sw $(\{*\},<)$	sw $(<,<)$	sw $(<,<)$
eq $(=,=)$	n $(=,>)$	e $(>,=)$	s $(=,<)$	w $(<,=)$	ne $(>,>)$	nw $(<,>)$	se $(>,<)$	sw $(<,<)$	eq $(=,=)$

Table 2.8. The composition table of the Cardinal Direction Calculus. By $\{*\}$ we represent the universal relation of the time point calculus.

For example the closure of *se* is $\{s, e, se, eq\}$; the topological closure of each basic relation can again be computed component-wise. In fact in the case of the time-point calculus the closure of $<$ and $>$ are $\leq$ and $\geq$ respectively. Ligozat in his work on cardinal directions proposed a *lattice* representation of the relations adapting the technique used in the temporal case and presented by Ligozat in [90]. The idea is to number the zones identified by a point considering that 0 corresponds to $<$, 1 to $=$ and 2 to $>$ in order to obtain a partially ordered set of the nine basic relation. Each disjunctive relation can be identified with a subset of the lattice (see the lattice in Figure 2.14).

Definition 2.12 (Convexity). *Given a lattice, a relation is convex iff it corresponds to an interval in the lattice.*

Definition 2.13 (Pre-Convexity). *A relation r is pre-convex iff the topological closure of its associated region R is a region associated to a convex relation.*

The total number of pre-convex relations is 141 which represents slightly more than 25% of all relations. Pre-convex relations are closed under conversion, intersection and composition.

2.4.2 The Rectangle Algebra

Most approaches to the representation of cardinal relations between extended spatial entities use approximations of spatial regions or limit the choice of admissible spatial regions. The *Rectangle Algebra* is one of the most referenced approach to represent cardinal relations among rectangles whose sides are parallel to the axes determined by the frame of

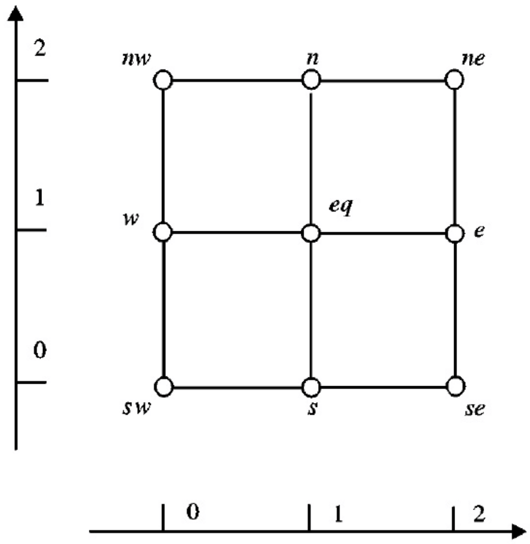

Fig. 2.14. The lattice representation of cardinal direction relations.

reference [67], [108], [15]. In this approach regions can be represented by their projections on the axes and this corresponds to have Allen's interval algebra [3] for each axis separately. In other words each relation on the plane is a pair of two interval relations; this corresponds to a set of 13×13 different basic relations whose formal semantics are provided by the interval algebra.

Balbiani et al. in [15], [16] studied the formal properties of the rectangle algebra and of its *n*-dimensional generalization (the *block algebra*). Balbiani et al. stated a result of NP-completeness (from the interval algebra) but they identified in [16] a tractable fragment of this algebra introducing the notions of *convex* and *preconvex* relations. This technique was heavily influenced by the work of Ligozat on the Cardinal Direction Calculus. The research of a tractable subset of relations required some other definitions of preconvexity: Balbiani et al. distinguished between weak and strong preconvexity proving that strongly preconvex relations are tractable in the sense that path-consistency implies consistency. The basic objects considered in the rectangle algebra are the rectangles whose sides are parallel to the axes of an orthogonal basis in a two-dimensional Euclidean space. The basic relations between these objects are defined from the basic relations of the Allen's interval algebra (represented in Figure 2.15) as follows

$$\mathcal{B}_{rec} = \{(A, B) : A, B \in \mathcal{B}_{int}\}.$$

The example in Figure 2.16 taken from [16] represents two rectangles related by the (m, p) relation. $\mathcal{B}_{rec}$ is the exhaustive list of the relations which can hold between any couple of rational rectangles. The set of relations of the rectangle algebra is defined as the power set of $\mathcal{B}_{rec}$. Each relation $R \in 2^{\mathcal{B}_{rec}}$ can be seen as the union of its basic relations and can be represented by the two projections R_1, R_2 defined as follows:

$$R_1 = \{A : (A, B) \in R\} \quad R_2 = \{B : (A, B) \in R\}$$

Relation	Symbol	Reverse	Meaning	Dim
precedes	p	pi		2
meets	m	mi		1
overlaps	o	oi		2
starts	s	si		1
during	d	di		2
finishes	f	fi		1
equals	eq	eq		0

Fig. 2.15. The set $\mathcal{B}_{int}$ of basic relations of Allen's interval algebra

Following the approach proposed by Ligozat for the Cardinal Direction Calculus, Bal-

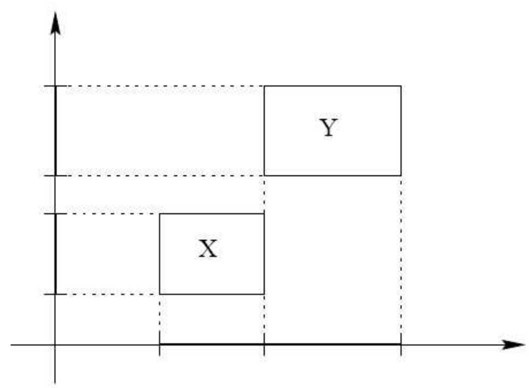

Fig. 2.16. Two rectangles satisfying the (m, p) relation.

biani et al. introduced in [16] a *lattice representation* of the set of basic rectangle relations by the definition of a partial order $\preceq$ on $\mathcal{B}_{rec}$. The result is based on the definition given by Ligozat in [90] of the *interval lattice* build on the set of basic interval relations by means of a partial order function $\leq$ (see the interval lattice in Figure 2.17). We recall here the main definitions from [16].

Definition 2.14 (Rectangle Lattice). *Given a partial order function $\leq$ on $\mathcal{B}_{int}$, a partial order function $\preceq$ on $\mathcal{B}_{rec}$ can be defined as follows:*

$$(A, B) \preceq (C, D) \textit{ iff } A \leq B \textit{ and } B \leq D \textit{ with } (A, B), (C, D) \in \mathcal{B}_{rec}.$$

$(\mathcal{B}_{rec}, \preceq)$ *defines a lattice called the* rectangle lattice.

Definition 2.15 (Convexity). *A relation $R \in 2^{\mathcal{B}_{rec}}$ is convex iff it corresponds to an interval in the rectangle lattice. A convex relation R is equal to the Cartesian product of its projections, which are convex relations w.r.t. the Interval Algebra.*

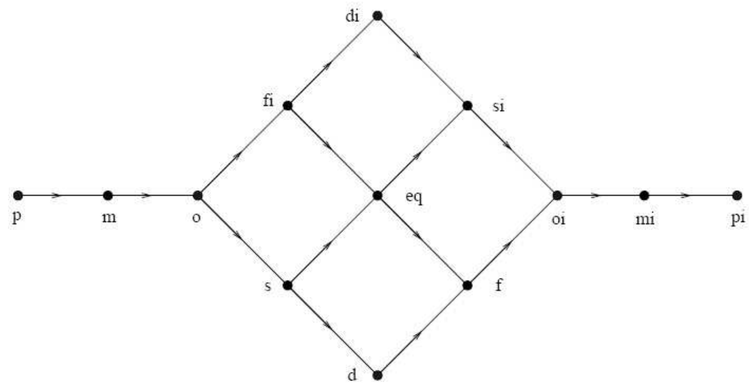

Fig. 2.17. The lattice representation of IA basic relations $(\mathcal{B}_{int}, \leq)$.

Definition 2.16 (Dimension of IA relations). *The* dimension *of a basic relation R denoted by $dim(R)$ corresponds to a dimension of its representation in the plane (0, 1 or 2). The dimension of a "complex" relation is the maximal dimension of its primitives.*

Definition 2.17 (Dimension of RA relations). *The* dimension *of a basic relation of $\mathcal{B}_{rec}$ is the maximal number of endpoint equalities that a basic relation can impose between the orthogonal axes projections of two rectangles (i.e. 4) minus the number of endpoint equalities imposed by the basic relation.*

Definition 2.18 (Closure of IA relations). *Given a basic relation R, its topological closure denoted by $C(R)$ is the relation which corresponds to the topological closure of the region associated to R. For a "complex" relation its closure is the union of the topological closure of its basic relations.*

Definition 2.19 (Closure of RA relations). *The closure of rectangle relations is defined as follows:*

- *let* $(A, B) \in \mathcal{B}_{rec}$ *be* $C((A, B)) = C(A) \times C(B)$*;*
- *let* $R \in 2^{\mathcal{B}_{rec}}$ *be* $C(R) = \bigcup \{C((A, B)) : (A, B) \in R\}$.

For each relation R of IA and RA there exists a smallest convex relations of IA and RA respectively which contains R and which is denoted by I(R) and called the *convex closure* of R.

Definition 2.20 (Weak-Preconvexity and Strong-Preconvexity). *Given $R, S \in 2^{\mathcal{B}_{rec}}$, R is a weakly-preconvex relation iff $dim(I(R) \setminus R) < dim(R)$ and S is a strongly-preconvex relation iff for all convex relations S_c, $S \cap S_c$ is a weakly-preconvex relation.*

As pointed out by Balbiani et al. in [15] and [16] the consistency problem for a constraint network defined over RA relations is NP-complete. The subclass generated by the set of the strongly-preconvex relations is the biggest known tractable set of RA which contains the 169 atomic relations and the consistency problem for a constraint network is polynomial.

2.4.3 Projection-based Directional Relations for Extended Objects

The model of Projection-based Directional Relations (PDR) studied by Goyal and Egenhofer [62], [63] and Skiadopoulos and Koubarakis [124], [125] and [126] is currently one of the most expressive models for qualitative reasoning with cardinal directions, although as it will be made clear in the rest of this chapter the calculus which results quite cumbersome and impractical, because of its size in terms of basic relations. The first version of the formalism presented in [62], [63], [124], [125] deals with extended regions that are connected with connected boundaries. An extended version of the system was presented first by Skiadopoulos and Koubarakis in 2002 and then in [126] in 2005. The extension allows regions to be disconnected and have holes. The name "Projection-based Directional Relations" has been proposed recently by Skiadopoulos et al. in [127], [128] to distinguish it from the later approach called "Cone-based Directional Relations" that proposes a set of relations which is not closed under composition. The investigation of this last approach is preliminary and for this reason we do not consider it in this thesis. We consider only the well investigated PDR approach in both different versions: the first PDR approach for regular extended regions and the wider PDR$^+$ for multi-dimensional spatial objects.

The PDR approach

The PDR approach is based on models in which we can express the cardinal direction relation of a region a w.r.t. a region b, by approximating b using a minimum bounding box while using the exact shape of a. The idea behind the formalism is to divide the space around the reference region b by means of its minimum bounding box into nine areas and record the areas where the primary region a falls into (see Figure 2.18). Skiadopoulos

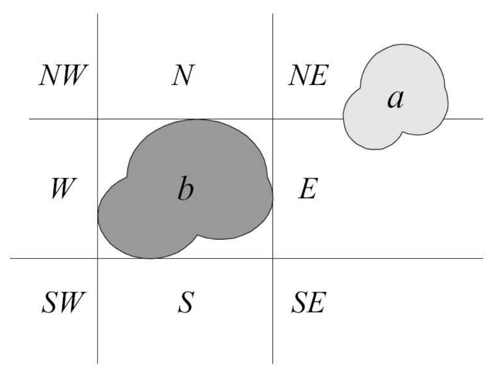

Fig. 2.18. The model of Cardinal Direction Constraints: b is the *reference* region and a is the *primary* region. The example is taken from [126].

and Koubarakis considered the Euclidean space $\mathbb{R}^2$ and defined regions as non-empty and bounded sets of points in $\mathbb{R}^2$. The projection on the x-axis (or y-axis) of a disconnected region is, in general, a bounded set of real numbers. If a region is connected then its projection on the x-axis (respectively y-axis) forms a single interval on the x-axis (respectively y-axis). Let a be a region, we will denote the greatest lower bound and the greatest upper bound of the projection of region a on the x-axis by $inf_x(a)$ and $sup_x(a)$ respectively. Similarly, we will denote $inf_y(a)$ and $sup_y(a)$ the greatest lower bound and the greatest upper bound on the y-axis respectively. The minimum bounding box of a region a, denoted by $mbb(a)$, is the box formed by the straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$ and $y = sup_y(a)$ (see Figure 2.19).

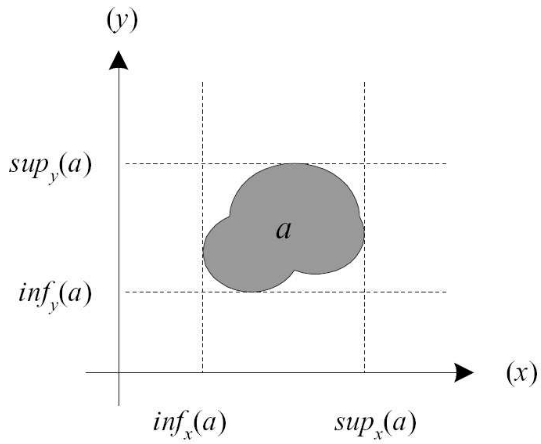

Fig. 2.19. A region and its Minimum Bounding Box [126].

The authors considered two main sets of regions:

- *REG* is the set of regions homeomorphic to the closed unit disk $\{(x, y) : x^2 + y^2 \geq 1\}$, which are closed, connected and with connected boundaries;
- *REG** extends *REG* with disconnected regions and regions with holes as follows: a region a belongs to *REG** iff there exists a finite set of regions $a_1, \ldots, a_n \in REG$ such that $a = a_1 \cup \ldots \cup a_n$.

Given a reference region b and a primary region a both in *REG**, the axes forming the minimum bounding box of b divide the space into 9 tiles (Figure 2.20a). the central tile denoted by $B(b)$ corresponds to the bounding box of the reference region while the peripheral tiles correspond to the eight cardinal direction relations *south, southwest, west, northwest, north, northeast, east, southeast* and will be denoted by S(b), SW(b), W(b), NW(b), N(b), NE(b), E(b), SE(b) respectively. If a primary region a is included (in the set-theoretic sense) in tile $S(b)$ of some reference region b (Figure 2.20b) then we say that a *is south* of b and we write $a\ S\ b$. If a lies partly in the area $NE(b)$ and partly in the area $E(b)$ of a region b then *a is partly northeast and partly east of b* and it is denoted by $a\ NE : E\ b$. We recall now the formal definition by Skiadopoulos and Koubarakis of basic cardinal direction relation from [126].

Definition 2.21 (Basic Cardinal Direction Relation). *A* basic cardinal direction relation *is an expression* $R_1 : \ldots : R_k$ *where*

(a) $1 \leq k \leq 9$,
(b) $R_1, \ldots, R_k \in \{B, S, SW, W, NW, N, NE, E, SE\}$,

(c) $R_i \neq R_i$ *for every* i, j *such that* $1 \leq i, j \leq k$ *and* $i \neq j$.

A basic cardinal direction relation $R_1 : \ldots : R_k$ *is called* single-tile *if* $k = 1$*; otherwise it is called* multi-tile*.*

Examples of multi-tile relations can be found in Figure 2.20c ($NE : E$) and 2.20d ($B : S : SW : W : NW : N : E : SE$). The set of basic cardinal direction relations in our model

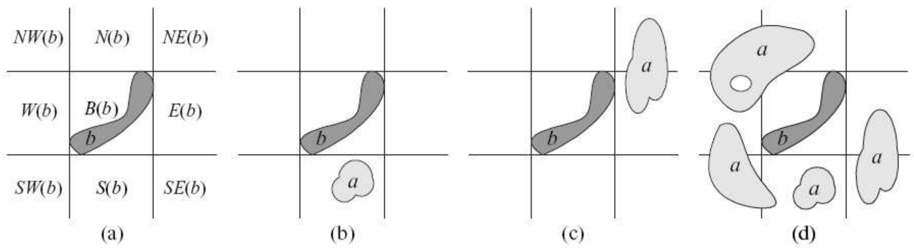

Fig. 2.20. The reference tiles (a) and cardinal relations between the reference region b and the primary region a: (b) is a single tile basic relation, (c) and (d) are multi-tile basic relations with a connected and a disconnected region respectively. [126]

contains $\sum_{i=1}^{9} \binom{9}{i} = 511$ elements. The set $\mathcal{D}^*$ contains the 511 basic relations which are jointly exhaustive and pairwise disjoint. The original PDR approach studied by Goyal and Egenhofer in [62], [63], [64], [65] applied to the REG set (let us denote it PDR$^-$) has only 218 basic relations and Zhang et al. in [147] presented recently an algorithm to check consistency of basic networks with time complexity $O(n^3)$. Using the basic relations we can defined the disjunctive cardinal direction relation which are represented by the set $2^{\mathcal{D}^*}$. The notation for a disjunctive relation is $R_1, \ldots, R_k$ with $1 < k \leq \mathcal{D}^*$. We recall now the formal definition of cardinal direction constraint from [126].

Definition 2.22 (Cardinal Direction Constraint). *A* cardinal direction constraint *is a formula* $a\ R\ b$ *where* $a,\ b$ *are variables ranging over regions in* REG^* *and* R *is cardinal direction relations from the set* $2^{\mathcal{D}^*}$*. Moreover a cardinal direction constraint in called* single-tile *(respectively* multi-tile, basic*) is* R *is a single-tile (respectively multi-tile, basic) cardinal direction relation.*

Obviously, a basic cardinal direction constraint is non-disjunctive while, in general, a cardinal direction constraint can either be disjunctive or non-disjunctive.

In their work Skiadopoulos and Koubarakis proposed an algorithm called "Algorithm Consistency" and proved the following results.

Theorem 2.23. *Deciding the consistency of a set of* basic cardinal direction constraints *in* n *region variables can be done using Algorithm Consistency in* $O(n^5)$ *time.*

Theorem 2.24. *Deciding the consistency of a set of cardinal direction constraints is* $\mathcal{NP}$*-complete.*

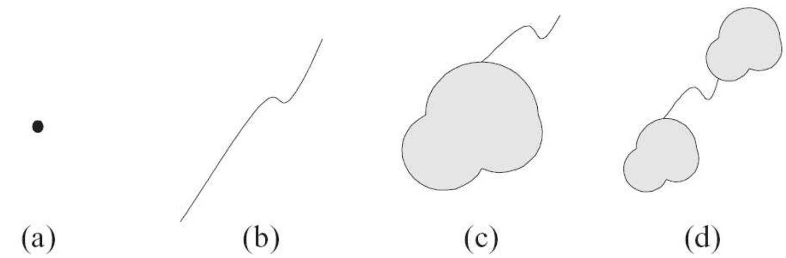

Fig. 2.21. Regions not in REG: points (a), lines (b) and regions with emanating lines (c-d). Taken from [126]

The PDR$^+$ approach

This variation of the PDR model also considers points, lines and regions with emanating lines (see Figure 2.21). Such regions have been excluded from REG* (they are not homeomorphic to the unit disk) but they can be easily included by dividing the space around the reference region b into the following 25 areas (see also Figure 2.22):

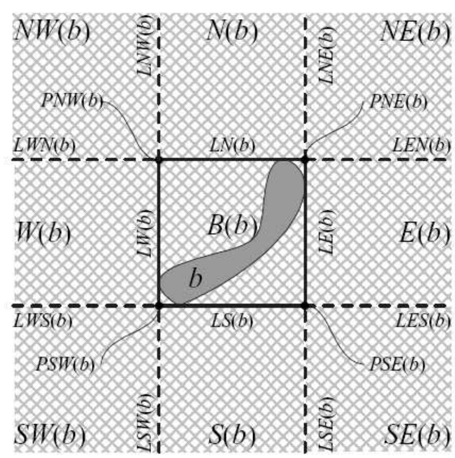

Fig. 2.22. Space partition into 25 tiles. Taken from [126]

- 9 two-dimensional areas $(B(b)$, $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$, $SE(b))$. These areas are formed by the axis (not included) of the bounding box of the reference region b (grey shaded areas of Figure 2.22).
- 8 semi-lines $(LSW(b)$, $LWS(b)$, $LWN(b)$, $LNW(b)$, $LNE(b)$, $LEN(b)$, $LES(b)$, $LSE(b))$. These semi-lines are formed by the vertical and horizontal lines that start from the corners of the bounding box of the reference region b (dotted lines of Figure 2.22).
- 4 line segments $(LS(b)$, $LW(b)$, $LN(b)$, $LE(b))$. These lines segments correspond to the sides of the bounding box of the reference region b (solid lines of Figure 2.22).

- 4 points ($PSW(b)$, $PNW(b)$, $PNE(b)$, $PSE(b)$). These points correspond to the corners of the bounding box of the reference region b.

The new set of relations denoted by $\mathcal{D}^{\mathbb{R}^2}$ contains $\sum_{i=1}^{25} \binom{25}{i} = 33,554,431$ jointly exhaustive and pairwise disjoint cardinal direction relations. Skiadopoulos and Koubarakis proposed a new version of the Algorithm Consistency in order to handle the consistency checking of a given set of cardinal direction constraints involving relations of $\mathcal{D}^{\mathbb{R}^2}$ and provided a proof of correctness for this version of the algorithm.

2.4.4 Combining Topological and Directional Information

Most earlier research on topological and directional relations focuses on one single aspect which can be often topological or directional. In natural language and many practical applications, topological and directional relations are used together. A promising approach of integration of topological and directional reasoning has been proposed recently by Li in [88]. Li observed that human representation of spatial information combines both topological and directional features in a natural way. An example can be the description of the location of Titisee, a famous tourist sight in Germany, we might say Titisee is in the Black Forest and is east of the town of Freiburg. In his work Li extended the RCC-8 constraint language to deal with topological as well as directional information represented by the Rectangle Algebra. The most important reasoning problem is to decide when a network of topological and directional constraints is satisfiable (or consistent). This is not trivial since topological and directional information are not independent and the enriched network may be unsatisfiable despite both the topological and the directional networks are satisfiable. Li's main result states that, if topological constraints are all in one of the three maximal tractable subclasses of RCC-8 [116], then the satisfiability of the joint network can be determined by considering the satisfiability of two related networks in, respectively, RCC-8 and the Rectangle Algebra (RA) [15].

As recalled in the sections above there is a strict connection between the RA and Allen's Interval Algebra (IA) generated by a set $\mathcal{B}_{int}$ of 13 basic relations between time intervals (see Figure 2.15 page 27) closed under composition. Nebel and Bürckert in [106] identified a maximal tractable subclass $\mathcal{H}$ of IA, called the ORD-Horn subclass, and showed that applying an algorithm for path-consistecy is sufficient for deciding satisfiability for $\mathcal{H}$. The RA is the two-dimensional counterpart of IA and it is generated by a set $\mathcal{B}_{rec}$ of 169 JEPD relations between rectangles:

$$\mathcal{B}_{rec} = \{\alpha \otimes \beta : \alpha, \beta \in \mathcal{B}_{rec}\}$$

As a final remark we recall that if $\mathcal{S}$ is a tractable subclass of IA, then $\mathcal{S} \otimes \mathcal{S} = \{\alpha \otimes \beta : \alpha, \beta \in \mathcal{S}\}$ is also tractable in RA.

For the mereo-topological part of the formalism Li referred to the Region Connection Calculus [113] where a region is a nonempty regular closed subset of the real plane. As recalled in section 2.3.1 RCC-8 comprises a set of eight spatial relations EC , DC , PO , EQ , TPP , TPPI , NTPP , NTPPI (represented in figure 2.7 page 13)denoted by Li as $\mathcal{B}_{rec}$. Li used the relation P as shorthand for the union of TPP $\cup$ NTPP $\cup$ EQ and PP for TPP $\cup$ NTPP. The corresponding RCC-8 algebra is denoted by $\langle\mathcal{B}_{rec}\rangle$. Renz in [114] showed that there are only three maximal tractable subclasses of RCC8 that contain all basic relations denoted by $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, $\mathcal{Q}_8$. For these subclasses the path-consistency algorithm is sufficient for deciding the satisfiability of a constraint network and there is an $O(n^2)$ algorithm for finding an atomic refinement of any path-consistent network.

We can now introduce the algebra called DIV9 generated by a subset of RA relations with some preliminary definitions.

Definition 2.25 (minimum bounding rectangle). *For a bounded plane region a, define*

$$sup_x(a) = sup\,\{x \in \mathbb{R} : (\exists y)(x, y) \in a\}$$

$$inf_x(a) = inf\,\{x \in \mathbb{R} : (\exists y)(x, y) \in a\}$$

$$sup_y(a) = sup\,\{y \in \mathbb{R} : (\exists x)(x, y) \in a\}$$

$$inf_y(a) = inf\,\{y \in \mathbb{R} : (\exists x)(x, y) \in a\}$$

We call $\mathcal{M}(a) = I_x(a) \times I_y(a)$ the minimum bounding rectangle *(or box) of a where $I_x(a) = [inf_x(a), sup_x(a)]$ and $I_y(a) = [inf_y(a), sup_y(a)]$ are the projection of the region a on the axes.*

For two bounded regions a, b if $sup_x(a) < inf_x(b)$ we say that a is *west* of b and b is *east* of a (written $a\mathsf{W}b$ and $b\mathsf{E}a$ respectively). If $sup_y(a) < inf_y(b)$ then we say a is *south* of b and b is *north* of a (written $a\mathsf{S}b$ and $b\mathsf{N}a$ respectively). When a is neither west nor east of b, then $I_x(a) \cap I_x(b) \neq \emptyset$ and we say that a is in *x-contact* with b (written $a\mathsf{Cx}b$). Similarly, if a is neither south nor north of b, we say that a is in *y-contact* with b (written $a\mathsf{Cy}b$). Li considered the following IA convex relations in order to define the RA subalgebra:

$$\begin{aligned} \Cap &= \cup\{\mathsf{m}, \mathsf{o}, \mathsf{s}, \mathsf{f}, \mathsf{d}, \mathsf{eq}, \mathsf{di}, \mathsf{fi}, \mathsf{si}, \mathsf{oi}, \mathsf{mi}\} \\ \Subset &= \cup\{\mathsf{s}, \mathsf{d}, \mathsf{eq}, \mathsf{f}\} \\ \Supset &= \cup\{\mathsf{fi}, \mathsf{di}, \mathsf{eq}, \mathsf{si}\} \end{aligned}$$

DIR9 is the Boolean algebra generated by the relations $\mathsf{N}, \mathsf{S}, \mathsf{W}, \mathsf{E}$, has nine atoms (see Table 2.9) and it corresponds to the algebra $\mathcal{A}_3 \otimes \mathcal{A}_3$ where $\mathcal{A}_3$ is the algebra $\mathcal{B}^3_{int}$ of JEPD interval relations $\{\mathsf{p}, \Cap, \mathsf{pi}\}$ studied by Golumbic and Shamir in [59]. They also proved that $\mathcal{H}_3 = \{\mathsf{p}, \Cap, \mathsf{pi}, \mathsf{p} \cup \Cap, \Cap \cup \mathsf{pi}, \mathsf{T}\}$ is the maximal tractable subclass of $\mathcal{A}_3$ since $\mathcal{H}_3 = \mathcal{A}_3 \cap \mathcal{H}$. The class of relations given by $\mathcal{H}_3 \otimes \mathcal{H}_3$ is a tractable subclass of $\mathcal{A}_3 \otimes \mathcal{A}_3$ and hence a tractable subclass of DIR9.

Definition 2.26 (Combined Constraint Network). *Suppose $V = \{v_i\}_{i=1}^n$ is a collection of spatial variables, $\Theta = \{\theta_{ij}\}_{i=1}^n$ and $\Delta = \{\delta_{ij}\}_{i=1}^n$ are respectively a topological (RCC-8) and a directional (DIR9) constraint network over V. A combined constraint network is the joint network denoted by $\Theta \uplus \Delta$.*

Li noted that topological and directional constraints are not independent and proved that deciding the satisfiability of $\Theta \uplus \Delta$ is of cubic complexity if Θ (RCC-8 network) is either one of $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, $\mathcal{Q}_8$ and Δ (DIR9 network) is over $\mathcal{H}_3 \otimes \mathcal{H}_3$, since applying PCA (path-consistency) is sufficient for deciding satisfiability for RCC-8 subclasses and for RA subclass. The conclusion of Li's work is that the consistency of atomic networks in the hybrid calculus can be decided in polynomial time. Checking consistency of a joint network requires a refinement of the two network components that preserves satisfiability. The transformation rules for both directional and topological information are recalled in the following definition, where P and PP, resp., for $\mathsf{TPP} \cup \mathsf{NTPP} \cup \mathsf{EQ}$ and $\mathsf{TPP} \cup \mathsf{NTPP}$.

Definition 2.27. *Given a topological network Θ and a directional network Δ, refinement rules to get $\overline{\Theta}$ and $\overline{\Delta}$ are defined as follows:*

$$\overline{\theta}_{ij} = \begin{cases} \theta_{ij} \cap \mathsf{DC}, & \textit{if } \mathsf{CC} \cap \delta_{ij} = \emptyset; \\ \theta_{ij}, & \textit{otherwise} \end{cases}$$

$$\overline{\delta}_{ij} = \begin{cases} \delta_{ij} \cap \mathsf{eq} \otimes \mathsf{eq}, & \textit{if } \delta_{ij} = \mathsf{EQ}; \\ \delta_{ij} \cap \Subset \otimes \Subset, & \textit{if } \mathsf{EQ} \neq \delta_{ij} \subseteq \mathsf{P}; \\ \delta_{ij} \cap \Supset \otimes \Supset, & \textit{if } \mathsf{EQ} \neq \delta_{ij} \subseteq \mathsf{P}^{\mathsf{i}}; \\ \delta_{ij} \cap \Cap \otimes \Cap, & \textit{if } \mathsf{DC} \cap \delta_{ij} = \emptyset; \\ \delta_{ij}, & \textit{otherwise} \end{cases}$$

So $\overline{\Theta}$ is a refinement of Θ in the same class of relations and $\overline{\Delta}$ is a refinement of Δ in RA, since the relation given by $\Supset \otimes \Supset$ is not in DIV9.

Lemma 2.28. *$\Theta \uplus \Delta$ is satisfiable if and only if $\overline{\Theta} \uplus \overline{\Delta}$.*

Li states the following main result.

Theorem 2.29. *Let Θ be a path-consistent RCC8 network over $\widehat{\mathcal{H}}_8$ (or $\mathcal{C}_8$, $\mathcal{Q}_8$), and let Δ be a* DIV9 *network. Then $\Theta \uplus \Delta$ is satisfiable if and only if $\overline{\Theta}$ and $\overline{\Delta}$ are satisfiable.*

Symbol	**Relation**	**Meaning**	**RA**
northwest	NW	$a\mathsf{N}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \mathsf{pi}$
north and x-contact	NC	$a\mathsf{N}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \mathsf{pi}$
northeast	NE	$a\mathsf{N}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \mathsf{pi}$
y-contact and west	CW	$a\mathsf{Cy}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \Cap$
y-contact and x-contact	CC	$a\mathsf{Cy}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \Cap$
y-contact and east	CE	$a\mathsf{Cy}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \Cap$
southwest	SW	$a\mathsf{S}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \mathsf{p}$
south and x-contact	SC	$a\mathsf{S}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \mathsf{p}$
southeast	SE	$a\mathsf{S}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \mathsf{p}$

Table 2.9. The atoms of DIR9.

2.5 A "Naïve" Classification of Spatial KR Systems

The aim of this work is a classification of formal knowledge representation systems for spatial information. As pointed out at the beginning of this chapter, we followed one of the most important research trends in considering a qualitative approach to the problem of spatial information handling, that has shown to be computationally equivalent but cognitively more adequate than any specific quantitative approach. In the previous sections we presented the most important formal systems for representing and reasoning over mereotopological and directional information. For topological relations we presented standard Region Connection Calculus (RCC) [40], [112] and [113], the variation of RCC represented by the Boolean RCC (BRCC) [143], [144], [84], [52], [53], and the Dimension

Extended Method (DEM) [33], [32]. For directional information we considered the Cardinal Direction Calculus (CDC) [89], the Rectangle Algebra (RA) [67], [108], [15], the Projection-Based Directional Relations (PDR) [62], [63], [124], [125], [126] for regions only or considering as primitives also points and lines (multi-dimensional PDR^{+}) and a combined approach to topological and directional relations (DIR9-RCC8) [88]. In Figure 2.23 we show a "naïve" classification of the qualitative formalisms for spatial reasoning presented in the previous sections. The classification is based on spatial elements available in each formalism. The choice of the list of spatial elements for the classification is coherent to the considerations made about commonsense reasoning on spatial matters.

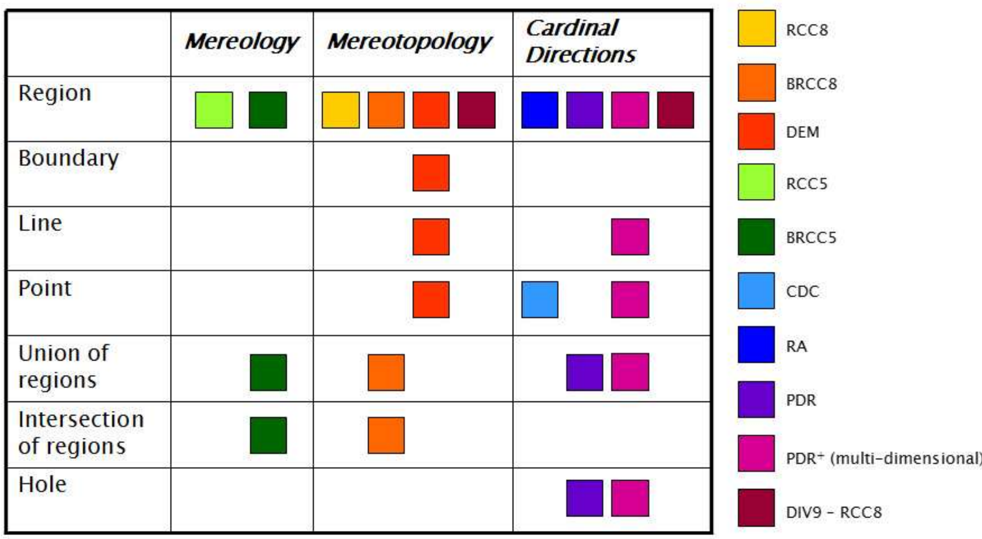

Fig. 2.23. Formal KR systems for Spatial Knowledge classified by spatial primitives and spatial relations.

In Figure 2.24 we provide a table to summarize computational properties of the considered spatial reasoning frameworks. We will see in the following chapters how these properties of QSRR formalisms effect computational properties of the final hybrid description logics.

	RCC8	*BRCC8*	*DEM*	*RCC5*	*BRCC5*
Decidability	Yes	Yes	NA	Yes	Yes
Complexity	NP-complete	NP-complete	NA	NP-complete	NP-complete

	CDC	*RA*	*PDR*	*PDR+*	*DIV9-RCC8*
Decidability	Yes	Yes	Yes	Yes	Yes
Complexity	NP-complete	NP-complete	NP-complete	NP-complete	Polynomial under restrictions

Fig. 2.24. Computational properties of formal QSRR formalisms.

3

Introduction to Description Logics

3.1 Introduction

Description Logics evolved from the knowledge representation formalism of Semantic Networks that became popular around 1966. This approach presented many different problems including vagueness and inconsistency in the meaning of various constructs: to solve these difficulties, Brachman et al. proposed the KL-ONE system [29], that is considered to be the first model of terminological language and is accepted as the ancestor of modern description logic systems. KL-one has been the transition language from Semantic Networks to more well-founded terminological logics. In fact the main result of the work of Brachman et al. was a new knowledge representation framework that allowed application-independent representations and inference procedures.

The term Description Logics, henceforth a DL, describes a family of Knowledge Representation formalisms that are provided with formal, logic-based semantics. These formalisms aim at representing a knowledge domain first defining the main concepts involved in it, and then using this structure to make assertions on the individuals that populate the domain. DLs are used in knowledge-based systems to represent and reason about conceptual and terminological knowledge of a problem domain. DLs are based on the notions of concepts (classes, unary predicates) and roles (binary relations) and are characterized mainly by set operators that allow complex concepts and roles to be built from atomic ones. A DL system allows concept descriptions to be interrelated and implicit knowledge can be derived from the explicitly represented knowledge using inference services. In particular a DL system must infer knowledge about classification of concepts and individuals.

The concept classification determines a hierarchy on the set of concepts: the subsumption problem is the paradigmatic query to DLs terminologies, and consists in checking if a concept is more general than another one. The classification of individuals determines whether a given individual is an instance of a concept. As pointed out by Baader et al. in [10] decidability and complexity of the inference problems are important issues and depend on the expressive power of the DL. In fact, very expressive DLs may be even undecidable while weaker DLs may be efficient but too poor to represent the major concepts that characterize the problem domain. For this reason ”investigating this tradeoff between the expressiveness of Description Logics and the complexity of their reasoning problems has been one of the most important issues in DL research” ([10] chap. 2).

3.2 The $\mathcal{AL}$ -family

3.2.1 Syntax and Semantics

We now briefly recall the syntax of $\mathcal{ALC}$ and the general semantics of description logics. A DL is a logical language that is limited to the syntax specified in below. Admissible formulae include at least *definitions*, and are formed by a left part (the defined object) and a right part (the definition). In general, in a description logic, we have two fundamental categories of objects that are definable: *concepts* and *roles*. Concepts are used to represent classes of objects named *individuals*, whilst roles are used to represent *relations* among a set of individuals. A single theory $\mathcal{T}$ respecting the syntax of a DL is named a *terminology*. A terminology or *TBox* is a set of terminological assertions like definitions.
Concepts and roles are either *atomic* or *descriptions*. If a concept or role is *atomic*, then it is either *defined* or *primitive*. A concept is atomic if and only if it is mentioned in a terminology, and is primitive if and only if it does not appear on the left part of a definition. Every syntactic formula of a DL is named a description. The right part of a definition is a description.

The above illustrated principles apply to any level of complexity of a description language. In below we provide the syntax and the semantics of the family of $\mathcal{AL}$ *-languages*. $\mathcal{AL}$(= attributive language) has been introduced by Schmidt-Schauss and Smolka in [121] as a minimal language; the other languages of this family are extensions of $\mathcal{AL}$.
The syntax of the basic language $\mathcal{AL}$defines a concept by means of the operators in below, where C and D are concepts and R is a (primitive) role.

$$
\begin{array}{rlr}
\text{C, D} \rightarrow & \text{A} & \text{(atomic concept)} \\
& \top & \text{(universal concept)} \\
& \bot & \text{(empty concept)} \\
& \neg\, \text{A} & \text{(atomic negation)} \\
& \text{C} \sqcap \text{D} & \text{(intersection)} \\
& \forall R.\text{C} & \text{(value restriction)} \\
& \exists R.\top & \text{(limited existential quantifier)}
\end{array}
$$

The set-theoretic à la Tarski semantics interprets any concept as a collection of elements of a reference set named the *universe of discourse* Δ, and roles as binary relations in Δ. The assignment of elements to the interpretation of concepts is subject to the (arbitrary) interpretation of primitive concepts. Every interpretation $\mathcal{J}$ of the primitive concepts of a terminology is called a *base* interpretation, whilst any interpretation over the same universe of discourse $\mathcal{I}$ where primitive concepts are interpreted as in $\mathcal{J}$ is called an *extension* of $\mathcal{J}$.

Descriptions, where C and D are concepts and R is a role, are interpreted as follows:

$$
\begin{aligned}
\top^{\mathcal{I}} &= \Delta \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg A^{\mathcal{I}} &= \Delta \backslash \text{A}^{\mathcal{I}} \\
(\text{C} \sqcap \text{D})^{\mathcal{I}} &= \text{C}^{\mathcal{I}} \cap \text{D}^{\mathcal{I}} \\
(\forall R.\text{C})^{\mathcal{I}} &= \{x | \forall y[(x,y) \in R^{\mathcal{I}} \rightarrow y \in \text{C}^{\mathcal{I}}]\} \\
(\exists R.\top)^{\mathcal{I}} &= \{x | \exists y[(x,y) \in R]\}
\end{aligned}
$$

Roles are interpreted as a subset of $\varDelta^2$. In $\mathcal{AL}$, as well as in $\mathcal{ALC}$, roles are primitive. An interpretation $\mathcal{I}$ of a given terminology $\mathcal{T}$ is said to be a *model iff* for every concept C defined in $\mathcal{T}$ by the description δ, the interpretation $C^{\mathcal{I}}$ is the same produced by the interpretation of $\delta^{\mathcal{I}}$ by means of the rules (3.1) to (3.1).
The immediate extensions for $\mathcal{ALC}$ are union ($\sqcup$), the full negation, the full existential quantifier. The syntax of $\mathcal{ALC}$ is as in below.

$$
\begin{array}{lll}
\mathrm{C, D} \rightarrow & \mathrm{A} & \text{(atomic concept)} \\
& \top & \text{(universal concept)} \\
& \bot & \text{(empty concept)} \\
& \neg\, \mathrm{C} & \text{(negation)} \\
& \mathrm{C} \sqcap \mathrm{D} & \text{(intersection)} \\
& \mathrm{C} \sqcup \mathrm{D} & \text{(union)} \\
& \forall R.\mathrm{C} & \text{(value restriction)} \\
& \exists R.C & \text{(full existential quantifier)}
\end{array}
$$

The interpretation rules for $\mathcal{ALC}$ are:

$$
\begin{aligned}
\top^{\mathcal{I}} &= \varDelta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg C^{\mathcal{I}} &= \varDelta \backslash \mathrm{C}^{\mathcal{I}} \\
(\mathrm{C} \sqcap \mathrm{D})^{\mathcal{I}} &= \mathrm{C}^{\mathcal{I}} \cap \mathrm{D}^{\mathcal{I}} \\
(\mathrm{C} \sqcup \mathrm{D})^{\mathcal{I}} &= \mathrm{C}^{\mathcal{I}} \cup \mathrm{D}^{\mathcal{I}} \\
(\forall R.\mathrm{C})^{\mathcal{I}} &= \{x | \forall y[(x,y) \in R^{\mathcal{I}} \rightarrow y \in \mathrm{C}^{\mathcal{I}}]\} \\
(\exists R.\mathrm{C})^{\mathcal{I}} &= \{x | \exists y[(x,y) \in R^{\mathcal{I}} \wedge y \in \mathrm{C}^{\mathcal{I}}]\}
\end{aligned}
$$

A further extension of $\mathcal{ALC}$ of interest is given by the *inverse role* construct which allows one to denote the inverse of a given relation. One can, for example, state that $\exists \mathsf{child}^{-}.\mathsf{Researcher}$ that there exists someone that has a parent who is a researcher by mean of the inverse of the role child. In a language without the inverse role construct one must use two distinct roles, like child and parent, that cannot be put in the proper inverse relation. The presence of the inverse role construct is specified by the letter $\mathcal{I}$ in the name of the language ($\mathcal{ALCI}$). The *inverse* construct is not the only operator definable over roles: in addition to the usual concept forming constructs it is possible to extend a language providing regular expressions as role formation operators. For the aim of the thesis we present here the logic $\mathcal{ALCI}_{reg}$ which is an extension of the language $\mathcal{ALC}$ by means of the following role constructs:

$$R, R' \rightarrow P \mid R \sqcup R' \mid R \circ R' \mid R^* \mid id(C) \mid R^-$$

Where R^* denotes the reflexive-transitive closure of the binary relation R, and $R \circ R'$ denote the chaining of the binary relations R_1 and R_2. The semantics is defined as follows:

$$
\begin{aligned}
P^{\mathcal{I}} &\subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \\
R \sqcup R'I &= R^{\mathcal{I}} \cup R^{\mathcal{I}} \\
R \circ R'I &= R^{\mathcal{I}} \circ R^{\mathcal{I}} \\
(R^{*})^{\mathcal{I}} &= (R^{\mathcal{I}})^{*} \\
id(C)^{\mathcal{I}} &= \big\{(o,o) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} |\, o \in C^{\mathcal{I}}\big\} \\
(R^{-})^{\mathcal{I}} &= \big\{(o,o') \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} |\, (o',o) \in R^{\mathcal{I}}\big\}
\end{aligned}
$$

The architecture of a knowledge representation system built on a DL comprises two disjoint components: the TBox (see the example in Figure 3.1) and the Abox (see the example in Figure 3.2). The TBox defined with the DL introduces the terminology to talk about the problem domain, while the ABox contains assertions about specific individuals expressed according to the vocabulary given by the terminology in the TBox.

$$
\begin{aligned}
&\mathrm{Woman} \equiv \mathrm{Person} \sqcap \mathrm{Female} \\
&\mathrm{Man} \equiv \mathrm{Person} \sqcap \neg\mathrm{Female} \\
&\mathrm{Mother} \equiv \mathrm{Woman} \sqcap \exists \mathrm{hasChild.Person} \\
&\mathrm{Father} \equiv \mathrm{Man} \sqcap \exists \mathrm{hasChild.Person} \\
&\mathrm{Parent} \equiv \mathrm{Father} \sqcup \mathrm{Mother}
\end{aligned}
$$

Fig. 3.1. A simple terminology (TBox)

$$
\begin{aligned}
&\mathrm{Mother(CAROL)} \\
&\mathrm{Father(JOHN)} \\
&\mathrm{hasChild(CAROL, SARAH)} \\
&\mathrm{hasChild(CAROL, JOHN)} \\
&\mathrm{hasChild(JOHN, MARY)}
\end{aligned}
$$

Fig. 3.2. A set of assertions (ABox)

Often it is required interpretations to respect the *unique name assumption (UNA)* on individual names, according to the following definition.

Definition 3.1 (Unique Name Assumption (UNA)). *An interpretation $\mathcal{I}$ respects the* unique name assumption (UNA) *on individual names if and only of for all $a, b \in \mathsf{N_I}$, $a^{\mathcal{I}} = b^{\mathcal{I}}$ implies $a = b$.*

Note that, since the set of individual names $\mathsf{N_I}$ is countably infinite, interpretations that respect the *UNA* on individuals names must have infinite domains. Another important syntactic notion is the *negation normal form* of a concept which consists in pushing the negation signs as far as possible into the description using DeMorgan's rules and the usual rules for quantifiers.

Definition 3.2 (Negation Normal Form (NNF)). *A concept C is said to be in Negation Normal Form (NNF), if the negation operator $\neg$ only appears in front of* concept names. *Each concept can be brought into NNF by "pushing in" the negation sign by exploiting* *the equivalences* $\neg\neg C \equiv C$, $\neg(C \sqcap D) \equiv (\neg C) \sqcup (\neg D)$, $\neg(C \sqcup D) \equiv (\neg C) \sqcap (\neg D)$, $\neg(\exists R.C) \equiv \forall R.\neg(C)$, $\neg(\forall R.C) \equiv \exists R.\neg(C)$.

In description logics, TBoxes are used to capture the background knowledge about the world and TBoxes can exhibit different features. The most simple, *acyclic TBoxes*, introduce defined concept names as *abbreviations* for complex concepts, while the *general TBoxes* introduce constraints like *"for all domain elements where C holds, D holds as well"*.

Definition 3.3 (Concept Definition, Acyclic TBox). *A* concept definition *denoted by* $A \doteq C$*, where* A *is a concept name and* C *is a concept. A* TBox *is a finite set of concept definitions with unique left-hand sides.* A directly uses *a concept name* B *w.r.t.* $\mathcal{T}$ *if there is a concept definition* $A \doteq C \in \mathcal{T}$ *with* B *occurring in* C*. The relation* uses *is defined as the transitive closure of* directly uses*. Finally a TBox* $\mathcal{T}$ *is* acyclic *if no concept name uses itself w.r.t.* $\mathcal{T}$*. Otherwise it is called cyclic.*

We call a concept name A *defined in* a TBox $\mathcal{T}$ if it occurs on the left-hand side of a concept definition in $\mathcal{T}$ and *primitive in* $\mathcal{T}$ otherwise. In the case of acyclic TBoxes, interpretations of primitive concept names and role names uniquely determine interpretations of defined concept names, which is not the case for cyclic TBoxes.

Definition 3.4 (GCI, General TBox). *A* general inclusion axiom (GCI) *is an expression of the form* $C \sqsubseteq D$ *where* C, D *are concepts. A* general TBox $\mathcal{T}$ *is a finite set of GCIs. An interpretation* $\mathcal{I}$ *satisfies a GCI* $C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$*, this is denoted by* $\mathcal{I} \models C \sqsubseteq D$*. An interpretation* $\mathcal{I}$ *is a* model *of a general TBox* $\mathcal{T}$ *if* $\mathcal{I}$ *satisfies all GCIs in* $\mathcal{T}$*.*

General TBoxes are highly desirable in knowledge representation, as they can capture complex constraints and dependencies in the application domain. However, they usually increase the complexity of reasoning, and may introduce semantic and computational problems.

An interesting feature of $\mathcal{DL}$s is the possibility to allow *individual names* called *nominals* not only in the ABox, but also in the description language. In literature there exist two main concept constructors. The first is the *one-of* constructor denoted by

$$\{a_1, \ldots, a_n\}$$

where $a_1, \ldots, a_n$ are individual names. It will be interpreted as

$$\{a_1, \ldots, a_n\}^{\mathcal{I}} = \left\{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\right\}.$$

The other constructor is the one called *fills* constructor

$$R\ :\ a,$$

for a role R and an individual a. The semantics of this is defined as

$$(R\ :\ a)^{\mathcal{I}} = \left\{d \in \Delta^{\mathcal{I}} | (d, a^{\mathcal{I}}) \in R^{\mathcal{I}}\right\}$$

This constructor stands for the set of those objects that have a as a filler of the role R.

3.2.2 Reasoning Services for TBoxes

A DL system provides also reasoning services to infer implicit knowledge on the terminology and the assertions. The main reasoning tasks on a TBox are:

- **Satisfiability**: a concept C is *satisfiable* with respect to $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}}$ is nonempty.
- **Subsumption**: a concept D *subsumes* another concept C with respect to $\mathcal{T}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.
- **Equivalence**: two concepts C and D are *equivalent* with respect to $\mathcal{T}$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.
- **Disjointness**: two concepts C and D are *disjoint* with respect to $\mathcal{T}$ if $C^{\mathcal{I}} \cap D^{\mathcal{I}} = \emptyset$ for every model $\mathcal{I}$ of $\mathcal{T}$.

Usually DL systems only provide a reasoning mechanism for the subsumption problem. In fact any other inference can be reduced to subsumption: from the viewpoint of worst-case complexity, subsumption is the most general inference for any $\mathcal{AL}$ *-language*.

Proposition 3.5. *Given two concepts C and D*

1. *C is unsatisfiable* $\Leftrightarrow$ *C is subsumed by* $\bot$*;*
2. *C and D are equivalent* $\Leftrightarrow$ *C is subsumed by D and D is subsumed by C;*
3. *C and D are disjoint* $\Leftrightarrow$ $C \cap D$ *is subsumed by* $\bot$.

On the other hand, for the best-case complexity calculus we can show that unsatisfiability is a special case of each of the other problems.

Proposition 3.6. *Let C be a concept. Then the following assertions are equivalent:*

1. *C is unsatisfiable*
2. *C is subsumed by* $\bot$*;*
3. *C and* $\bot$ *are equivalent;*
4. *C and* $\top$ *are equivalent;*

In order to obtain upper and lower complexity bounds for inference on concepts in $\mathcal{AL}$ *-languages*, we must look at the lower bound for unsatisfiability and the upper bound for subsumption. In fact, the lower bound for the unsatisfiability problem is the lower bound also for the complexity of the subsumption, the equivalence and the disjointness problem. Moreover, the upper bound for the complexity of the subsumption problem holds also for any other inference.

An important feature of $\mathcal{DL}$ is the fact that every TBox con be "internalized" into a single concept, i.e., it is possible to build a concept that expresses all the axioms of the TBox. This operation relies on the ability to build a universal role linking all individuals in a connected model. A universal role can be defined by means of regular expressions over roles as union of roles and the transitive closure. The possibility of internalizing the TBox when dealing with expressive description logics means that for such description logics reasoning with TBoxes, that is a logical implication, is no harder that reasoning with a single concept. The computational complexity of the above reasoning tasks increases with the expressivity of the underlying $\mathcal{DL}$. $\mathcal{ALC}$-concept satisfiability is a PSpace-complete problem (see the work of Schmidt-Schauss and Smolka [121]), and various decidable extensions of $\mathcal{ALC}$ are PSpace-, ExpTime-, and NExp-Time-complete. Concept satisfiability and logical implication in $\mathcal{ALC}_{reg}$ are ExpTime-hard. Concept satisfiability and logical implication in $\mathcal{ALC}_{reg}$ and $\mathcal{ALCI}_{reg}$ can be decided in deterministic exponential time.

3.2.3 Reasoning Services for ABoxes

An ABox contains two different kinds of assertions: concept assertions and role assertions. A concept assertion $C(a)$ states that the individual named a belongs to the interpretation of the concept C; a role assetion $R(b, c)$ states that the individual named c if a filler of the role R for the individual named b. In the example of Figure 3.2, CAROL belongs to the interpretation of the concept Mother, JOHN belongs to Father, but JOHN is a filler of the role hasChild for SARAH and simultaneously he is the father of MARY, that is MARY is a filler of the role hasChild for JOHN.

Languages that are constructed only upon a Tbox are named pure terminological languages, while those that include an Abox are named hybrid languages. An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ for hybrid languages differs from interpretations for pure terminological languages, because it does not only map atomic concepts and roles to sets and relations, but also each individual name a that is mentioned in the Abox to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The interpretation $\mathcal{I}$ satisfies a concept assertion $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and it satisfies a role assertion $R(b, c)$ if $(b^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}}$. An Interpretation $\mathcal{I}$ is a *model* for an ABox $\mathcal{A}$ if $\mathcal{I}$ satisfies all concept and role assertions.

One of the main reasoning tasks for ABoxes is to check the consistency of the set of assertions. An Abox is *consistent* with respect to a TBox $\mathcal{T}$, if there exists an interpretation $\mathcal{I}$ that is a model for both $\mathcal{I}$ and $\mathcal{T}$.

Another important aspect is the need to query the Tbox and the Abox about relationships between concepts, roles and individuals. In particular, DL users are interested in checking if an assertion α is entailed by an ABox $\mathcal{A}$, written $\mathcal{A} \models \alpha$, that is if every model of $\mathcal{A}$ satisfies α. The instance checking problem can be reduced to the consistency problem of the ABox. In fact, if we consider an assertion $\alpha = C(a)$, $\mathcal{A} \models \alpha \iff A \cup \neg C(a)$ is inconsistent for every individual a.

Two more reasoning tasks are the retrieval problem and its dual, the realization problem. The first service must return all individuals that are instances of a given concept. The second, given an individual and a set of concepts, returns the most specific concept that has that individual as an instance. Actually, there are various other services and inference problems that a $\mathcal{DL}$-knowledge representation system might offer. We have only sketched here the most important standard inference services and refer the reader to the "Description Logics Handbook" [10] for further details.

3.3 Description Logics with Concrete Domains

3.3.1 Overview

For several relevant applications of $\mathcal{DL}$s, there is a need for $\mathcal{DL}$s that can represent and reason about information of a more "concrete" nature, such as weights, amounts, durations, and spatial extensions. A drawback that all standard Description Logics share, as pointed out by Baader et al. in [12] is that all the knowledge must be represented at the abstract logical level. The need for such a language extension to concrete domains and predefined predicates was already evident to the designers of early $\mathcal{DL}$systems such as MESON (see [42] and [110]), K-REP (see [103], [104]) and CLASSIC (see [28], [26]). In addition to abstract individuals these systems also allow one to refer to "concrete" individuals such as numbers and strings. However these approaches are restricted to a specific collection of concrete objects. In 1991 Baader and Hanschke [11] proposed a different schema for integrating generic concrete domains into $\mathcal{DL}$s and presented tableau algorithm for deciding consistency of $\mathcal{ALC}(\mathcal{D})$-ABoxes. The algorithm is independent of a particular concrete domain $\mathcal{D}$ requiring only that $\mathcal{D}$ is admissible. The admissibility of a domain $\mathcal{D}$ means that deciding satisfiability of predicate conjunctions in $\mathcal{D}$ is decidable. They showed that two and more admissible concrete domains can be combined into one, retaining admissibility. More formally, a *concrete domain* $\mathcal{D}$ is a pair consisting of a set of concrete objects and predicates over this set. The language $\mathcal{ALC}(\mathcal{D})$ is obtained from $\mathcal{ALC}$ by augmenting it with:

- *abstract features*, i.e. roles interpreted as functional relations;
- *concrete features*, i.e. partial functions from the logical domain into the concrete domain;
- a *concept constructor* that allows to describe constraints on concrete values using predicates from the concrete domain.

The standard way of integrating numbers or other concrete datatypes into $\mathcal{DL}$s is to divide the set of logical objects into two disjoint sets, one containing *abstract* objects and the other containing *concrete* objects. Abstract objects can be related to concrete ones via functional concrete features, such as has_age, and relations between concrete objects can be described by a set of domain-specific predicates. A sequence of *abstract features*, i.e. functional roles, followed by a single concrete feature $(f_1 \dots f_k g)$ is called a *path*. *Concrete domain concept constructors* are of the form $\exists u_1, \dots, u_n.P$ or $\forall u_1, \dots, u_n.P$, where $u_1, \dots, u_n$ are paths and P is an n-ary predicate from the concrete domain. The intended interpretation is that u_i-values are related by the concrete predicate P.

The computational properties of the family of $\mathcal{DL}$s extended with concrete domains has been studied in the last year particularly by Lutz in [92], [93] and [98] (a survey can be found in [96]). In [92] Lutz combined $\mathcal{ALC}$ with the concrete domain of rational numbers with equality and inequality predicated and proved that concrete domains together with general inclusion axioms lead to undecidability. Lutz in [95] proved that there is a connection between the complexity of reasoning with an extended $\mathcal{DL}$ and the complexity of the concrete domain: reasoning in $\mathcal{ALC}(\mathcal{D})$ is PSpace-complete if reasoning with the concrete domain $\mathcal{D}$ is in PSpace. Various extensions of $\mathcal{ALC}(\mathcal{D})$ were investigated in [69], [131], [93], [95], including extensions with acyclic concept definitions, feature agreements and a role-forming concrete domain operator. Even slight extensions were shown to lead to NExpTime-complete reasoning and some of them even to undecidability. In [98] Lutz proved that combining concrete domains with general TBoxes easily leads to undecidability: the example investigated was $\mathcal{ALC}$ extended with general TBoxes and a rather inexpressive concrete domain based on the natural numbers with equality and incrementation predicate. Despite of the first discouraging results, some useful concrete domains that can be combined with general TBoxes in a decidable $\mathcal{DL}$ have been presented by Lutz in [97] and [94]: a temporal one based on the Allen relations for interval-based temporal reasoning, and a numerical one based on the real numbers equipped with various unary and binary predicates, such as $\leq, \geq, \neq$. Always in [97] and [94] Lutz proved that reasoning in $\mathcal{ALC}$ extended with these concrete domains and general Tboxes is decidable and EXPTIME-complete.

3.3.2 $\mathcal{ALC}(\mathcal{D})$

The formalism denoted $\mathcal{ALC}(\mathcal{D})$ is an extension of a $\mathcal{DL}$ by means of concrete domains and it is obtained incrementally by augmenting the language of $\mathcal{ALC}$ with *abstract features* (roles interpreted as functional relations), *concrete features* (partial functions from the logical layer to the concrete domain) and a new *concept constructor* (describing constraints between objects). We recall here the original proposal by Baader and Hanschke [11] presented widely in Lutz's survey on Description Logics with concrete domains [96].

Definition 3.7 (Concrete Domain). *A* concrete domain $\mathcal{D}$ *consists of a set* $\Delta^{\mathcal{D}}$*, the domain of* $\mathcal{D}$*, and a set* $pred(\mathcal{D})$*, the predicate names of* $\mathcal{D}$*. Each predicate name* $P \in pred(\mathcal{D})$ *is associated with an arity* n*, and an* n*-ary predicate* $P^{\mathcal{D}} \subseteq (\Delta^{\mathcal{D}})^n$

A practical example of a concrete domain is $\mathcal{N} = (\mathbb{N}, \{<_n, \leq_n, \geq_n, >_n, <, \leq, \geq, >\})$ where $\Delta^{\mathcal{N}}$ is the set of all nonnegative integers and $pred(\mathcal{N})$ consists of the binary predicate names $<, \leq, \geq, >$ as well as the unary predicate names $<_n, \leq_n, \geq_n, >_n$ for $n \in \mathcal{N}$, which are interpreted by predicates on $\mathbb{N}$ in the obvious way. In addition to numerical domains, the definition of concrete domain captures more abstract domains as, for instance, a relational database DB that can seen as a concrete domain $\mathcal{DB}$ whose domain is the set of atomic values occurring in DB and whose predicates are the relations that can be defined over DB using a query language (such as SQL). The decidability of the extended language is not guaranteed by the general notion of concrete domain and requires some restrictions. The first is related to the calculus of the Negation Normal Form of a concept. To be able to compute the negation normal form of concepts in the extended language, we must require that the set of predicate names of the concrete domain is *closed under negation*, i.e. if P is an n-ary predicate name in $pred(\mathcal{D})$ then there must exist a predicate name Q in $pred(\mathcal{D})$ such that $Q^{\mathcal{D}} = (\Delta^{\mathcal{D}})^n \backslash P^{\mathcal{D}}$. This predicate will be denoted by $\bar{P}$. In addition, we need a unary predicate name to denote the predicate $\Delta^{\mathcal{D}}$. The other restriction is related to the satisfiability of conjunctions defined as follows. Let $P_1, \ldots, P_k$ be k not necessarily different predicate names in $pred(\mathcal{D})$ of arities $n_1, \ldots, n_k$, we consider the conjunction

$$\bigwedge_{i=1}^{k} P_i(\bar{x}^{(i)})$$

where $\bar{x}^{(i)}$ stands for an n_i-tuple $(x_1^{(i)}, \ldots, x_{n^i}^{(i)})$. Such a conjunction is said to be *satisfiable* if and only if there exists an assignment of elements of $\Delta^{\mathcal{D}}$ to the variables such that the conjunction becomes true in $\mathcal{D}$. The problem of deciding satisfiability of finite conjunctions of this form is normally called *satisfiability problem* for $\mathcal{D}$.

Definition 3.8 (Admissibility). *The concrete domain* $\mathcal{D}$ *is said to be* admissible *iff*

(a) *the set of its predicate names is closed under negation and contains a symbol* $\top_{\mathcal{D}}$ *for* $\Delta^{\mathcal{D}}$*,*

(b) *the satisfiability problem for $\mathcal{D}$ is decidable.*

The language$\mathcal{ALC}(\mathcal{D})$ extends $\mathcal{ALC}$ in two respects: first, the set of role names is partitioned into functional roles and ordinary roles, second there is a new constructor (*existential predicate restriction*) added to the syntax rules for $\mathcal{ALC}$. We recall here the formal definitions of the syntax and the semantics of $\mathcal{ALC}(\mathcal{D})$.

Definition 3.9 ($\mathcal{ALC}(\mathcal{D})$ Syntax). *Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_{cF}}$ be pairwise disjoint and countably infinite sets of* concept names, role names *and* concrete features *(also called functional roles) respectively. Let $\mathsf{N_{aF}}$, the set of* abstract features, *be a countably infinite subset of $\mathsf{N_R}$. A* path *u is a composition $f_1 \dots f_n g$ of n abstract features f_i with $0 \leq i \leq n$ and a concrete feature g. The syntax of $\mathcal{ALC}(\mathcal{D})$ is defined by adding to the syntax rules for $\mathcal{ALC}$ the rule:*

$$C, D \rightarrow \exists(u_1, \dots, u_n).P$$

where P is an n-ary predicate of $\mathcal{D}$ and $u_1, \dots, u_n$ are chains of abstract features.

Definition 3.10 ($\mathcal{ALC}(\mathcal{D})$ Semantics). *An* interpretation *$\mathcal{I}$ for $\mathcal{ALC}(\mathcal{D})$ consists of a set $\Delta^{\mathcal{I}}$, the abstract domain of the interpretation and an interpretation function. The abstract domain and the given concrete domain must be disjoint, $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{D}} = \emptyset$. As before, the interpretation function associates each concept name with a subset of $\Delta^{\mathcal{I}}$ and each ordinary role name a binary relation on $\Delta^{\mathcal{I}}$. The new feature is that the functional roles are now interpreted by partial functions from $\Delta^{\mathcal{I}}$ into $\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{D}}$. If $u = f_1 \circ \dots \circ f_n$ is a chain of abstract features, then $u^{\mathcal{I}}$ denotes the composition $f_1^{\mathcal{I}} \circ \dots \circ f_n^{\mathcal{I}}$ of the partial functions $f_1^{\mathcal{I}}, \dots, f_n^{\mathcal{I}}$. the semantics of the usual $\mathcal{ALC}$-constructors is defined as before, while the existential predicate restriction is interpreted as follows:*

$$(\exists(u_1, \dots, u_n).P)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \text{ there exist } r_1, \dots, r_n \in \Delta^{\mathcal{D}} \text{ such that}$$
$$u_1^{\mathcal{I}}(x) = r_1, \dots, u_n^{\mathcal{I}}(x) = r_n \text{ and } (r_1, \dots, r_n) \in P^{\mathcal{D}}\}.$$

In their paper [11], Baader and Hanschke presented a tableau algorithm capable of deciding the problem of $\mathcal{ALC}(\mathcal{D})$-concept satisfiability and hence capable of deciding also the problem of concept subsumption. This decidability result is rather general since it applies to any concrete domain that satisfies the admissibility condition. The complexity of reasoning with concrete domain has been investigated by Lutz in [93] and [95] stating that the complexity of reasoning with $\mathcal{ALC}(\mathcal{D})$ clearly depends on the complexity of $\mathcal{D}$-satisfiability stating that pure $\mathcal{ALC}(\mathcal{D})$-concept satisfiability and subsumption are PSPACE-complete if $\mathcal{D}$ is admissible and $\mathcal{D}$-satisfiability is in PSPACE.

The $\mathcal{DL}$s presented in the following sections are extensions of $\mathcal{ALC}(\mathcal{D})$ considered "standard" in the area of Description Logics: $\mathcal{ALCRP}(\mathcal{D})$ which allow the definition of complex roles with reference to concrete domain predicates and $\mathcal{ALC}(\mathcal{C})$ which is defined over a particular kind of concrete domains called constraint systems.

3.3.3 $\mathcal{ALCRP}(\mathcal{D})$

One of the most expressive $\mathcal{DL}$ extended with concrete domains is $\mathcal{ALCRP}(\mathcal{D})$ which is obtained by using the concrete domain not only to define concepts, but also allowing the definition of complex roles related to concrete domain predicates. This $\mathcal{DL}$ was first presented in [131] by Haarslev, Lutz and Möller. The central notion of this formalism is the *concrete domain role* defined as follows.

Definition 3.11 (Concrete Domain Role). *A* concrete domain role *is an expression of the form*

$$\exists(u_1, \ldots, u_n), (v_1, \ldots, v_m).P$$

where $u_1, \ldots, u_n, v_1, \ldots, v_m$ are paths and P is an $n+m$-ary predicate. The semantic is given as follows:

$$\begin{aligned}(\exists(u_1, \ldots, u_n), (v_1, \ldots, v_m).P)^{\mathcal{I}} := \\ \{(d, e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \textit{There exist } x_1, \ldots, x_n \textit{ and } y_1, \ldots, y_m \\ \textit{such that } u_i^{\mathcal{I}}(d) = x_i \textit{ for } 1 \le i \le n,\ v_i^{\mathcal{I}}(d) = y_i \textit{ for} \\ 1 \le i \le m,\ \textit{and } (x_1, \ldots, x_n, y_1, \ldots, y_m) \in P^{\mathcal{D}}\}\end{aligned}$$

The language $\mathcal{ALCRP}(\mathcal{D})$ is obtained from $\mathcal{ALC}(\mathcal{D})$ by allowing the use of concrete domain roles inside the $\exists R.C$ and $\forall R.C$ constructors but not inside the concrete domain concept constructor. In [100] Lutz and Möller prove that there exists a concrete domain $\mathcal{D}$ such that the satisfiability of $\mathcal{ALCRP}(\mathcal{D})$-concepts is undecidable.

Theorem 3.12 (Lutz, Möller). *For concrete domains $\mathcal{D}$ such that (i) $\mathbb{N} \subseteq \Delta^{\mathcal{D}}$ and (ii) $pred(\mathcal{D})$ provides a unary predicate for equality with 0, a binary equality predicate, and a binary predicate for incrementation, pure $\mathcal{ALCRP}(\mathcal{D})$-concept satisfiability and subsumption are undecidable.*

In [131] Haarslev et al. identify a decidable fragment of the formalism introducing some syntactic restrictions presented in the following definition.

Definition 3.13 (Restricted Concept Term). *A concept term X is called* restricted *w.r.t. a TBox $\mathcal{T}$ iff the equivalent X' unfolded w.r.t. $\mathcal{T}$ and in NNF fulfills the following conditions:*

1. *For any subconcept term C of X' that is of the form $\forall R_1.D$ where R_1 is a complex role term, D does not contain any terms of the form $\exists R_2.E$ where R_2 is also a complex role term.*
2. *For any subconcept term C of X' that is of the form $\exists R_1.D$ where R_1 is a complex role term, D does not contain any terms of the form $\forall R_2.E$ where R_2 is also a complex role term.*
3. *For any subconcept term C of X' that is of the form $\exists R.D$ or $\forall R.D$ where R is a complex role term, D contains only predicate exists restrictions that (i) quantify over feature chains of length 1 and (ii) are not contained inside any value and exists restrictions that are also contained in $\mathcal{D}$.*

A terminology is called restricted iff all concept terms occurring on the right-hand side of terminological axioms in $\mathcal{T}$ are restricted w.r.t. $\mathcal{T}$. An ABox $\mathcal{A}$ is called restricted w.r.t. a TBox $\mathcal{T}$ iff $\mathcal{T}$ is restricted and all concept terms used in $\mathcal{A}$ are restricted w.r.t. the terminology $\mathcal{T}$. The syntactical restrictions ensure that the finite model property holds on the restricted formalism and that the standard reasoning problems are decidable considering only restricted concept terms.

In [91] Lutz investigated the complexity of reasoning with concrete domains and showed the correspondence between the complexity of a concrete domain and the complexity of the reasoning with the $\mathcal{DL}$ extended by that domain. Lutz proved the following result.

Theorem 3.14 (Lutz, [96]). *Let $\mathcal{D}$ be a concrete domain. If $\mathcal{D}$ is admissible and $\mathcal{D}$-satisfiability is in NP, then pure satisfiability of restricted $\mathcal{ALCRP}(\mathcal{D})$-concepts can be decided in* NEXPTIME.

3.3.4 $\mathcal{ALC}(\mathcal{C})$

Another restriction to the standard $\mathcal{ALC}(\mathcal{D})$ which is of some interest for this thesis is $\mathcal{ALC}(\mathcal{C})$, presented in recent years by Lutz and Miličić in [99]. $\mathcal{ALC}(\mathcal{C})$ is a restriction of $\mathcal{ALC}(\mathcal{D})$ in the sense of the considered concrete domains. The authors concentrate on concrete domains which are Constraint Systems based on relation algebras. In this section we recall the definitions of [99] which provide a meaningful framework for spatial reasoning applications as proved by the authors which consider the constraint language given by RCC-8 as an example of constraint system. In Lutz and Miličić introduce a new framework that apply a Description Logic with concrete domain and general TBoxes ($\mathcal{ALC}(D)$) to a constraint system $\mathcal{C}$, the combination produces a system, $\mathcal{ALC}(\mathcal{C})$, for which Lutz and Miličić provide a Tableau decision algorithm that establishes the consistency of a terminology.

The use of concrete domains allows the integration of "concrete qualities" into abstract description logic concepts. This is accomplished via *concrete features* which can be seen as a link between an abstract definition included in a wider terminology and a concrete object which, in this case, has a spatial "existence". For instance we can consider the abstract definition of city linked via the *abstract feature* loc (interpreted as "to have a location in the concrete space") to the concrete constraints imposed on that class of concrete spatial objects. An example of such constraints can be that a city must belong to a state and that cannot overlap another city.

The tableau algorithm proposed by the researchers applies to the system under three very general assumptions: decidability of $\mathcal{C}$, the *patchwork* property and the *compactness* property of the system. When these properties hold they propose to describe the system as ω-admissible, and prove that every ω-admissible constraint system $\mathcal{C}$ makes $\mathcal{ALC}(\mathcal{C})$ decidable as well.

$\mathcal{ALC}(\mathcal{C})$ is essentially a reduction of $\mathcal{ALCRP}(\mathcal{D})$ where the operators are those of a constraint algebra and the concrete domain is, indeed, the constraint system itself. In particular, a constraint system is defined by Lutz and Miličić as a concrete domain that only has binary predicates, which are interpreted as jointly exhaustive and pairwise disjoint (JEPD) relations. We refer here to the definitions 1 and 2 of [99].

Definition 3.15 (Rel-network). *Let Var be a countably infinite set of variables and Rel a finite set of binary relation symbols. A Rel-constraint is an expression $(x\ r\ y)$ with $x, y \in$Var and $r \in$Rel. A Rel-network is a (finite or infinite) set of Rel-constraints. For N a Rel-network, we use V_N to denote the variables used in N. We say that N is complete if, for all $x, y|inV_N$, there is exactly one constraint $(x\ r\ y) \in N$*

Definition 3.16 (Model, Constraint System). *Let N be a Rel-network and N' a complete Rel-network. We say that N' is a model of N if there is a mapping $\tau : V_N \rightarrow V_{N'}$ such that $(x\ r\ y) \in N$ implies $(\tau(x)\ r\ \tau(y)) \in N'$.*

A constraint system $\mathcal{C} = \langle \mathsf{Rel}, \mathfrak{M} \rangle$ consists of a finite set of relation symbols Rel and a set $\mathfrak{M}$ of complete Rel-networks (the models of $\mathcal{C}$). A Rel-network N is satisfiable in $\mathcal{C}$ if $\mathfrak{M}$ contains a model of N.

We trivially say that a constraint system is decidable when there is an algorithm that, for every constraint network (or, in other formalizations, every finite set of constraint expressions), in a finite time establishes whether the network is satisfiable or not.
The **patchwork property** is enjoyed by those constraint systems such that the following implication holds: if two complete networks identical for intersection parts are satisfiable, then the composition (that is the union) of these networks is satisfiable as well. The **compactness property**, finally, holds for those systems such that networks (possibly infinite) are satisfiable if and only if every finite sub-network is satisfiable.

When a system $\mathcal{C}$ is decidable and enjoys both the patchwork property and the compactness property then we say that $\mathcal{C}$ is ω-admissible. $\mathcal{ALC}(\mathcal{C})$ augmented with a concrete domain $\mathcal{C}$ ω-admissible is still decidable and provides much more expressive power respect to the simple Description Logic.

Lutz and Miličić defined in [99] a framework for reasoning about topological relations, proving that the constraint system (RCC-8)$_{\mathbb{R}^2}$ is ω-admissible and therefore that the combination of $\mathcal{ALC}$ and the *Region Connection Calculus* is decidable. We now recall the syntax and the semantics of $\mathcal{ALC}(\mathcal{C})$ as defined in [99].

Definition 3.17. *Let $\mathcal{C} = (\mathsf{Rel}; \mathfrak{M})$ be a constraint system, and let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_{cF}}$ be mutually disjoint and countably infinite sets of concept names, role names, and concrete features. We assume that $\mathsf{N_R}$ is partitioned into two countably infinite subsets $\mathsf{N_{aF}}$* (abstract features) *and $\mathsf{N_{sR}}$* (standard roles). *A path of length $k+1$ with $k \geq 0$ is a sequence $R_1 \ldots R_k g$ where $R_1, \ldots, R_k \in \mathsf{N_R}$ and $g \in \mathsf{N_{cF}}$. A path $R_1 \ldots R_k g$ with $R_1, \ldots, R_k \in \mathsf{N_{aF}}$ is called* feature path. *The set of $\mathcal{ALC}(\mathcal{C})$-concepts is the smallest set such that:*

- *every concept name $A \in \mathsf{N_C}$ is a concept;*
- *if C and D are concepts and $R \in \mathsf{N_R}$, then $\neg C$, $C \sqcup D$, $C \sqcap D$, $\forall R.C$, and $\exists R.C$ are concepts;*
- *if u_1 and u_2 are feature paths and $r_1, \ldots, r_k \in \mathsf{Rel}$, then the following are also concepts:*
$$\exists u_1, u_2.(r_1 \vee \ldots \vee r_k) \text{ and } \forall u_1, u_2.(r_1 \vee \ldots \vee r_k)$$
- *if U_1 and U_2 are paths of length at most two and $r_1, \ldots, r_k \in \mathsf{Rel}$, then the following are also concepts:*
$$\exists U_1, U_2.(r_1 \vee \ldots \vee r_k) \text{ and } \forall U_1, U_2.(r_1 \vee \ldots \vee r_k)$$

A concept inclusion is an expression of the form $C \sqsubseteq D$, where C and D are concepts. We use $C := D$ as abbreviation for the two concept inclusions $C \sqsubseteq D$ and $D \sqsubseteq C$. An infinite set of concept inclusions is called a (general) TBox.

Definition 3.18. *An* interpretation *$\mathcal{I}$ is a tuple $(\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}}, M_{\mathcal{I}})$, where $\Delta_{\mathcal{I}}$ is a set called the* domain, *$M_{\mathcal{I}} \in \mathfrak{M}$ and the* interpretation function *$\cdot^{\mathcal{I}}$ maps*

- *each concept name C to a subset $C^{\mathcal{I}}$ of $\Delta_{\mathcal{I}}$;*
- *each role name R to a subset $R^{\mathcal{I}}$ of $\Delta_{\mathcal{I}} \times \Delta_{\mathcal{I}}$;*
- *each abstract feature f to a partial function $f^{\mathcal{I}}$ from $\Delta_{\mathcal{I}}$ to $\Delta_{\mathcal{I}}$;*
- *each concrete feature g to a partial function $g^{\mathcal{I}}$ from $\Delta_{\mathcal{I}}$ to the set of variables $V_{M_{\mathcal{I}}}$ of $M_{\mathcal{I}}$.*

If $\mathbf{r} = r_1 \vee \ldots \vee r_k$ *where* $r_1, \ldots, r_k \in \mathsf{Rel}$, *we write* $M_{\mathcal{I}} \models (x\ \mathbf{r}\ y)$ *iff there exists an* $i \in \{1 \ldots k\}$ *such that* $(x\ \mathbf{r_i}\ y) \in M_{\mathcal{I}}$. *The interpretation function can be extended to arbitrary concepts as follows:*

$$
\begin{aligned}
\neg C^{\mathcal{I}} &:= \Delta \backslash C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &:= \{x \in \Delta_{\mathcal{I}} |\ \forall y[(x,y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &:= \{x \in \Delta_{\mathcal{I}} |\ \exists y[(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\} \\
(\exists U_1, U_2.\mathbf{r})^{\mathcal{I}} &:= \{x \in \Delta_{\mathcal{I}} |\ \exists v_1 \in U_1^{\mathcal{I}}(d) \text{ and} \\
&\quad\ \ v_2 \in U_2^{\mathcal{I}}(d) \text{ with } M_{\mathcal{I}} \models (v_1\ \mathbf{r}\ v_2)\} \\
(\forall U_1, U_2.\mathbf{r})^{\mathcal{I}} &:= \{x \in \Delta_{\mathcal{I}} |\ \forall v_1 \in U_1^{\mathcal{I}}(d) \text{ and} \\
&\quad\ \ v_2 \in U_2^{\mathcal{I}}(d) \text{ with } M_{\mathcal{I}} \models (v_1\ \mathbf{r}\ v_2)\}
\end{aligned}
$$

where for a path $U = R_1 \ldots R_k g$ *and* $x \in \Delta_{\mathcal{I}}$, $U^{\mathcal{I}}(d)$ *is defined as*

$$
\{v \in V_{M_{\mathcal{I}}} |\ \exists e_1, \ldots, e_{k+1}\ :\ x = e_1, (e_i, e_{i+1}) \in R_i^{\mathcal{I}}
$$

$$
for\ 1 \leq i \leq k,\ and\ g^{\mathcal{I}}(e_{k+1}) = v\}.
$$

An interpretation $\mathcal{I}$ *is a* model *of a concept* C *iff* $C^{\mathcal{I}} \neq \emptyset$. $\mathcal{I}$ *is a* model *of a TBox* $\mathcal{T}$ *iff it satisfies* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *for all concept inclusions* $C \sqsubseteq D$ *in* $\mathcal{T}$.

In the following chapters of the thesis we will present some hybridization techniques between the QSRR formalisms introduced in Chapter 2 and the Description Logics presented in the present Chapter.

4

QSR with Description Logics with fixed RBox

4.1 The $\mathcal{ALCI}_{\mathsf{RCC}}$ family

4.1.1 Syntax and Semantics of the $\mathcal{ALCI}_{\mathsf{RCC}}$ family

An important family of $\mathcal{ALC}$-logics is given by the extension to the *inverse role* construct. $\mathcal{ALCI}$ is a description logic that includes such an *inverse role* construct, which allows one to denote the inverse of a given relation. Recent research carried out by Haarslev and Ding aims to improve the efficiency of reasoners for a logic with the inverse role operator (see [145] and [146]). $\mathcal{ALCI}$ is more expressive than $\mathcal{ALC}$; nevertheless in order to model adequately qualitative spatial concepts the expressiveness of this logic is still insufficient. In particular there exists a specific family of inverse role logics denoted by $\mathcal{ALCI}_{\mathcal{RCC}}$ and described by Wessel in [140], [139] developed specifically for QSR.

In 2001 Wessel presented an overview of various $\mathcal{ALC}$-extensions with composition based role inclusion axioms of the form $S \circ T \sqsubseteq R_1 \sqcup ... \sqcup R_n$ enforcing $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup ... \cup R_n^{\mathcal{I}}$ on the models $\mathcal{I}$ (see [138]). A set of this role axioms is called a role box and the resulting logic was called $\mathcal{ALC}_{\mathcal{RA}\ominus}$. Concept satisfiability in this logic and in smaller sub languages is undecidable. Wessel et al. [141] also enforced disjointness on all roles considering that in a spatial environment we have the JEPD property, but even this new logic called $\mathcal{ALC}_{\mathcal{RA}}$ turned out to be undecidable [138]. Obviously since $\mathcal{ALC}_{\mathcal{RA}}$ is undecidable, the extension given by inverse roles $\mathcal{ALCI}_{\mathcal{RA}}$ is undecidable too.

However they investigated if undecidability arises even if certain classes of role boxes are considered, especially the role boxes obtained from translating the composition table of $\mathcal{RCC}$. Depending on the exploited $\mathcal{RCC}$ composition table Wessel in [140] defines some $\mathcal{ALCI}_{\mathcal{RA}}$ specializations called $\mathcal{ALCI}_{\mathcal{RCC}}$.

More formally, the syntax and semantics of the family of $\mathcal{ALCI}_{\mathcal{RCC}}$ logics are defined as follows. Let $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_{\mathcal{R}}$ be two disjoint sets of symbols: the set of *concept names* or atomic concepts and the set of *role names* or atomic roles. Each $\mathcal{ALCI}$ concept is a valid $\mathcal{ALCI}_{\mathcal{RCC}}$ concept and the syntax and the semantics are the same.

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ has a function $.^{\mathcal{I}}$ that maps every concept name to a subset of the domain of the interpretation $\Delta^{\mathcal{I}}$ and every role name to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ by the usual rules.

Wessel introduced a new notation for the finite disjunction of roles: if $R = S_1, ..., S_n$ is a disjunction of roles S_i, we write $\forall S_1, ..., S_n.C$ ($\exists S_1, ..., S_n.C$) as a shorthand for

$\forall S_1.C \sqcap ... \sqcap \forall S_n.C$ ($\exists S_1.C \sqcap ... \sqcap \exists S_n.C$). Henceforth given a concept θ we will denote $\tilde{\theta}$ the *equivalent form* with all the shorthands expanded in the corresponding disjunctive formulae.

According to the different sets of roles $\mathcal{N_R}$ corresponding to different $\mathcal{RCC}$ formalisms, we define the following logics:

- $\mathcal{ALCI}_{\mathcal{RCC}8}$: $\mathcal{N_R} = \{DC, EC, PO, EQ, TPP, TPPI, NTPP, NTPPI\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}5}$: $\mathcal{N_R} = \{DR, PO, EQ, PP, PPI\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}3}$: $\mathcal{N_R} = \{DR, ONE, EQ\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}2}$: $\mathcal{N_R} = \{ONE, O\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}1}$: $\mathcal{N_R} = \{SR\}$.

An important feature of these logics with respect to $\mathcal{ALC_{RA}}$ is that $\mathcal{N_R}$ is a fixed and finite and an arbitrary countable set. The *role boxes* of $\mathcal{ALCI}_{\mathcal{RCC}1} \dots \mathcal{ALCI}_{\mathcal{RCC}8}$ contain the role axioms that can be obtained exploiting the RCC composition tables (see Tables 2.1, 2.2, 2.3, 2.4 and 2.5 pages 15-16). The semantics of this family of $\mathcal{DL}$ deals with the problem of the description of the properties of spatial relations. $\mathcal{DL}$ do not offer any instrument to formally describe properties like reflexiveness or transitivity of roles. These kinds of properties are fundamental for correct reasoning on spatial matters. For this reason Wessel adopts the notion of *frame* from modal logics and poses side-conditions on spatial roles to guarantee the soundness w.r.t. the RCC reasoning. Wessel considers a *frame* as an interpretation which only fixes the extensions of the role names and a *frame condition* as a semantic requirement that must hold on the extensions of the role names. Frame conditions cannot be embedded into the $\mathcal{DL}$-language (the same situation as in modal logic languages) and must be imposed as meta-level constraints. The following frame conditions guarantee the same properties of the RCC relation algebra.

Definition 4.1 (General RCC Frame Conditions). *Given an interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *the following* frame conditions *must hold:*

- One Cluster *requirement:* $\forall x, y \in \Delta^{\mathcal{I}} \;: < x, y > \in \bigcup_{R \in \mathcal{N_R}} R^{\mathcal{I}}$
- Disjointness *requirement:* $\forall R, S \in \mathcal{N_R}$ *with* $R \neq S$: $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
- Converse *requirement:* $R^{\mathcal{I}} = (S^{\mathcal{I}})^{-1}$ *iff* $R = \mathsf{inv}(S)$
- Role composition *requirement:* $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup ... \cup R_n^{\mathcal{I}}$ *iff* $S \circ T \sqsubseteq R_1 \sqcup ... \sqcup R_n$ *is an entry of the corresponding* RCC *composition table.*

The union of the *One Cluster* condition with the *Disjointness* condition guarantees the JEDP property to the set $\mathcal{N_R}$ of spatial roles. The *Converse* condition guarantees that the set of spatial roles is closed under the inverse operation, while the *Role Composition* requirement imposes the correspondence between the RBox and the composition table of the considered RCC formalism. Some more conditions are posed by Wessel on the equivalence relations which must behave as the identity-element of the corresponding set of spatial roles. While in the case of coarser RCC formalisms there is a single possible condition to ensure the existence of the identity-element, for the more expressive RCC3, RCC5, RCC-8 there exist two possible different semantics.

Definition 4.2 (Equivalence Frame Condition). *Given the* identity relation $Id(\Delta^{\mathcal{I}}) =_{def} \{< x, x > | x \in \Delta^{\mathcal{I}}\}$ *following frame conditions holds according to the* RCC *-formalism:*

- $\mathcal{ALCI}_{\mathcal{RCC}1}$: $Id(\Delta^{\mathcal{I}}) \subseteq SR^{\mathcal{I}}$

- $\mathcal{ALCI}_{\mathcal{RCC}2}$: $Id(\Delta^{\mathcal{I}}) \subseteq O^{\mathcal{I}}$
- $\mathcal{ALCI}_{\mathcal{RCC}3}$, $\mathcal{ALCI}_{\mathcal{RCC}5}$, $\mathcal{ALCI}_{\mathcal{RCC}8}$:
  - $Id(\Delta^{\mathcal{I}}) \subseteq EQ^{\mathcal{I}}$ *(weak EQ-semantics)*
  - $Id(\Delta^{\mathcal{I}}) = EQ^{\mathcal{I}}$ *(strong EQ-semantics)*

The *strong EQ-semantics* and the *weak EQ-semantics* have immediate consequences on the models and on the satisfiability of concepts, as the availability of nominals or the possibility to impose a maximal cardinality of the admissible models. We refer the reader to the following section for formal definitions and further details on expressivity and complexity.

4.1.2 Expressivity

Model Properties

One of the main problems pointed out by Wessel is that both $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ have not the *Finite Model Property* (FMP). Standard Description Logics like $\mathcal{ALCI}$ allow only concepts with finite models; on the contrary these extended Description Logics admit concepts that can be represented only with non-finite models. The FMP is an important property because it guarantees the decidability of the formalism. These two logics do not present this property and this makes more difficult to prove that they are decidable. The lack of the FMP derives from two peculiar aspects of these formalisms: the disjointness of spatial roles and the fact that the *part roles* are transitive. The Proper-Part role (and the Inverse Proper-Part role) induces a strict partial order on the universe of spatial objects. The concept that follows presented by Wessel is satisfiable only by non-finite models, because it induces an infinite chain of containment of spatial objects.

$$(\exists \mathcal{PP}.C \sqcap \forall \mathcal{PP}.\exists \mathcal{PP}.D)(x) \tag{4.1}$$

A property important for computational implications is the *Tree Model Property* (TMP). Description Logics that have the TMP allow the modeling of concepts that are satisfiable by interpretations that correspond to tree structures. The fact that RCC relations between objects are represented with complete graphs means that Description Logics based on jointly exhaustive relations (i.e. with the *one cluster* requirement) have not the TMP and it is well known that algorithmic graph problems exhibit high computational complexity.

Many relevant observations have been made by Wessel on the direct consequences of the adopted semantics. We recall here some of these observations in the following propositions.

Proposition 4.3. *Each $\mathcal{ALCI}_{\mathcal{RCC}}$ logic has a* universal role $R^* =_{def} \mathcal{N}_{\mathcal{R}}$ *that corresponds to the disjunction of all spatial roles and refers to all spatial objects in the model including the point itself. The existence of the universal role is due to the* one cluster *requirement and allows to encode whole TBoxes into single concept expressions by a process called* internalization*.*

Proposition 4.4. *The* strong EQ-semantics *implies the existence of the* difference role $R^* =_{def} \mathcal{N}_{\mathcal{R}} \setminus \{EQ\}$ *which refers to all other points in the model excluding the point itself. The* difference role *has the expressive power to encode* nominals*.*

Proposition 4.5. *The* weak EQ-semantics *means that the $EQ^{\mathcal{I}}$ relation is a superset of the identity relation: $EQ^{\mathcal{I}}$ is a congruence relation for the roles. The* weak EQ-semantics *can be made strong adding for each relevant concept name $CN \in \mathcal{N}_{\mathcal{C}}$ to the original knowledge base the following axiom:*

$$CN \sqsubseteq \forall EQ.CN$$

The nodes in an EQ-clique can be collapsed into a single reflexive node, resulting in a model under the strong EQ-semantics.

It is clear that the satisfiability under the *weak EQ-semantics* implies the satisfiability under the the *strong EQ-semantics*. As pointed out in Proposition 4.4, the availability of the *difference role* in the *strong EQ-semantics* in all superlogics of $\mathcal{ALCI}_{\mathcal{RCC}3}$ allows the encoding of *nominals* which are concept names interpreted as singletons, therefore representing a single individual in $\Delta^{\mathcal{I}}$. An important consequence of the presence of nominals is the possibility to translate whole knowledge bases (ABoxes included) into concept expressions. Another relevant direct consequence of the semantics is stated in the following

Proposition 4.6. *Providing a finite set of nominals for each individual in the model, it is possible to limit the* maximal cardinality *of models enforcing on a knowledge base finite models of maximal cardinality.*

Para-decidability

Following the same approach of Baader and Nutt in [10], we define a tableau-based satisfiability algorithm for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ to test concept satisfiability. Let C_0 be a $\mathcal{ALCI}_{\mathcal{RCC}5}$ (or $\mathcal{ALCI}_{\mathcal{RCC}8}$) concept in *normal form* (i.e. negation occurs only in front of concept names and all the shorthands have already been expanded in the corresponding disjunctive formulae). The algorithm starts with the ABox $\mathcal{A}_0 = \{C_0(x_0)\}$ and applies the transformation rules (see Table 4.1) to the ABox until no more rules apply. If at least one ABox obtained does not contain any contradiction then $\mathcal{A}_0$ is consistent (therefore C_0 is satisfiable).
The rules presented in Table 4.1 are standard tableau rules for generic Description Logics (see the tableau-based algorithm for $\mathcal{ALCN}$ presented by Baader and Nutt, Chapter 2, [10]) except for the $t\exists$-Rule. For our purpose we split the rule for the existential quantification depending on the kind of role: we define a rule for standard (non-spatial) roles, that is $R \notin \mathcal{N}_{\mathcal{R}}$, and a specific rule for roles that correspond to RCC relationships. If R is a spatial role, the introduction of a new individual y implies a set of implicit relationships: one relationship for every spatial object already in the ABox. Moreover it could mean a modification of existing constraints between old spatial objects. For this reason the rule:

- defines a temporary ABox $\mathcal{A}^{\mathrm{tmp}}$ by introducing a new individual y such that $R(x, y)$ and $C(y)$;
- extracts the spatial constraint network given by all spatial role assertions in the temporal ABox: $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})$;
- calculates the deductive closure of the constraint network $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*$;
- finds all consistent scenarios of the deductive closure, in other words it checks all possible configurations of spatial relations among the objects according to the deductive closure: $\{SC_1((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*), \ldots, SC_n((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*)\}$;

$\sqcap$-Rule:
Condition: $\mathcal{A}$ contains $(C_1 \sqcap C_2)(x)$, but it does not contain both $C_1(x)$ and $C_2(x)$
Action: $\mathcal{A}' = \mathcal{A} \cup \{C_1(x), C_2(x)\}$

$\sqcup$-Rule:
Condition: $\mathcal{A}$ contains $(C_1 \sqcup C_2)(x)$, but it does not contain either $C_1(x)$ nor $C_2(x)$
Action: $\mathcal{A}' = \mathcal{A} \cup \{C_1(x)\}$, $\mathcal{A}'' = \mathcal{A} \cup \{C_2(x)\}$

$\forall$-Rule:
Condition: $\mathcal{A}$ contains $(\forall R.C)(x)$ and $R(x, y)$, but it does not contain $C(y)$
Action: $\mathcal{A}' = \mathcal{A} \cup \{C(y)\}$.

$nt\exists$-Rule:
Condition: $\mathcal{A}$ contains $(\exists R.C)(x)$ with $R \notin \mathcal{N}_{\mathcal{R}}$, but there is no individual name z such that $C(z)$ and $R(x, z)$ are in $\mathcal{A}$
Action: $\mathcal{A}' = \mathcal{A} \cup \{R(x, y), C(y)\}$ where y is an individual name not occurring in $\mathcal{A}$.

$t\exists$-Rule:
Condition: $\mathcal{A}$ contains $(\exists R.C)(x)$ with $R \in \mathcal{N}_{\mathcal{R}}$, but there is no individual name z such that $C(z)$ and $R(x, z)$ are in $\mathcal{A}$
Action: $\mathcal{A}^{\text{tmp}} = \mathcal{A} \cup \{R(x, y), C(y)\}$ where y is an individual name not occurring in $\mathcal{A}$.
Given the set of topological role assertions of $\mathcal{A}^{\text{tmp}}$, $(\mathcal{A}^{\text{tmp}}\text{Sp})$,
find the deductive closure of the constraint network $(\mathcal{A}^{\text{tmp}}\text{Sp})^*$.
Find all consistent scenarios of the deductive closure:
$\{SC_1((\mathcal{A}^{\text{tmp}}\text{Sp})^*), \ldots, SC_n((\mathcal{A}^{\text{tmp}}\text{Sp})^*)\}$.
Generate all the new ABoxes
$\mathcal{A}'_i = (\mathcal{A}^{\text{tmp}} \backslash \mathcal{A}\text{Sp}) \cup SC_i((\mathcal{A}^{\text{tmp}}\text{Sp})^*)$.

Table 4.1. Transformation rules of the satisfiability algorithm for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$.

- generates a new ABox for each consistent scenario replacing the constraint network given by old spatial role assertions: $\mathcal{A}'_i = (\mathcal{A}^{\text{tmp}} \backslash \mathcal{A}\text{Sp}) \cup SC_i((\mathcal{A}^{\text{tmp}}\text{Sp})^*)$.

It is worth noticing that the tableau procedure relies in the decidability of the RCC - constraint networks. The $t\exists$-Rule calls a sort of "oracle" to compute the deductive closure and all consistent scenarios. This is obviously possible thanks to the computational properties of the considered QSRR formalism. We will see that there are formalisms like the DEM by Clementini [33] for which we cannot guarantee the termination even for the single $t\exists$-Rule and so for the full tableau procedure. Now we want to prove that the tableau-based algorithm is sound and complete. In order to do so we prove that the transformation rules preserve the satisfiability of concept expressions.

Lemma* 4.7 *The transformation rules preserve ABox satisfiability:*

(a) *if $\mathcal{A}$ is satisfiable then the Abox $\mathcal{A}'$ obtained with the $\sqcap$-Rule is still satisfiable;*
(b) *if $\mathcal{A}$ is satisfiable then at least one of the ABoxes $\mathcal{A}'$, $\mathcal{A}''$ obtained with the $\sqcup$-Rule is still satisfiable;*
(c) *if $\mathcal{A}$ is satisfiable then the Abox $\mathcal{A}'$ obtained with the $\forall$-Rule is still satisfiable;*
(d) *if $\mathcal{A}$ is satisfiable then the Abox $\mathcal{A}'$ obtained with the $nt\exists$-Rule is still satisfiable;*

(e) *if $\mathcal{A}$ is satisfiable then at least one of the Aboxes $\mathcal{A}'_i$ obtained with the $t\exists$-Rule is still satisfiable;*

Proof

(a) $\sqcap$**-Rule:** Let us consider the case that $\mathcal{A}$ is satisfiable and $\mathcal{A}'$ is not satisfiable. Then it must be that either $C_1(x)$ or $C_2(x)$ is not satisfiable. This implies that $(C_1 \sqcap C_2)(x)$ is not satisfiable; this is absurd because it belongs to $\mathcal{A}$ that is satisfiable by hypothesis.
(b) $\sqcup$**-Rule:** If $\mathcal{A}$ were satisfiable while both $\mathcal{A}'$ and $\mathcal{A}''$ are not satisfiable then $C_1(x)$ and $C_2(x)$ would be non satisfiable. This implies that $(C_1 \sqcup C_1)(x)$ is not satisfiable too: this is absurd because it is in $\mathcal{A}$.
(c) $\forall$**-Rule:** If $\mathcal{A}'$ were non-satisfiable because $C(y)$ is non-satisfiable, then also $(\forall R.C)(x)$ would be non-satisfiable: absurd.
(d) $nt\exists$**-Rule:** If $\mathcal{A}$ is satisfiable and $\mathcal{A}'$ is non-satisfiable then either $R(x, y)$ or $C(y)$ is non-satisfiable. This implies that there exists no individual in the interpretation of the concept C such that it is in relation R with the individual x. In other words the formula $(\exists R.C)(x)$ is not satisfiable, but this is absurd because it is in $\mathcal{A}$.
(e) $t\exists$**-Rule:** Let us consider, by contradiction, the case that $\mathcal{A}$ is satisfiable but there is no satisfiable $\mathcal{A}'_i$. In other words there is no consistent scenario for the deductive closure of the constraint network, that is the set of these role assertions is not satisfiable. The deductive closure of a constraint network corresponds to the minimal network. It is well-known in literature that the calculus of the minimal network preserves the consistency. For this reasons if the deductive closure (or minimal network) is not consistent then the full constraint network is not consistent, in other words this set of role assertions is not satisfiable. This is absurd because this set belongs to $\mathcal{A}^{\mathrm{tmp}}$ that must be satisfiable for the same reasons exposed for the previous $nt\exists$-Rule.

$\square$

The tableau procedure provides a model for finitely satisfiable concept expressions and a counterexample for unsatisfiable concept expressions. The problem is that we have no blocking condition and no infinity checker to ensure the termination of the procedure. According to Proposition 4.6 and choosing the *strong EQ-semantics* which provides nominals it is possible to enforce a maximal cardinality on models to guarantee the decidability of the language. Actually, the finite model property combined with a procedure that provides a model for finitely satisfiable concept expressions and counterexample for unsatisfiable concept expressions implies the satisfiability of the formalism. This proves the following

Theorem* 4.8 (Para-decidability) *The $\mathcal{DL}$ $\mathcal{ALCI}_{\mathcal{R}CC5}$ and $\mathcal{ALCI}_{\mathcal{R}CC8}$ are satisfiable under the* strong EQ-semantics *with a condition of finite models with maximal cardinality.*

As pointed out by Wessel a finite model restriction could be appealing from an application point of view, since there is not much practical use of concepts that only allow infinite or huge models in modeling real world spatial phenomena on a qualitative level. In the next paragraph we summarize the complexity results for this family of Description Logics.

4.1.3 Computational Results

From the complexity point of view, the analysis carried out by Wessel in [140] proved that $\mathcal{ALCI}_{\mathcal{R}CC1}$, $\mathcal{ALCI}_{\mathcal{R}CC2}$ and $\mathcal{ALCI}_{\mathcal{R}CC3}$ are decidable. He also showed that more expressiveness results in computational problems. The following theorems proved by Wessel hold.

Theorem 4.9 ([140, Wessel, pg 36]). *$\mathcal{ALCI}_{\mathcal{RCC}5}$ is PSPACE-hard as lower bound.*

Theorem 4.10 ([140, Wessel, pg 37]). *$\mathcal{ALCI}_{\mathcal{RCC}8}$ is EXPTIME-hard as lower bound.*

Wessel showed that these two logics are "close" to being undecidable: slight extensions would make the logic undecidable and the given complexity bounds are lower bounds.

From the beginning of terminological reasoning it was clear the strong connection with other logical languages, for instance the Description Logic $\mathcal{ALC}$ corresponds to the fragment of first-order logic obtained by restricting the syntax to formulas containing two variables. Since $\mathcal{DL}$ are languages formed by unary and binary predicates, they are very closely related to modal languages if one regards roles as accessibility relations. Schild in [118] pointed out that $\mathcal{ALC}$ is also the multi-modal version of the logic **K** investigated by Halpern and Moses in [71]. $\mathcal{ALC}$-concepts can be immediately translated into multi-modal **K**-formulas and vice-versa. Moreover, an $\mathcal{ALC}$-concept is satisfiable if and only if the corresponding **K**-formula is satisfiable. This relationship allows to borrow well known complexity results and techniques from modal logics to Description Logics. Nevertheless we must consider that $\mathcal{DL}$ can present features (as concrete domains) that have no counterpart in modal logics and for this reason it is not always possible to inherit complexity results. This is the case of the $\mathcal{ALCI}_{\mathcal{RCC}}$ in 2003 when Wessel presented the original idea. In a more recent work [101] Lutz and Wolter define some modal logics for topological relations and prove them to be undecidable. We show in Section 4.2 the correspondence between $\mathcal{ALCI}_{\mathcal{RCC}}$ for RCC-8 and RCC-5 and the family of modal logics of Lutz and Wolter proving undecidability even for these $\mathcal{DL}$. In the following sections we define a labeled deduction system built on the language presented by Lutz and Wolter in [101] and prove it to be sound and complete with the intended semantic of frames based on RCC relations. Wessel in [140] provide a translation of $\mathcal{ALCI}_{\mathcal{RCC}}$ to a hybrid multi-modal logic without complexity results. The goal is to borrow computational results from modal logics, so we consider the following mapping provided by Wessel in [140] from $\mathcal{ALCI}_{\mathcal{RCC}}$ to a hybrid modal logic that provides nominals denoted by propositional letters.

The translation τ from description logics concepts to modal logics formulas is defined by the following conditions.

- The set of concept names $\mathcal{N}_{\mathcal{C}}$ corresponds to the set of propositional letters *VAR* of the modal logic: we assume $\mathcal{N}_{\mathcal{C}}$ =*VAR*.
- Each role name in the role-box $\mathcal{R} \in \mathcal{N}_{\mathcal{R}}$ corresponds to an accessibility relation connected to a necessity modal operator $[\mathsf{r}]$.
- The translation τ of a concept C is defined inductively as follows:
  - $\tau(C) := C$ if C is a concept name;
  - $\tau(\neg C) := \neg\tau(C)$;
  - $\tau(C \sqcap D) := \tau(C) \wedge \tau(D)$;
  - $\tau(C \sqcup D) := \tau(C) \vee \tau(D)$;
  - $\tau(\exists R.C) := \langle\mathsf{r}\rangle\, \tau(C)$;
  - $\tau(\forall R.C) := [\mathsf{r}]\tau(C)$.

This is a syntactic transformation that maps a description logic onto the basic multi-modal logic K_m is closed under *necessitation* and *modus ponens* and is defined by:

- all propositional axioms schemas,
- $[\mathsf{r}](p \to q) \to ([\mathsf{r}]p \to [\mathsf{r}]q)$ the K-axiom for all box operators,
- $\langle \mathsf{r} \rangle\, p \leftrightarrow \neg[\mathsf{r}]\neg p$ for each accessibility relation R.

The previous definition expresses a base formalism that does not capture the requirements that we need to impose on the interpretation of roles. It is possible to enforce these frame conditions by adding appropriate frame axioms sound and complete w.r.t. the intended class of frames.

Such hybrid modal logic provides a *satisfaction operator* $@i$ that corresponds to the truth function:

$$@i\varphi \text{ is equal to } \mathcal{M}, i \models \varphi$$

In [107], pp 437 it is shown that adding a set of *pure axioms* (hybrid formulae that do not mention propositional letters) to the basic hybrid modal logic K produces a logic which is sound and complete w.r.t. the intended class of frames.

Wessel in [140] provided the following pure axiomatic system AS that defines the frame class for the topological relations of RCC:

(a) "One Cluster" : $\bigvee_{R \in \mathcal{N}_\mathcal{R}} @i \langle \mathsf{r} \rangle\, j$
(b) "Strong EQ-semantics": $\neg @i \langle \mathsf{eq} \rangle\, j$
(c) "Weak EQ-semantics": $i \to \langle \mathsf{eq} \rangle\, i$
(d) "Disjointness": $\forall \mathsf{R} \in \mathcal{N}_\mathcal{R}, @i \langle \mathsf{r} \rangle\, j \to @i \bigwedge_{\mathsf{S} \in \{\mathcal{N}_\mathcal{R} - \mathsf{R}\}} \neg \langle \mathsf{s} \rangle\, j$
(e) "Converses": $\forall \mathsf{R} \in \mathcal{N}_\mathcal{R}, @i \langle \mathsf{r} \rangle\, j \to @j \langle \mathsf{inv(r)} \rangle\, i$
(f) "Compositions": for all role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \ldots \sqcup R_n$ from the corresponding RCC composition table, add $@i \langle \mathsf{s} \rangle \langle \mathsf{t} \rangle\, j \to @i \langle \mathsf{r_1} \rangle\, j \vee \ldots \vee @i \langle \mathsf{r_n} \rangle\, j$

The axioms (a) and (d) of the system state the JEPD property of RCC base-relations which are jointly exhaustive and pairwise disjoint. The axioms (b) and (c) are equivalent to a strong reflexiveness such that each object can and must be in the EQ relation strictly with itself. The axiom (e) states that each relation has a converse: it is true for all RCC relations because EQ, DC, EC, PO and DR are symmetric while for part relations the inverse is explicitly defined. The last axiom schema is compliant with standard composition tables defined for RCC. For readers not familiar to hybrid multi-modal logics we refer to Section 4.2 for the complete definition of this theoretical framework. In the same chapter we prove the labeled deduction system built on the language of Lutz and Wolter sound and complete even w.r.t. the modal mapping of Wessel DLs family. This proves the following result.

Theorem* 4.11 *The satisfiability problem for concepts in $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ is undecidable.*

Proof The proof of this result comes from some results provided in the following section, where we show that there exist some undecidability results in modal logics (recalled in propositions 4.20 and 4.21) that can be borrowed to proved undecidability of terminological languages $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$. This can be achieved thanks to Lemma 4.47 that states a correspondence between spacial modal logics and the considered description logics. □

Actually it is not possible to define a syntactic restriction for the elimination of concept expressions with infinite models. For this reason, for a practical use of this language for qualitative spatial reasoning, it is important to consider the para-decidability that can be gained by enforcing the finite models of maximal cardinality on the strong EQ-semantics provided in the previous section.

4.2 Correspondence Theory: from $\mathcal{ALCI}_{\mathcal{RCC}}$ to Modal Logics

A well-known technique for complexity investigation of DLs is to map a DL on a modal logic with known complexity results. In early 1990's Schild investigated the relation between DLs and modal logics, in particular Schild pointed out that $\mathcal{ALC}$ is a syntactic variant of multi-modal K , the base logic K with several necessity operators. This relation between these two families of formal languages was used by Schild in [119], [120] and by De Giacomo e Lenzerini in [57], [58] to transfer decidability and complexity results from ML to DLs. The aim of this chapter is the definition of a labeled deduction system built on the language presented by Lutz and Wolter in [101]. We will prove this deduction system sound and complete with the intended semantic of frames based on RCC relations. Finally we will consider a mapping from $\mathcal{ALCI}_{\mathcal{RCC}}$ to a hybrid multi-modal logic provided by Wessel in [140] and prove the labeled deduction system built on the language of Lutz and Wolter sound and complete even w.r.t. the modal mapping of Wessel DLs family.
The definition of a multi-modal language for spatial reasoning depends on the set of spatial relations. Lutz and Wolter in their work describe two families of modal languages: a family defined over the RCC8 relations and the other over the RCC5 relations. In section 4.2.2 we will introduce the base concepts of multi-modal framework and then present the languages defined by Lutz and Wolter with the corresponding complexity results. Then in section 4.2.3 we will define the labeled deduction system and prove it to be sound and complete with respect to the multi-modal languages defined by Lutz and Wolter. Finally in the last section 4.2.4 we will prove the labeled system to be sound and complete with both the formalism of Wessel and the formalism of Lutz and Wolter to transfer complexity results from the latter to the former.

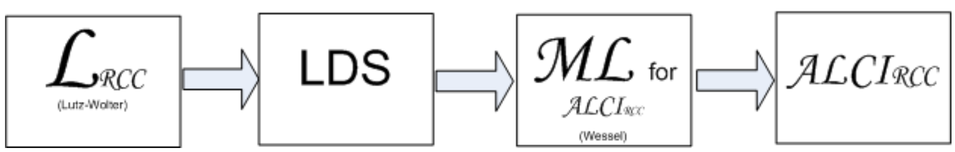

Fig. 4.1. Porting complexity results from a modal logic to the Description Logic $\mathcal{ALCI}_{\mathcal{RCC}}$: from the modal logic language $\mathcal{L}_{\mathsf{RCC}}$ (with known complexity results) to a labeled deduction system (LDS) embedding the first modal language into a completely axiomatized system, to $\mathcal{ML}$ a modal logic obtained from $\mathcal{ALCI}_{\mathcal{RCC}}$ with a (Schild) mapping, to $\mathcal{ALCI}_{\mathcal{RCC}}$ with unknown complexity results.

4.2.1 Preliminary Definitions: The multi-modal framework

Multi-modal logics have been previously investigated by Catach in [31], Guasquet in [56], Governatori in [61] and Baldoni et al. in [18], [19]. Multi-modal languages provide a formal environment to represent all those complex frameworks that require more than a single relation to describe the conceptual universe. An example can be easily found in the in the family of Tense Logics. Hereafter we define the general syntax and semantics for multi-modal languages based on propositional logic.

Definition 4.12 (Syntax). *The language of* ***propositional multi-modal logic*** *consists of a denumerable infinite set of propositional variables, the Boolean connectives* $\neg$ *and* $\wedge$ *and a set of modal operators* $[\cdot]$. *More formally the the alphabet contains:*

- *a non-empty countable set VAR of propositional variables;*
- *a non-empty countable set MOD (modal alphabet), such that VAR and MOD are disjoint;*
- *the Boolean connectives* $\neg$, $\wedge$ *and* $\rightarrow$, $\vee$ *defined the usual way;*
- *a modal operator constructor* $[\cdot]$*;*
- *left and right parentheses " (" , ") ".*

The set FOR of formulae of multi-modal propositional language $\mathcal{L}$ *is defined to be the least set that satisfies the following conditions:*

- *VAR* $\subseteq$ *FOR;*
- *if* $\varphi_1, \varphi_2 \in$ *FOR then* $(\neg\varphi_1), (\varphi_1 \vee \varphi_2), (\varphi_1 \wedge \varphi_2), (\varphi_1 \subset \varphi_2) \in$ *FOR;*
- *if* $\varphi \in$ *FOR and* $t \in$ *MOD then* $([t]\varphi) \in$ *FOR.*

For readability, we omit parentheses if they are unnecessary: we give "$\wedge$*" and "*$\vee$*" the same precedence; lower that "*$\neg$*" but higher than "*$\rightarrow$*". We will use standard abbreviation* $\langle t \rangle \varphi$ *for* $\neg[t]\neg\varphi$.

According to usual axiomatization of modal logics in current literature we may require our system to meet the following axiom schemas (1, 2) and inference rules (3, 4) given in Hilbert-style, and to be closed under modus ponens and necessitation:

1. all axiom schemas for propositional multi-modal logic;
2. the axiom schema K: $[\cdot](A \rightarrow B) \rightarrow ([\cdot]A \rightarrow [\cdot]B)$;
3. modus ponens: if $A \rightarrow B$ and A are theorems, then so is B ;
4. necessitation: if A is a theorem, then so is $[\cdot]A$.

Multi-modal formulas are a generalization of unimodal formulas in which the modal connectives $\Box$ and $\Diamond$ are replaced by a number n of pairs of connectives $[i]$ and $\langle i \rangle$. A multi-modal frame is a tuple $(W, R_1, R_2, \ldots, R_n)$, where W is a non-empty set of worlds and every R_i is a binary accessibility relation on W. According to modal logics theory of *possible-worlds semantic* developed by Kripke in [85] and [86] we give here the multi-modal definitions of frame and interpretation.

Definition 4.13 (Kripke Frame). *Given a language* $\mathcal{L}$*, an ordered pair* $(W, \mathcal{R}_t |\, t \in MOD)$*, consisting of a non-empty set W of "possible worlds" and a set of "binary relations"* $\mathcal{R}_t$ *(one for each* $t \in MOD$*) on W, is called frame.*

Frames with an infinite number of possible words are admitted. We say that w' is accessible from *w by means of* $\mathcal{R}_t$ if $(w, w') \in \mathcal{R}_t$, $\mathcal{R}_t$ is the accessibility relation of the modality $[t]$. We denote with $\mathcal{F}_{\mathcal{L}}$ the class of frames based on the language $\mathcal{L}$.

Definition 4.14 (Kripke Interpretation). *Given a language* $\mathcal{L}$*, a Kripke interpretation M is an ordered triple* $\langle W, \mathcal{R}_t |\, t \in MOD, V \rangle$*, where:*

- $(W, \mathcal{R}_t |\, t \in MOD)$ *is a frame of* $\mathcal{F}_{\mathcal{L}}$*;*
- V *is a valuation function, a mapping from* $W \times VAR$ *to the set* $\{\mathbf{T}, \mathbf{F}\}$.

We say that M is based on the frame $(W, \mathcal{R}_t | \ t \in MOD)$.

The meaning of a formula belonging to $\mathcal{L}$ is given by means of the satisfiability relation $\vDash$. We can now introduce a more formal definition of the semantic for a multi-modal logic.

Definition 4.15 (Semantics). *Let* $M = \langle W, \mathcal{R}_t | \ t \in MOD, V \rangle$ *be a Kripke interpretation,* w *a world in W and* φ *a formula, then, we say that* φ *is satisfiable in the Kripke interpretation* M *at* w*, denoted by* $M, w \vDash \varphi$*, according to the following inductive definition:*

- $M, w \vDash \varphi$ *and* $\varphi \in VAR$ *iff* $V(w, \varphi) = \mathbf{T}$*;*
- $M, w \vDash \neg\varphi$ *iff* $M, w \vDash \varphi$ *for every* $w \in W$*;*
- $M, w \vDash \varphi_1 \wedge \varphi_2$ *iff* $M, w \vDash \varphi_1$ *and* $M, w \vDash \varphi_2$*;*
- $M, w \vDash [t]\varphi$ *iff for all* $w' \in W$ *such that* $(w, w') \in \mathcal{R}_t$ *implies* $M, w' \vDash \varphi$.

Given a Kripke interpretation M we say that a formula φ is:

- *satisfiable* in M if $M, w \vDash \varphi$ for some world $w \in W$;
- *valid* in M if R $\neg\varphi$ is not satisfiable in M;
- *satisfiable w.r.t. a class* $\mathcal{M}$ of Kripke interpretations if φ is satisfiable in some interpretation in $\mathcal{M}$;
- *valid w.r.t.* $\mathcal{M}$ if it is valid in all interpretation in $\mathcal{M}$.

4.2.2 Multi-modal Logics for the RCC

The family of $\mathcal{L}_{\mathsf{RCC8}}$ logics

The definition of an expressive language RCC-compliant requires multi-modal logics previously investigated by Catach in [31], Guasquet in [56], Governatori in [61] and Baldoni et al. in [18], [19].

The modal language $\mathcal{L}_{\mathsf{RCC8}}$ [101] extends the propositional logic with countably many variables $p_1, p_2, \ldots$ and the Boolean connective $\neg$ and $\wedge$ by means of unary modal operators $[\mathsf{dc}], [\mathsf{ec}], [\mathsf{po}], [\mathsf{tpp}], [\mathsf{ntpp}], [\mathsf{tppi}], [\mathsf{ntppi}], [\mathsf{eq}]$ for each topological relation.

Lutz and Wolter defined in [101] Kripke frames (see Definition 4.13) in term of *concrete region structure* and *general region structure*.

Definition 4.16 (Concrete Region Structure [101]). *Let us consider a topological space* $\mathfrak{T}$ *and a set* $U_{\mathfrak{T}}$ *of regions defined over the topological space. A* ***concrete region structure*** *induced over a* $(\mathfrak{T},\ U_{\mathfrak{T}})$ *is a tuple given by a set of regions and a list of accessibility relations:*

$$\mathfrak{R}(\mathfrak{T},\ U_{\mathfrak{T}}) := \left\langle U_{\mathfrak{T}}, \mathsf{dc}^{\mathfrak{T}}, \mathsf{ec}^{\mathfrak{T}}, \mathsf{po}^{\mathfrak{T}}, \mathsf{eq}^{\mathfrak{T}}, \mathsf{tpp}^{\mathfrak{T}}, \mathsf{ntpp}^{\mathfrak{T}}, \mathsf{tppi}^{\mathfrak{T}}, \mathsf{ntppi}^{\mathfrak{T}} \right\rangle$$

The following definition of *general region structure* is a first-order characterization of concrete region structure, tailored to represent exactly the properties of RCC8 relations.

Definition 4.17 (General Region Structure [101]). *We call a* ***general region structure*** *the following tuple*

$$\mathfrak{R} := \left\langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \mathsf{po}^{\mathfrak{R}}, \mathsf{eq}^{\mathfrak{R}}, \mathsf{tpp}^{\mathfrak{R}}, \mathsf{ntpp}^{\mathfrak{R}}, \mathsf{tppi}^{\mathfrak{R}}, \mathsf{ntppi}^{\mathfrak{R}} \right\rangle$$

where W is a non-empty set of worlds/regions and $r^{\mathfrak{R}}$ *are binary relations on W that are mutually disjoint, jointly exhaustive and satisfied the following conditions:*

- $\mathsf{eq}^{\mathfrak{R}}$ *is the identity* Id *on W;*
- $\mathsf{dc}^{\mathfrak{R}}$, $\mathsf{ec}^{\mathfrak{R}}$ *and* $\mathsf{po}^{\mathfrak{R}}$ *are symmetric;*
- $\mathsf{tppi}^{\mathfrak{R}}$, $\mathsf{ntppi}^{\mathfrak{R}}$ *are the inverse relations of* $\mathsf{tpp}^{\mathfrak{R}}$, $\mathsf{ntpp}^{\mathfrak{R}}$ *respectively;*
- *the rules of the composition table are satisfied in the sense that, for any entry* $\mathsf{q}_1, \dots, \mathsf{q}_\mathsf{k}$ *in row* r_1 *and column* r_2 *the first-order sentence*

  $$\forall x \forall y \forall z ((\mathsf{r}_1(x,y) \land \mathsf{r}_2(y,z)) \ \rightarrow (\mathsf{q}_1(x,z) \land \ldots \land \mathsf{q}_2(x,z)))$$

  is valid.

In the following definition we will present formally the syntax and the semantics of $\mathcal{L}_{\mathsf{RCC8}}$.

Definition 4.18 (Syntax of $\mathcal{L}_{\mathsf{RCC8}}$ [101]). *Given a set VAR of propositional variables, the set of* well-formed $\mathcal{L}_{\mathsf{RCC8}}$ formulas *is defined by the following Backus-Naur-form presentation, where* $p \in$ *VAR:*

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \land \varphi \mid [\mathsf{ec}]\varphi \mid [\mathsf{dc}]\varphi \mid [\mathsf{eq}]\varphi \mid [\mathsf{po}]\varphi \mid [\mathsf{tpp}]\varphi \mid [\mathsf{ntpp}]\varphi \mid [\mathsf{tppi}]\varphi \mid [\mathsf{ntppi}]\varphi \,.$$

We will consider the usual abbreviations: $\varphi_1 \rightarrow \varphi_2$ *for* $\neg\varphi_1 \lor \varphi_2$ *and* $\langle r \rangle\, \varphi$ *for* $\neg[r]\neg\varphi$.

Definition 4.19 (Semantics of $\mathcal{L}_{\mathsf{RCC8}}$ [101]). *A* ***region model*** $\mathfrak{M} = \langle \mathfrak{R}, \mathrm{VAR} \rangle$ *for* $\mathcal{L}_{\mathsf{RCC8}}$ *consists of a region structure* $\mathfrak{R} = \left\langle W, \mathsf{ec}^{\mathfrak{R}}, \ldots \right\rangle$ *and an interpretation function that maps each propositional variable on a subset of W.*
A formula φ *is either true at a region* $s \in W$ *(written* $\mathfrak{M}, s \models \varphi$*) or false at* s *(written* $\mathfrak{M}, s \not\models \varphi$*), the inductive definition being as follows:*

- *if* φ *is a propositional variable, then* $\mathfrak{M}, s \models \varphi$ *iff* $s \in \varphi^{\mathfrak{M}}$*;*
- $\mathfrak{M}, s \models \neg\varphi$ *iff* $\mathfrak{M}, s \not\models \varphi$*;*
- $\mathfrak{M}, s \models \varphi_1 \land \varphi_2$ *iff* $\mathfrak{M}, s \models \varphi_1$ *and* $\mathfrak{M}, s \models \varphi_2$*;*
- $\mathfrak{M}, s \models [\mathsf{r}]\varphi$ *iff, for all* $t \in W$, $(s,t) \in r^{\mathfrak{R}}$ *implies* $\mathfrak{M}, t \models \varphi$.

We can now define some syntactic sugar according to the modal logics "tradition" to point out the expressive power of the language. First we can define the *difference modality* $\Box_d\varphi$, investigated for example in [41]

$$\mathfrak{M}, s \models \Box_d\varphi \text{ iff } \mathfrak{M}, s \models \varphi \text{ for all } t \in W \text{ such that } t \neq s.$$

Such a difference modality can be defined only upon the existence of an equivalence relation with a strong semantics equal to the identity function over the set of regions (worlds). Second we define the *universal box* $\Box_u\varphi$ previously investigated in [60] which implies the validity of the boxed formula

$$\mathfrak{M}, s \models \Box_u\varphi \text{ iff } \mathfrak{M}, s \models \varphi \text{ for all } t \in W.$$

Finally this language can express *nominals* by writing that a formula holds in precisely one region

$$\mathsf{nom}(\varphi) = \diamond_u(\varphi \land \Box_d\neg\varphi).$$

The availability of nominals depends on the strong EQ semantics and allows the introduction of names for regions (for instance $\mathsf{nom}(Verona)$, $\mathsf{nom}(Italy)$).

Complexity Results for RCC -Modal Logics

In [101] Lutz and Wolter proposed a systematic investigation of region structures that correspond to the usual notion of frames. In particular they consider region structures based on:

- the set $\mathbb{T}_{\mathsf{reg}}$ of all non-empty regular closed subsets of some topological space $\mathbb{T}$, as $(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ for some $n > 0$;
- the set $\mathbb{R}^n_{\mathsf{conv}}$ of non-empty convex regular closed subsets of $\mathbb{R}^n$ for some $n > 0$;
- the set $\mathbb{R}^n_{\mathsf{rect}}$ of non-empty closed hyper-rectangular subsets of $\mathbb{R}^n$ for some $n > 0$;

They also define some relevant classes of region structures as $\mathcal{RS}$ which is the class of all general region structures (defined explicitly on RCC semantics) and $\mathcal{TOP}$ which denoted the class of all full region structures where each regular closed set is a region. The investigation made by Lutz and Wolter is very accurate and the formulation of the main results is somewhat technical. For the aim of this thesis we recall in the following proposition the relevant part of the general undecidability results presented in [101].

Proposition 4.20 (Undecidability of $\mathcal{L}_{\mathsf{RCC8}}$, Lutz and Wolter [101]). *The multi-modal language $\mathcal{L}_{\mathsf{RCC8}}$ is undecidable for region structures (or frames) with the following characteristics:*

- *axiomatized on the usual RCC-8 semantics with identity function as EQ relation,*
- *based on the usual topology on a two-dimensional space,*
- *considering the set of regions given by all regular closed sets of the topology.*

Lutz and Wolter apply the same technique to the set of relations corresponding to the mereological part of the Region Connection Calculus RCC-5 in order to define a "coarser" version of the language $\mathcal{L}_{\mathsf{RCC8}}$. In the same paper they proposed computational results for the multi-modal $\mathcal{L}_{\mathsf{RCC5}}$ and state undecidability even in this case.

Proposition 4.21 (Undecidability of $\mathcal{L}_{\mathsf{RCC5}}$, Lutz and Wolter [101]). *The multi-modal language $\mathcal{L}_{\mathsf{RCC5}}$ is undecidable for region structures (or frames) with the following characteristics:*

- *axiomatized on the usual RCC-5 semantics with identity function as EQ relation,*
- *based on the usual topology on a two-dimensional space,*
- *considering the set of regions given by all regular closed sets of the topology.*

In the following we will use these undecidability results to prove undecidability of the description logic $\mathcal{ALCI}$, but it is worth to clear some apparent contradictions. Researchers proved undecidability (i) in particular for a strong EQ semantics, (ii) even for finite region structures. The first point (i) refers to the semantic requirement on the EQ relation that must coincide with the identity function. The undecidability result expressed by Lutz and Wolter is rather general, nevertheless the strong semantics requirement would not be a restriction on the validity of the result. As noticed by Wessel in [140] every model in weak EQ semantics can be a model in the strong EQ semantics collapsing each EQ-clique into a single node. This means that the undecidability of the strong semantics implies the undecidability of the weak semantics. The other point (ii) that can raise some perplexity is the fact that Lutz and Wolter prove undecidability even for finite region structure and this apparently contradicts the decidability of $\mathcal{ALCI}$ with strong semantics and maximal cardinality of admissible models. The problem of satisfiability of a concept w.r.t. a finite but unbounded region structure is different from the problem of satisfiability w.r.t. a model with a maximal finite cardinality. For this reason there is no contradiction between these results.

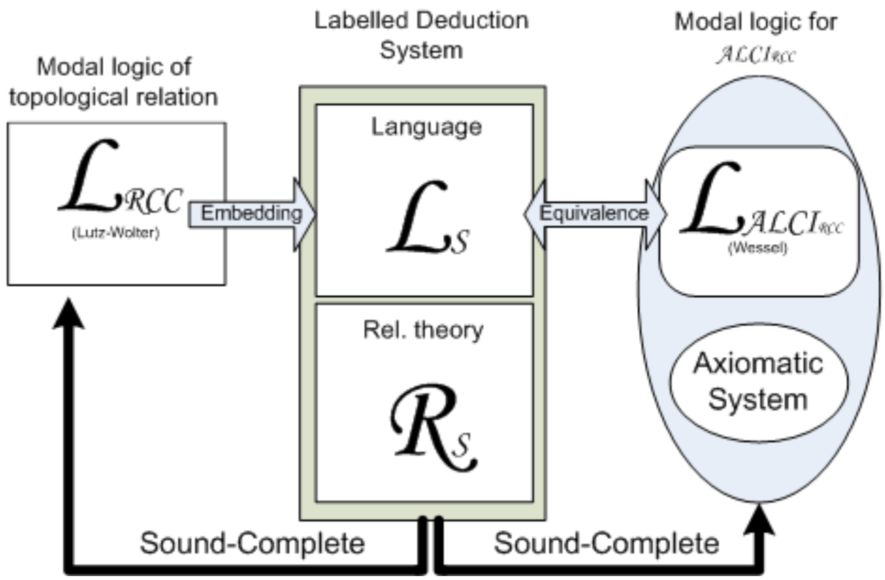

Fig. 4.2. Schema of the architecture for proving equivalence of $\mathcal{L}_{\mathsf{RCC}}$ to $\mathcal{ALCI}_{\mathcal{RCC}}$: we build on the modal logic $\mathcal{L}_{\mathsf{RCC}}$ a labeled system proved to be sound and complete with the modal translation of the description logic

4.2.3 Labeled System for Spatial Multi-modal Logics

The aim of this section is to prove the undecidability of $\mathcal{ALCI}_{\mathcal{RCC}8}$ and $\mathcal{ALCI}_{\mathcal{RCC}5}$. For this reason we will provide a connection between these two description logics and the modal logics of topological relations defined by of Lutz and Wolter [101]. In this paper the researchers provide important undecidability results for this family of multi-modal propositional languages. Our purpose is to define a *labeled deduction system for multi-modal propositional language* LDS sound and complete with respect to both the description logics defined by Wessel and the modal logics defined by Lutz and Wolter. This will prove that the undecidability results for modal logics hold for the description logics too.

The family of *labeled deduction systems* ([21], [47], [51], [135]) represent a method to provide a uniform and more formal description for those systems that associate a logical language to complex accessibility relations. An example of labeled systems for natural deduction can be found in [102] .The labeled framework allows to better understand the behavior of formalisms in particular in the case of undecidability results.

A labeled deduction system is composed by two different parts: a base system $\mathrm{N}(\mathcal{L})$ tailored to a modal language $\mathcal{L}$ and a system of relational theories $\mathrm{N}(\mathcal{R})$ that axiomatize properties of accessibility relations.

The aim of this section is the definition of a labeled deduction system ($lds := \mathrm{N}(\mathcal{L}_s)+\mathrm{N}(\mathcal{R}_s)$) for a propositional multi-modal logic for spatial reasoning. For this purpose we refer to the general framework for propositional modal logics presented by Luca Viganó in Chapter 2 of [135] and to the family of modal logics of topological relations defined by Lutz and Wolter in [101].

The base system $\mathrm{N}(\mathcal{L}_s)$

The base system $\mathrm{N}(\mathcal{L}_s)$ for a labeled deduction system LDS is given by a formal modal language $\mathcal{L}_s$, provided with a labeling function and some deduction rules. For our purpose we will adopt $\mathcal{L}_s := \mathcal{L}_{\mathsf{RCC8}}$ as the modal language of topological relations defined by Lutz and Wolter in [101] completed with natural deduction rules.

In the following definition we will define the set of well-formed formulas of the language $\mathcal{L}_{\mathsf{RCC8}}$, according to the definition 4.18, in term of two different connectives (the *local falsum* $\perp$ and *implication* $\rightarrow$).

Definition* 4.22 (well-formed formulas) *Let us considered the set of $\mathcal{L}_{\mathsf{RCC8}}$ well-formed formulas given by the following definition:*

$$A ::= p \mid \perp \mid A \rightarrow A \mid [\mathsf{ec}]A \mid [\mathsf{dc}]A \mid [\mathsf{eq}]A \mid [\mathsf{po}]A \mid [\mathsf{tpp}]A \mid [\mathsf{ntpp}]A \mid [\mathsf{tppi}]A \mid [\mathsf{ntppi}]A.$$

Where p is a propositional variable and all other connectives and modal operators, e.g. $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\diamond$ (possibility), can be defined the usual manner, e.g. $(\neg A) =_{def} (A \rightarrow \perp)$ and $\diamond A =_{def} (\neg(\Box(\neg A)))$. The modal connectives correspond to a set of accessibility relations denoted as follows:

$$\langle \mathsf{EC}, \mathsf{DC}, \mathsf{EQ}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPPI}, \mathsf{NTPPI} \rangle$$

.

Now we can define the notion of *labeled formula* merging the set of well-formed formulas with the labeling function that connects a label to a set of regions. The underlying idea is to express explicitly the fact that a well-formed formula holds in some specific regions (worlds).

Definition* 4.23 (labeled well-formed formulas) *We define the set Φ_l of* labeled well-formed formulas *(lwff) as the set of all formulas of the form*

$$x : A$$

where A is a well-formed formula of the modal language $\mathcal{L}_{\mathsf{RCC8}}$.

Definition* 4.24 *The* grade *of a lwff $x : A$, in symbols grade$(x : A)$, is the number of occurrences of $\rightarrow$ and $\Box$ in the formula A.*

For the rest of the chapter we will assume that the variables $A, B, \ldots$ will range over $\mathcal{L}_{\mathsf{RCC8}}$-formulas, x, y, z over the set of labels LAB and v, w over the set of regions W.

A natural deduction system is a collection of rules formalizing the process of proof under assumption. The rule define the behavior of logical operator describing how an

$$\dfrac{\begin{array}{c}[x : A \rightarrow \perp]\\ \vdots \\ y : \perp\end{array}}{x : A}\, RAA_\perp \qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots \\ x : B\end{array}}{x : A \rightarrow B}\rightarrow I \qquad \dfrac{x : A \rightarrow B \quad x : A}{x : B}\rightarrow E$$

Fig. 4.3. Natural Deduction rules for labeled propositional logics: $RAA_\perp$ is the *reductio ad absurdum*, $\rightarrow I$ is the rule for implication introduction and $\rightarrow E$ for implication elimination.

$$\dfrac{\begin{array}{c}[x : A]\\ \vdots \\ x : \perp\end{array}}{x : \neg A}\,\neg\mathrm{I} \qquad \dfrac{x : \neg A \quad x : A}{x : \perp}\,\neg\mathrm{E}$$

Fig. 4.4. An example of derived Natural Deduction rules: rules for negation introduction and negation elimination.

$$\dfrac{\begin{array}{c}[x\mathsf{R}y]\\ \vdots \\ y : A\end{array}}{x : [\mathsf{r}]A}\,[\mathsf{r}]\mathrm{I}* \qquad \dfrac{x : [\mathsf{r}]A \quad x\mathsf{R}y}{y : A}\,[\mathsf{r}]\mathrm{E}$$

$$\dfrac{y : A \quad x\mathsf{R}y}{x : \langle\mathsf{r}\rangle\, A}\,\langle\mathsf{r}\rangle\,\mathrm{I} \qquad \dfrac{x : \langle\mathsf{r}\rangle\, A \qquad \begin{array}{c}[y : A][x\mathsf{R}y]\\ \vdots \\ \lambda(z) : B\end{array}}{\lambda(z) : B}\,\langle\mathsf{r}\rangle\,\mathrm{E}$$

Fig. 4.5. Rules for box $[\mathsf{r}]$ introduction and elimination for a general accessibility relation R and the corresponding derived rules for the diamond modality. *In the $[\mathsf{r}]$I rule y is different from x and does not occur in any assumption on which $(y : A)$ depends other than the discarded assumption $(x\mathsf{R}y)$.

instance of a connective can be eliminated or introduced. For this reason, a deduction system presents for each connective an *introduction* rule and an *elimination* rule, except *falsum* ($\perp$) for which only an elimination rule is given. For primitive propositional logics connectives ($\perp$ and $\rightarrow$) we adopt standard inference rules, provided in Figure 4.3, and any other rule can be derived from these. An example of derived rules can be found in Figure 4.4. According to the definition of labeled system for unimodal propositional modal logic given by Viganó in [135], we do not enforce Prawitz's side condition on the rule $RAA_\perp$ that $A \neq \perp$. We will call the *conclusion* of a rule the formula below the line and *premises* of the rule all formulas above the line. There are rules, such as rules of box introduction, in which the conclusion becomes independent of some (or all) assumptions, e.g. $x\mathsf{R}y$. In this case we *discharge* these premises displaying this case with square brackets. The remaining assumptions, if any, are called *open* assumptions.

In Figure 4.5 we provide two generic rules which hold for all necessity operators. The reason is that the behavior of a box operator, from a deduction point of view, does not depend on the properties of the accessibility relation. In fact both the introduction and the elimination rules are independent from the relation. For this reason for multi-modal logics it is possible to define generic deduction rules which hold for all modal operators. The derived rules for the diamond modality enjoy the same independence from accessibility relation properties since each diamond operator are defined in terms of the corresponding box operator and of the negation.

In the following section we will introduce formally the set of *relational theories* which axiomatize the properties of all accessibility relations.

The relational system $\mathrm{N}(\mathcal{R}_s)$

Definition* 4.25 (relational well-formed formulas) *The set Φ_r of* relational well-formed formulas *(relational formulas or rwffs, for short) is defined as follows:*

$$\rho ::= x\mathsf{EQ}y \mid x\mathsf{DC}y \mid x\mathsf{EC}y \mid x\mathsf{TPP}y \mid x\mathsf{NTPP}y \mid x\mathsf{TPPI}y \mid x\mathsf{NTPPI}y \mid x\mathsf{PO}y \mid \emptyset \mid \rho \sqsupset \rho \mid \forall x.\, \rho\,.$$

Where the symbol $\emptyset$ represents the global falsum*.*

As usual we can define abbreviations $\sim$, $\sqcap$, $\sqcup$ for the negation, the conjunction, and the disjunction in the relational language.

Definition* 4.26 *The **grade** of a rwff $x\mathsf{R}y$, in symbols grade$(x\mathsf{R}y)$, is the number of occurrences of $\sqsupset$ and $\forall$ in the formula.*

The idea of *global falsum* is important to provide a *falsum* for the relational theory, which is different from the *falsum* of the logical fragment of the labeled deduction system. The definition of two different symbols requires two specific rules for the propagation of absurdum. In Figure 4.6 we present the basic set of relational rules. The rules $uf1$ and $uf2$ export falsum (and we thus call it a *universal falsum*) from the labeled sub-system to the relational one, and vice versa.

The base relational rules in Figure 4.6 allow us to derive rwffs from other rwffs only. The rules $RAA_\emptyset$, $\sqsupset I$, and $\sqsupset E$ are reductio ad absurdum and implication introduction and elimination for rwffs, while $\forall I$ and $\forall E$ are the standard rules for universal quantification.

As pointed out by Viganó in Chapter 2 of his book [135], modal logics are traditionally presented by extending a Hilbert system for propositional classical logic with a collection of axioms schemas and inference rules. See for an example the general definition 4.12 of syntax for multi-modal propositional logics. The definition of *relational theories* represents a different approach to the axiomatization of accessibility relations properties respect to Hilbert-style which exhibits some disadvantages in computational properties investigation.

We can now introduce rules that axiomatize the basic properties of the accessibility relations according to the theory of the Region Connection Calculus with a strong-EQ semantics. Note that the EQ relation will be axiomatized not as a standard generic congruence relation (reflexive, symmetric and transitive), which corresponds to a weak semantics.

- EQ

$$\frac{}{\forall x.y.\, x\mathsf{EQ}x}\ \textit{refl}^{+}$$

- DC

$$\frac{}{\forall x.y.\, x\mathsf{DC}y \sqsupset y\mathsf{DC}x}\ \textit{simm}$$

$$\dfrac{\begin{matrix}[\rho \sqsupset \emptyset] \\ \vdots \\ \emptyset\end{matrix}}{\rho}\ RAA_{\emptyset} \qquad \dfrac{\begin{matrix}[\rho_1] \\ \vdots \\ \rho_2\end{matrix}}{\rho_1 \sqsupset \rho_2}\ \sqsupset I \qquad \dfrac{\rho_1 \sqsupset \rho_2 \quad \rho_1}{\rho_2}\ \sqsupset E$$

$$\dfrac{\rho}{\forall x.\, \rho}\ \forall I^{*} \qquad \dfrac{\forall x.\, \rho}{\rho[y/x]}\ \forall E \qquad \dfrac{x : \perp}{\emptyset}\ uf1 \qquad \dfrac{\emptyset}{x : \perp}\ uf2$$

Fig. 4.6. The base relational rules. *In the $\forall I$ rule, x must not occur in any open assumption in which ρ depends.

- EC

$$\overline{\forall x.y.\, x\mathsf{EC}y \sqsupset y\mathsf{EC}x}\ simm$$

- for any part relation R, e.g. TPP, NTPP, TPPI, NTPPI

$$\overline{\forall x.\ \sim (x\mathsf{R}x)}\ irr \qquad \overline{\forall x.y.\, x\mathsf{R}y \sqsupset \sim (y\mathsf{R}x)}\ asimm$$

$$\overline{\forall x.y.z.\, (x\mathsf{R}y \sqcap y\mathsf{R}z) \sqsupset x\mathsf{R}z}\ trans$$

- for pairs of part relations $(\mathsf{R}, \mathsf{RI})$: ($\mathsf{TPP}$, TPPI) and (NTPP, NTPPI)

$$\overline{\forall x.y.\, (x\mathsf{R}y \sqsupset y\mathsf{RI}x) \sqcap (y\mathsf{RI}x \sqsupset x\mathsf{R}y)}\ inv$$

- PO

$$\overline{\forall x.y.\, x\mathsf{PO}y \sqsupset y\mathsf{PO}x}\ simm$$

- the relations join the "connection" property (they are jointly exhaustive).:

$$\overline{\forall x.y.\, (x\mathsf{EQ}y \sqcup x\mathsf{EC}y \sqcup x\mathsf{DC}y \sqcup x\mathsf{TPP}y \sqcup x\mathsf{NTPP}y \sqcup x\mathsf{TPPI}y \sqcup x\mathsf{NTPPI}y \sqcup x\mathsf{PO}y)}\ conn$$

  In other words this rule states that between two labels must exist at least one relation.

We must add rules to axiomatize the pairwise disjunction of spatial relations. In other words the following rules state that between two labels must exist at most one relation. These rules combined with the connection rule state that between two labels there will be exactly one and only one relation.

- EQ
  Let $\varphi = \sim ((x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\overline{\forall x.y.\, (x\mathsf{EQ}y \sqsupset \varphi)}\ disEQ$$

- DC
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\overline{\forall x.y.\, (x\mathsf{DC}y \sqsupset \varphi)}\ disDC$$

- EC
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{EC}y \sqsupset \varphi)}\ disEC$$

- TPP
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{TPP}y \sqsupset \varphi)}\ disTPP$$

- NTPP
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{NTPP}y \sqsupset \varphi)}\ disNTPP$$

- TPPI
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{NTPPI}y) \sqcup (x\mathsf{PO}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{TPPI}y \sqsupset \varphi)}\ disTPPI$$

- NTPPI
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{PO}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{NTPPI}y \sqsupset \varphi)}\ disNTPPI$$

- PO
  Let $\varphi = \sim ((x\mathsf{EQ}y) \sqcup (x\mathsf{DC}y) \sqcup (x\mathsf{EC}y) \sqcup (x\mathsf{TPP}y) \sqcup (x\mathsf{NTPP}y) \sqcup (x\mathsf{TPPI}y) \sqcup (x\mathsf{NTPPI}y))$

$$\frac{}{\forall x.y.\,(x\mathsf{PO}y \sqsupset \varphi)}\ disPO$$

To profile the behavior of spatial relations compliant with the RCC composition table, we must introduce a rule for each entry of the table that cannot be derived from the previous rules. For instance we will skip the rule for the entry corresponding to $(EQ \circ EQ) = EQ$ because it is equivalent to the transitive property of the EQ relation. Here below are some examples of composition rules.

$$\frac{}{\forall x.y.z.\,(x\mathsf{EC}y \sqcap y\mathsf{TPPI}z) \sqsupset (x\mathsf{EC}z \sqcup x\mathsf{DC}z)}\ \mathsf{EC-TPPI}$$

$$\frac{}{\forall x.y.z.\,(x\mathsf{EC}y \sqcap y\mathsf{NTPPI}z) \sqsupset (x\mathsf{DC}z)}\ \mathsf{EC-NTPPI}$$

Soundness and Completeness

In this section we will introduce the semantics for this labeled deduction system LDS and prove it to be sound and complete with respect to the language $\mathcal{L}_{\mathsf{RCC}}$. For the proof techniques followed in this sections we refer the reader to the work of Viganó and Volpe [102] where they define a labeled deduction system for temporal multi-modal logics proved sound and complete w.r.t. the intended temporal semantics.

Definition* 4.27 *A Kripke* frame *for* LDS *is a relational structure*

$$(\mathcal{W}, \mathsf{EC}^{\mathcal{M}}, \mathsf{DC}^{\mathcal{M}}, \mathsf{EQ}^{\mathcal{M}}, \mathsf{PO}^{\mathcal{M}}, \mathsf{TPP}^{\mathcal{M}}, \mathsf{NTPP}^{\mathcal{M}}, \mathsf{TPPI}^{\mathcal{M}}, \mathsf{NTPPI}^{\mathcal{M}})$$

where $\mathcal{W}$ is a non-empty set of regions and each relation R *is a subset of $\mathcal{W} \times \mathcal{W}$.*

A Kripke model *is a tuple*

$$(\mathcal{W}, \mathsf{EC}^{\mathcal{M}}, \mathsf{DC}^{\mathcal{M}}, \mathsf{EQ}^{\mathcal{M}}, \mathsf{PO}^{\mathcal{M}}, \mathsf{TPP}^{\mathcal{M}}, \mathsf{NTPP}^{\mathcal{M}}, \mathsf{TPPI}^{\mathcal{M}}, \mathsf{NTPPI}^{\mathcal{M}}, \mathcal{V})$$

where $(\mathcal{W}, \mathsf{EC}^{\mathcal{M}}, \mathsf{DC}^{\mathcal{M}}, \mathsf{EQ}^{\mathcal{M}}, \mathsf{PO}^{\mathcal{M}}, \mathsf{TPP}^{\mathcal{M}}, \mathsf{NTPP}^{\mathcal{M}}, \mathsf{TPPI}^{\mathcal{M}}, \mathsf{NTPPI}^{\mathcal{M}})$ *is a* LDS *frame and the valuation $\mathcal{V}$ is a function that maps an element of $\mathcal{W}$ and a propositional variable to a truth value (*0 *or* 1*).*

In order to give a semantics for this labeled system, we need to define explicitly an interpretation of labels as worlds. In order to have a strong EQ-semantics that guarantees the correspondence between the EQ relation and the *Id* function on the set of worlds $\mathcal{W}$ we must impose the injection property for the labelling function. The injection property forces the worlds to be equivalent only by means of the EQ relation that has been axiomatized according to a strong semantics that allows the definition of *nominals* under the *unique name assumption*.

Definition* 4.28 *Given a set of labels L and a model $\mathcal{M}$, an* interpretation *is an injective function $\lambda : L \to \mathcal{W}$ that maps every label in L to a different world in $\mathcal{W}$.*

Given a model $\mathcal{M}$ and an interpretation λ on it, truth *for an rwff or lwff φ is the smallest relation $\models^{\mathcal{M},\lambda}$ satisfying the following conditions where* R *is a $maths f RCC8$-relation and* $\mathsf{R}^{\mathcal{M}}$ *is the corresponding set in the relational structure of a* LDS *model:*

$\models^{\mathcal{M},\lambda} x\mathsf{R}y$ *iff* $(\lambda(x), \lambda(y)) \in \mathsf{R}^{\mathcal{M}}$*;*
$\models^{\mathcal{M},\lambda} \rho_1 \sqsupset \rho_2$ *iff* $\models^{\mathcal{M},\lambda} \rho_1$ *implies* $\models^{\mathcal{M},\lambda} \rho_2$*;*
$\models^{\mathcal{M},\lambda} \forall x.\, \rho$ *iff for all* y, $\models^{\mathcal{M},\lambda} \rho[y/x]$*;*

$\models^{\mathcal{M},\lambda} x : p$ *iff* $\mathcal{V}(\lambda(x), p) = 1$*;*
$\models^{\mathcal{M},\lambda} x : A \to B$ *iff* $\models^{\mathcal{M},\lambda} x : A$ *implies* $\models^{\mathcal{M},\lambda} x : B$*;*
$\models^{\mathcal{M},\lambda} x : [\mathsf{R}]A$ *iff for all* y, $\models^{\mathcal{M},\lambda} x\mathsf{R}y$ *implies* $\models^{\mathcal{M},\lambda} y : A$.

It is obvious that $\not\models^{\mathcal{M},\lambda} x :\perp$ *and* $\not\models^{\mathcal{M},\lambda} \emptyset$*. When* $\models^{\mathcal{M},\lambda} \varphi$*, we say that φ is* true *in $\mathcal{M}$ according to the interpretation λ.*

Truth for lwffs and rwffs built using other connectives or operators can be defined in the usual manner. Furthermore *truth* for lwffs is related to the standard truth relation for unlabeled modal logics by observing that $\models^{\mathcal{M}} x : A$ iff $\models^{\mathcal{M}}_{x} A$.

As pointed out by Viganó, the explicit embedding of properties of the models and the capability of reasoning about them, via rwffs and relational rules, require us to prove soundness and completeness also for rwffs. In other worlds we must show that it possible to deduce a relational formula if and only if it is true with respect to the set Δ of rwffs.

Now we must introduce the concept of *derivation* and *proof* for our labeled deduction system.

Definition* 4.29 (Derivations and proofs) *A* derivation *of a formula (lwff or rwff)* φ *from a proof context* (Γ, Δ) *in* LDS *is a tree formed using the rules in* LDS*, ending with* φ *and depending only on a finite subset of* $\Gamma \cup \Delta$*. We then write* $\Gamma, \Delta \vdash \varphi$*. A derivation of* φ *in* LDS *depending on the empty set,* $\vdash \varphi$*, is a* proof *of* φ *in* LDS *and we then say that* φ *is a theorem of* LDS*.*

A deduction system is sound and complete with respect to the intended semantics if

Definition* 4.30 (Soundness) $\mathsf{LDS} = \mathrm{N}(\mathcal{L}_s) + \mathrm{N}(\mathcal{R}_s)$ *is* sound *if it holds:*

(i) $\Gamma, \Delta \vdash \rho$ *implies* $\Gamma, \Delta \models^{\mathcal{M},\lambda} \rho$ *for every model* $\mathcal{M}$ *and every interpretation* λ*;*
(ii) $\Gamma, \Delta \vdash x : A$ *implies* $\Gamma, \Delta \models^{\mathcal{M},\lambda} x : A$ *for every model* $\mathcal{M}$ *and every interpretation* λ*.*

Theorem* 4.31 LDS *is sound with the intended semantics.*

Proof

(i) The proof is by induction on the structure of the derivation of ρ. We consider that from the assumptions we can derive formulas above the derivation line, under this hypothesis we have to show the last step of each derivation rule. We will prove soundness for the relational theory only. In fact it is trivial to see that the derivation rules for the lwffs are standard with respect to classic propositional multi-modal logics. We will show here that the relational derivation system is sound w.r.t. the intended semantics. The base case when $\rho \in \Delta$ is trivial. There is one step case for every axiom or rule. The axioms irr, $rifl$, $simm$, $asimm$, $trans$, $conn$ and all disjunction rules directly refer to the properties of transitivity, reflexivity, irreflexivity, symmetry, asymmetry and JEPD property of LDS models (Definition 4.27) and so they are sound by construction.
Consider the case of an application of $RAA_\emptyset$

$$\dfrac{\begin{array}{c}\Gamma\ \Delta\ [\rho \sqsupset \emptyset]^1\\ \pi\\ \emptyset\end{array}}{\rho}\ RAA_\emptyset^1$$

where $\Delta_1 = \Delta \cup \{\rho \sqsupset \emptyset\}$. By the induction hypothesis, $\Gamma, \Delta_1 \models^{\mathcal{M},\lambda} \emptyset$ for every model $\mathcal{M}$ and every interpretation λ. Let us consider an arbitrary model $\mathcal{M}$ and an arbitrary interpretation λ; we assume $\models^{\mathcal{M},\lambda} (\Gamma, \Delta)$ and prove $\models^{\mathcal{M},\lambda} \rho$. Since $\not\models^{\mathcal{M},\lambda} \emptyset$ because of the definition of the Truth function and since from the induction hypothesis we obtain $\not\models^{\mathcal{M},\lambda} (\Gamma, \Delta_1)$, that, given the assumption $\models^{\mathcal{M},\lambda} (\Gamma, \Delta)$, leads to $\not\models^{\mathcal{M},\lambda} \rho \sqsupset \emptyset$, i.e. $\models^{\mathcal{M},\lambda} \rho$ and $\not\models^{\mathcal{M},\lambda} \emptyset$ by Definition 4.28.
The cases for $\sqsupset I$, $\sqsupset E$, $\forall I$ and $\forall E$ follow by simple adaptations of the standard proofs for classical logic.
Finally, consider the case of an application of $uf1$

$$\dfrac{\begin{array}{c}\Gamma\ \Delta\\ \pi\\ x{:}\bot\end{array}}{\emptyset}\ uf1$$

for a proof context (Γ, Δ) and some label x. By the induction hypothesis, we have $\Gamma, \Delta \models^{\mathcal{M},\lambda} x : \bot$ for every $\mathcal{M}$ and every λ. Given a generic model $\mathcal{M}$ and a generic interpretation λ, we can write $\not\models^{\mathcal{M},\lambda} x : \bot$; it follows that $\not\models^{\mathcal{M},\lambda} (\Gamma, \Delta)$ and then also $\Gamma, \Delta \models^{\mathcal{M},\lambda} \emptyset$ by Definition 4.28. The same proof in the opposite direction holds for the rule $uf2$.

(ii) As in (i), by induction on the structure of the derivation of $x : A$. The base case is trivial and there is a step case for every rule of the labeled system. The cases of introduction and elimination of connectives and that of universal falsum are as in (i).
Consider an application of the rule $[\mathsf{r}]\mathrm{I}$

$$\dfrac{\begin{array}{c}\Gamma\ \Delta\ [x\mathsf{R}y]^1\\ \pi\\ y{:}A\end{array}}{x{:}[\mathsf{r}]A}\ [\mathsf{r}]\mathrm{I}^1$$

where $\Gamma, \Delta_1 \vdash y : A$ with y fresh and with $\Delta_1 = \Delta \cup \{x\mathsf{R}y\}$. By the induction hypothesis, for every model $\mathcal{M}$ and every interpretation λ it holds $\Gamma, \Delta \models^{\mathcal{M},\lambda} y : A$. We let λ be any interpretation such that $\models^{\mathcal{M},\lambda} (\Gamma, \Delta)$ and show that $\models^{\mathcal{M},\lambda} x : [\mathsf{r}]A$. Let w be any world such that $\lambda(x)\mathsf{R}^{\mathcal{M}}w$. Since λ can be trivially extended to another interpretation (still called λ for simplicity) by setting $\lambda(y) = w$, the induction hypothesis yields $\models^{\mathcal{M},\lambda} y : A$, and thus $\models^{\mathcal{M},\lambda} x : [\mathsf{r}]A$.
Finally, consider an application of the rule $[\mathsf{r}]E$

$$\dfrac{\begin{array}{c}\Gamma_1\ \Delta_1\\ \pi_1\\ x{:}[\mathsf{r}]A\end{array}\qquad \begin{array}{c}\Gamma_2\ \Delta_2\\ \pi_2\\ x\mathsf{R}y\end{array}}{y{:}A}\ [\mathsf{r}]\mathrm{E}\,.$$

Let $\mathcal{M}$ be an arbitrary model and λ an arbitrary interpretation. If we assume $\models^{\mathcal{M},\lambda} (\Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2)$, then from the induction hypotheses we obtain $\models^{\mathcal{M},\lambda} x : [\mathsf{r}]A$ and $\models^{\mathcal{M},\lambda} x\mathsf{R}y$, and thus $\models^{\mathcal{M},\lambda} y : A$ by Definition 4.28. Rules for the diamond modality are sound because they are derived from rules for the box modality proved sound above.

□

Definition* 4.32 (Completeness) $\mathsf{LDS} = \mathrm{N}(\mathcal{L}_s) + \mathrm{N}(\mathcal{R}_s)$ *is* complete *if it holds:*

(i) $\Gamma, \Delta \models^{\mathcal{M},\lambda} \rho$ *implies* $\Gamma, \Delta \vdash \rho$ *for every model* $\mathcal{M}$ *and every interpretation* λ*;*
(ii) $\Gamma, \Delta \models^{\mathcal{M},\lambda} x : A$ *implies* $\Gamma, \Delta \vdash x : A$ *for every model* $\mathcal{M}$ *and every interpretation* λ.

In order to prove completeness we will use a Henkin-style proof based on the concept of *canonical model*

$$\mathcal{M}^C = (\mathcal{W}^C, \mathsf{EQ}^C, \mathsf{DC}^C, \mathsf{EC}^C, \mathsf{TPP}^C, \mathsf{NTPP}^C, \mathsf{TPPI}^C, \mathsf{NTPPI}^C, \mathsf{PO}^C, \mathcal{V}^C)$$

is built from a proof context (Γ, Δ) to show that $(\Gamma, \Delta) \nvdash \varphi$ implies $\Gamma, \Delta \not\models^{\mathcal{M}^C, \lambda^C} \varphi$ for every formula φ. For this reason we will introduce some new definitions based on the notion of proof-context (Γ, Δ). As a notational remark, we will write $\varphi \in (\Gamma, \Delta)$ whenever $\varphi \in \Gamma$ or $\varphi \in \Delta$, and write $x \in (\Gamma, \Delta)$ whenever the label x occurs in some $\varphi \in (\Gamma, \Delta)$.

Definition 4.33. *A proof context (Γ, Δ) is* consistent *iff $\Gamma, \Delta \nvdash x :\perp$ for every x, and it is* inconsistent *otherwise.*

Proposition 4.34 ([102]). *Let (Γ, Δ) be a consistent proof context. Then:*

(i) for every x and every A, either $(\Gamma \cup \{x : A\}, \Delta)$ is consistent or $(\Gamma \cup \{x :\sim A\}, \Delta)$ is consistent;
(ii) for every relational formula ρ , either $(\Gamma, \Delta \cup \{\rho\})$ is consistent or $(\Gamma, \Delta \cup \{\neg\rho\})$ is consistent.

For the proof of this proposition see [102].

Definition 4.35. *A proof context (Γ, Δ) is* maximally consistent *iff the following three conditions hold:*

1. (Γ, Δ) is consistent,
2. for every relational formula ρ, either $\rho \in \Delta$ or $\neg\rho \in \Delta$,
3. for every x and every A, either $x : A \in \Gamma$ or $x :\sim A \in \Gamma$.

The proof of the following lemma is written according to the proof technique proposed in [135] in the Chapter 2 of propositional unimodal logic.

Lemma 4.36. *Every consistent proof context (Γ, Δ) can be extended to a maximally consistent proof context (Γ^*, Δ^*).*

Proof Consider the extended language of LDS obtained adding infinitely many new constants for witness terms and for witness worlds. Let t range over the original terms, s range over the new constants for witness terms, and r range over both; further, let w range over labels, v range over the new constants for witness worlds, and u range over both. All these may be subscripted. Let φ_1, φ_2, ... be an enumeration of all lwffs and rwffs in the extended language; when φ_i is $u : A$, we write $\sim \varphi_i$ for $u :\sim A$.

We iteratively build a sequence of consistent proof contexts by defining $(\Gamma_0, \Delta_0) = (\Gamma, \Delta)$ and $(\Gamma_{i+1}, \Delta_{i+1})$ inductively as follows:

- (Γ_i, Δ_i), if $(\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_i)$ is inconsistent; else
- $(\Gamma_i \cup \{u :\sim [\mathsf{r}]A, v :\sim A\}, \Delta_i \cup \{v\mathsf{R}u\})$ for a v not occurring in $(\Gamma_i \cup \{u :\sim [\mathsf{r}]A\}, \Delta_i)$ if φ_{i+1} is $u :\sim [\mathsf{r}]A$; else
- $(\Gamma_i, \Delta_i \cup \{\neg\forall x.\, \rho, \neg\rho[s/x]\})$ for an s not occurring in $(\Gamma_i, \Delta_i \cup \{\neg\forall x.\, \rho\})$ if φ_{i+1} is $\sim \forall x.\, \rho$; else
- $(\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_i)$ if φ_{i+1} is an lwff or $(\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\})$ if φ_{i+1} is an rwff.

Now define the sum of all previously built proof contexts:

$$(\Gamma^*, \Delta^*) = (\bigcup_{i\geq 0} \Gamma_i, \bigcup_{i\geq 0} \Delta_i)\,.$$

We show that the proof context (Γ^*, Δ^*) is maximally consistent, i.e. it verifies the three conditions of Definition 4.35.

(i) The construction rules preserve consistency: every (Γ_i, Δ_i) is consistent. We prove non trivial cases when φ_{i+1} is $\sim [\mathsf{r}]A$, or $\neg\forall x.\,\rho$. We only consider the first case, since the second is very similar.
If $(\Gamma_i \cup \{u : \sim [\mathsf{r}]A\}, \Delta_i)$ is consistent, then so is $(\Gamma_i \cup \{u : \sim [\mathsf{r}]A, v : \sim A\})$ for a v not occurring in $(\Gamma_i \cup \{u : \sim [\mathsf{r}]A\}, \Delta_i)$. By contraposition, suppose that

$$\Gamma_i \cup \{u : \sim [\mathsf{r}]A, v : \sim A\}\,,\ \Delta_i \cup \{u\mathsf{R}v\} \vdash u_j : \bot$$

by a derivation π (where v does not occur in $(\Gamma_i \cup \{u : \sim [\mathsf{r}]A\}, \Delta_i)$). Then in LDS we can have a derivation like the following:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma_i \quad \Delta_i \quad u{:}\neg[\mathsf{r}]A \quad [v{:}\neg A]^1 \quad [u\mathsf{R}v]^2\\ \pi\\ u_j{:}\bot\end{array}}{v{:}A}\,RAA_\bot{}^{1}}{u{:}[\mathsf{r}]A}\,[\mathsf{r}]I^2 \qquad u{:}\neg[\mathsf{r}]A}{u{:}\bot}\,\neg E$$

This shows that $(\Gamma_i \cup \{u : \neg[\mathsf{r}]A\}, \Delta_i)$ is inconsistent, which is not the case.

(ii) Consider an rwff ρ. Suppose that both $\rho \notin \Delta^*$ and $\sim \rho \notin \Delta^*$ hold. Let ρ be φ_{i+1} for some i in our enumeration of formulas and $\sim \rho$ be φ_{j+1}. Now suppose $i < j$ (the other case is symmetric). $\rho \notin \Delta^*$ implies that $(\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\})$ is inconsistent. Given that in our inductive construction we only add formulas to the proof context, i.e. $\Delta_i \subseteq \Delta_j$, we have that $(\Gamma_j, \Delta_j \cup \{\varphi_{i+1}\})$ is also inconsistent. Then, by Proposition 4.33(ii), $(\Gamma_j, \Delta_j \cup \{\varphi_{j+1}\})$ has to be consistent and φ_{j+1} is added by definition to Δ_j. This implies $\varphi_{j+1} \in \Delta^*$, i.e. $\sim \rho \in \Delta^*$.

(iii) The proof for labeled formulas is the same as in the previous case and proceeds by contraposition by using Proposition 4.33(i).

□

Lemma* 4.37 *Let (Γ, Δ) be a maximally consistent proof context. Then:*

(i) $\Gamma, \Delta \vdash \varphi$ *iff* $\varphi \in (\Gamma, \Delta)$*;*
(ii) $\rho_1 \sqsupset \rho_2 \in \Delta$ *iff* $\rho_1 \in \Delta$ *implies* $\rho_2 \in \Delta$*;*
(iii) $\forall x.\,\rho \in \Delta$ *iff* $\rho[y/x] \in \Delta$ *for all* y*;*
(iv) $u : A \to B \in \Gamma$ *iff* $u : A \in \Gamma$ *implies* $u : B \in \Gamma$*;*
(v) $u_1 : [\mathsf{r}]A \in \Gamma$ *iff* $u_1\mathsf{R}u_2 \in \Delta$ *implies* $u_2 : A \in \Gamma$ *for all* u_2*;*

Proof Only some cases are proved, all other cases are similar and follow by maximality and consistency of (Γ, Δ).

(i) The proof is analogous for rwffs and lwffs, we see the first case.
$(\Leftarrow)$ If $\varphi \in (\Gamma, \Delta)$, then trivially $\Gamma, \Delta \vdash \varphi$.
$(\Rightarrow)$ Consider an rwff φ such that $\varphi \notin (\Gamma, \Delta)$. Then, by Definition 4.35, $\neg\varphi \in (\Gamma, \Delta)$. It follows trivially that $\Gamma, \Delta \vdash \neg\varphi$ holds. By hypothesis, $\Gamma, \Delta \vdash \varphi$ and thus by using $\neg E$ we get $\Gamma, \Delta \vdash \emptyset$, that contradicts the consistency of (Γ, Δ).

(v) $(\Leftarrow)$ Suppose $u_1 : [\mathsf{r}]A \notin \Gamma$ and $u_2 : A \in \Gamma$ for every u_2 such that $u_1\mathsf{R}u_2 \in \Delta$. Then, by maximality of (Γ, Δ), $u_1 : \neg[\mathsf{r}]A \in \Gamma$. Now suppose there exists a u_3 such that $u_1\mathsf{R}u_3 \in \Delta$ and $u_3 : \neg A \in \Gamma$. Then, by hypothesis, we know $u_3 : A \in \Gamma$ and

this leads to a contradiction. Otherwise, if such a u_3 does not exist, we can conclude $u_1 : [\mathsf{r}]A \in \Gamma$ that leads to a contradiction as well.
($\Rightarrow$) We show it by contraposition. Suppose $u_1 : [\mathsf{r}]A \in \Gamma$, $u_1 \mathsf{R} u_2 \in \Delta$ and $u_2 : A \notin \Gamma$. By maximality of (Γ, Δ), we have $u_2 : \neg A \in \Gamma$. Then the following is an LDS proof that shows (Γ, Δ) is inconsistent.

$$\dfrac{\dfrac{u_1{:}[\mathsf{r}]A \qquad u_1 \mathsf{R} u_2}{u_2{:}A}\,[\mathsf{r}]E \qquad u_2{:}\neg A}{u{:}\bot}\,\neg E$$

□

Definition* 4.38 (Canonical Model) *Let $C = (\Gamma, \Delta)$ be a maximally consistent proof context and L^C be the set of labels occurring in it. We define the* canonical model

$$\mathcal{M}^C = (\mathcal{W}^C, \mathsf{EQ}^C, \mathsf{DC}^C, \mathsf{EC}^C, \mathsf{TPP}^C, \mathsf{NTPP}^C, \mathsf{TPPI}^C, \mathsf{NTPPI}^C, \mathsf{PO}^C, \mathcal{V}^C)$$

as follows:

- $\mathcal{W}^C = \{u \mid u \in L^C\}$*;*
- $(u_i, u_j) \in \mathsf{R}^C$ *iff* $u_i \mathsf{R} u_j \in \Delta$*;*
- $\mathcal{V}^C(u, p) = 1$ *iff* $u : p \in \Gamma$*.*

We define the canonical interpretation $\lambda^C : L^C \to \mathcal{W}^C$ *as follows:*

$$\lambda^C(u) = \lambda(u) \textit{ for every } u \in L^C.$$

The following proposition follows from the construction of the maximally consistent proof context.

Proposition* 4.39 *Let $C = (\Gamma, \Delta)$ be a maximally consistent proof context then $u_i \mathsf{R} u_j \in \Delta$ iff $\Delta \models^{\mathcal{M}^C, \lambda^C} u_i \mathsf{R} u_j$.*

From proposition 4.39 and 4.37 it follows that:

Proposition* 4.40 *Let $C = (\Gamma, \Delta)$ be a maximally consistent proof context then $u : A \in \Delta$ iff $\Gamma \Delta \models^{\mathcal{M}^C, \lambda^C} u : A$.*

Theorem* 4.41 LDS *is complete with the intended semantics, i.e. it holds:*

(i) if $\Gamma, \Delta \nvdash w : A$, then there exist a LDS *model $\mathcal{M}^C$ and an interpretation λ^C such that $\Gamma, \Delta \not\models^{\mathcal{M}^C, \lambda^C} w : A$;*
(ii) if $\Gamma, \Delta \nvdash \rho$, then there exist a LDS *model $\mathcal{M}^C$ and an interpretation λ^C such that $\Gamma, \Delta \not\models^{\mathcal{M}^C, \lambda^C} \rho$.*

The proof of the completeness is the same as the one formulated by Viganó and Volpe for the labeled system for tense logics in [102] Theorem 30.

4.2.4 Proving Undecidability for $\mathcal{ALCI}_{\mathcal{RCC}8}$ and $\mathcal{ALCI}_{\mathcal{RCC}5}$

Schild's Mapping between $\mathcal{DL}$ and Modal Logics

The discovery of the correspondence between Description Logics and modal logics was one of the most important steps in the field of terminological reasoning. The importance of this relation is well summarized by Baader and Lutz in Chapter 2 of the "Handbook of Modal Logics" [13] and deals with the investigation of complexity and expressivity of terminological languages. Description Logics [10] are a family of knowledge representation languages that can be used to formalize and structure knowledge about a conceptual domain. DLs differ from other knowledge representation systems because of a formal, *logic based* semantics, which can be given by a translation into first-order predicate logic. For instance we can consider the following concept description and the corresponding first-order formula.

Example 4.42.

$$Man \sqcap \exists married.Italian \sqcap \forall child.Male$$

$$Man(x) \wedge \exists y(married(x,y) \wedge Italian(y)) \wedge \forall y(child(x,y) \rightarrow Male(y))$$

This concept description express the concept of "a man who is married to an Italian and, in the case he has children, all children are male". The relation between description logics and first-order logics is quite intuitive, in fact the description logic semantics is expressed in terms of first-order logic. On the contrary the relation between DLs and modal logics (henceforth MLs) is less obvious but of much importance for the computational complexity investigation.

The first investigation on the connection between DLs and MLs was carried out by Schild in early 1990's. In [118] Schild noticed that $\mathcal{ALC}$ is a syntactic variant of multimodal K (defined more formally in one of the following section), that is the basic modal logic of Kripke frames with several accessibility relations. The translations of $\mathcal{ALC}$ and K into first-order predicate logic yield the same class of first-order formulae. This connection between DLs and MLs was used by Schild in [119], [120] and by De Giacomo e Lenzerini in [57], [58] to transfer decidability and complexity results from ML to DLs. Moreover tableau-based algorithms were developed for DLs (see [14] for an overview) and complex and optimized implementation of these algorithms [77] turn out to behave well on artificial benchmarks from the modal logic environment [109] and also in practice [70].

The base description logic $\mathcal{ALC}$ introduced by Schmidt-Schauß and Smolka in [121] can be extended in order to offer more expressive reasoning. $\mathcal{ALC}$ is provided with a countably infinite set of concept names (A, B, ...), a set of role names (r, s, ...) and a finite set of operators that allows the definition of complex concepts. As pointed out by Schild [118] $\mathcal{ALC}$ is a notational variant of the multi-modal K . In fact, concept names are equivalent to propositional variables and role names can be considered as the accessibility relations associated with a pair of modal operators.

The $\mathcal{ALC}$ semantics is based on *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the non empty set $\Delta^{\mathcal{I}}$ is the *domain* of the interpretation and $\cdot^{\mathcal{I}}$ is the interpretation function that maps each concept name to a set of individuals of the domain and each role name to a set of pairs that corresponds to a binary relation on the domain of the interpretation. Description

Logics interpretations correspond to Kripke structures with $\Delta^{\mathcal{I}}$ the set of possible-worlds and $\cdot^{\mathcal{I}}$ providing both the accessibility relations and the propositional variables evaluation.

An important extension to the basic $\mathcal{ALC}$ is the *inverse* role constructor $\cdot^{-}$. Roles of the form r^{-} correspond to the backward modalities and allow to define the converse parent of the role child or, in the spatial reasoning framework, part-of as converse of has-part. Another important feature is represented by *nominals* that are concept names that are required to be interpreted as singleton set. The name has been adopted from the context hybrid modal logics. For a complete overview see Chapter "Hybrid Logics" of "Handbook of Modal Logics" [8] and [5], [55] and [7].

The concept language is only one part of description logics. To manage the terminological knowledge about a conceptual domain we must be able to organize and interrelate multiple concept descriptions. This can be achieved with a terminological formalism. For this reason we consider a TBox (terminological box) which is a set of concept definitions

$$A \equiv C$$

where A is a concept name and C is a concept description, such that no concept name appears on the left-hand side of a concept definition in the TBox. In other words a concept can be defined only once. Schild [120] noted that for TBox with fixpoint semantics there exists a correspondence with the Vardi and Wolper's version of the propositional μ-calculus [132].

The third part of description logics is the *assertional formalism* which allows the formalization of a part of the world by means of specific assertions about individual of the domain of the interpretation. The union of terminological and assertional knowledge is commonly called *knowledge base*. Assuming the availability of a countably infinite supply of individual names, then an ABox (assertional box) is a finite set of assertions of the form

$$C(a) \qquad (concept\ assertion)$$

$$r(a,b) \qquad (role\ assertion)$$

where *a* and *b* are individual names, *C* a concept description and *r* a role description. the definition of the ABox requires the extension of the interpretation to individual names into individuals of the domain.

From Modal logics to $\mathcal{ALCI}_{\mathcal{RCC}}$

In [140] Wessel proposed an axiomatization of the $\mathcal{ALCI}_{\mathcal{RCC}}$ family according to the technique of Schild's mapping. The purpose of this investigation was to analyze what could be possible to gain from the analysis of the corresponding logic formalism. No result was found by mean of this analysis since no complexity result about the modal logic translation of $\mathcal{ALCI}_{\mathcal{RCC}}$ was known in literature.

As pointed out in section 4.1 we are interested in description logics with a *strong EQ-semantics* (the relation EQ is the identity relation between spatial objects) because of possible applications on GISs and the expressivity capable to impose a maximal cardinality constraint over models that guarantees the *finite model property*. The availability of *nominals*, given by the strong EQ-semantics, allows to:

- translate whole ABoxes into concept expressions reasoning directly on relevant spatial entities (e.g. Verona);

- translate whole knowledge bases (TBox+ABox) into concept expressions;
- switch to a closed domain reasoning mode for inference tasks that reference an ABox.

Recently Lutz and Wolter in [101] defined some multi-modal logics of topological relations introduced in section 4.2.2 and proved some undecidability results. In this paragraph we will prove a correspondence between $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ with the strong EQ-semantics and the labeled deduction system built on Lutz-Wolter modal logics.

The translation τ from description logics concepts to modal logics formulas is defined by the following conditions.

- The set of concept names $\mathcal{N_C}$ corresponds to the set of propositional letters *VAR* of the modal logic: we assume $\mathcal{N_C}$ =*VAR*.
- Each role name in the role-box $\mathcal{R} \in \mathcal{N_C}$ corresponds to an accessibility relation connected to a necessity modal operator $[\mathsf{r}]$.
- The translation τ of a concept C is defined inductively as follows:
  – $\tau(C) := C$ if C is a concept name;
  – $\tau(\neg C) := \neg\tau(C)$;
  – $\tau(C \sqcap D) := \tau(C) \wedge \tau(D)$;
  – $\tau(C \sqcup D) := \tau(C) \vee \tau(D)$;
  – $\tau(\exists R.C) := \langle \mathsf{r} \rangle\, \tau(C)$;
  – $\tau(\forall R.C) := [\mathsf{r}]\tau(C)$.

This is a syntactic transformation that maps a description logic onto the basic multi-modal logic K_m is closed under *necessitation* and *modus ponens* and is defined by:

- all propositional axioms schemas,
- $[\mathsf{r}](p \to q) \to ([\mathsf{r}]p \to [\mathsf{r}]q)$ the K-axiom for all box operators,
- $\langle \mathsf{r} \rangle\, p \leftrightarrow \neg[\mathsf{r}]\neg p$ for each accessibility relation R.

The previous definition expresses a base formalism that does not capture the requirements that we need to impose on the interpretation of roles. It is possible to enforce these frame conditions by adding appropriate frame axioms sound and complete w.r.t. the intended class of frames.

In order to axiomatize all frame conditions and to define a formal system equivalent to the description logic we must start from a hybrid modal logic that provides nominals denoted as propositional letters. Such a hybrid modal logic provides a *satisfaction operator* $@i$ that corresponds to the truth function previously defined:

$$@i\varphi \text{ is equal to } \mathcal{M}, i \models \varphi$$

Definition 4.43. *A* pure formula *is a hybrid formula that does not mention propositional letters.*

In [107], pp 437 it is shown the following

Proposition 4.44. *Adding a set of pure axioms to the basic hybrid modal logic K produces a logic which is sound and complete w.r.t. the intended class of frames.*

Wessel in [140] provided the following pure axiomatic system AS that defines the frame class for the topological relations of RCC:

(a) "One Cluster" : $\bigvee_{R \in \mathcal{N}_\mathcal{R}} @i \langle \mathsf{r} \rangle j$
(b) "Strong EQ-semantics": $\neg @i \langle \mathsf{eq} \rangle j$
(c) "Weak EQ-semantics": $i \rightarrow \langle \mathsf{eq} \rangle i$
(d) "Disjointness": $\forall \mathsf{R} \in \mathcal{N}_\mathcal{R}, @i \langle \mathsf{r} \rangle j \rightarrow @i \bigwedge_{\mathsf{S} \in \{\mathcal{N}_\mathcal{R} - \mathsf{R}\}} \neg \langle \mathsf{s} \rangle j$
(e) "Converses": $\forall \mathsf{R} \in \mathcal{N}_\mathcal{R}, @i \langle \mathsf{r} \rangle j \rightarrow @j \langle \mathsf{inv(r)} \rangle i$
(f) "Compositions": for all role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \ldots \sqcup R_n$ from the corresponding RCC composition table, add $@i \langle \mathsf{s} \rangle \langle \mathsf{t} \rangle j \rightarrow @i \langle \mathsf{r_1} \rangle j \vee \ldots \vee @i \langle \mathsf{r_n} \rangle j$

The axioms (a) and (d) of the system state the JEPD property of RCC base-relations which are jointly exhaustive and pairwise disjoint. The axioms (b) and (c) are equivalent to a strong reflexiveness such that each object can and must be in the EQ relation strictly with itself. The axiom (e) states that each relation has a converse: it is true for all RCC relations because EQ, DC, EC, PO and DR are symmetric while for part relations the inverse is explicitly defined. The last axiom schema is compliant with standard composition tables defined for RCC.

Theorem 4.45. *The axiomatic system* AS *is sound and complete w.r.t. the class of frame built on RCC base-relations.*

This result follows from proposition 4.44 and the fact that AS is a pure axiomatic system that describes exactly the behavior of RCC base-relations. From this theorem comes the following

Theorem* 4.46 *The multi-modal logic K extended by the axiomatic system* AS*, $(K_m +$ AS$)$ is equivalent to the labeled deduction system* LDS $= \mathrm{N}(\mathcal{L}_s) + \mathrm{N}(\mathcal{R}_s)$ *for RCC-8 and RCC-5 .*

Proof It is easy to see that by definition that the language $\mathrm{N}(\mathcal{L}_s)$ is built on $\mathcal{L}_{\mathsf{RCC8}}$ (resp. RCC-5) which are propositional multi-modal logics K_m defined for Kripke frames based on RCC topological relations. In particular both the axiomatic system AS and the relational theory $\mathrm{N}(\mathcal{R}_s)$ are proved to be sound and complete w.r.t. the same class of frames. It is now trivial to prove that, by definition of LDS, each axioms schemas in AS has a counterpart in $\mathrm{N}(\mathcal{R}_s)$, for instance consider the equivalence between:

- the "disjunction" axiom schema and the rules $disEQ$, $disDC$, $disEC$, $disTPP$, $disNTPP$, $disTPPI$, $disNTPPI$, $disPO$;
- the two "Strong EQ-semantics" axioms and the strong reflexiveness property of the EQ relation ($refl^+$).

□

Lemma* 4.47 *The description logic $\mathcal{ALCI}_{\mathcal{RCC}}$ exhibits the same computational complexity of the multi-modal language $\mathcal{L}_{\mathsf{RCC}}$.*

Proof This result comes from the correspondence between the Labeled Deduction System built on the language $\mathcal{L}_{\mathsf{RCC}}$ and the hybrid multi-modal logic defined by Wessel both sound and complete w.r.t. the intended class of frames. The correspondence via Schild's mapping between the hybrid logic and the language $\mathcal{ALCI}_{\mathcal{RCC}}$ allows to borrow the computational result from the modal formalism to the terminological one. □

4.3 Generalizing the $\mathcal{ALCI}_{\mathsf{C}}$ approach

4.3.1 The Idea

The idea presented in [140] by Wessel to consider a finite set of JEPD relations is very interesting. Wessel also provides a translation of the $\mathcal{DL}$ into a hybrid multi-modal logic which in 2003 was not yet investigated. One of the most important problems when dealing with relations is the formal definition of properties like reflexiveness or transitivity. In this case, the formal inclusion in the role box of the composition table translated into composition-based role axioms guarantees some coherence respect to those properties which are deducible from composition rules. For instance the transitivity of the *part relation* in RCC-5 which is an order relation is guaranteed by the rule that states "$\mathsf{PP} \circ \mathsf{PP} = \mathsf{PP}$". Nevertheless there are properties like irreflexiveness that can be described only at a semantic level imposing a disjunction constraint with the relation corresponding to *equality*. In order to axiomatize properties that make the set of spatial accessibility relations sound and complete w.r.t. the intended class of frames, Wessel in [140] defines an axiomatic system containing only *pure hybrid axioms*. This is a good technique to get soundness and completeness, but it is not the only possible solution. As proved in the previous section it is possible to model formally the properties of accessibility relations even with a labelled deduction system defined for the intended class of frames. The generalized idea is: we translate (thanks to Schild's mapping technique) the description logic that we want to study, denoted by $\mathcal{DL}_{\mathsf{C}}$, into a modal logic $\mathcal{ML}$ then we can ensure the soundness and the completeness w.r.t. the class of frames corresponding to the constraint language C defining a proper labelled deduction system.

In this section we will present a generalization of this technique defining some new $\mathcal{DL}$ built over QSR formalism considered as constraint languages built on relation algebras with the JEPD property for the set of relations. For the aim of this work we consider as good candidates those formalisms which can be translated into well-known calculi such as the Cardinal Direction Calculus (CDC) of Ligozat [89], the Rectangle Algebra (RA) by Balbiani et al. [15] and the combination of RCC-8 and a subalgebra of the Rectangle Algebra (DIR9-RCC8) investigated by Li [88]. This choice comes from the strong connection between the constraint language embedded in the $\mathcal{DL}$ and the class of frames defined over a modal logic. There is a big difference between the formal definitions of the set of spatial relations and of the universe of individuals given by a constraint language or a class of frames and the logical structure that allows complex reasoning with those relations as a description logic or a modal logic. In the following we will state the general syntax and semantics for the hybrid description logic and present examples from QSR.

4.3.2 General Syntax and Semantics

The aim of the thesis is to analyze hybridization techniques between Description Logics and QSR formalisms. For this reason we present here a generalization of the logic $\mathcal{ALCI}_{\mathsf{RCC}}$ proposed by Wessel. Let us consider the case of constraint languages such as Allen's Interval Algebra [3] or RCC by Randell, Cui and Cohn [113]: these formalism are based on finite relation algebras whose relations form a set of JEPD elements and are closed under the inverse operator. A generic constraint language with the same characteristics can be embedded into a language $\mathcal{ALCI}_{\mathsf{C}}$ encoding all composition rules into a finite role box. In Definition 4.48 below, we provide the notion of Constraint Language, that is formalized in pure algebraic terms, differently from the definition commonly used in the current literature that is purely logical. We adopt this version here for the sake of linguistic coherence with the rest of the thesis.

Definition* 4.48 (Constraint Language) *A* Constraint Language $\mathsf{C} = \{\mathfrak{U}, \mathfrak{R}\}$ *is given by a universe of individuals* $\mathfrak{U}$ *and a finite binary relation algebras* $\mathfrak{R} = \{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$ *whose relations are defined on the universe of individuals, are closed under the inverse operator, has the identity element w.r.t. the operation of composition and are jointly exhaustive and pairwise disjoint (JEPD).*

We will consider also the composition table $\mathfrak{T} : \mathfrak{R} \times \mathfrak{R} \rightarrow 2^{\mathfrak{R}}$ of the constraint language which states the rules of composition of all relations in $\mathfrak{R}$ as a function going from the set of relations pairs to the set of all possible disjunctions of relations.

Definition* 4.49 ($\mathcal{ALCI}_\mathsf{C}$ Syntax) *Let* $\mathcal{N}_\mathcal{C}$ *and* $\mathcal{N}_{\mathcal{R}_\mathcal{S}}$ *be two disjoint sets of symbols of* concept names *and* spatial role names *respectively. The syntax is defined inductively and is borrowed from* $\mathcal{ALCI}$*:*

- *every concept name* $C \in \mathcal{N}_\mathcal{C}$ *is a concept;*
- *if* R *is in* $\mathcal{N}_{\mathcal{R}_\mathcal{S}}$ *and* C *and* D *are concepts, then also* $\neg C$, $C \cup D$, $C \cap D$, $\exists R.C$, $\forall R.C$, $\exists \mathsf{inv}(R).C$, $\forall \mathsf{inv}(R).C$ *are concepts.*

We keep the notation introduced by Wessel for the finite disjunction of role names: if $R = S_1, ..., S_n$ is a disjunction of disjunctive roles $S_i \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}$, we write $\forall S_1, ..., S_n.C$ ($\exists S_1, ..., S_n.C$) as a shorthand for $\forall S_1.C \sqcap ... \sqcap \forall S_n.C$ ($\exists S_1.C \sqcap ... \sqcap \exists S_n.C$). Henceforth given a concept θ we will denote by $\tilde{\theta}$ the *equivalent form* with all the shorthands expanded in the corresponding disjunctive formulae. The logic $\mathcal{ALCI}_\mathsf{C}$is defined according to the considered constraint language C with the requirement that the set of spatial role names has a role name for each relation in $\mathfrak{R}$: the fact that $\mathfrak{R} = \{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$ implies that $\mathcal{N}_{\mathcal{R}_\mathcal{S}} = \{R_1, \ldots, R_n\}$. As seen with $\mathcal{ALCI}_\mathsf{RCC}$ the soundness w.r.t. the original spatial framework is given by semantic requirements. It is interesting that the soundness can be imposed only via semantics as in modal logics where the properties of accessibility relations are declared semantically. The labelled deduction systems for modal logics allows the formal definition of relation properties, we wonder if it could be possible to define a similar formal structure to embed the descriptions of role properties into the formalism.

Definition* 4.50 ($\mathcal{ALCI}_\mathsf{C}$ Semantics) *An* interpretation $\mathcal{I} =_{def} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty set* $\Delta^\mathcal{I}$*, called the domain of* $\mathcal{I}$ *and an interpretation function* $\cdot^\mathcal{I}$*, that maps every concept name* C *to a subset of* $\Delta^\mathcal{I}$ *and every role name* R *to a subset of* $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ *according to the following rules:*

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg C^{\mathcal{I}} &= \Delta \backslash C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x | \forall y [(x,y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &= \{x | \exists y [(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\} \\
(\forall \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \forall y [(y,x) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \exists y [(y,x) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}
\end{aligned}$$

We must note that until the set of spatial roles is closed under inverse operation it is not necessary to use the $\mathsf{inv}(R)$ concept expressions for roles in $\mathcal{N}_{\mathcal{R}_\mathcal{S}}$. As pointed out previously the soundness w.r.t. a constraint language requires some semantic conditions and since we consider the language extended by a role box it is necessary to impose in particular a restriction on composition based role inclusion axioms.

Definition* 4.51 (Role Composition Requirement) *Given a constraint language* $\mathsf{C} = \{\mathfrak{U}, \mathfrak{R}\}$ *with the composition table* $\mathfrak{T} : \mathfrak{R} \times \mathfrak{R} \rightarrow 2^{\mathfrak{R}}$ *which states the rules of composition of all relations in* $\mathfrak{R}$*, the role box of the description language* $\mathcal{ALCI}_\mathsf{C}$*will contain a composition based role inclusion axiom of the form* $S \circ T \sqsubseteq R_1 \sqcup ... \sqcup R_n$ *(enforcing* $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup ... \cup R_n^{\mathcal{I}}$ *on the models* $\mathcal{I}$*) if and only if the composition table states that* $(\mathsf{S}, \mathsf{T}) \overset{\mathfrak{T}}{\longmapsto} \{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$ *(where* $\{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$ *is intended as disjunction of disjunctive relations).*

We have also to guarantee the soundness with the other characteristics of the relation algebra of the constraint language, which are the closeness w.r.t. the inverse operator and the JEPD property for the set of relations. These conditions must be formalized as semantics requirements:

Definition* 4.52 (Frame Conditions) *Given an interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *the following* frame conditions *must hold:*

- One Cluster *requirement:* $\forall x, y \in \Delta^{\mathcal{I}} \; : < x, y > \in \bigcup_{R \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}} R^{\mathcal{I}}$
- Disjointness *requirement:* $\forall R, S \; \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}$ *with* $R \neq S : \; R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
- Inverse *requirement:* $R \in \mathcal{N}_{\mathcal{R}_\mathcal{S}} \Rightarrow \mathsf{inv}(R) \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}$

Wessel in his work proposed two different semantics for the equivalence relation of expressive languages. The possibility of differentiating equivalence semantics relies on the expressive power of the constraint languages, in particular on the presence of a real equivalence relation between spatial objects. Another aspect to take into account to evaluate the equivalence semantics is the choice of the universe of discourse. As we seen in Chapter 2 there are formalisms for QSRR that consider approximations of regions. In these cases the semantics of the equivalence relation must be studied carefully. In the following sections we will investigate the hybridization of $\mathcal{ALCI}_\mathsf{C}$ with different constraint languages.

We write formally the translation from a $\mathcal{ALCI}_\mathsf{C}$ language into a multi-modal language.

Definition 4.53 (Modal Translation, Wessel [140]). *The translation τ from description logics concepts to modal logics formulas is defined by the following conditions.*

- *Given a $\mathcal{DL}$ language $\mathcal{ALCI}_{\mathsf{C}}$, the corresponding modal logic $\mathcal{L}_{\mathsf{C}}$ is obtained with the following mapping. The set of concept names $\mathcal{N}_{\mathcal{C}}$ corresponds to the set of propositional letters* VAR *of the modal logic: we assume $\mathcal{N}_{\mathcal{C}}$ =*VAR.
- *Each role name in the role-box $\mathcal{R} \in \mathcal{N}_{\mathcal{R}_{\mathcal{S}}}$ corresponds to an accessibility relation connected to a necessity modal operator* $[\mathsf{r}]$.
- *The translation τ of a concept C is defined inductively as follows:*
  - $\tau(C) := C$ *if C is a concept name;*
  - $\tau(\neg C) := \neg\tau(C)$*;*
  - $\tau(C \sqcap D) := \tau(C) \wedge \tau(D)$*;*
  - $\tau(C \sqcup D) := \tau(C) \vee \tau(D)$*;*
  - $\tau(\exists R.C) := \langle \mathsf{r} \rangle \, \tau(C)$*;*
  - $\tau(\forall R.C) := [\mathsf{r}]\tau(C)$.

This is a syntactic transformation that maps a description logic onto the basic multi-modal logic K_m is closed under necessitation *and* modus ponens *and is defined by:*

- *all propositional axioms schemas,*
- $[\mathsf{r}](p \rightarrow q) \rightarrow ([\mathsf{r}]p \rightarrow [\mathsf{r}]q)$ *the K-axiom for all box operators,*
- $\langle \mathsf{r} \rangle \, p \leftrightarrow \neg[\mathsf{r}]\neg p$ *for each accessibility relation R.*

The previous definition expresses the basic multi-modal language but does not provide the formal definition of the properties of accessibility relations associated to modal operators. To enhance soundness and correctness w.r.t. the intended class of frames there are two possible ways: the definition of a pure hybrid axiomatic system as Wessel proposed in his work or the definition of a labelled deduction system with explicit deduction rule for the set of accessibility relations. In the following sections we will present examples that can be reduced to the fusion of well-known temporal logics inheriting this way all computational properties for the corresponding modal logics, in these cases the mapping will not require the definition of a hybrid multi-modal logics because of the presence in literature of many results on temporal logics.

4.3.3 $\mathcal{ALCI}_{\mathsf{CDC}}$

An interesting application of the $\mathcal{ALCI}_{\mathsf{C}}$ approach is the hybridization with the Cardinal Direction Calculus of Ligozat [89]. It is easy to define formally the constraint language given by the set of cardinal relations between points in a two-dimensional space. This set of relations is a finite relation algebra whose relations are closed under the inverse operator, have the operation of composition with an identity element and have the JEPD property.

Definition* 4.54 (CDC Constraint Language) *The* Cardinal Direction Calculus Constraint Language CDC *is defined as* $\{\mathbb{R}^2, \{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}, \mathsf{EQ}\}\}$ *where the universe is the set of points in* $\mathbb{R}^2$ *and* $\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}, \mathsf{EQ}\}$ *is the set of Ligozat's projection-based cardinal direction relations.*

The composition table $\mathfrak{T}$ is recalled in Table 2.8 in page 25. As already pointed out, the composition can computed componentwise because each projection-base cardinal direction can be seen as a pair of independent relations of the time point calculus $\{<, =, >\}$. In other words a relation r in set of CDC relations can be seen as $(\mathsf{r_x}, \mathsf{r_y})$ where each component belongs to a relation algebra with JEPD relations closed to the inverse operation and with a composition operation whose composition table is recalled in Table 2.7 (page 24). Let us state it formally in the following proposition.

Proposition* 4.55 *Each relation* $\mathsf{r} \in \{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}, \mathsf{EQ}\}$ *can be equivalently denoted as* $(\mathsf{r_x}, \mathsf{r_y})$ *where* $\mathsf{r_x} \in \{<_x, =_x, >_x\}$ *and* $\mathsf{r_y} \in \{<_y, =_y, >_y\}$*. Each set* $\{<_i, =_i, >_i\}$ *with* $i \in \{x, y\}$ *is a relation algebra, has the JEPD property, is closed under inverse operator and has a composition operation that for basic relations corresponds to the entry of the composition table 2.7.*

Definition* 4.56 ($\mathcal{ALCI}_{\mathsf{CDC}}$ Syntax) *Let* $\mathcal{N_C}$ *and* $\mathcal{N_{R_S}}$ *be two disjoint sets of symbols of* concept names *and* spatial role names *respectively and let*

$$\mathcal{N_{R_S}} = \{N, NE, E, SE, S, SW, W, NW, EQ\}.$$

The syntax is defined inductively and is borrowed from $\mathcal{ALCI}$*:*

- *every concept name* $C \in \mathcal{N_C}$ *is a concept;*
- *if* R *is in* $\mathcal{N_{R_S}}$ *and* C *and* D *are concepts, then also* $\neg C$*,* $C \cup D$*,* $C \cap D$*,* $\exists R.C$*,* $\forall R.C$*,* $\exists \mathsf{inv}(R).C$*,* $\forall \mathsf{inv}(R).C$ *are concepts.*

We note that the alternative notation for cardinal relations introduced in Proposition 4.55 have no consequences either from a syntactical or from a semantic point of view. It is just a different syntactic formalization which will be helpful below in mapping to a double modal logic. The notation given by the set of nine relations $\{\mathsf{N}, \mathsf{NE}, \mathsf{E}, \mathsf{SE}, \mathsf{S}, \mathsf{SW}, \mathsf{W}, \mathsf{NW}, \mathsf{EQ}\}$ can be seen as a set of concise labels for the set of all possible pairs deriving from the *fusion* of two equivalent, disjoint and independent relation algebras:

$$\{(\mathsf{r_x}, \mathsf{r_y}) | \mathsf{r_x} \in \{<_x, =_x, >_x\} \text{ and } \mathsf{r_y} \in \{<_y, =_y, >_y\}\}$$

The semantics is defined exactly as in the general case:

Definition* 4.57 ($\mathcal{ALCI}_{\mathsf{CDC}}$ Semantics) *An* interpretation $\mathcal{I} =_{def} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a non-empty set* $\Delta^{\mathcal{I}}$*, called the domain of* $\mathcal{I}$ *and an interpretation function* $\cdot^{\mathcal{I}}$*, that maps every concept name* C *to a subset of* $\Delta^{\mathcal{I}}$ *and every role name* R *to a subset of* $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *according to the following rules:*

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg C^{\mathcal{I}} &= \Delta \backslash C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x | \forall y[(x, y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &= \{x | \exists y[(x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\} \\
(\forall \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \forall y[(y, x) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \exists y[(y, x) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}
\end{aligned}$$

We defined formally in Definitions 4.51 and 4.52 some requirements that must generally hold for constraint languages, in this case for CDC. This constraint language is expressive enough to provide both the strong and the weak semantics for the EQ relation. The choice of a weak semantics has much sense if we consider reasoning on regions approximated by points (for instance the centroid of the region). The EQ relation provided with a weak semantics corresponds to a congruence between spatial objects and in the case of approximated regions a congruence would have much more sense than an identity relation. Nevertheless a strong EQ-semantics can be considered the right choice in case of punctual representation of spatial objects. As already pointed out, a punctual representation of spatial objects is common in qualitative reasoning: in the GIS environment such a choice can have much sense depending on the granularity of information (for instance towns with respect to a continental framework).

Following Wessel's approach based on Schild's correspondence theory between $\mathcal{DL}$ and modal logics we define the modal language L_{CDC} sound and complete with the intended class of frames. In the previous Section 2.4.1 we recall that all the projection based CDC-relations can be represented as pairs of relations of the *Time Point Calculus* [136]) and that the composition of CDC-relations can be computed componentwise. We can then consider the RBox axioms sound and complete w.r.t. the CDC composition table and the semantic requirements for the JEPD and inverse properties to be fulfilled by the hybrid language $\mathcal{ALCI}_{\mathsf{C}}$. As in the case of the original Cardinal Direction Calculus, we can consider either each spatial role as an alias, or as a label for a pair of relations as pointed out in Proposition 4.55. For this reason we can map each component of spatial relations into a multi-modal logic and then we con consider the fusion of the two modal logics. According to Schild's mapping we can defined a first modal logic.

Definition 4.58 (Modal Translation $\mathcal{L}_{\mathsf{CDC}}$). *The translation τ from $\mathcal{ALCI}_{\mathsf{C}}$ concepts to modal logics $\mathcal{L}_{\mathsf{CDC}}$ formulas is defined by the following conditions.*

- *Given a $\mathcal{DL}$ language $\mathcal{ALCI}_{\mathsf{CDC}}$, the corresponding modal logic $\mathcal{L}_{\mathsf{CDC}}$ is obtained with the following mapping. The set of concept names $\mathcal{N_C}$ corresponds to the set of propositional letters* VAR *of the modal logic: we assume $\mathcal{N_C}$ =*VAR*.*
- *Each role name in the role-box $\mathcal{R} \in \mathcal{N}_{\mathcal{R_S}}$ corresponds to an accessibility relation connected to a necessity modal operator* $[\mathsf{r}]$.
- *The translation τ of a concept C is defined inductively as follows:*
  - $\tau(C) := C$ *if C is a concept name;*
  - $\tau(\neg C) := \neg\tau(C)$*;*
  - $\tau(C \sqcap D) := \tau(C) \wedge \tau(D)$*;*
  - $\tau(C \sqcup D) := \tau(C) \vee \tau(D)$*;*
  - $\tau(\exists R.C) := \langle \mathsf{r} \rangle\, \tau(C)$*;*
  - $\tau(\forall R.C) := [\mathsf{r}]\tau(C)$.

This is the basic multi-modal logic K_m is closed under necessitation *and* modus ponens *and is defined by:*

- *all propositional axioms schemas,*
- $[\mathsf{r}](p \to q) \to ([\mathsf{r}]p \to [\mathsf{r}]q)$ *the K-axiom for all box operators,*
- $\langle \mathsf{r} \rangle\, p \leftrightarrow \neg[\mathsf{r}]\neg p$ *for each accessibility relation R.*

We saw previously that it is possible the definition of an axiomatic system to guarantee soundness and completeness w.r.t. the CDC framework.

(a) "One Cluster" : $\bigvee_{R \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}} @i \langle \mathsf{r} \rangle j$
(b) "Strong EQ-semantics": $\neg @i \langle \mathsf{eq} \rangle j$
(c) "Weak EQ-semantics": $i \rightarrow \langle \mathsf{eq} \rangle i$
(d) "Disjointness": $\forall \mathsf{R} \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}, @i \langle \mathsf{r} \rangle j \rightarrow @i \bigwedge_{\mathsf{S} \in \{\mathcal{N}_{\mathcal{R}_\mathcal{S}} - \mathsf{R}\}} \neg \langle \mathsf{s} \rangle j$
(e) "Converses": $\forall \mathsf{R} \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}, @i \langle \mathsf{r} \rangle j \rightarrow @j \langle \mathsf{inv(r)} \rangle i$
(f) "Compositions": for all role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \ldots \sqcup R_n$ from the corresponding CDC composition table, add $@i \langle \mathsf{s} \rangle \langle \mathsf{t} \rangle j \rightarrow @i \langle \mathsf{r_1} \rangle j \vee \ldots \vee @i \langle \mathsf{r_n} \rangle j$

Moreover, such a definition is also possible with a change of notation to translate the modal logic $\mathcal{L}_{\mathsf{CDC}}$ into the fusion of two distinct *hybrid temporal logics* investigated by Areces et al. in [6].

In fact we already noted that each spatial role in $\mathcal{R} \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}$ is a label for a pair of linear relations as follows:

$$\begin{aligned}
N &\equiv\ (=_x, >_y) \\
E &\equiv\ (>_x, =_y) \\
S &\equiv\ (=_x, <_y) \\
W &\equiv\ (<_x, =_y) \\
NE &\equiv\ (>_x, >_y) \\
NW &\equiv\ (<_x, >_y) \\
SE &\equiv\ (>_x, <_y) \\
SW &\equiv\ (<_x, <_y) \\
EQ &\equiv\ (=_x, =_y)
\end{aligned}$$

In the rest of the section we will recall the definition of hybrid temporal logic, define the fusion of two temporal logics proved to be decidable and then show a simple reduction from the hybrid modal logic $\mathcal{L}_{\mathsf{CDC}}$ to the fusion of hybrid tense logics. The following definitions are based on definitions provided in [6].

Definition 4.59 (Syntax of Hybrid Temporal Logic $Kl_@$). *Given a countable set $\mathcal{P}$ of propositional variables and a countable set of nominals $\mathcal{N}$ disjoint from $\mathcal{P}$, the set of* atoms *$\mathcal{A}$ is the union of $\mathcal{P}$ and $\mathcal{N}$. The set of* well-formed $Kl_@$ formulas *is defined by the following Backus-Naur-form presentation, where $p \in \mathcal{A}$ and $i \in \mathcal{N}$:*

$$A ::= \ p \mid \perp \mid \ A \rightarrow A \mid \mathsf{G}A \mid \mathsf{H}A \mid @_i A\,.$$

Truth of a tense formula is relative to a world in a model, so, intuitively, $\mathsf{G}A$ holds at a world iff A always holds in the future, and $\mathsf{H}A$ holds at a world iff A always holds in the past. The given syntax uses a minimal set of connectives, operators, and quantifiers. As usual, we can introduce abbreviations and use, e.g., $\sim$, $\wedge$, $\vee$ for the negation, the conjunction, and the disjunction: for instance "$\sim A \equiv A \rightarrow \perp$". It is also possible to define other temporal operators "$\mathsf{F}A \equiv \sim \mathsf{G} \sim A$" to express that A holds sometime in the future and "$\mathsf{F}A \equiv \sim \mathsf{H} \sim A$" for A holds sometime in the past and the *necessity equivalence modality* "$[\mathsf{Eq}]A \equiv \sim \mathsf{G}A \wedge \sim \mathsf{H}A$" and the corresponding *possibility equivalence modality* "$\langle \mathsf{Eq} \rangle A \equiv \sim [\mathsf{Eq}] \sim A$". Let us recall now the semantics for this modal logic starting from the definition of *frame* and *model*. Ordinary unsorted modal languages are interpreted on Kripke models while hybrid languages are interpreted on *hybrid models*, which are Kripke models defined on evaluations that ensure to each nominal name a unique world (called also *state*).

Definition 4.60 ($Kl_@$ Frame and Model). *A* $Kl_@$ frame *is a pair* $(\mathcal{W}, \prec)$*, where* $\mathcal{W}$ *is a non-empty set of worlds and* $\prec \subseteq \mathcal{W} \times \mathcal{W}$ *is a binary relation that satisfies the properties of irreflexivity, transitivity and connectedness, i.e. for all* $(x, y) \in \mathcal{W}^2$ *we have* $x = y$ *or* $(x, y) \in \prec$ *or* $(y, x) \in \prec$*.*

A $Kl_@$ hybrid model *is a triple* $(\mathcal{W}, \prec, \mathcal{V})$*, where* $(\mathcal{W}, \prec)$ *is a* Kl *frame and the valuation* $\mathcal{V} : \mathcal{A} \to 2^{\mathcal{W}}$ *is a function that maps atoms into subsets of the set of worlds such that for all* $i \in \mathcal{A}, \mathcal{V}(i)$ *is a singleton subset of* $\mathcal{W}$*.*

It is important to notice that this class of frames corresponds to the framework of the Time Point Calculus which describes all possible relations among points projected onto a line. The class of frames denoted by $(\mathcal{W}, \prec)$ is equivalent to and more concise than the notation $(\mathcal{W}, \prec, \approx, \succ)$ which explicits the set of the JEDP relations of the Time Point Calculus.

Definition 4.61 (Semantics of Hybrid Temporal Logic $Kl_@$). *Let* $\mathcal{M} = (\mathcal{W}, \prec, \mathcal{V})$ *be a model and* $m \in \mathcal{W}$*, then the* satisfaction relation *is the smallest relation satisfying the following conditions:*

$\mathcal{M}, m \models p$	*iff* $m \in \mathcal{V}(p), p \in \mathcal{A}$
$\mathcal{M}, m \models A \to B$	*iff* $\mathcal{M}, m \models A$ *implies* $\mathcal{M}, m \models B$
$\mathcal{M}, m \models \mathsf{G}A$	*iff for all* $n \in \mathcal{W}$*,* $(m \prec n)$ *implies* $\mathcal{M}, n \models A$
$\mathcal{M}, m \models \mathsf{H}A$	*iff for all* $n \in \mathcal{W}$*,* $(m \succ n)$ *(equiv.* $(n \prec m)$*) implies* $\mathcal{M}, n \models A$
$\mathcal{M}, m \models @_i A$	*iff* $\mathcal{M}, n \models A$ *where* $n = \mathcal{V}(i)$ *and* $i \in \mathcal{N}$

A formula A is satisfiable if there is a hybrid model $\mathcal{M}$ and a world $m \in \mathcal{W}$ such that $\mathcal{M}, m \models A$. Because hybrid valuations assign singletons to nominals, each nominal is satisfied at exactly one state in any model. As we saw in case of the multi-modal language of Lutz and Wolter in [101] when we are considering JEPD relations with a difference modality it is possible to define nominals as syntactic sugar; in these cases the hybrid modality results as variation of notation and does not offer any extra expressive power. Nevertheless in this case the definition of the hybrid temporal logic is interesting because the objective is proving the equivalence between the logic obtained from $\mathcal{ALCI}_{\mathsf{C}}$ with the Schild's mapping technique, that can be extended as hybrid to be easily proved sound and complete w.r.t. the intended class of frames.

Definition 4.62 (Axiomatization of $Kl_@$).

(a) *"One Cluster" :* $@i\mathsf{G}j \lor @i\mathsf{H}j \lor @i[\mathsf{Eq}]\ j$
(b) *"Duality":* $(\sim \mathsf{H} \sim \mathsf{G}A \to A) \land (\sim \mathsf{G} \sim \mathsf{H}A \to A)$
(c) *"Transitivity":* $\mathsf{G}A \to \mathsf{GG}A$
(d) *"Necessitation": if* A *is a theorem, then so are* $\mathsf{G}A$ *and* $\mathsf{H}A$
(e) *"Modus Ponens": if* $A \to B$ *and* A *are theorems, then so is* B *;*

It is important to notice that in this case the composition rules are stated by *transitivity*, *duality* and *one cluster* axioms and do not require to explicit the corresponding composition table: this class of frames represents exactly the Time Point Calculus. From the computational point of view we can refer to a result by Areces et al. [6].

Theorem 4.63 (Areces et al., [6]). *Let* S *be a subclass of the class of strict total orders. The complexity of the* S*-satisfiability problem is (up to a polynomial factor) the same for nominal tense logic (hybrid temporal logics) as it is for Priorean tense logics*

As pointed out by Areces et al. we do not pay any computational cost because hybridization over strict total orders does not increase the expressive power at our disposal. We know from a work of Spaan [129] that the satisfiability problem for Priorean tense logics over arbitrary frames is PSPACE-complete, and that it remains PSPACE-complete over transitive frames. We can now define a new "double" temporal logics using the formation of *fusion*. In the following we base the *fusion* strategy to combine logics on the definition given by Agi Kurucz in [87] where she presents some techniques of combining modal logics.

Definition* 4.64 (Double Temporal Logic) *Let* $Kl^x_@$ *be the logic* $Kl_@$ *considered on the frame* $\mathcal{F}_x = (\mathbb{R}^2, \prec_x)$ *where the* $\prec_x$ *relation holds for points on a two-dimensional space such that two points* $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *are related by* $P_1 \prec_x P_2$ *if and only if* $x_1 < x_2$ *where* $<$ *is the usual strict total order relation on* $\mathbb{R}$*.*
Let $Kl^y_@$ *be the logic* $Kl_@$ *considered on the frame* $\mathcal{F}_y = (\mathbb{R}^2, \prec_y)$ *where the* $\prec_y$ *relation holds for points on a two-dimensional space such that two points* $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *are related by* $P_1 \prec_y P_2$ *if and only if* $y_1 < y_2$ *where* $<$ *is the usual strict total order relation on* $\mathbb{R}$*.*
The fusion of the two logics denoted by $Kl^x_@ \otimes Kl^y_@$ *is the smallest modal logic with the union of the two sets of modal operators and it is characterized by the frame:*

$$\mathcal{F}_x \otimes \mathcal{F}_y = (\mathbb{R}^2, \prec_x, \prec_y)$$

It can be useful to underline here some features of the fusion of modal logics w.r.t. the results presented by Kurucz [87]:

- if $Kl^x_@$ and $Kl^y_@$ are axiomatized by a set of axioms (Σ^x, Σ^y respectively) then $Kl^x_@ \otimes Kl^y_@$ is axiomatized by the union $\Sigma^x \cup \Sigma^y$ (in other words in a fusion the modal operators of the component logics are independent and do not interact);
- the fusion of consistent modal logics is a conservative extension of the components;
- the fusion of two modal logics is conservative of the finite model property if both logics have it and the logic $Kl_@$ has the finite model property;
- if $Kl^x_@$ and $Kl^y_@$ are both decidable then $Kl^x_@ \otimes Kl^y_@$ is decidable as well (by a more general result by Wolter [142]).

Hence we have a decidability results of the new logic $Kl^x_@ \otimes Kl^y_@$, based on the same class of frames as the logic $\mathcal{L}_{\mathsf{CDC}}$ which is equivalent to the description language $\mathcal{ALCI}_{\mathsf{CDC}}$according to Schild's mapping. We can define a function to map each formula in $\mathcal{L}_{\mathsf{CDC}}$ to a formula in $Kl^x_@ \otimes Kl^y_@$.

Proposition* 4.65 *Given the two hybrid multi-modal logics* $\mathcal{L}_{\mathsf{CDC}}$ *and* $Kl^x_@ \otimes Kl^y_@$ *it is possible to map* $\varphi \in \mathcal{L}_{\mathsf{CDC}}$ *on a formula* $\varphi^\mu \in Kl^x_@ \otimes Kl^y_@$*, where* μ *is the mapping from formulas in* $\mathcal{L}_{\mathsf{CDC}}$ *to formulas in* $Kl^x_@ \otimes Kl^y_@$ *defined inductively as follows:*

- *the set of* $\mathcal{L}_{\mathsf{CDC}}$ *variables* $\mathsf{VAR}_{\mathcal{L}_{\mathsf{CDC}}}$ *must be a subset of the set of atoms of* $Kl^x_@ \otimes Kl^y_@$

$$\mathsf{VAR}_{\mathcal{L}_{\mathsf{CDC}}} \mapsto \mathsf{VAR}^\mu \subseteq \mathsf{A}_{Kl^x_@} \cup \mathsf{A}_{Kl^y_@} \tag{4.2}$$

- *the set of non modal operators are equivalent and the mapping is trivial*
- *given a formula φ and its mapping φ^μ the set of modal operators can be translated as follows*

$$
\begin{aligned}
[\mathsf{n}]\varphi &\mapsto [\mathsf{Eq}]_x\varphi^\mu \wedge \mathsf{G}_y\varphi^\mu\\
[\mathsf{e}]\varphi &\mapsto \mathsf{G}_x\varphi^\mu \wedge [\mathsf{Eq}]_y\varphi^\mu\\
[\mathsf{s}]\varphi &\mapsto [\mathsf{Eq}]_x\varphi^\mu \wedge \mathsf{H}_y\varphi^\mu\\
[\mathsf{w}]\varphi &\mapsto \mathsf{H}_x\varphi^\mu \wedge [\mathsf{Eq}]_y\varphi^\mu\\
[\mathsf{ne}]\varphi &\mapsto \mathsf{G}_x\varphi^\mu \wedge \mathsf{G}_y\varphi^\mu\\
[\mathsf{nw}]\varphi &\mapsto \mathsf{H}_x\varphi^\mu \wedge \mathsf{G}_y\varphi^\mu\\
[\mathsf{se}]\varphi &\mapsto \mathsf{G}_x\varphi^\mu \wedge \mathsf{H}_y\varphi^\mu\\
[\mathsf{sw}]\varphi &\mapsto \mathsf{H}_x\varphi^\mu \wedge \mathsf{H}_y\varphi^\mu\\
[\mathsf{eq}]\varphi &\mapsto [\mathsf{Eq}]_x\varphi^\mu \wedge [\mathsf{Eq}]_y\varphi^\mu
\end{aligned}
$$

From this it follows that each formula of $\mathcal{L}_{\mathsf{CDC}}$ is a formula in the fusion of hybrid temporal logics that is decidable. This proves the following result.

Theorem* 4.66 (Decidability of $\mathcal{L}_{\mathsf{CDC}}$) *The multi-modal logic $\mathcal{L}_{\mathsf{CDC}}$ is decidable.*

Proof The decidability of each component $Kl_@$ is preserved by fusion, so $Kl_@^x \otimes Kl_@^y$ is decidable as well. We provide in Proposition 4.65 a mapping that proving that the set of $\mathcal{L}_{\mathsf{CDC}}$-formulas is a subset of $Kl_@^x \otimes Kl_@^y$-formulas and that there is a correspondence between the intended class of frames. Hence the modal language $\mathcal{L}_{\mathsf{CDC}}$ is decidable. □

Theorem* 4.67 (Decidability of $\mathcal{ALCI}_{\mathsf{CDC}}$) *The description logic $\mathcal{ALCI}_{\mathsf{CDC}}$ is decidable.*

Proof The decidability of the description logic follows from the correspondence between $\mathcal{ALCI}_{\mathsf{CDC}}$ and $\mathcal{L}_{\mathsf{CDC}}$ according to the correspondence theory between $\mathcal{DL}$s and modal logics by Schild [118]. □

4.3.4 $\mathcal{ALCI}_{\mathsf{RA}}$ and $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$

In this section we consider other possible applications of the hybridization technique represented by $\mathcal{ALCI}_{\mathsf{C}}$ and by the computational investigation technique given by the mapping into modal logics. In the previous section we presented the description logic $\mathcal{ALCI}_{\mathsf{CDC}}$ which is the basic $\mathcal{ALCI}$ with finite role box fixed on the set of Cardinal Direction Calculus by Ligozat [89]. This language turned out to be decidable because of a correspondence with a “double” temporal logic defined on points. As a matter of fact the basic tense logic is decidable and the fusion of two tense logics is decidable as well. An other interesting candidate for hybridization is the Rectangle Algebra [67], [108] investigated in particular by Balbiani et al. in [15] which is the combination of two disjoint Interval Algebras of Allen [3]. We refer the reader to section 2.4.2 for further details on the QSRR formalism.

$\mathcal{ALCI}_{\mathsf{RA}}$

We can give a brief but formal definition of the constraint language behind the hybridized system and of course of the syntax and the semantics.

Definition* 4.68 (RA Constraint Language) *The* Constraint Language RA *is defined on the real plane by the product (denoted by* $\otimes$*) of the set of Interval Algebra relations (*$\mathsf{Rel}_{\mathsf{IA}}$*) for itself. So given* $\mathsf{Rel}_{\mathsf{IA}} = \{\mathsf{P}, \mathsf{Pi}, \mathsf{M}, \mathsf{Mi}, \mathsf{O}, \mathsf{Oi}, \mathsf{S}, \mathsf{Si}, \mathsf{D}, \mathsf{Di}, \mathsf{F}, \mathsf{Fi}, \mathsf{Eq}\}$ *the constraint language is defined as follows:*

$$\mathsf{RA} = \{\mathbb{R}^2, \mathsf{Rel}_{\mathsf{RA}} \otimes \mathsf{Rel}_{\mathsf{IA}}\}.$$

The composition table $\mathfrak{T}$ for the Rectangle Algebra is given by composition table of the Interval Algebra since the composition of Rectangle relations can computed component-wise: each projection-base cardinal direction can be seen as a pair of independent relations of the interval algebra relation.

$$\mathsf{r} \in \mathsf{Rel}_{\mathsf{RA}} \equiv (\mathsf{r_x}, \mathsf{r_y}) \in \mathsf{Rel}_{\mathsf{IA}} \otimes \mathsf{Rel}_{\mathsf{IA}}.$$

It is worth noticing the similarity between the constraint language CDC and RA: the "product" nature of the set $\mathsf{Rel}_{\mathsf{RA}}$ is equivalent to the proposition 4.55 for CDC. Given the number of resulting relations of the Rectangle Algebra it is worth adopting a change of notation w.r.t. the language $\mathcal{ALCI}_{\mathsf{CDC}}$ denoting each RA relation with the corresponding pair of IA relations. The formal syntax of the language is defined as follows:

Definition* 4.69 ($\mathcal{ALCI}_{\mathsf{RA}}$ Syntax) *Let* $\mathcal{N}_{\mathcal{C}}$ *and* $\mathcal{N}_{\mathcal{R}_{\mathcal{S}}}$ *be two disjoint sets of symbols of* concept names *and* spatial role names *respectively. Let us consider the the set* $rel = \{P, Pi, M, Mi, O, Oi, S, Si, D, Di, F, Fi, Eq\}$ *be the set of names for IA relations and let*

$$\mathcal{N}_{\mathcal{R}_{\mathcal{S}}} = rel \otimes rel.$$

The syntax is defined inductively and is borrowed from $\mathcal{ALCI}$*:*

- *every concept name* $C \in \mathcal{N}_{\mathcal{C}}$ *is a concept;*
- *if* R *is in* $\mathcal{N}_{\mathcal{R}_{\mathcal{S}}}$ *and* C *and* D *are concepts, then also* $\neg C$, $C \cup D$, $C \cap D$, $\exists R.C$, $\forall R.C$, $\exists \mathsf{inv}(R).C$, $\forall \mathsf{inv}(R).C$ *are concepts.*

Definition* 4.70 ($\mathcal{ALCI}_{\mathsf{RA}}$ Semantics) *An* interpretation $\mathcal{I} =_{def} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a non-empty set* $\Delta^{\mathcal{I}}$*, called the domain of* $\mathcal{I}$ *and an interpretation function* $\cdot^{\mathcal{I}}$*, that maps every concept name* C *to a subset of* $\Delta^{\mathcal{I}}$ *and every role name* R *to a subset of* $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *according to the following rules:*

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg C^{\mathcal{I}} &= \Delta \backslash C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x | \forall y[(x, y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &= \{x | \exists y[(x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\} \\
(\forall \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \forall y[(y, x) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \exists y[(y, x) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}
\end{aligned}$$

As for $\mathcal{ALCI}_{\mathsf{CDC}}$, the language $\mathcal{ALCI}_{\mathsf{RA}}$ is expressive enough to provide both a weak and a strong semantics. The choice should be made considering the "approximating" nature of the restriction to rectangular regions. In fact, the Rectangle Algebra can be seen as the set of relations between regions approximated by their *minimum bounding box* and in this case it would not be correct to require a strong semantics for the (Eq, Eq) relation. In this case a correspondence of two bounding boxes can imply an identity relations on regions. A weak semantics for the (Eq, Eq) relation means a congruence between "equal" spatial objects. Unless we do not consider other than directional relations among spatial objects it can be acceptable to identify a unique rectangle by a certain position and specific dimensions.

In the case of this description language the corresponding modal logics according to Schild's mapping is a "double" interval temporal logics augmented eventually with nominals for a sound and complete axiomatization of the intended class of frames. In this case we omit the modal translation because the technique to be applied is exactly the same as for $\mathcal{ALCI}_{\mathsf{CDC}}$. We refer the reader to [134] and [76] for a complete survey on temporal logics and recall here the basic definition of interval temporal logic. Allens relations give rise to respective unary modal operators, thus defining the modal logic of time intervals HS introduced by Halpern and Shoham in [72]. Almost all of these modal operators are definable in terms of others and it suffices to choose as basic the modalities corresponding to the relations *begin*, *end* and their inverses. The formulas of the logics denoted by HS are generated given a set of propositional variables by the following abstract Backus-Naur syntax:

$$\varphi ::= \; p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle \mathsf{B} \rangle\, \varphi \mid \langle \mathsf{E} \rangle\, \varphi \mid \langle \bar{\mathsf{B}} \rangle \varphi \mid \langle \bar{\mathsf{E}} \rangle \varphi\,.$$

The formal semantics of these modal operators is defined as follows:
$\mathcal{M}, [d_0, d_1] \models \langle \mathsf{B} \rangle\, \varphi$ iff there exists d_2 s.t. $d_0 \leq d_2 < d_1$ and $\mathcal{M}, [d_0, d_2] \models \varphi$
$\mathcal{M}, [d_0, d_1] \models \langle \mathsf{E} \rangle\, \varphi$ iff there exists d_2 s.t. $d_0 < d_2 \leq d_1$ and $\mathcal{M}, [d_2, d_1] \models \varphi$
$\mathcal{M}, [d_0, d_1] \models \langle \bar{\mathsf{B}} \rangle \varphi$ iff there exists d_2 s.t. $d_1 < d_2$ and $\mathcal{M}, [d_0, d_2] \models \varphi$
$\mathcal{M}, [d_0, d_1] \models \langle \bar{\mathsf{E}} \rangle \varphi$ iff there exists d_2 s.t. $d_2 < d_0 \leq d_1$ and $\mathcal{M}, [d_2, d_1] \models \varphi$

HS is a highly undecidable logic. In [72] the authors have obtained important results about non-axiomatizability, undecidability and complexity of the satisfiability in HS for many natural classes of models. From these premises follows that

Theorem* 4.71 ($\mathcal{ALCI}_{\mathsf{RA}}$ Undecidability) *The language $\mathcal{ALCI}_{\mathsf{RA}}$ is undecidable.*

Proof As for the case of the modal logic for $\mathcal{ALCI}_{\mathsf{CDC}}$, even in this case there is a correspondence between the fusion of two interval modal logics and the Schild's mapping of the description language. Given two logics HS_x and HS_y we can easily defined the fusion $\mathsf{HS}_x \otimes \mathsf{HS}_y$ corresponding to the language L_{RA}. In this case the undecidability of the HS component implies the undecidability of the corresponding L_{RA}. This proves the undecidability of $\mathcal{ALCI}_{\mathsf{RA}}$. $\square$

For the sake of readability we omitted the extended definitions of the fusion $\mathsf{HS}_x \otimes \mathsf{HS}_y$ and of the mapping, since they are analogous to the case of CDC relations.

$\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$

The hybrid language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ is the only $\mathcal{DL}$ that provides the expressive power for both mereo-topological and directional reasoning. The corresponding QSR formalism presented by Li in 2007 [88] is a first attempt to combine notions that in natural language and in many practical applications are used together. As seen in previous sections the basic idea of a qualitative calculi, even if it is expressed by a decidable formalism, can loose decidability when embedded into a formal structure for reasoning that might be either a modal logic or a description language. We saw for instance RCC-8 that when embedded into $\mathcal{ALCI}_{\mathcal{RCC}8}$ results in an undecidable language. The language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ is the union of $\mathcal{ALCI}_{\mathcal{RCC}8}$ and a sub-language of $\mathcal{ALCI}_{\mathsf{RA}}$ and from this fact arises the undecidability of the formalism. Nevertheless we can consider this language of some interest because we will see that under certain restrictions on the maximal cardinality of models it becomes decidable. Now for the sake of completeness we define syntax and semantics of this language whose role box is given by the combination of two disjoint relation algebras.

Definition* 4.72 (DIV9-RCC8 Constraint Language) *The* DIV9-RCC8 Constraint Language *is defined as* $\mathsf{DIV9} - \mathsf{RCC8} = \{\mathbb{R}^2, \mathsf{Rel}_{\mathsf{DIV9-RCC8}}\}$ *where the universe is the set of regular regions in* $\mathbb{R}^2$ *and* $\mathsf{Rel}_{\mathsf{DIV9-RCC8}}$ *is the union of the set of RCC-8 relations* $\mathsf{Rel}_{\mathsf{RCC8}} = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}, \mathsf{TPPI}, \mathsf{NTPP}, \mathsf{NTPPI}\}$ *with the sub algebra DIV9 of the Rectangle Algebra* $\mathsf{Rel}_{\mathsf{DIV9}} = \{\mathsf{NW}, \mathsf{NC}, \mathsf{NE}, \mathsf{CW}, \mathsf{CC}, \mathsf{CE}, \mathsf{SW}, \mathsf{SC}, \mathsf{SE}\}$.

It is worth noticing that while RCC-8 works with regions, the relations in DIV9 works with the approximation of regions given by the *minimum bounding box*. Li in his work pointed out that topological and directional constraints are not independent. This affects the computational technique to find out satisfiability of a combined constraint network, but does not require any restriction on the set of definable constraints and therefore neither on the definition of admissible $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ -concepts. In the following we will omit the definition of composition tables and composition-based role axioms, since the two relation algebras are disjoint and keep the original properties of the corresponding QSR formalism.

Definition* 4.73 ($\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ Syntax) *Let* $\mathcal{N}_{\mathcal{C}}$ *and* $\mathcal{N}_{\mathcal{R}_{\mathcal{S}}}$ *be two disjoint sets of symbols of* concept names *and* spatial role names *respectively and let*

$$\begin{aligned}\mathcal{N}_{\mathcal{R}_{\mathcal{S}}} = \{&DC, EC, PO, EQ, TPP, TPPI, NTPP, NTPPI\} \cup \\ &\{NW, NC, NE, CW, CC, CE, SW, SC, SE\}.\end{aligned}$$

The syntax is defined inductively and is borrowed from $\mathcal{ALCI}$*:*

- *every concept name* $C \in \mathcal{N}_{\mathcal{C}}$ *is a concept;*
- *if* R *is in* $\mathcal{N}_{\mathcal{R}_{\mathcal{S}}}$ *and* C *and* D *are concepts, then also* $\neg C$, $C \cup D$, $C \cap D$, $\exists R.C$, $\forall R.C$, $\exists \mathsf{inv}(R).C$, $\forall \mathsf{inv}(R).C$ *are concepts.*

Definition* 4.74 ($\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ Semantics) *An* interpretation $\mathcal{I} =_{def} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a non-empty set* $\Delta^{\mathcal{I}}$*, called the domain of* $\mathcal{I}$ *and an interpretation function* $\cdot^{\mathcal{I}}$*, that maps every concept name* C *to a subset of* $\Delta^{\mathcal{I}}$ *and every role name* R *to a subset of* $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *according to the following rules:*

$$
\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
\neg C^{\mathcal{I}} &= \Delta \backslash C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x | \forall y[(x,y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists R.C)^{\mathcal{I}} &= \{x | \exists y[(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\} \\
(\forall \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \forall y[(y,x) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\} \\
(\exists \mathsf{inv}(R).C)^{\mathcal{I}} &= \{x | \exists y[(y,x) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}
\end{aligned}
$$

Considering the JEPD properties for both the sets of relations, it is trivial that between any two spatial objects hold both a mereo-topological and a directional relation as in the case of two disjoints constraint networks defined over the same set of variables. For this reason we can define the set of semantic requirements as the "One Cluster" requirements as the union of the requirements of $\mathcal{ALCI}_{\mathcal{RCC}8}$ and $\mathcal{ALCI}_{\mathsf{RA}}$. The investigation of the corresponding modal logic according to Schild's mapping is no longer interesting since the undecidable nature of $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$-concepts is clear. In the following chapter we will define some terminological languages extended with spatial concrete domains, as an alternative to the definition of $\mathcal{DL}$ with fixed role box.

5

QSR with Description Logics extended by Concrete Domains

5.1 Introduction

Description Logics extended by concrete domains have been defined for overcoming the necessity of representing concrete qualities in the terminological reasoning. Numerical concrete domains was the first to be considered by researchers, who tried to give an adequate definition for commonsense and quite simple concepts as Woman. As pointed out by Baader et al. in [12] Woman could be defined as "Human $\sqcap$ Female", but it would be inaccurate since a newborn female baby would probably not be considered a woman. We should require a female human being to be old enough to be called a woman. Standard $\mathcal{DL}$ do not provide the expressive power to state such a condition in an efficient way: we can introduce the atomic concept AtLeast18 to describe the property of being at least 18 years old but this strategy is inefficient since for each age that we need to model, we must introduce a new atomic concept and all the required inclusion axioms to guarantee the appropriate subsumption between all "numerical" atomic concepts. A more elegant solution is represented by the use of the concrete domain of nonnegative integers equipped usual predicates as $\leq$ or $>$. This allows to express age properties by introducing a new functional role hasAge that binds via numerical predicates a concept to a number:

$$\mathsf{Human} \sqcap \mathsf{Female} \sqcap \exists \mathsf{hasAge}.\ \geq_{18}$$

where $\geq_{18}$ stands for the unary predicate $\{n|n \geq 18\}$ of all nonnegative integers greater or equal to 18. An important application for concrete domains is surely Qualitative Spatial Reasoning.

In Section 3.3 we provided the formal definition of the general framework of description logics with concrete domains. In this Chapter we focus on the definition of spatial concrete domains and the analysis of their properties, in order to decide possible hybridizations. As pointed out by Lutz in [93] and [96] the complexity of the satisfiability of constraint networks defined over a concrete domain influenced the complexity of the concept-satisfiability problem for the $\mathcal{DL}$ extended with that concrete domain. In the following we presented some concrete domains already known in the literature and define other domains based on the QSRR formalism recalled in Chapter 2.

The main properties for concrete domains are: satisfiability, admissibility, patchwork property, compactness and ω-admissibility. The admissibility of a concrete domain implies the possibility of decidable hybridization both with the basic $\mathcal{ALC}(\mathcal{D})$ and

$\mathcal{ALCRP}(\mathcal{D})$, while the ω-admissibility allows one to include the constraint system in $\mathcal{ALC}(\mathcal{C})$ with general TBox keeping decidability. We first recall briefly the formal definitions of these properties and then analyze the referenced QSRR formalisms considering good candidates for hybridization.

Definition 5.1 (D-satisfiability, Baader et al. [12]). *Let $P_1, \ldots, P_k$ be k not necessarily different predicate names in $pred(\mathcal{D})$ of arities $n_1, \ldots, n_k$, we consider the conjunction*

$$\bigwedge_{i=1}^{k} P_i(\bar{x}^{(i)})$$

where $\bar{x}^{(i)}$ stands for an n_i-tuple $(x_1^{(i)}, \ldots, x_{n^i}^{(i)})$. Such a conjunction is said to be satisfiable *if and only if there exists an assignment of elements of $\Delta^{\mathcal{D}}$ to the variables such that the conjunction becomes true in $\mathcal{D}$. The problem of deciding satisfiability of finite conjunctions of this form is normally called the* satisfiability problem *for $\mathcal{D}$.*

According to the definition 3.8 of Section 3.3 the admissibility of a concrete domain is formalized as follows

Definition 5.2 (Admissibility, Baader et al. [12]). *The concrete domain $\mathcal{D}$ is said to be* admissible *iff*

(a) *the set of its predicate names is closed under negation and contains a symbol $\top_{\mathcal{D}}$ for $\Delta^{\mathcal{D}}$,*
(b) *the satisfiability problem for $\mathcal{D}$ is decidable.*

Among concrete domains there are particular domains based on finite relations algebras that have received a particular attention by researchers. In recent years, Lutz and Miličić [99] gave a general notion of *constraint system* that is intended to capture standard constraint systems based on a set of jointly-exhaustive and pairwise-disjoint (JEPD) binary relations.

Definition 5.3 (Rel-network, Lutz and Miličić [99]). *Let Var be a countably infinite set of variables and Rel a finite set of binary relation symbols. A Rel-constraint is an expression $(x\ r\ y)$ with $x, y \in \mathsf{Var}$ and $r \in \mathsf{Rel}$. A Rel-network is a (finite or infinite) set of Rel-constraints. For N a Rel-network, we use V_N to denote the variables used in N. We say that N is complete if, for all $x, y \in V_N$, there is exactly one constraint $(x\ r\ y) \in N$*

Definition 5.4 (Model, Constraint System, Lutz and Miličić [99]). *Let N be a Rel-network and N' a complete Rel-network. We say that N' is a model of N if there is a mapping $\tau : V_N \to V_{N'}$ such that $(x\ r\ y) \in N$ implies $(\tau(x)\ r\ \tau(y)) \in N'$.*

A constraint system $\mathcal{C} = \langle \mathsf{Rel}, \mathfrak{M} \rangle$ consists of a finite set of relation symbols Rel and a set $\mathfrak{M}$ of complete Rel-networks (the models of $\mathcal{C}$). A Rel-network N is satisfiable in $\mathcal{C}$ if $\mathfrak{M}$ contains a model of N.

The patchwork property ensures that satisfiable networks based on the constraint system can be "patched" together to a joint network that is also satisfiable.

Definition 5.5 (Patchwork Property, Lutz and Miličić [99]). *Let $\mathcal{C} = \langle \mathsf{Rel}, \mathfrak{M} \rangle$ be a constraint system, and N, M be finite complete* Rel*-networks such that, for the intersection parts*

$$I_{N,M} := \{(x\ r\ y)|x, y \in V_N \cap V_M \text{ and } (x\ r\ y) \in N\}$$
$$I_{M,N} := \{(x\ r\ y)|x, y \in V_N \cap V_M \text{ and } (x\ r\ y) \in M\}$$

we have $I_{N,M} = I_{M,N}$. *Then the* composition *of* N *and* M *is defined as* $N \cup M$. *We say that* $\mathcal{C}$ *has the* patchwork property *if the following holds: if* N *and* M *are satisfiable then* $N \cup M$ *is satisfiable.*

The compactness property ensures that if we patch together an infinite number of satisfiable networks, the resulting infinite network is still satisfiable.

Definition 5.6 (Compactness Property, Lutz and Miličić [99]). *Let* $\mathcal{C} = \langle \mathsf{Rel}, \mathfrak{M} \rangle$ *be a constraint system. If* N *is a* Rel*-network and* $V \subseteq V_N$*, we write* $N_{|_V}$ *to denote the network* $\{(x\ r\ y) \in N|x, y \in V\} \subseteq N$. *Then* $\mathcal{C}$ *has the* compactness property *if the following holds: a* Rel*-network N with* V_N *infinite is satisfiable in* $\mathcal{C}$ *if and only if, for every finite* $V \subseteq V_N$*, the network* $N_{|_V}$ *is satisfiable in* $\mathcal{C}$*.*

The property called ω-admissibility summarizes these last properties as follows.

Definition 5.7 (ω-admissibility, Lutz and Miličić [99]). *Let* $\mathcal{C} = \langle \mathsf{Rel}, \mathfrak{M} \rangle$ *be a constraint system. We say that* $\mathcal{C}$ *is* ω*-admissible if and only if the following holds:*

(a) *satisfiability of finite* $\mathcal{C}$*-networks is decidable;*
(b) $\mathcal{C}$ *has the patchwork property;*
(c) $\mathcal{C}$ *has the compactness property.*

In the rest of the Chapter we will see which properties hold for each of the QSRR formalisms.

5.2 Concrete domains based on RCC-8

The set of mereo-topological relations represented by RCC-8 has been greatly investigated by researchers that aim at a terminological spatial reasoning. We already recalled the general definition of concrete domain and the specialization given by "constraint systems" which are concrete domains based on finite relation algebras. In this section as in the rest of the chapter, when possible we will analyze for each QSRR formalism these two different concrete domains.

5.2.1 The concrete domain $\mathcal{S}_2$

In 1999 Haarslev et al. [131] introduced the first concrete domain for spatial reasoning, $\mathcal{S}_2$. We recall here the basic definitions about point set topology given in [131] by Haarlsev et al. based on the book Algebraic Topology by Spanier [130].

Definition 5.8 (Topology and Topological Space, Haarslev et al. [131]). *Let* $\mathcal{U}$ *be a set, a* topology *on* $\mathcal{U}$ *is a family* T *of subsets of* $\mathcal{U}$*, with*

(a) *if* $O_1, O_2 \in T$ *then* $O_1 \cap O_2 \in T$*,*
(b) *if* $O_i \in T$ *for* $i \in I$*, then* $\bigcup O_i \in T$*,*
(c) $\emptyset, \mathcal{U} \in T$

The pair $\langle \mathcal{U}, T \rangle$ *is called a* topological space. *the elements of* T *are called* open subsets *of* $\mathcal{U}$.

Definition 5.9 (Haarslev et al. [131]). *Let* $\langle \mathcal{U}, T \rangle$ *be a topological space, let* M *be a subset of* $\mathcal{U}$*, and let* x *be a point in* $\mathcal{U}$.

- M *is* closed *if* $\mathcal{U} \setminus M$ *is open.*
- *A set* $N \subset \mathcal{U}$ *is called* neighborhood *of* x *if there is an open subset* $O \subset \mathcal{U}$ *such that* $x \in O \subset \mathcal{U}$.
- x *is called an* interior point *of* M *if there is a neighborhood* N *of* x *contained in* M*. The set of all interior points of* M *is called the* interior *of* M *and is denoted by* $i(M)$.
- x *is called an* exterior point *of* M *if there is a neighborhood* N *of* x *that contains no point in* M*. The set of all exterior points of* M *is called the* exterior *of* M *and is denoted by* $e(M)$.
- x *is called a* boundary point *of* M *if every neighborhood* N *of* x *contains at least one point in* M *and one point not in* M*. The set of all boundary points of* M *is called the* boundary *of* M *and is denoted by* $b(M)$.
- *the* closure *of* M *is the smallest closed set which contains* M *and is denoted by* $\widehat{M}$.
- M *is* regular closed *if* $\widehat{i(M)}$ *is equal to* M.

The RCC-8 theory is an axiomatic theory for QSRR that describes all possible relationships that may hold between any two subsets of a topological space. Each of the RCC-8 relations are defined in terms of interior and boundary of regions in a topological space (we refer the reader to Section 2.3.1 for further details). We recall that an RCC8-formula is an expression of the form XRY where R is one the RCC-8 relations or a disjunction of such relations. A set of RCC8-formulas N is called RCC8-network and the set of variables used in N is denoted by $Var(N)$. An RCC8-network N is *satisfiable* if and only if there exist a topological space $\langle \mathcal{U}, T \rangle$ and a mapping δ from $Var(N)$ to the set of all non-empty, regular closed subsets of $\mathcal{U}$ such that for all RCC8-formulas XRY in N we have $(\delta(X), \delta(Y)) \in R$.

Definition 5.10 ($\mathcal{S}_2$, Haarslev et al. [131]). *The concrete domain* $\mathcal{S}_2$ *is defined w.r.t. the topological space* $\left\langle \mathbb{R}^2, 2^{\mathbb{R}^2} \right\rangle$*. The domain* $\Delta_{\mathcal{S}_2}$ *contains all non-empty, regular closed subsets of* $\mathbb{R}^2$*. The elements of* $\Delta_{\mathcal{S}_2}$ *are called* regions*. The set* $\Phi_{\mathcal{S}_2}$ *contains predicates which are defined as follows:*

- *A unary predicate* is-region *with* $\text{is-region}^{\mathcal{S}_2} = \Delta_{\mathcal{S}_2}$ *and its negation* is-no-region *with* $\text{is-no-region}^{\mathcal{S}_2} = \emptyset$
- *A binary predicate* inconsistent-relation *with* $\text{inconsistent-relation}^{\mathcal{S}_2} = \emptyset$.
- *The eight basic predicates* dc, ec, po, tpp, ntpp, tppi, ntppi, eq *correspond to the RCC-8 relations and are defined as follows. Let* r_1 *and* r_2 *be two regions. We have* $(r_1, r_2) \in \text{dc}^{\mathcal{S}_2}$ *iff* $(r_1, r_2) \in DC$, $(r_1, r_2) \in \text{ec}^{\mathcal{S}_2}$ *iff* $(r_1, r_2) \in EC, \ldots$
- *For each distinct set* $\{p_1, \ldots, p_n\}$ *of basic predicates, where* $n \geq 2$ *an additional predicate of arity 2 is defined. The predicate has the name* $\mathsf{p_1}\text{-}\ldots\text{-}\mathsf{p_n}$ *and we have* $(r_1, r_2) \in \mathsf{p_1}\text{-}\ldots\text{-}\mathsf{p_n}$ *iff* $(r_1, r_2) \in \mathsf{p_1}$ *or* $\ldots$ $(r_1, r_2) \in \mathsf{p_n}$ *is true.*

In [131] the following proposition is proved.

Proposition 5.11 (Haarslev et al. [131]). *The concrete domain* $\mathcal{S}_2$ *is admissible.*

5.2.2 The constraint system $\mathsf{RCC8}_{\mathbb{R}^2}$

A variation w.r.t. $\mathcal{S}_2$ is given by the constraint system denoted by $\mathsf{RCC8}_{\mathbb{R}^2}$ defined by Lutz and Miličić in [99]. The formal definition is recalled in the following and corresponds to a simplified version of $\mathcal{S}_2$. The JEPD property of the set of relations allows to omit in the definition of the constraint system the unary predicate for the universal relation and the binary predicate for the inconsistent relation. The following definition is based on the original definition by Lutz and Miličić but for the sake of readability we try to standardize notation w.r.t. the general concrete domain definition.

Definition 5.12 ($\mathsf{RCC8}_{\mathbb{R}^2}$ **constraint system, Lutz and Miličić [99]).** *The constraint system* $\mathsf{RCC8}_{\mathbb{R}^2}$ *is defined w.r.t. the topological space* $\left\langle \mathbb{R}^2, 2^{\mathbb{R}^2} \right\rangle$. *The domain* $\Delta_{\mathsf{RCC8}_{\mathbb{R}^2}}$ *contains all non-empty, regular closed subsets of* $\mathbb{R}^2$. *The set* $\Phi_{\mathsf{RCC8}_{\mathbb{R}^2}}$ *contains the eight basic predicates* $\mathsf{dc}, \mathsf{ec}, \mathsf{po}, \mathsf{tpp}, \mathsf{ntpp}, \mathsf{tppi}, \mathsf{ntppi}, \mathsf{eq}$ *that correspond to the RCC-8 relations. The constraint system is denoted by*

$$\mathsf{RCC8}_{\mathbb{R}^2} = \langle \mathsf{RCC8}, \mathfrak{M}_{\mathbb{R}^2} \rangle$$

where $\mathfrak{M}_{\mathbb{R}^2} := \{N_{\mathbb{R}^2}\}$ *is the set of models for constraint networks on RCC-8 , where* $N_{\mathbb{R}^2}$ *is defined by fixing a variable* $v_s \in \mathsf{Var}$ *for every regular closed set* $s \in \Delta_{\mathsf{RCC8}_{\mathbb{R}^2}}$ *and setting*

$$N_{\mathbb{R}^2} := \{(v_s\ r\ v_t) | r \in \mathsf{RCC8}, s, t \in \Delta_{\mathsf{RCC8}_{\mathbb{R}^2}} \text{ and } (s,t) \in r^{\mathsf{RCC8}_{\mathbb{R}^2}}\}$$

In [99] Lutz and Miličić proved that the constraint system based on RCC-8 relations has both the patchwork property and the compactness property. since the formalism of RCC-8 is decidable the corresponding constraint system is ω-admissible.

5.3 Concrete domain based on BRCC8

We now build a concrete domain for the Boolean Region Connection Calculus. This is obtained as follows. We consider the set Δ, formed by the same regions admitted as models of the region connection calculus. Over Δ we define the relations of the region connection calculus (exactly as in the concrete domain for RCC) as predicates, and then add the predicates to represent the operations of intersection, union and complement of regions in the following way:

- $\cap(X, Y, Z)$ means that X and Y have Z as intersection;
- $\cup(X, Y, Z)$ means that X and Y have Z as union;
- $\neg(X, Y)$ means that X have Y as complement (and viceversa).

We also add the unary predicate $\top$ to the domain Δ to represent Δ itself. On the logical expressions involving literals (positive and negative) formed by the above mentioned predicates we consider, as newly formed predicates all the skolemised expressions obtained by binding variables on the expressions themselves. The above mentioned domain is named Δ_{BRCC}. Lemma 5.13, whose proof is a straightforward consequence of the definition of Δ_{BRCC}.

Lemma 5.13. Δ_{BRCC} *is admissible.*

Let us now consider the extension to Constraint Networks proposed by Zakharyaschev and Wolter [143], in which, we recall, vertices are labeled by Boolean expressions on region variables, and edges by expressions of the Region Connection Calculus. Without loss of generality we can assume that the Boolean expressions appearing in each vertex are in Disjunctive Normal Form, namely are unions of intersection of regions or complements of regions. To translate from an extended constraint network C onto a set of expressions S in Δ_{BRCC} we proceed as follows:

- For every expression ϕ labeling a vertex in C we add to S an expression ϵ, that is the conjunctions of expressions recursively derived from ϕ as follows:
  - For every disjunct d we add to the expression ϵ the conjunction of the expressions directly obtained from the conjuncts in d;
  - For each disjunct with n conjunctions within we provide $n-1$ new variables used to name the conjuncts in a stream of pairs from the first pair to the conjunction of the result of that pair to the third element until we reach the last element of the disjunct;
  - We subsequently add the expressions representing union of variables in the same stream defined by the above construction;
- We finally add to each expression a prefix formed by existential quantifiers for each variable labeling a vertex in C.

The counter-translation is trivially obtained by reversing the above operation. The translation sketched above is named τ. We can show the following lemma.

Lemma 5.14. *Given an extended constraint network C on BRCC, the set of expressions $\tau(C)$ is expressed in the constraint language of Δ_{BRCC}.*

Given a set of expressions S of the constraint language Δ_{BRCC} the translation $\tau^{-1}(S)$ is an extended constraint network on BRCC.

Given the two lemmas above we can prove the following theorem.

Theorem 5.15. *Δ_{BRCC} is an admissible concrete domain.*

Since the BRCC is not a simple constraint language it is not possible to define the corresponding constraint system.

5.4 Concrete domains based on CDC

In this Section we investigate properties of concrete domains based on the Cardinal Direction Calculus by Ligozat [89] referring the reader to Section 2.4.1 for the wider explanation of the formalism. As for the case of RCC-8 relations will define both the standard concrete domain $\mathcal{S}_{\mathsf{CDC}}$ and the constraint system $\mathsf{CDC}_{\mathbb{R}^2}$.

5.4.1 The concrete domain $\mathcal{S}_{\mathsf{CDC}}$

Definition* 5.16 ($\mathcal{S}_{\mathsf{CDC}}$) *The concrete domain $\mathcal{S}_{\mathsf{CDC}} = (\Delta_{\mathcal{S}_{\mathsf{CDC}}}, \Phi_{\mathcal{S}_{\mathsf{CDC}}})$ is defined on the two-dimensional space $\mathbb{R}^2$. The domain $\Delta_{\mathcal{S}_{\mathsf{CDC}}}$ contains points in $\mathbb{R}^2$. The set $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ contains predicates which are defined as follows:*

- *A unary predicate* is-point *with* is-point$^{\mathcal{S}_{\mathsf{CDC}}} = \Delta_{\mathcal{S}_{\mathsf{CDC}}}$ *and its negation* is-no-point *with* is-no-point$^{\mathcal{S}_{\mathsf{CDC}}} = \emptyset$
- *A binary predicate* inconsistent-relation *with* inconsistent-relation$^{\mathcal{S}_{\mathsf{CDC}}} = \emptyset$.
- *The nine basic predicates* n, ne, e, se, s, sw, w, nw, eq *correspond to the Cardinal Direction Calculus relations and are defined as follows. Let p_1 and p_2 be two points. We have $(p_1, p_2) \in \mathsf{n}^{\mathcal{S}_{\mathsf{CDC}}}$ iff $(p_1, p_2) \in N$, $(p_1, p_2) \in \mathsf{ne}^{\mathcal{S}_{\mathsf{CDC}}}$ iff $(p_1, p_2) \in NE, \ldots$*
- *For each distinct set $\{\pi_1, \ldots, \pi_n\}$ of basic predicates, where $n \geq 2$ an additional predicate of arity 2 is defined. The predicate has the name π_1-$\ldots$-π_n and we have $(p_1, p_2) \in \pi_1$-$\ldots$-π_n iff $(p_1, p_2) \in \pi_1$ or $\ldots$ $(p_1, p_2) \in \pi_n$ is true.*

For the next proofs, we follows the proof-schema proposed by Haarslev et al. in [131].

Lemma* 5.17 *The satisfiability of finite conjunctions of predicates from $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ is decidable.*

Proof The problem of checking the satisfiability of infinite conjunctions of predicate can be reduced to checking the consistency of CDC networks. Let a finite conjunction $C = \pi_1(x_1^1, \ldots, x_{n_1}^1) \wedge \ldots \wedge \pi_k(x_1^k, \ldots, x_{n_k}^k)$ of predicates from $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ be given. Its satisfiability can be decided as follows.

- If for any $i = 1 \ldots k$, π_i is either is-no-point or inconsistent-relation, then return *unsatisfiable*.
- Remove any conjunct with $\pi_i =$ is-point. All predicates in the remaining conjunction have arity 2.
- Translate the remaining conjunction C' into a CDC network N as follows: The variables $Var(N)$ are exactly the variables occurring in C'. Consider any conjunct $\pi_i(x_1^i, x_2^i)$ from C' separately. The predicate π_i has the form π_1-$\ldots$-π_n with $n \geq 1$. Let $R = R_1 \wedge \ldots R_n$ be the corresponding disjunction of CDC relations. Make a case distinction as follows: (i) if there is no CDC formula $x_1^i S x_2^i$ in N, then add $x_1^i R x_2^i$; (ii) let there be a CDC formula $x_1^i S x_2^i$ in N, where S is a disjunction of CDC relations. Let $(R \cap S)$ denote the disjunction of those relations that appear in both R and S. If there is no such relation, return *inconsistent*. Otherwise, remove the formula $x_1^i S x_2^i$ from N and add the new formula $x_1^i (R \cap S) x_2^i$.
- For all pairs $(r_1, r_2) \in Var(N)$, for which there is no formula $r_1 R r_2$ in N, add the formula $r_1 * r_2$ where $*$ is universal relation given by the disjunction of all CDC relations.
- Check the satisfiability of the set N and return the result.

In [89] Ligozat proved that deciding the consistency of CDC networks is decidable. The proof that C is satisfiable iff N is satisfiable is trivial and follows immediately from the definition of N and the definition of predicates in $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$. □

Proposition* 5.18 *The concrete domain $\mathcal{S}_{\mathsf{CDC}}$ is admissible.*

Proof CDC relations are jointly exhaustive and pairwise disjoint [89]. Given this, we can verify that $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ is closed under negation. Let P be the set of basic predicates, for any set of predicates $\{\pi_1, \ldots, \pi_n\} \subseteq P$ with $1 \leq n < 9$, we have that $\overline{\pi_1\text{-}\ldots\text{-}\pi_n} = \mathsf{s}_1$-$\ldots$-$\mathsf{s}_n$ where $\{s_1\text{-}\ldots\text{-}s_n\}$ is defined as $P \backslash \{\pi_1\text{-}\ldots\text{-}\pi_n\}$. The predicate s_1-$\ldots$-s_n is in $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ and the negation of the disjunctive combination of all basic predicates in inconsistent-relation and vice versa. The fact that $\Phi_{\mathcal{S}_{\mathsf{CDC}}}$ is closed under negation together with the lemma 5.17 implies the admissibility of the concrete domain. □

5.4.2 The constraint system $\mathsf{CDC}_{\mathbb{R}^2}$

In [38] Cristani and Gabrielli investigate the possibility of applying the $\mathcal{DL}$ $\mathcal{ALC}(\mathcal{C})$ to the constraint system given by Ligozat's Cardinal Direction Relations. The paper [38] provide an investigation of properties of the constraint system denoted $\mathsf{CDC}_{\mathbb{R}^2}$ in the following. Hereafter its extended definition.

Definition* 5.19 ($\mathsf{CDC}_{\mathbb{R}^2}$) *The constraint system* $\mathsf{CDC}_{\mathbb{R}^2}$ *is defined on the two-dimensional space* $\mathbb{R}^2$. *The domain* $\Delta_{\mathsf{CDC}_{\mathbb{R}^2}}$ *contains points in* $\mathbb{R}^2$. *The set* $\Phi_{\mathsf{CDC}_{\mathbb{R}^2}}$ *contains the nine basic predicates* $\mathsf{n}, \mathsf{ne}, \mathsf{e}, \mathsf{se}, \mathsf{s}, \mathsf{sw}, \mathsf{w}, \mathsf{nw}, \mathsf{eq}$ *that correspond to the* CDC *relations. The constraint system is denoted by*

$$\mathsf{CDC}_{\mathbb{R}^2} = \langle \mathsf{CDC}, \mathfrak{M}_{\mathbb{R}^2} \rangle$$

where $\mathfrak{M}_{\mathbb{R}^2} := \{N_{\mathbb{R}^2}\}$ *is the set of models for constraint networks on* CDC*, where* $N_{\mathbb{R}^2}$ *is defined by fixing a variable* $v_s \in \mathsf{Var}$ *for every point* $s \in \Delta_{\mathsf{CDC}_{\mathbb{R}^2}}$ *and setting*

$$N_{\mathbb{R}^2} := \{(v_s \; r \; v_t) | r \in \mathsf{CDC}, s, t \in \Delta_{\mathsf{CDC}_{\mathbb{R}^2}} \textit{ and } (s,t) \in r^{\mathsf{CDC}_{\mathbb{R}^2}}\}$$

We trivially say that a constraint system is decidable when there is an algorithm that, for every constraint network (or, in other formalizations, every finite set of constraint expressions), in a finite time establishes whether the network is satisfiable or not.

Lemma 5.20 (Ligozat, [89]). *The Cardinal Direction Calculus is NP-complete.*

From decidability of the QSRR formalism follows the decidability of the constraint system. The proof of this property is given by Ligozat who even identified a maximal tractable subclass of *pre-convex* relations with the property that the path-consistency implies the consistency. The further properties of the Cardinal Direction Calculus that are needed for guaranteeing that it can be combined with $\mathcal{ALC}$ preserving decidability are the patchwork property and the compactness property. They can be both proved in a rather simple way. The **patchwork property** is enjoyed by those constraint systems that meet the following condition: if two complete networks of base relations such that for the intersection parts are identical are satisfiable, then the composition (that is the union) of these networks is satisfiable as well. The **compactness property**, finally, holds for those systems such that networks (possibly infinite) are satisfiable if and only if every finite sub-network is satisfiable.

Lemma* 5.21 (Cristani and Gabrielli, [38]) *The constraint system* $\mathsf{CDC}_{\mathbb{R}^2}$ *enjoys the Patchwork Property.*

Proof In order to prove that the $\mathsf{CDC}_{\mathbb{R}^2}$ enjoys the patchwork property we need to specify how two networks of constraints compose in such a framework. The composition is determined by the union of the vertices and the intersection of the labels on the edges. Moreover, every network $\mathcal{N}$ of base relations of $\mathcal{CDC}$ can be seen as a pair of networks ($\mathcal{N}_x$, $\mathcal{N}_y$) both defined on the Point Algebra (PA) of Vilain, Kautz and Van Beek [136]. The first of these networks represent the Cardinal relations North and South, whilst the second one represents the relations West to East. Note that since the Point Algebra can be seen as a sub-algebra of Allen's Interval algebra (IA), and IA has been shown [99] to have the patchwork property, then PA has the patchwork property as well.

Now consider two networks $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ defined on $\mathcal{CDC}$, that are satisfiable. The four derived networks $\mathcal{N}_x^{(1)}$, $\mathcal{N}_x^{(2)}$, $\mathcal{N}_y^{(1)}$ and $\mathcal{N}_y^{(2)}$ are all PA networks and the composition of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ is $\mathcal{N}$. Obviously, $\mathcal{N}_x$ is the result of the composition of $\mathcal{N}_x^{(1)}$ and $\mathcal{N}_x^{(2)}$, whilst $\mathcal{N}_y^{(1)}$ and $\mathcal{N}_y^{(2)}$ compose to $\mathcal{N}_y$. Because of the patchwork property of PA, satisfiability of $\mathcal{N}$ is guaranteed by the satisfiability of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$, which means that $\mathcal{CDC}$ enjoys the patchwork property. □

The last property we need to exhibit is the compactness property.

Lemma* 5.22 (Cristani and Gabrielli, [38]) *The constraint system* $\mathsf{CDC}_{\mathbb{R}^2}$ *enjoys the Compactness Property.*

Proof It is easy to see that satisfiability of an infinite CDC-network N implies the satisfiability of every finite sub-network $N|_V$ defined over a finite arbitrary subset V of the set of variables V_N of N. To show the converse we give a translation of a CDC-network to a set $\Gamma(N)$ of first-order sentences that preserves the satisfiability. We consider a binary predicate $<$ for the ordering on $\mathbb{R}$ and constants $(x_a)_{a\in\mathrm{Var}}$ and $(y_a)_{a\in\mathrm{Var}}$ denoting the two components of the point corresponding to a in the plane. The set of first-order sentences $\Gamma(N)$ is build according to the following rule: one sentence for each constraint in N, translating the relations of the CDC as in the example.

$$(a\ n\ b)\ \text{ becomes }\ (x_a\ =\ x_b) \wedge (y_a\ >\ y_b)$$

Each CDC-network N is satisfiable in $\mathrm{CDC}_{\mathbb{R}^2}$ iff $\Gamma(N)$ is satisfiable in a structure $(\mathbb{R}^2, <_x, <_y, P_1^{\mathfrak{M}}, P_2^{\mathfrak{M}}, ...)$ where the relations $<_x$ and $<_y$ denote, respectively, the total ordering established between points based on each coordinate (x and y represent the two directions of the vector space $\mathbb{R}^2$) and the $P_i^{\mathfrak{M}}$ represent the assignments of variables in the constraint system to the values of the model $\mathfrak{M}$. □

The above lemmas prove the following claims.

Theorem* 5.23 (Cristani and Gabrielli, [38]) *The constraint system* $\mathsf{CDC}_{\mathbb{R}^2}$ *is ω-admissible.*

Proof The ω-admissibility of $\mathsf{CDC}_{\mathbb{R}^2}$ follows immediately by definition from Lemma 5.20, Lemma 5.21 and Lemma 5.22. □

5.5 Concrete domains based on RA

In this Section we investigate properties of concrete domains based on the Rectangle Algebra investigated in [67], [108] and [15]. We refer the reader to Section 2.4.2 for a full explanation of the formalism. As for the case of RCC-8 relations we will define both the standard concrete domain $\mathcal{S}_{\mathsf{RA}}$ and the constraint system $\mathsf{RA}_{\mathbb{R}^2}$.

5.5.1 The concrete domain $\mathcal{S}_{\mathsf{RA}}$

Definition* 5.24 ($\mathcal{S}_{\mathsf{RA}}$) *The concrete domain $\mathcal{S}_{\mathsf{RA}} = (\Delta_{\mathcal{S}_{\mathsf{RA}}}, \Phi_{\mathcal{S}_{\mathsf{RA}}})$ is defined on the two-dimensional space $\mathbb{R}^2$. The domain $\Delta_{\mathcal{S}_{\mathsf{RA}}}$ contains rectangles in $\mathbb{R}^2$ whose sides are parallel to the axes. The set $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ contains predicates which are defined as follows:*

- *A unary predicate* $\mathsf{is\text{-}rectangle}$ *with* $\mathsf{is\text{-}rectangle}^{\mathcal{S}_{\mathsf{RA}}} = \Delta_{\mathcal{S}_{\mathsf{RA}}}$ *and its negation* $\mathsf{is\text{-}no\text{-}rectangle}$ *with* $\mathsf{is\text{-}no\text{-}rectangle}^{\mathcal{S}_{\mathsf{RA}}} = \emptyset$
- *A binary predicate* $\mathsf{inconsistent\text{-}relation}$ *with* $\mathsf{inconsistent\text{-}relation}^{\mathcal{S}_{\mathsf{RA}}} = \emptyset$.
- *The 169 basic predicates given by the product of two distinct sets of Allen's interval relations* $\mathsf{IA} = \{\mathsf{P}, \mathsf{Pi}, \mathsf{M}, \mathsf{Mi}, \mathsf{O}, \mathsf{Oi}, \mathsf{S}, \mathsf{Si}, \mathsf{D}, \mathsf{Di}, \mathsf{F}, \mathsf{Fi}, \mathsf{Eq}\}$

$$\mathsf{IA} \otimes \mathsf{IA}$$

  correspond to the Rectangle Algebra relations and are defined as follows. Let r_1 *and* r_2 *be two rectangles. We have*

$$(r_1, r_2) \in (\mathsf{P_x}, \mathsf{P_y})^{\mathcal{S}_{\mathsf{RA}}} \text{ iff } (r_1, r_2) \in (P_x, P_y),$$
$$(r_1, r_2) \in (\mathsf{P_x}, \mathsf{M_y})^{\mathcal{S}_{\mathsf{RA}}} \text{ iff } (r_1, r_2) \in (P_x, M_y), \ \ldots$$

- *For each distinct set* $\{(\pi_x^1, \pi_y^1), \ldots, (\pi_x^n, \pi_y^n)\}$ *of basic predicates, where* $n \geq 2$ *an additional predicate of arity 2 is defined. The predicate has the name* $(\pi_{\mathsf{x}}^1, \pi_{\mathsf{y}}^1)\text{-}\ldots\text{-}(\pi_{\mathsf{x}}^{\mathsf{n}}, \pi_{\mathsf{y}}^{\mathsf{n}})$ *and we have* $(r_1, r_2) \in (\pi_{\mathsf{x}}^1, \pi_{\mathsf{y}}^1)\text{-}\ldots\text{-}(\pi_{\mathsf{x}}^{\mathsf{n}}, \pi_{\mathsf{y}}^{\mathsf{n}})$ *iff* $(r_1, r_2) \in (\pi_{\mathsf{x}}^1, \pi_{\mathsf{y}}^1)$ *or* $\ldots$ $(r_1, r_2) \in (\pi_{\mathsf{x}}^{\mathsf{n}}, \pi_{\mathsf{y}}^{\mathsf{n}})$ *is true.*

Lemma* 5.25 *The satisfiability of finite conjunctions of predicates from* $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ *is decidable.*

Proof The problem of checking the satisfiability of infinite conjunctions of predicate can be reduced to checking the consistency of RA networks, following the same technique used for Lemma 5.17. The correspondence between the problem of satisfiability of infinite conjunctions and on the consistency of the RA network follows immediately from the definition of predicates in $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ and the correctness of the reduction strategy used in the previous cases (CDC and RCC8). In [16] Balbiani et al. pointed out that deciding the consistency of RA networks is decidable. This proves the lemma. □

Proposition* 5.26 *The concrete domain* $\mathcal{S}_{\mathsf{RA}}$ *is admissible.*

Proof RA relations are jointly exhaustive and pairwise disjoint. Given this, we can verify that $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ is closed under negation. Let P be the set of basic predicates, for any set of predicates $\{(\pi_x^1, \pi_y^1), \ldots, (\pi_x^n, \pi_y^n)\} \subseteq P$ with $1 \leq n < 169$, we have that $\overline{(\pi_{\mathsf{x}}^1, \pi_{\mathsf{y}}^1)\text{-}\ldots\text{-}(\pi_{\mathsf{x}}^{\mathsf{n}}, \pi_{\mathsf{y}}^{\mathsf{n}})} = (\mathsf{s_x^1}, \mathsf{s_y^1})\text{-}\ldots\text{-}(\mathsf{s_x^n}, \mathsf{s_y^n})$ where $\{(s_x^1, s_y^1)\text{-}\ldots\text{-}(s_x^n, s_y^n)\}$ is defined as $P \setminus \{(\pi_x^1, \pi_y^1)\text{-}\ldots\text{-}(\pi_x^n, \pi_y^n)\}$. The predicate $(\mathsf{s_x^1}, \mathsf{s_y^1})\text{-}\ldots\text{-}(\mathsf{s_x^n}, \mathsf{s_y^n})$ is in $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ and the negation of the disjunctive combination of all basic predicates in inconsistent-relation and vice versa. The fact that $\Phi_{\mathcal{S}_{\mathsf{RA}}}$ is closed under negation together with the lemma 5.25 implies the admissibility of the concrete domain. □

5.5.2 The constraint system $\mathsf{RA}_{\mathbb{R}^2}$

In [99] Lutz and Miličić investigated the properties of the constraint system $\mathsf{Allen}_{\mathbb{R}}$ based on Allen's Interval Algebra proving its ω-admissibility. They proved the patchwork property, the compactness, then using the well known result of decidability for set of constraints on Allen's relations deduced the ω-admissibility of the constraint system. In this section we will provide the ω-admissibility result for the constraint system based on RA relations, using the independence between relations referred to different axes and the ω-admissibility of each component.

Definition* 5.27 ($\mathsf{RA}_{\mathbb{R}^2}$) *The constraint system* $\mathsf{RA}_{\mathbb{R}^2}$ *is defined on the two-dimensional space* $\mathbb{R}^2$*. The domain* $\Delta_{\mathsf{RA}_{\mathbb{R}^2}}$ *contains rectangles in* $\mathbb{R}^2$ *whose sides are parallel to the axes. The set* $\Phi_{\mathsf{RA}_{\mathbb{R}^2}}$ *contains 169 basic predicates given by the product of two distinct sets of Allen's interval relations* $\mathsf{IA} = \{\mathsf{P}, \mathsf{Pi}, \mathsf{M}, \mathsf{Mi}, \mathsf{O}, \mathsf{Oi}, \mathsf{S}, \mathsf{Si}, \mathsf{D}, \mathsf{Di}, \mathsf{F}, \mathsf{Fi}, \mathsf{Eq}\}$

$$\mathsf{RA} = \mathsf{IA} \otimes \mathsf{IA}$$

correspond to the Rectangle Algebra relations and are defined as follows. Let r_1 *and* r_2 *be two rectangles. We have*

$$(r_1, r_2) \in (\mathsf{P_x}, \mathsf{P_y})^{\mathcal{S}_{\mathsf{RA}}} \textit{ iff } (r_1, r_2) \in (P_x, P_y),$$
$$(r_1, r_2) \in (\mathsf{P_x}, \mathsf{M_y})^{\mathcal{S}_{\mathsf{RA}}} \textit{ iff } (r_1, r_2) \in (P_x, M_y),\ \ldots$$

The constraint system is denoted by

$$\mathsf{RA}_{\mathbb{R}^2} = \langle \mathsf{RA}, \mathfrak{M}_{\mathbb{R}^2} \rangle$$

where $\mathfrak{M}_{\mathbb{R}^2} := \{N_{\mathbb{R}^2}\}$ *is the set of models for constraint networks on* RA*, where* $N_{\mathbb{R}^2}$ *is defined by fixing a variable* $v_s \in \mathsf{Var}$ *for every rectangle* $s \in \Delta_{\mathsf{RA}_{\mathbb{R}^2}}$ *and setting*

$$N_{\mathbb{R}^2} := \{(v_s\ r\ v_t) | r \in \mathsf{RA}, s, t \in \Delta_{\mathsf{RA}_{\mathbb{R}^2}} \textit{ and } (s, t) \in r^{\mathsf{RA}_{\mathbb{R}^2}}\}$$

As already pointed out in Section 2.4.2, the problem of checking the consistency of a constraint network for RA relations is NP-complete and from decidability of the QSRR formalism follows the decidability of the constraint system. In the following we check the main properties of the constraint system based on RA relations.

Lemma* 5.28 *The constraint system* $\mathsf{RA}_{\mathbb{R}^2}$ *enjoys the Patchwork Property.*

Proof In order to prove that the $\mathsf{RA}_{\mathbb{R}^2}$ enjoys the patchwork property we need to specify how two networks of constraints compose in such a framework. The composition is determined by the union of the vertices and the intersection of the labels on the edges. Moreover, every network $\mathcal{N}$ of base relations of $\mathcal{RA}$ can be seen as a pair of networks ($\mathcal{N}_x$, $\mathcal{N}_y$) both defined on Allen's Interval Algebra [3]. The first of these networks represent the relations North and South, whilst the second one represents the relations West to East. Note that Allen's Interval algebra (IA) has been shown [99] to have the patchwork property.

Now consider two networks $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ defined on $\mathcal{RA}$, that are satisfiable. The four derived networks $\mathcal{N}_x^{(1)}$, $\mathcal{N}_x^{(2)}$, $\mathcal{N}_y^{(1)}$ and $\mathcal{N}_y^{(2)}$ are all IA networks and the composition of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ is $\mathcal{N}$. Obviously, $\mathcal{N}_x$ is the result of the composition of $\mathcal{N}_x^{(1)}$ and $\mathcal{N}_x^{(2)}$, whilst $\mathcal{N}_y^{(1)}$ and $\mathcal{N}_y^{(2)}$ compose to $\mathcal{N}_y$. Because of the patchwork property of IA, satisfiability of $\mathcal{N}$ is guaranteed by the satisfiability of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$, which means that $\mathcal{CDC}$ enjoys the patchwork property. □

The last property we need to exhibit is the compactness property.

Lemma* 5.29 *The constraint system* $\mathsf{RA}_{\mathbb{R}^2}$ *enjoys the Compactness Property.*

Proof It is easy to see that the compactness property of both the independent components in $\mathsf{Allen}_{\mathbb{R}}$ of a RA constraint network implies that the same property holds for $\mathsf{RA}_{\mathbb{R}^2}$ as well. □

The above lemmas prove the following claims.

Theorem* 5.30 *The constraint system* $\mathsf{RA}_{\mathbb{R}^2}$ *is* ω*-admissible.*

Proof The ω-admissibility of $\mathsf{RA}_{\mathbb{R}^2}$ follows immediately by definition from the decidability of the consistency problem for RA networks and from Lemma 5.28 and Lemma 5.29. □

5.6 Concrete domains based on PDR and PDR$^+$

In this section we consider the model of Projection-based Directional Relations (PDR) studied by Goyal and Egenhofer [62], [63] and Skiadopoulos and Koubarakis [124], [125] and [126], which is actually one of the most expressive models for qualitative reasoning with cardinal directions. In particular we refer to the paper [126] for the definition of the constraint language and the proof that the consistency problem for constraint networks on PDR relations can be solved polynomially for basic non-disjunctive relations and is NP-complete in the general case. In [126] Skiadopoulos et al. extended the same framework including either lines and points. As already pointed out in section 2.4.3, from the computational perspective the extension implies a huge increase of the number of relations but do not effect the complexity of the problem of checking consistency of constraints networks. The high number of possible basic relations is quite discouraging, nevertheless this set of JEPD relations allows one to describe exactly any possible situation involving a reference region and an other spatial element in the set of complex regular set augmented by lines and points.

In this case we will define only the standard concrete domain $\mathcal{S}_{\mathsf{PDR}}$ unless we have no proof of ω-admissibility for the constraint system $\mathsf{PDR}_{\mathbb{R}^2}$. We will not define explicitly the concrete domains corresponding to the PDR^+, since from a definitorial point of view they are equivalent except for the "universe" of discourse.

5.6.1 The concrete domain $\mathcal{S}_{\mathsf{PDR}}$

Definition* 5.31 ($\mathcal{S}_{\mathsf{PDR}}$**)** *The concrete domain* $\mathcal{S}_{\mathsf{PDR}} = (\Delta_{\mathcal{S}_{\mathsf{PDR}}}, \Phi_{\mathcal{S}_{\mathsf{PDR}}})$ *is defined on the two-dimensional space* $\mathbb{R}^2$*. The domain* $\Delta_{\mathcal{S}_{\mathsf{PDR}}}$ *is equal to the set* REG* *which contains:*

- *regions homeomorphic to the closed unit disk* $\{(x, y) : x^2 + y^2 \geq 1\}$*, which are closed, connected and with connected boundaries in* $\mathbb{R}^2$*;*
- *disconnected regions and regions with holes as follows: a region* a *in* $\mathbb{R}^2$ *belongs to* REG* *iff there exists a finite set of regions* $a_1, \ldots, a_n \in REG$ *such that* $a = a_1 \cup \ldots \cup a_n$

The set $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ *contains predicates which are defined as follows:*

- *A unary predicate* is-region *with* $\text{is-region}^{\mathcal{S}_{\mathsf{PDR}}} = \Delta_{\mathcal{S}_{\mathsf{PDR}}}$ *and its negation* is-no-region *with* $\text{is-no-region}^{\mathcal{S}_{\mathsf{PDR}}} = \emptyset$
- *A binary predicate* inconsistent-relation *with* $\text{inconsistent-relation}^{\mathcal{S}_{\mathsf{PDR}}} = \emptyset$.
- *The 511 basic predicates corresponding to the basic PDR relations* P *defined in Definition 2.21. Let* r_1 *and* r_2 *be two regions and let* P *be in* B_p *the set of 511 basic predicates corresponding to the set of basic relations* B_r*. We have*

$$(r_1, r_2) \in \mathsf{P}^{\mathcal{S}_{\mathsf{PDR}}} \textit{ iff } (r_1, r_2) \in P, \textit{ where } P \in B_r$$

- *For each distinct set* $\{\mathsf{P_1}, \ldots, \mathsf{P_n}\}$ *of basic predicates, where* $n \geq 2$ *an additional predicate of arity 2 is defined. The predicate has the name* $\mathsf{P_1}\text{-}\ldots\text{-}\mathsf{P_n}$ *and we have* $(r_1, r_2) \in \mathsf{P_1}\text{-}\ldots\text{-}\mathsf{P_n}$ *iff* $(r_1, r_2) \in \mathsf{P_1}$ *or* $\ldots$ $(r_1, r_2) \in \mathsf{P_n}$ *is true.*

Lemma* 5.32 *The satisfiability of finite conjunctions of predicates from* $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ *is decidable.*

Proof The problem of checking the satisfiability of infinite conjunctions of predicate can be reduced to checking the consistency of PDR networks, following the same technique used for Lemma 5.17. The correspondence between the problem of satisfiability of infinite conjunctions and on the consistency of the PDR network follows immediately from the definition of predicates in $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ and the correctness of the reduction strategy used in the previous cases (CDC and RCC8). The problem of checking consistency for PDR constraint networks is NP-complete 2.24 taken from [126]. This proves the lemma. □

Proposition* 5.33 *The concrete domain* $\mathcal{S}_{\mathsf{PDR}}$ *is admissible.*

Proof PDR relations are jointly exhaustive and pairwise disjoint (see Section 2.4.3). Given this, we can verify that $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ is closed under negation. Let P be the set of basic predicates, for any set of predicates $\{P_1, \ldots, P_n\} \subseteq B_r$ with $1 \leq n < 511$, we have that $\overline{\mathsf{P_1}\text{-}\ldots\text{-}\mathsf{P_n}} = \mathsf{S_1}\text{-}\ldots\text{-}\mathsf{S_n}$ where $\{S_1\text{-}\ldots\text{-}S_n\}$ is defined as $B_r \setminus \{P_1\text{-}\ldots\text{-}P_n\}$. The predicate $\mathsf{S_1}\text{-}\ldots\text{-}\mathsf{s_n}$ is in $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ and the negation of the disjunctive combination of all basic predicates in inconsistent-relation and vice versa. The fact that $\Phi_{\mathcal{S}_{\mathsf{PDR}}}$ is closed under negation together with the lemma 5.32 implies the admissibility of the concrete domain. □

5.7 Concrete domains based on DIV9 − RCC8

In Section 2.4.4 we presented an attempt investigated by Li in [88] to combine topological and directional information for qualitative spatial reasoning. He observed that topological and directional information is not independent, since a combined constraint network may be unsatisfiable despite that both of the components are satisfiable. In section 4.3 we investigate computational properties of the language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ fixing the set of role names and the role box w.r.t. the DIV9 − RCC8 constraint language. An undecidability result arises immediately from the undecidability of the RCC-8 fragment of the language proved in Section 4.1. In the literature there exist several tractable subclasses of RCC nevertheless, it is not possible to investigate them in the framework of fixed role boxes, since it is not possible to impose syntactic restrictions to limit the disjunctive relations definable by a user. However dealing with an external concrete domain allows one to impose a particular set of predicates corresponding to a sub-algebra of the the set of relations. The case of DIV9 − RCC8 is a good example of predicate restriction. Li in his work states that

Proposition 5.34 (Li, [88]). *If topological constraints are all in one of the three maximal tractable subclasses of RCC8 [114], then the satisfiability of the joint network can be determined by considering the satisfiability of two related networks respectively in RCC8 and the rectangle algebra (RA) [16].*

From this it follows that we can define a special concrete domain based on two disjoint sub-algebras one for topological information and one for directional relations defined as the "sum" of two different concrete domains sharing variables. In the rest of the section we do not consider a particular RCC-8 sub-algebra among the three maximal tractable subclasses of RCC8 that contain all basic relations, $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, $\mathcal{Q}_8$ ([114]). For this reason we will use the notation Top to refer to one of the tractable sub-algebras.

5.7.1 The concrete domain $\mathcal{S}_{\mathsf{DIV9-RCC8}}$

Definition* 5.35 **($\mathcal{S}_{\mathsf{DIV9-RCC8}}$)** *The concrete domain* $\mathcal{S}_{\mathsf{DIV9-RCC8}} = (\Delta_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}, \Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}})$ *is defined on the two-dimensional space* $\mathbb{R}^2$*. The domain* $\Delta_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *is equal to the set of regular closed sets w.r.t. the topological space* $\left\langle \mathbb{R}^2, 2^{\mathbb{R}^2} \right\rangle$*. The set* $\Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *contains predicates which are defined as follows:*

- *A unary predicate* is-region *with* $\text{is-region}^{\mathcal{S}_{\mathsf{DIV9-RCC8}}} = \Delta_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *and its negation* is-no-region *with* $\text{is-no-region}^{\mathcal{S}_{\mathsf{DIV9-RCC8}}} = \emptyset$
- *A binary predicate* inconsistent-relation *with* $\text{inconsistent-relation}^{\mathcal{S}_{\mathsf{DIV9-RCC8}}} = \emptyset$.
- *The set* B_p *of* $8+9$ *basic predicates corresponding to the union of the set of RCC-8 relations* $\mathsf{RCC8} = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}, \mathsf{TPPI}, \mathsf{NTPP}, \mathsf{NTPPI}\}$ *with the sub algebra DIV9 of the Rectangle Algebra* $\mathsf{DIV9} = \{\mathsf{NW}, \mathsf{NC}, \mathsf{NE}, \mathsf{CW}, \mathsf{CC}, \mathsf{CE}, \mathsf{SW}, \mathsf{SC}, \mathsf{SE}\}$. *Let* r_1 *and* r_2 *be two regions and let* P *be in* B_p*. We have*

$$(r_1, r_2) \in \mathsf{P}^{\mathcal{S}_{\mathsf{DIV9-RCC8}}} \text{ iff } (r_1, r_2) \in P, \text{ where } P \in B_r$$

- *For each distinct set* $\{\mathsf{P}_1, \ldots, \mathsf{P}_n\}$ *of basic predicates in* DIV9*, where* $n \geq 2$ *an additional predicate of arity 2 is defined. The predicate has the name* $\mathsf{P}_1\text{-}\ldots\text{-}\mathsf{P}_n$ *and we have* $(r_1, r_2) \in \mathsf{P}_1\text{-}\ldots\text{-}\mathsf{P}_n$ *iff* $(r_1, r_2) \in \mathsf{P}_1$ *or* $\ldots (r_1, r_2) \in \mathsf{P}_n$ *is true.*
- *For each distinct set of basic predicates* $\{\mathsf{P}_1, \ldots, \mathsf{P}_n\}$ *in* Top*, where* $n \geq 2$ *an additional predicate of arity 2 is defined. The predicate has the name* $\mathsf{P}_1\text{-}\ldots\text{-}\mathsf{P}_n$ *and we have* $(r_1, r_2) \in \mathsf{P}_1\text{-}\ldots\text{-}\mathsf{P}_n$ *iff* $(r_1, r_2) \in \mathsf{P}_1$ *or* $\ldots (r_1, r_2) \in \mathsf{P}_n$ *is true. No other set of basic predicates is considered.*

We will see that this concrete domain is admissible, since the set of predicates $\Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ is closed under negation and the satisfiability problem of finite predicate conjunctions is decidable.

Lemma* 5.36 *The satisfiability of finite conjunctions of predicates from* $\Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *is decidable.*

Proof It is possible to reduce the problem of satisfiability of finite conjunctions of predicates to the consistency of constraint networks. Let a finite conjunction $C = P_1(x_1^1, \ldots, x_{n_1}^1) \wedge \ldots \wedge P_k(x_1^k, \ldots, x_{n_k}^k)$ of predicates from $\Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ be given. Its satisfiability can be decided as follows.

- If for any $i = 1 \ldots k$, P_i is either is-no-region or inconsistent-relation, then return *unsatisfiable*.
- Remove any conjunct with $P_i =$ is-region. All predicates in the remaining conjunction have arity 2.

- Translate the remaining conjunction C' into a $\mathsf{DIV9} - \mathsf{RCC8}$ network N as follows: The variables $Var(N)$ are exactly the variables occurring in C'. Consider any conjunct $P_i(x_1^i, x_2^i)$ from C' separately.
  – For each predicate P_i with the form $\mathsf{P_1}\text{-}\ldots\text{-}\mathsf{P_n}$ where $n \geq 1$ and $\mathsf{P_1}$ is in DIV9, let $R = R_1 \wedge \ldots R_n$ be the corresponding disjunction of DIV9 relations. Make a case distinction as follows: (i) if there is no DIV9 formula $x_1^i S x_2^i$ in N, then add $x_1^i R x_2^i$; (ii) let there be a DIV9 formula $x_1^i S x_2^i$ in N, where S is a disjunction of DIV9 relations. Let $(R \cap S)$ denote the disjunction of those relations that appear in both R and S. If there is no such relation, return *inconsistent*. Otherwise, remove the formula $x_1^i S x_2^i$ from N and add the new formula $x_1^i (R \cap S) x_2^i$.
  – For each predicate P_i with the form $\mathsf{P_1}\text{-}\ldots\text{-}\mathsf{P_n}$ where $n \geq 1$ and $\mathsf{P_1}$ is in Top, let $R = R_1 \wedge \ldots R_n$ be the corresponding disjunction of Top relations. Make a case distinction as follows: (i) if there is no Top formula $x_1^i S x_2^i$ in N, then add $x_1^i R x_2^i$; (ii) let there be a Top formula $x_1^i S x_2^i$ in N, where S is a disjunction of Top relations. Let $(R \cap S)$ denote the disjunction of those relations that appear in both R and S. If there is no such relation, return *inconsistent*. Otherwise, remove the formula $x_1^i S x_2^i$ from N and add the new formula $x_1^i (R \cap S) x_2^i$.
- For all pairs $(r_1, r_2) \in Var(N)$, for which there is no formula $r_1 R r_2$ in N, add the formula $r_1 * r_2$ where $*$ is universal relation given by the disjunction of all DIV9 relations conjuncted with the disjunction of all RCC8 relations.
- Check the satisfiability of the set N according to the transformation rules in Definition 2.27 and return the result.

In [88] Li proved that deciding the consistency of $\mathsf{DIV9} - \mathsf{RCC8}$ networks is decidable w.r.t. the conditions expressed on Proposition 5.34 on the considered set of topological relations. The proof that C is satisfiable iff N is satisfiable follows from the fact that N is given by the sum of two distinct constraint networks. The correspondence between finite predicate conjunctions and constraint networks follows from the technique described above both in the case of RCC8 and of RA relations in previous sections. □

Proposition* 5.37 *The concrete domain $\mathcal{S}_{\mathsf{DIV9}-\mathsf{RCC8}}$ is admissible.*

Proof We can verify that $\Phi_{\mathcal{S}_{\mathsf{DIV9}-\mathsf{RCC8}}}$ is closed under negation. From the JEPD property of both DIV9 and Top it follows that, given a set P of basic predicates in $\Phi_{\mathcal{S}_{\mathsf{DIV9}-\mathsf{RCC8}}}$, for any set of predicates $\{p_1, \ldots, p_n\} \subseteq P$, we have that $\overline{\mathsf{p_1}\text{-}\ldots\text{-}\mathsf{p_n}} = \mathsf{s_1}\text{-}\ldots\text{-}\mathsf{s_n}$ where $\{s_1\text{-}\ldots\text{-}s_n\}$ is defined as $P \setminus \{p_1\text{-}\ldots\text{-}p_n\}$. The predicate $\mathsf{s_1}\text{-}\ldots\text{-}\mathsf{s_n}$ is in $\Phi_{\mathcal{S}_{\mathsf{DIV9}-\mathsf{RCC8}}}$ and the negation of the disjunctive combination of all basic predicates in inconsistent-relation and vice versa.

The fact that $\Phi_{\mathcal{S}_{\mathsf{DIV9}-\mathsf{RCC8}}}$ is closed under negation together with the lemma 5.17 implies the admissibility of the concrete domain. □

In the following we investigate the properties of the corresponding constraint system.

5.7.2 The constraint system $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$

In [88] Li proved that the consistency of a topological and directional constraint network can be computed componentwise under some restrictions on the class of topological relations (see Proposition 5.34) and a refinement on disjunctive relations. The point is that the restriction of topological relations to a tractable subclass of a relation algebra is not possible on standard $\mathcal{DL}$, but as already seen for $\mathcal{S}_{\mathsf{DIV9-RCC8}}$ can be imposed on external concrete domains.

Definition* 5.38 ($\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$) *The constraint system* $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ *is defined on the two-dimensional space* $\mathbb{R}^2$*. The domain* $\Delta_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *is equal to the set of regular closed sets w.r.t. the topological space* $\left\langle \mathbb{R}^2, 2^{\mathbb{R}^2} \right\rangle$*. The set* $\Phi_{\mathcal{S}_{\mathsf{DIV9-RCC8}}}$ *contains* $8+9$ *basic predicates corresponding to the union of the set of* ***RCC-8*** *relations* $\mathsf{RCC8} = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}, \mathsf{TPPI}, \mathsf{NTPP}, \mathsf{NTPPI}\}$ *with the sub algebra* ***DIV9*** *of the Rectangle Algebra* $\mathsf{DIV9} = \{\mathsf{NW}, \mathsf{NC}, \mathsf{NE}, \mathsf{CW}, \mathsf{CC}, \mathsf{CE}, \mathsf{SW}, \mathsf{SC}, \mathsf{SE}\}$*. The constraint system is denoted by*

$$\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2} = \langle \mathsf{DIV9} \uplus \mathsf{Top}, \mathfrak{M}_{\mathbb{R}^2} \rangle$$

where $\mathfrak{M}_{\mathbb{R}^2} := \{N_{\mathbb{R}^2}\}$ *is the set of models for constraint networks on* $\mathsf{DIV9} - \mathsf{RCC8}$*, where* $N_{\mathbb{R}^2}$ *is defined by fixing a variable* $v_s \in \mathsf{Var}$ *for every region* $s \in \Delta_{\mathsf{DIV9-RCC8}_{\mathbb{R}^2}}$ *and setting*

$$N_{\mathbb{R}^2} := \{(v_s\ r\ v_t) | r \in \mathsf{DIV9} - \mathsf{RCC8}, s, t \in \Delta_{\mathsf{DIV9-RCC8}_{\mathbb{R}^2}} \textit{ and } (s,t) \in r^{\mathsf{DIV9-RCC8}_{\mathbb{R}^2}}\}$$

Lemma 5.39 (Li, [88]). *Let the topological component be labelled with relations of tractable subclasses* $\widehat{\mathcal{H}}_8$*,* $\mathcal{C}_8$*,* $\mathcal{Q}_8$*, and let the directional component be defined by* $\mathcal{H}_3 \otimes \mathcal{H}_3$ *relations. Then deciding the satisfiability of the joint network is of cubic complexity.*

In the following we will see that the patchwork and the compactness properties descend from the same properties of RCC-8 and RA.

Lemma* 5.40 *The constraint system* $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ *enjoys the Patchwork Property.*

Proof In order to prove the patchwork property we need first to prove that there exists a sort of distributive property between the operation of "patching" compatible networks, denoted by $\cup$ (Definition 5.5 of Patchwork Property, and the operation of joining distinct spatial networks, denoted by $\uplus$ (Definition 2.26 of Combined Constraint Network). Let us consider a constraint network N as a couple $(V_N, \{(x \mathsf{\ r\ } y)|x, y \in V_N\})$ where V_N is a set of variables and $\{(x \mathsf{\ r\ } y)|x, y \in V_N\}$ is a set of constraint defined over V_N.
According to Lutz and Miličić notion of "patching" we consider the union of two constraint networks N_1 and N_2 as

$$N_1 \cup N_2 = ((V_{N_1} \cup V_{N_2}), \{(x \mathsf{\ r\ } y) \in N_1 \mid x, y \in V_{N_1}\} \cup \{(x \mathsf{\ r\ } y) \in N_2 \mid x, y \in V_{N_2}\}).$$

According to Li's notion of Combined Constraint Network we denote a joint network as

$$\Theta \uplus \Delta = (V, \{(x \mathsf{\ r\ } y) \in \Theta \mid x, y \in V\} \cup \{(x \mathsf{\ r\ } y) \in \Delta \mid x, y \in V\}).$$

Now we prove that given two joint networks $(\Theta_1 \uplus \Delta_1)$ and $(\Theta_2 \uplus \Delta_2)$

$$(\Theta_1 \uplus \Delta_1) \cup (\Theta_2 \uplus \Delta_2) \equiv (\Theta_1 \cup \Theta_2) \uplus (\Delta_1 \cup \Delta_2)$$

We can see that by definition of "patching"

$$\begin{aligned}(\Theta_1 \uplus \Delta_1) \cup (\Theta_2 \uplus \Delta_2) = ((V_1 \cup V_2), \{(x \mathsf{\ r\ } y) \in (\Theta_1 \uplus \Delta_1) \mid x, y \in V_1\} \cup \\ \{(x \mathsf{\ r\ } y) \in (\Theta_2 \uplus \Delta_2) \mid x, y \in V_2\})\end{aligned}$$

This can be rewritten by definition of joint network as

$$\begin{aligned}((V_1 \cup V_2), \{(x \mathsf{\ r\ } y) \in \Theta_1 \mid x, y \in V_1\} \cup \\ \{(x \mathsf{\ r\ } y) \in \Delta_1 \mid x, y \in V_1\} \cup \\ \{(x \mathsf{\ r\ } y) \in \Theta_2 \mid x, y \in V_2\} \cup \\ \{(x \mathsf{\ r\ } y) \in \Delta_2 \mid x, y \in V_2\})\end{aligned}$$

A reordering of the components of the union w.r.t. the definition of "patching" allows the following rewriting which corresponds to the joining of two networks

$$\begin{aligned}&((V_1 \cup V_2), \{(x \mathsf{\ r\ } y) \in (\Theta_1 \cup \Theta_2 \mid x, y \in (V_1 \cup V_2)\} \cup \\ &\qquad \{(x \mathsf{\ r\ } y) \in (\Delta_1 \cup \Delta_2 \mid x, y \in (V_1 \cup V_2)\}\ ) = (\Theta_1 \cup \Theta_2) \uplus (\Delta_1 \cup \Delta_2).\end{aligned}$$

In order to prove the patchwork property for DIC9 − RCC8, we must show that given two constraint networks $(\Theta_1 \uplus \Delta_1)$ and $(\Theta_2 \uplus \Delta_2)$, if they are compatible and both satisfiable then $(\Theta_1 \uplus \Delta_1) \cup (\Theta_2 \uplus \Delta_2)$ is satisfiable as well. According to Theorem 2.29 $(\Theta_1 \uplus \Delta_1)$ and $(\Theta_2 \uplus \Delta_2)$ are satisfiable if and only if $\overline{\Theta}_1$, $\overline{\Delta}_1$, $\overline{\Theta}_2$ and $\overline{\Delta}_2$ are satisfiable. We already saw that constraint networks built either with RCC-8 or RA relations enjoy the patchwork property. From this follows that

$$\begin{aligned}\overline{\Theta}_1, \overline{\Theta}_2 \text{ satisfiable } &\Rightarrow \overline{\Theta}_1 \cup \overline{\Theta}_2 \text{ satisfiable}\\ \overline{\Delta}_1, \overline{\Delta}_2 \text{ satisfiable } &\Rightarrow \overline{\Delta}_1 \cup \overline{\Delta}_2 \text{ satisfiable}\end{aligned}$$

By definition of "patching" and the equality of common parts descends immediately that $\overline{\Theta}_1 \cup \overline{\Theta}_2 \equiv \overline{\Theta_1 \cup \Theta_2}$ and from Lemma 2.28 it follows that $\overline{\Theta_1 \cup \Theta_2}, \overline{\Delta_1 \cup \Delta_2}$ are satisfiable iff $(\Theta_1 \cup \Theta_2) \uplus (\Delta_1 \cup \Delta_2)$ is satisfiable and as already proved $(\Theta_1 \cup \Theta_2) \uplus (\Delta_1 \cup \Delta_2) \equiv (\Theta_1 \uplus \Delta_1) \cup (\Theta_2 \uplus \Delta_2)$, hence the "patching" of two satisfiable networks is still satisfiable. This proves that the constraint system $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ enjoys the Patchwork Property. □

Lemma* 5.41 *The constraint system* $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ *enjoys the Compactness Property.*

Proof The compactness property states that a constraint network defined over as infinite set of variables $N|_{V_\infty}$ is satisfiable iff each finite subnetwork $N|_V$ with $V \subset V_\infty$ is satisfiable. We must show that given an infinite joint network $\Theta|_{V_\infty} \uplus \Delta|_{V_\infty}$ it is satisfiable iff each $\Theta|_V \uplus \Delta|_V$ is satisfiable. From Lemma 2.28 it follows that a finite network $\Theta|_V \uplus \Delta|_V$ is satisfiable iff $\overline{\Theta}|_V$, $\overline{\Delta}|_V$ are satisfiable. From the compactness property of both RCC-8 and RA it follows that for each finite $V \subset V_\infty$

$$\begin{aligned}\overline{\Theta}|_V \text{ satisfiable iff } \overline{\Theta}|_{V_\infty} \text{ satisfiable}\\ \overline{\Delta}|_V \text{ satisfiable iff } \overline{\Delta}|_{V_\infty} \text{ satisfiable}\end{aligned}$$

For the same Lemma 2.28 the satisfiability of both $\overline{\Theta}|_{V_\infty}$ and $\overline{\Delta}|_{V_\infty}$ implies and is implied by the satisfiability of the joint network $\Theta|_{V_\infty} \uplus \Delta|_{V_\infty}$. This proves the lemma. □

Theorem* 5.42 *The constraint system* $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ *is ω-admissible.*

Proof The ω-admissibility of $\mathsf{DIV9} - \mathsf{RCC8}_{\mathbb{R}^2}$ follows immediately from the decidability of the consistency problem for joint networks and from Lemma 5.40 and Lemma 5.41. □

6

Tradeoff Analysis of $\mathcal{DL}$s for Spatial Reasoning

6.1 Introduction

In the present Chapter we will present computational results about reasoning with $\mathcal{DL}$s extended by spatial operators. In previous chapters we introduced two different techniques for extending terminological languages in order to achieve the capability of qualitative spatial reasoning. The first technique to be investigated is based on the definition of a logic with composition-based role axioms that embeds the composition table of a QSRR formalism into a set of axioms. This family of languages denoted by $\mathcal{ALCI}_{\mathsf{C}}$ descends from the attempt made by Wessel in [140] and [139] to define a terminological language for spatial reasoning. The other technique considered in the previous chapter is given by the definition of spatial concrete domains based on qualitative spatial reasoning formalisms. In the rest of the Chapter we will present a systematic analysis of hybridizations between QSRR formalisms and $\mathcal{DL}$ with both techniques taking into account the formal properties of the spatial environment.

6.2 Hybridizations via fixed role-boxes

Wessel in [137] investigates concept satisfiability of $\mathcal{ALC}$ extended with arbitrary role boxes containing composition role axioms without any requirement of disjunction on role interpretation. This logic called $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$ turned out to be undecidable. The same logic added with role disjointness and denoted $\mathcal{ALC}_{\mathcal{RA}}$ turned out to be undecidable as well. Undecidability arises in the general case and Wessel tries to understand what happens if only certain classes of role boxes are considered, especially the role boxes which are obtained from translating the RCC composition tables. In Chapter 4 we define the generalization of Wessel's idea of fixed role box and proposed some hybridizations with constraint languages, that when embedded into the description logic $\mathcal{ALCI}$ have well-investigated counterparts in modal logics. In the following we will present the main computational results about description logics with spatial roles. We will also introduce some observations on the expressive power of these logics and other considerations on the possible restrictions to gain decidability from undecidable languages.

6.2.1 Computational properties of $\mathcal{ALCI}_{\mathsf{C}}$ logics

The main property of the family of formalisms denoted generically $\mathcal{ALCI}_{\mathsf{C}}$ is that each of these language relies on a particular spatial constraint language C (see the formal Definitions 4.48 of constraint language and 4.49 of the syntax of the embedded terminological language) that corresponds to a finite relation algebra whose relations enjoy the JEPD property. In other words the set of relations defined on a spatial universe of discourse described exhaustively and with no ambiguity all possible situations. We consider in particular qualitative spatial constraint languages such that the problem of consistency of constraint networks defined over the same set of relations is decidable. The decidability of the QSRR formalism does not obviously imply the decidability of the logical language which provides the possibility to quantify variables by existential and universal quantifications. The case of $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ are examples of the the undecidability of the terminological language in spite of the decidability of the corresponding QSRR formalism. As pointed out in Chapter 4 there are restrictions on the semantics of the EQ relation such that it is possible to impose a finite model property. This property together with a tableau procedure that provides a model or a counterexample implies a decidability result for the restricted language.

$\mathcal{ALCI}_{\mathcal{RCC}8}$ and $\mathcal{BRCC}$

The first spatial terminological language defined via fixed role box is $\mathcal{ALCI}_{\mathcal{RCC}}$, which actually is a family of description logics built over different versions of the Region Connection Calculus. Wessel in [140] proves the decidability of coarser versions of the terminological language and leaves as open the problem for the languages $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$. In Section 4.1 we recall the formal definition of the syntax and the semantics of the family of languages, while in Section 4.2 we provide a mapping with multi-modal logic proved by Lutz and Wolter in [101] to be undecidable w.r.t. the usual topology on $\mathbb{R}^2$ and the set of regular closed regions. We prove that the undecidability result holds for both logics defined over $\mathsf{RCC\text{-}8}$ and $\mathsf{RCC\text{-}5}$ frames. Wessel in [140] notes that the strong EQ semantics (i.e. the EQ relation corresponds to the identity function) allows one to define nominals and with nominal to impose a maximal cardinality on admissible models. Such a requirement is equivalent to a finite model property and guarantees the termination of the tableau procedure defined in (Section 4.1 page 57). We called this "para-decidability" because the tableau under the maximal cardinality restriction checks if a concept admits a model whose cardinality is under the limit or if there exists a counterexample which involves a number of individuals less that the limit. The maximal number of individuals in a model can be defined to be very high in order to manage complex concept definitions.

In a GIS environment the availability of nominals is very useful for the definition of concepts as "Italian lake" to describe lakes that are in "part" relation with Italy. We can consider the atomic concept of "lake" and relate it with the nominal "Italy" that must be always interpreted as a singleton (see the example in Figure 6.1). Without nominals we can start from the generic concept of "nation" in order to define the concept of "national lake" in the sense of "lake belonging to a single state". Exploiting the Abox features we can assert Lake Garda to belong to Italy and the system will deduce that Lake Garda is a national lake. This is obviously different from having in the TBox the definition of "Italian lake" that can be used to define other concepts. In the example in Figure 6.1 we

$$\begin{array}{ll}\mathsf{National\text{-}Lake} & \equiv \mathsf{Lake} \sqcap \exists \mathsf{PP.Nation} \\ \mathsf{Italian\text{-}Lake} & \equiv \mathsf{Lake} \sqcap \exists \mathsf{PP.Nation} \rightarrow [\mathsf{Italy}]\end{array}$$

Fig. 6.1. An example of concept definitions with or without nominals: the availability of nominals allows the definition of concepts that are not definable in languages which provide only a weak semantics for the EQ relation.

denote a nominal with square brackets [Nom]. Let us consider the example described in Section 2.2 about the Lessinia Park in the north of Italy. As already pointed out, RCC8 allows one to describe relations between regular regions (see Figure 6.2). The language $\mathcal{ALCI}_{\mathcal{RCC}8}$ does not allows us to define for instance the park as a union of municipalities or to refer to the Veronese part of the park as the intersection between the park and the district of Verona as "[Lessinia-Park] $\sqcap$ [Verona]". The semantics defined for description logics interpreted this operation as the intersection between the sets of individuals that are different singletons whose intersection is the emptyset. The problem is that union

$$\begin{array}{ll}\mathsf{Verona\text{-}municipality} & \equiv \mathsf{municipality} \sqcap \exists \mathsf{PP}.[\mathsf{Verona}] \\ \mathsf{Vicenza\text{-}municipality} & \equiv \mathsf{municipality} \sqcap \exists \mathsf{PP}.[\mathsf{Vicenza}] \\ \mathsf{Lessinia\text{-}Park\text{-}in\text{-}Verona} & \equiv \exists \mathsf{PP}.[\mathsf{Lessinia\text{-}Park}] \sqcap \exists \mathsf{PP}.[\mathsf{Verona}] \\ \mathsf{Vicenza\text{-}municipality}(\mathsf{Crespadoro}) & \\ \mathsf{Vicenza\text{-}municipality}(\mathsf{Altissimo}) & \\ \mathsf{Verona\text{-}municipality}(\mathsf{Erbezzo}) & \\ \mathsf{Verona\text{-}municipality}(\mathsf{Fumane}) & \\ \mathsf{PO}([\mathsf{Lessinia\text{-}Park}]^{\mathcal{I}}, [\mathsf{Verona}]^{\mathcal{I}}) & \\ \mathsf{EC}([\mathsf{Vicenza}]^{\mathcal{I}}, [\mathsf{Verona}]^{\mathcal{I}}) & \\ \mathsf{EC}(\mathsf{Crespadoro}, \mathsf{Altissimo}) & \\ \mathsf{PO}([\mathsf{Lessinia\text{-}Park}]^{\mathcal{I}}, \mathsf{Crespadoro} & \\ \mathsf{PO}([\mathsf{Lessinia\text{-}Park}]^{\mathcal{I}}, \mathsf{Altissimo} & \\ \mathsf{PO}([\mathsf{Lessinia\text{-}Park}]^{\mathcal{I}}, \mathsf{Erbezzo} & \\ \mathsf{PO}([\mathsf{Lessinia\text{-}Park}]^{\mathcal{I}}, \mathsf{Fumane} & \\ \ldots & \end{array}$$

Fig. 6.2. Concept definitions and individual assertions for describing in a simplified way the region of Lessinia Park in the North of Italy.

and intersection among regions can not be defined as usual disjunction and conjunction operators among definitions of concepts. The presence of "functional operators" in the BRCC formalism tells us that the Boolean Region Connection Calculus cannot be seen as a constraint language and so cannot be embedded into a $\mathcal{ALCI}_{\mathsf{C}}$ logics with role box with composition based axioms. Despite this limit in the expressive power of the logics $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, we have decidability under the adoption of the strong EQ semantics to ensure maximal cardinality of models. The last aspect we must consider is the complexity lower bounds identified by Wessel of PSPACE-hardness for $\mathcal{ALCI}_{\mathcal{RCC}5}$ (Theorem 4.9) and EXPTIME-hardness for $\mathcal{ALCI}_{\mathcal{RCC}8}$ (Theorem 4.10).

$\mathcal{ALCI}_{\mathsf{C}}$

In Section 4.3 we presented the generalization of the definition of description logics with fixed and finite role box. We require the definition of a logic $\mathcal{ALCI}_{\mathsf{C}}$ to be given under the condition to embed a constraint language built on a finite relation algebra. The first limit for the embedding is the impossibility of hybridization for the Boolean RCC, since it is not a "pure" constraint language. A more general limit is given by the coding of composition tables into sets of composition based role axioms. Constraint languages as the PDR and PDR^+ approaches by Skiadopoulos and Koubarakis do not provide composition tables. We must also consider that the high number of relations is rather discouraging in the case of a formalization of the corresponding composition rules. The main problem of this generalized approach is that to check decidability it is important to have an already investigated modal counterpart as in the languages proposed in Section 4.3. The first language

$$
\begin{aligned}
\mathsf{City\text{-}East\text{-}Greenwich} &\equiv \mathsf{City} \sqcap \exists\{\mathsf{E}, \mathsf{NE}, \mathsf{SE}\}.[\mathsf{GreenwichObservatory}] \\
\mathsf{City\text{-}West\text{-}Greenwich} &\equiv \mathsf{City} \sqcap \exists\{\mathsf{W}, \mathsf{NW}, \mathsf{SW}\}.[\mathsf{GreenwichObservatory}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{SE}([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, \mathsf{Paris}) \\
&\mathsf{SE}(\mathsf{Paris}, \mathsf{Rome}) \\
&\mathsf{NE}(\mathsf{Rome}, \mathsf{Wien})) \\
&\mathsf{N}(\mathsf{Rome}, \mathsf{Venice}) \\
&\mathsf{SW}([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, \mathsf{Nantes}) \\
&\mathsf{SW}(\mathsf{Nantes}, \mathsf{Madrid})
\end{aligned}
$$

Fig. 6.3. An example of concept definitions and role assertions with directional information coded w.r.t. the CDC relation Algebra. The composition-based axioms in the Rbox allows the system to deduce non-explicit information.

we investigated is $\mathcal{ALCI}_{\mathsf{CDC}}$, which is the hybridization with the Cardinal Direction Calculus by Ligozat [89]. The main peculiarity of this language is the correspondence with a doubled multi-modal temporal logic defined over two real lines. We formulated for this language a decidability result (see Theorem 4.67) for the problem of concept satisfiability checking. In Figure 6.3 we propose an example of a $\mathcal{ALCI}_{\mathsf{CDC}}$ Knowledge Base with hidden information. Figure 6.4 show explicit information given in the Knowledge base in Figure 6.3 denoted by black arrows with some non-explicit relations (red arrows) deducible from the given assertions and definitions. From explicit and hidden information it is possible to deduce which individuals (towns in this case) belongs to the concept City-East-Greenwich and which to City-West-Greenwich.

Two other hybridizations of $\mathcal{ALCI}_{\mathsf{C}}$ languages were considered in this thesis (Section 4.3): $\mathcal{ALCI}_{\mathsf{RA}}$ and $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$. Both these languages, which are respectively the hybridization with the Rectangle Algebra ([67], [108], [15]) and with the combined directional and topological framework proposed by Li [88], turned out to be undecidable borrowing the computational results from their modal counterparts. Since the Rectangle Algebra is a doubled Interval Algebra, the proof of undecidability for a formal language defined w.r.t. this class of frames relies on the undecidability of Halpern-Shoham interval modal logic [72]. The undecidability of the combined framework $\mathsf{DIV9} - \mathsf{RCC8}$ arises from the undecidability of the RCC component of the language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$. For

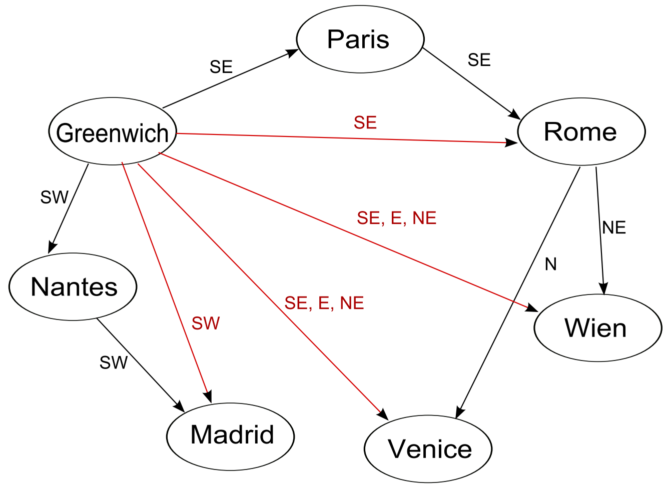

Fig. 6.4. Cardinal relations between spatial objects: black arrows denote explicit information and red arrows deduced information w.r.t. the Greenwich Observatory.

both this languages it is possible to adopt the strong EQ semantics and to enforce via nominals the finite model property in order to gain decidability. A problem with the strong semantics is related to approximation of regions by minimum bounding box: the equality between two rectangles is interpreted that they are the same rectangle. A congruence semantics for the EQ relation seems to be more appropriate to describe the situation of regions bounded by equivalent bounding boxes. In the following we provide an example to show the limit in expressivity derived from a strong EQ semantics. In Figure 6.5 we show the expressive power of the language $\mathcal{ALCI}_{\mathsf{RA}}$ providing an example of knowledge base. As in the case of Cardinal Direction Relations, even in this case the system can deduce hidden information as which are the spatial objects to the "left" of Greenwich and which are to the "right". It is worth to make clear how heavy is the limit set by the strong

$$\mathsf{European\text{-}City\text{-}East\text{-}Greenwich} \equiv \mathsf{European\text{-}City} \sqcap \exists(*, \mathsf{A}).[\mathsf{GreenwichObservatory}]$$
$$\mathsf{European\text{-}City\text{-}West\text{-}Greenwich} \equiv \mathsf{European\text{-}City} \sqcap \exists(*, \mathsf{B}).[\mathsf{GreenwichObservatory}]$$

$(\mathsf{A},\mathsf{B})([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, \mathsf{Paris})$
$(\mathsf{A},\mathsf{B})(\mathsf{Paris}, \mathsf{Rome})$
$(\mathsf{A},\mathsf{A})(\mathsf{Rome}, \mathsf{Wien}))$
$(\mathsf{Oi},\mathsf{A})(\mathsf{Rome}, \mathsf{Venice})$
$(\mathsf{B},\mathsf{B})([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, \mathsf{Nantes})$
$(\mathsf{B},\mathsf{B})(\mathsf{Nantes}, \mathsf{Madrid})$

Fig. 6.5. An example of concept definitions and role assertions with information coded w.r.t. the Rectangle Algebra. The composition-based axioms in the Rbox allows the system to deduce non-explicit information.

semantics for the EQ relation. The Rectangle Algebra is a set of relations between approx-

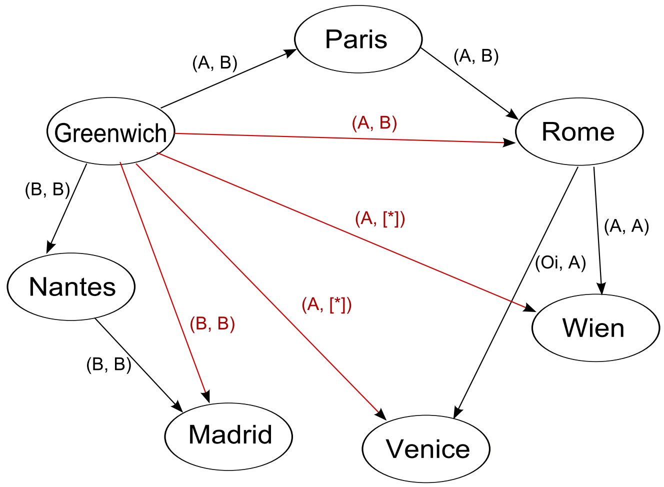

Fig. 6.6. RA relations between spatial objects: black arrows denote explicit information and red arrows deduced information w.r.t. the Greenwich Observatory.

imations of regions via minimum bounding box (MBB). Unless of a restriction of the set of regions to the set of rectangle whose sides are parallel to the axes, this approximation causes a flattening of the knowledge base, collapsing distinct regions to a common MBB. The approximative nature of the Rectangle Algebra leads to a congruence between regions bounded by the same rectangle. The strong semantics means that we cannot define two distinct regions bounded by the same rectangle. This allows one to define poor knowledge bases, given that there are relevant cases of distinct regions with a common MBB. We can consider for instance the state of Italy and the Italian peninsula which includes the Vatican state and the state of San Marino.

Analogous considerations can be done for the language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$, which has a component based on the Rectangle Algebra. One main aspect of $\mathsf{DIV9} - \mathsf{RCC8}$ is that it provides a single EQ relation taken from the set of $\mathsf{RCC\text{-}8}$ relations. This means that it makes sense to adopt a strong EQ semantics, because this calculus is based on the real shape of regions and not on their approximation via MBB. This language allows the definition of knowledge bases with both topological and directional information. In Figure 6.7 we present an example of combined information.

An interesting limit of the reasoning capability of this language is given by the absence in the role box of composition-based axioms that state a connection between the topological and the directional inference. Let us consider the case defined in Figure 6.7: "Rome is part of Italy" and "Italy is SE with respect to Greenwich". This language does not allow to deduce that Rome is SE Greenwich as well, Because there is no interaction between the two distinct sets of roles. The strong EQ semantics, allows a tableau procedure to check if a concept is satisfiable, but the lack of combined axioms implies the impossibility of cross-deductions. We leave the definition of a combined composition table as further work. In the following section we will introduce a generalized tableau for any $\mathcal{ALCI}_{\mathsf{C}}$ with strong EQ semantics and maximal cardinality.

$$
\begin{aligned}
&\mathsf{European\text{-}City} \equiv \mathsf{city} \sqcap \exists\{\mathsf{TPP}, \mathsf{NTPP}\}.\mathsf{Europe}\\
&\mathsf{Italian\text{-}City} \equiv \mathsf{city} \sqcap \exists\{\mathsf{TPP}, \mathsf{NTPP}\}.\mathsf{State} \longrightarrow [\mathsf{Italy}]\\
&\mathsf{Spanish\text{-}City} \equiv \mathsf{city} \sqcap \exists\{\mathsf{TPP}, \mathsf{NTPP}\}.\mathsf{State} \longrightarrow [\mathsf{Spain}]\\
&\mathsf{German\text{-}City} \equiv \mathsf{city} \sqcap \exists\{\mathsf{TPP}, \mathsf{NTPP}\}.\mathsf{State} \longrightarrow [\mathsf{Germany}]\\
&\mathsf{European\text{-}City\text{-}East\text{-}Greenwich} \equiv\\
&\qquad\qquad \mathsf{European\text{-}City} \sqcap \exists\{\mathsf{NE}, \mathsf{CE}, \mathsf{SE}\}.[\mathsf{GreenwichObservatory}]\\
&\mathsf{European\text{-}City\text{-}West\text{-}Greenwich} \equiv\\
&\qquad\qquad \mathsf{European\text{-}City} \sqcap \exists\{\mathsf{NW}, \mathsf{CW}, \mathsf{SW}\}.[\mathsf{GreenwichObservatory}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{Italian\text{-}City}(\mathsf{Rome})\\
&\mathsf{Spanish\text{-}City}(\mathsf{Madrid})\\
&\mathsf{German\text{-}City}(\mathsf{Berlin})\\
&\mathsf{DC}([\mathsf{Italy}]^{\mathcal{I}}, [\mathsf{Spain}]^{\mathcal{I}})\\
&\mathsf{DC}([\mathsf{Italy}]^{\mathcal{I}}, [\mathsf{Germany}]^{\mathcal{I}})\\
&\mathsf{DC}([\mathsf{Germany}]^{\mathcal{I}}, [\mathsf{Spain}]^{\mathcal{I}})
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{SE}([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, [\mathsf{Italy}]^{\mathcal{I}})\\
&\mathsf{SW}([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, [\mathsf{Spain}]^{\mathcal{I}})\\
&\mathsf{CE}([\mathsf{GreenwichObservatory}]^{\mathcal{I}}, [\mathsf{Germany}]^{\mathcal{I}})\\
&\mathsf{NC}([\mathsf{Italy}]^{\mathcal{I}}, [\mathsf{Germany}]^{\mathcal{I}})
\end{aligned}
$$

Fig. 6.7. An example of concept definitions and role assertions with information coded w.r.t. DIV9 – RCC8. The composition-based axioms in the RBox allows the system to deduce non-explicit information.

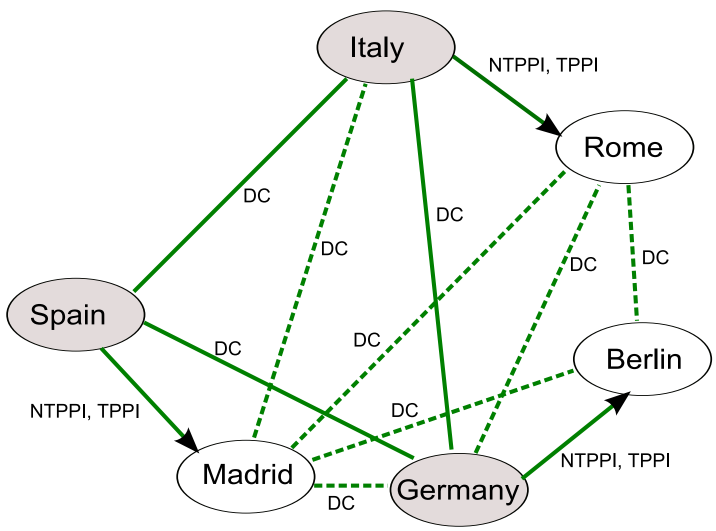

Fig. 6.8. RCC-8 relations between spatial objects: normal green arrows denote explicit information while broken arrows denote deduced information.

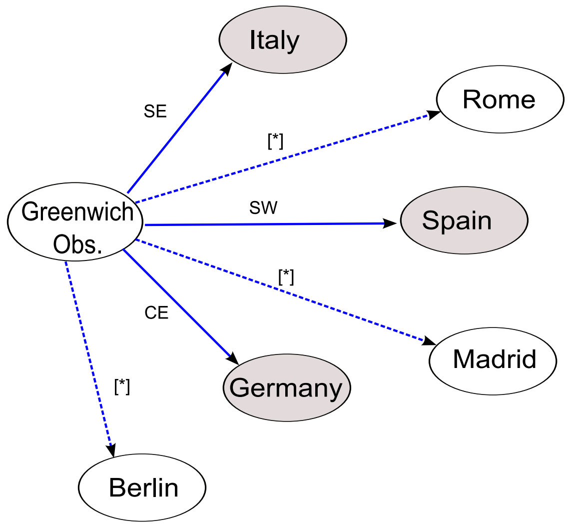

Fig. 6.9. DIV9 relations between spatial objects: normal blue arrows denote explicit information while broken blue arrows deduced information w.r.t. the Greenwich Observatory.

6.2.2 Generalizing Paraconsistency for undecidable $\mathcal{ALCI}_{\mathsf{C}}$

Description logics are a family of formalisms that exhibits usually a bad computational behavior, given the complexity of the reasoning task. As proved by Wessel in [137], extensions like arbitrary role boxes implies the undecidability even of base description logics. A restriction of the role box to a finite set of role axioms does not imply decidability of the language, as proved by the hybridization with RCC-8 , RCC-5 , RA and DIV9 − RCC8. In section 4.3 we provide a generalization of Wessel's idea of finite role box with the definition of a general hybrid language $\mathcal{ALCI}_{\mathsf{C}}$ based on a generic *constraint language*. In this section we propose a general tableau for the para-decidability of $\mathcal{ALCI}_{\mathsf{C}}$ languages with a strong EQ semantics and a limit on the number of individuals in a model. With a strong EQ semantics we can enforce "maximal cardinality reasoning" with nominals, ruling out all models with more than a given number of individuals. This technique states the para-decidability of a $\mathcal{ALCI}_{\mathsf{C}}$-concept checking if a concept admits model of cardinality smaller than the bound or if there exists a counterexample with the same limit on its cardinality.

Following the same approach of Baader and Nutt in [10], we define a tableau-based satisfiability algorithm for $\mathcal{ALCI}_{\mathsf{C}}$ to test concept satisfiability. In order to guarantee the termination for each rule in table 6.1 we must require the decidability of the constraint language C.

Definition* 6.1 *Given a decidable* constraint language C*, we denote by* Ω_{C} *a generic procedure to check consistency of constraint networks built with* C*.*

In the following we call Ω_{C} the *oracle*. Let C_0 be a $\mathcal{ALCI}_{\mathsf{C}}$ concept in *normal form* (i.e. negation occurs only in front of concept names and all the shorthands have already been expanded in the corresponding disjunctive formulae). The algorithm starts with the ABox $\mathcal{A}_0 = \{C_0(x_0)\}$ and applies the transformation rules (see Table 4.1) to the ABox until no

$\sqcap$-Rule:	
Condition:	$\mathcal{A}$ contains $(C_1 \sqcap C_2)(x)$, but it does not contain both $C_1(x)$ and $C_2(x)$
Action:	$\mathcal{A}' = \mathcal{A} \cup \{C_1(x), C_2(x)\}$
$\sqcup$-Rule:	
Condition:	$\mathcal{A}$ contains $(C_1 \sqcup C_2)(x)$, but it does not contain either $C_1(x)$ nor $C_2(x)$
Action:	$\mathcal{A}' = \mathcal{A} \cup \{C_1(x)\}$, $\mathcal{A}'' = \mathcal{A} \cup \{C_2(x)\}$
$\forall$-Rule:	
Condition:	$\mathcal{A}$ contains $(\forall R.C)(x)$ and $R(x, y)$, but it does not contain $C(y)$
Action:	$\mathcal{A}' = \mathcal{A} \cup \{C(y)\}$.
$\exists$-Rule:	
Condition:	$\mathcal{A}$ contains $(\exists R.C)(x)$ with $R \in \mathcal{N}_{\mathcal{R}_\mathcal{S}}$, but there is no individual name z such that $C(z)$ and $R(x, z)$ are in $\mathcal{A}$
Action:	$\mathcal{A}^{\mathrm{tmp}} = \mathcal{A} \cup \{R(x, y), C(y)\}$ where y is an individual name not occurring in $\mathcal{A}$. Given the set of topological role assertions of $\mathcal{A}^{\mathrm{tmp}}$, $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})$, find the deductive closure of the constraint network $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*$. Find all consistent scenarios of the deductive closure: $\{SC_1((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*), \ldots, SC_n((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*)\}$. Generate all the new ABoxes $\mathcal{A}'_i = (\mathcal{A}^{\mathrm{tmp}} \backslash \mathcal{A}\mathrm{Sp}) \cup SC_i((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*)$.

Table 6.1. Transformation rules of the satisfiability algorithm for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$.

more rules apply. If at least one ABox obtained does not contain any contradiction then $\mathcal{A}_0$ is consistent (and so C_0 is satisfiable).
When dealing with a spatial role R, the introduction of a new individual y implies a set of implicit relationships: one relationship for every spatial object already in the ABox. Moreover it could mean a modification of existing constraints between old spatial objects. For this reason the rule for the existential quantification:

- defines a temporary ABox $\mathcal{A}^{\mathrm{tmp}}$ by introducing a new individual y such that $R(x, y)$ and $C(y)$;
- extracts the spatial constraint network given by all spatial role assertions in the temporal ABox: $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})$;
- calculates the deductive closure of the constraint network $(\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*$;
- finds, calling the oracle Ω_{C}, all consistent scenarios of the deductive closure, in other words it checks all possible configurations of spatial relations among the objects according to the deductive closure: $\{SC_1((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*), \ldots, SC_n((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*)\}$;
- generates a new ABox for each consistent scenario replacing the constraint network given by old spatial role assertions: $\mathcal{A}'_i = (\mathcal{A}^{\mathrm{tmp}} \backslash \mathcal{A}\mathrm{Sp}) \cup SC_i((\mathcal{A}^{\mathrm{tmp}}\mathrm{Sp})^*)$.

As proved in Lemma 4.7 these transformation rules preserve ABox satisfiability. It is worth recalling the fact that this tableau provide a model for the concept or a counterexample. According to Proposition 4.6 and choosing the *strong EQ-semantics* which provides nominals it is possible to enforce a maximal cardinality on models to guarantee the decidability of the language. The finite model property combined with a procedure that provides a model for finitely satisfiable concept expressions and counterexample for unsatisfiable concept expressions implies the satisfiability of the formalism. This proves the following

Theorem* 6.2 (Para-decidability for $\mathcal{ALCI}_{\mathsf{C}}$) *A $\mathcal{DL}$ $\mathcal{ALCI}_{\mathsf{C}}$ is satisfiable under the* strong EQ-semantics *with a condition of finite models with maximal cardinality.*

In the following we will present the analysis of qualitative spatial reasoning with description logics with fixed and finite role boxes.

6.2.3 Tradeoff analysis for $\mathcal{ALCI}_{\mathsf{C}}$ logics

In this section we present a graphic summary of computational properties of hybridizations between $\mathcal{ALCI}_{\mathsf{C}}$ description languages and QSRR formalisms. In Figure 6.10 we show spatial elements provided by each QSRR formalism investigated in Chapter 4.

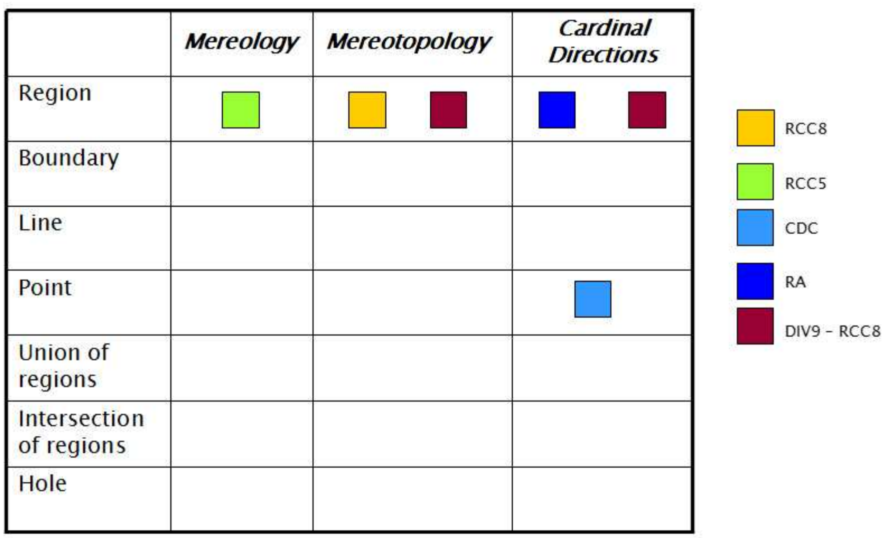

Fig. 6.10. Position of $\mathcal{ALCI}_{\mathsf{C}}$ hybridizations with QSRR formalisms respect to expressivity.

Figure 6.11 provides the summary of computational results given in Chapter 4. Our contribution is denoted by *. The only decidable hybridization with the technique of fixed role box turned out to be $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$, which is defined on Ligozat's Cardinal Direction Calculus. Nevertheless hybridizations with the Region Connection Calculus and the combined $\mathsf{DIV9} - \mathsf{RCC8}$ are para-decidable under the strong EQ semantics and the assumption of maximal cardinality for admissible models. The same restrictions over the hybridization with the Rectangle Algebra in order to gain para-decidability, lead to a very inexpressive language. We leave as an open problem if there exist extensions of the restricted hybrid language to overcome the limit of expressivity.

	Decidability	Strong EQ Semantics	Paradecidability*	Lower Bound	Upper Bound
$ALCI_{RCC8}$	Undec*	Expressive	Yes*	ExpTime-Hard	
$ALCI_{RCC5}$	Undec*	Expressive	Yes*	PSpace-Hard	
$ALCI_{CDC}$*	Dec*	Not needed	Yes*	N.A.	PSpace-complete*
$ALCI_{RA}$*	Undec*	Inexpressive	Yes*	N.A.	
$ALCI_{DIV9-RCC8}$*	Undec*	Expressive	Yes*	N.A.	

Fig. 6.11. Computational properties of $\mathcal{ALCI}_{\mathrm{C}}$ hybridizations with QSRR formalisms. We denote with * our contribution.

6.3 Hybridizations via Concrete Domains

Concrete domains are a way to extend a description logics using an external domain to increase the expressive power for a terminological language. The main problem with terminological reasoning is given by the high computational complexity of reasoning tasks. The complexity of reasoning with concrete domain has been investigated by Lutz in [93] and [95] stating that the complexity of reasoning with $\mathcal{ALC}(\mathcal{D})$ clearly depends on the complexity of the $\mathcal{D}$-satisfiability problem. It is shown in Chapter 5 how the $\mathcal{D}$-satisfiability problem is equivalent for the considered concrete domains to the consistency problem of corresponding constraint networks. For the aim of the thesis we consider three possible hybridizations based on the concrete domain technique:

- with the basic logic $\mathcal{ALC}(\mathcal{D})$,
- with the more expressive $\mathcal{ALCRP}(\mathcal{D})$ with concrete domain role constructors,
- with the basic logic $\mathcal{ALC}(\mathcal{C})$, defined with special concrete domains called constraint systems, with general TBoxes.

In the following sections we provide an analysis of hybridizations between logics extended by concrete domains and the spatial concrete domains investigated in Chapter [96]. We will also consider the conditions required in order to gain the decidability of spatial terminological languages and provide summary tables of the tradeoff between complexity and expressivity for all the considered languages.

6.3.1 Spatial Reasoning with $\mathcal{ALC}(\mathcal{D})$

Computational properties

$\mathcal{ALC}(\mathcal{D})$ is a family of logics based on the idea of concrete domain. A concrete domain is an interesting technique to increase the expressive power of a language allowing terminological descriptions that call, via "concrete features", relations and operators defined outside the language. In Chapter 5 we provided formal definitions of spatial concrete domains based on decidable formalisms for qualitative spatial reasoning. In Figure 6.12 we

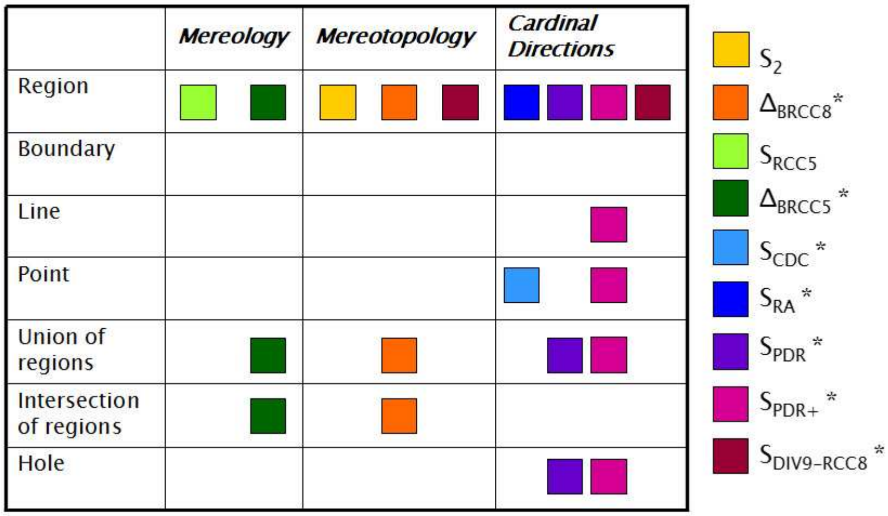

Fig. 6.12. Admissible spatial concrete domains for $\mathcal{ALC}(\mathcal{D})$ and $\mathcal{ALCRP}(\mathcal{D})$ hybridizations. We denote with * our contribution.

present a schema with the formalisms for qualitative spatial reasoning defined as concrete domains, used in the following to extend both $\mathcal{ALC}(\mathcal{D})$ and $\mathcal{ALCRP}(\mathcal{D})$.

The condition to gain decidability for the basic logic $\mathcal{ALC}(\mathcal{D})$ is the *admissibility* of the concrete domain (see Definition 5.2): a concrete domain $\mathcal{D}$ to be *admissible* must have a set of predicate names closed under negation and a symbol $\top_{\mathcal{D}}$ for $\Delta^{\mathcal{D}}$ and the $\mathcal{D}$-satisfiability problem must be decidable. Lutz in [96] recalls some relevant results which state how the complexity of reasoning with $\mathcal{ALC}(\mathcal{D})$ depends on the complexity of the $\mathcal{D}$-satisfiability problem (the problem of satisfiability of finite conjunctions of predicates of $\mathcal{D}$). We summarized here the relation between the complexity of the $\mathcal{D}$-satisfiability problem and the computational complexity of $\mathcal{ALC}(\mathcal{D})$:

- $\mathcal{D}$ is admissible and $\mathcal{D}$-satisfiability is in PSPACE $\Rightarrow$ $\mathcal{ALC}(\mathcal{D})$-concept satisfiability and subsumption are PSPACE-complete (see Theorem 7 in [96]);
- $\mathcal{D}$ is admissible and $\mathcal{D}$-satisfiability is in NP $\Rightarrow$ $\mathcal{ALC}(\mathcal{D})$-concept satisfiability w.r.t. acyclic TBoxes NEXPTIME (see Theorem 15 in [96])

Figure 6.13 provides a summary of computational properties of $\mathcal{ALC}(\mathcal{D})$ extended with spatial concrete domains related to the corresponding QSRR formalism. In the set of admissible concrete domains we include even the coarser mereological versions of $\mathcal{S}_2$, Δ_{BRCC8} denoted by $\mathcal{S}_{\mathsf{BRCC5}}$ and Δ_{BRCC5} respectively, since admissibility of a full concrete domain defined on a set of predicates implies intuitively the admissibility of concrete domains defined on sets of predicates. In fact it is easy to see that the conditions for admissibility are still fulfilled even collapsing NTPP and TPP relations into a single part relation.

	QSRR Complexity	ALC(D)	Decidability	Upper bound for pure concept-SAT	Upper bound for concept-SAT w.r.t. Acyclic TBoxes
RCC8	NP-complete	$ALC(S_2)$	Dec.	PSpace-complete	NExpTime
BRCC8	NP-complete	$ALC(\Delta_{BRCC8})$ *			
RCC5	NP-complete	$ALC(S_{RCC5})$			
BRCC5	NP-complete	$ALC(\Delta_{BRCC5})$ *			
CDC	NP-complete	$ALC(S_{CDC})$ *			
RA	NP-complete	$ALC(S_{RA})$ *			
PDR	NP-complete	$ALC(S_{PDR})$ *			
PDR^+	NP-complete	$ALC(S_{PDR+})$ *			
DIV9-RCC8	Polynomial	$ALC(S_{DIV9\text{-}RCC8})$ *			

Fig. 6.13. Computational properties of $\mathcal{ALC}(\mathcal{D})$ hybridizations with QSRR formalisms.

Expressivity of $\mathcal{ALC}(\mathcal{D})$

A relevant point is the difference between fixed role box languages and languages of the $\mathcal{ALC}(\mathcal{D})$ family. In Chapter 4 we presented the hybridization technique based on $\mathcal{ALCI}_{\mathsf{C}}$ logics considering some QSRR formalisms. Chapter 5 presented formal definitions of concrete domains still based on QSRR formalisms. In the following we compare the expressive power and the computational properties of the two hybridization techniques, considering a single set of relations. Nevertheless all the following observations are rather general and the examples of spatial concept definitions give the general idea of the expressiveness of every considered spatial concrete domain. We consider the case of mereo-topological hybridizations ($\mathcal{ALCI}_{\mathcal{RCC}8}$ and $\mathcal{ALC}(\mathcal{S})_2$) to highlight the differences in terms of computational results and expressive power between the two hybridization techniques. As seen in previous chapters the language $\mathcal{ALCI}_{\mathcal{RCC}8}$ turned out to be undecidable while $\mathcal{ALC}(\mathcal{S})_2$ is decidable, as shown on Figure 6.13. It is clear that $\mathcal{ALCI}_{\mathcal{RCC}8}$ pays a high computational cost w.r.t. the other language but it is not clear what is the expressivity gain of the embedding of spatial roles into the syntax. Let us go back to the meaning of "role" and consider a role name r and two concept names C, D. The semantics of $D := \exists \mathsf{r}.\mathsf{C}$ says that we denote as D each individual of the universe related via r with at least an individual which fulfills the C "condition". For instance

$$\mathsf{Car\text{-}Owner} := \exists \mathsf{owns}.\mathsf{Car}$$

denotes whoever is related via the "owning" relation with at least one object of the universe belonging to the set of "car"-individuals. If we consider a language with spatial roles we can describe spatial relations between concepts. We can define for instance the notion of sea-town as

$$\mathsf{Sea\text{-}Town} := \mathsf{Town} \sqcap \exists \mathsf{EC}.\mathsf{Sea}$$

considering all individuals that belong both to the set of "town"-individual and to set of individuals related via the spatial relation of "external connection" with at least one of the "sea"-individuals. This is a rather intuitive way to use spatial relations in concept definitions, nevertheless we pay a cost in terms of undecidability of the language. When we consider standard $\mathcal{ALC}$ extended with a spatial concrete domain we gain decidability, but we loose something in terms of expressivity. A concrete domain requires at least a concrete feature to link abstract descriptions to concrete objects. Since we consider in this example the spatial concrete domain $\mathcal{S}_2$ defined on the set of RCC-8 relations we choose the concrete feature rep to refer to the spatial representation of a given concept. Let us consider the case of the two simple definitions in Figure 6.14. The description of concept

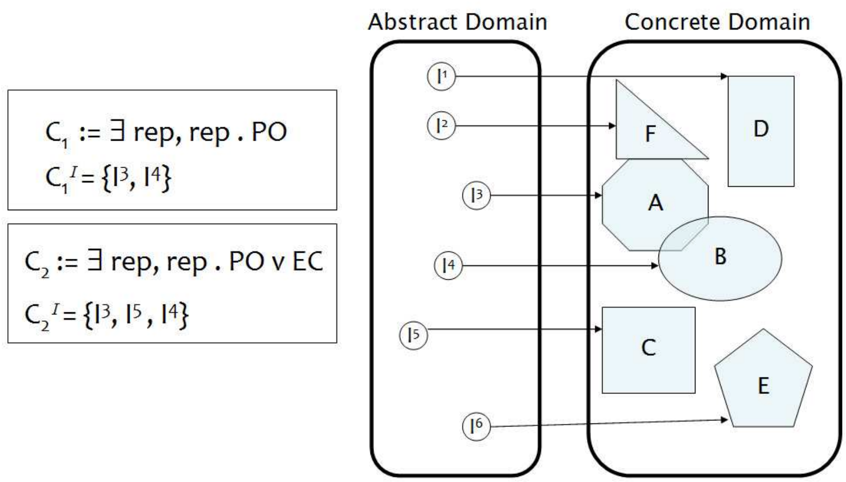

Fig. 6.14. An exemplification of $\mathcal{ALC}(\mathcal{D})$ expressivity via concrete domain: a concept description defined with the spatial feature rep is interpreted as a set of abstract individuals with a concrete (spatial) representation.

C_1 corresponds to each individual related to a spatial object which is in PO relation with the spatial representation of an other individual. The description of C_2 corresponds to each individual related to a spatial object which is in PO relation or in EC relation with the spatial representation of an other individual. The limit of the expressivity of $\mathcal{ALC}(\mathcal{D})$ is clear when we try to define more complex notions. Referring to the previous example of "sea-town" defined using $\mathcal{ALCI}_{\mathcal{RCC}8}$, we can try to define the same concept via $\mathcal{ALC}(\mathcal{D})$ as follows

$$\mathsf{Sea\text{-}Town} := \mathsf{Town} \sqcap \exists \mathsf{rep}, \mathsf{rep}.\mathsf{EC}$$

The problem is that it is not possible to restrict the object in EC relation with the town to be in the interpretation of the "sea" concept. The value restriction of the related object is rather complicated and requires the definition of special abstract roles in order to guarantee the right connection. Lutz and Miličić in [99] proposed the definition of Hotel that we simplify to show how a logic equipped with a concrete domain works. Consider the general concepts of "building" and "room", in order to express a spatial relation between these two concepts we must introduce an abstract role "has-room" as follows:

$$\mathsf{Building} := \forall \mathsf{has\text{-}room}.\mathsf{Room} \sqcap \exists \mathsf{has\text{-}room}.\mathsf{Room}$$

This definition states the relation in the abstract domain between "building"-individuals and "room"-individuals. Only now we can express the spatial relation that must hold between the representations in the concrete domain:

$$\begin{aligned}\mathsf{Building} := {} & \forall \mathsf{has\text{-}room}.\mathsf{Room} \sqcap \exists \mathsf{has\text{-}room}.\mathsf{Room} \\ & \sqcap \exists (\mathsf{has\text{-}room\ rep}), (\mathsf{rep}).\mathsf{TPP} \vee \mathsf{NTPP}\end{aligned}$$

The notation $(\mathsf{has\text{-}room\ rep})$ can be seen as a composition of relations which in this case refers to spatial representations of individuals in "has-room" relation with the considered concept of building.
Summarizing, the limit of the language is that in order to define a concept "A" spatially related with "B"-individuals it is necessary a specific abstract role that "summons" precisely all the "B" individuals. Referring to the example of sea-town the definition for instance of the abstract role being-sea-neighboring is required, even if counter intuitive. For each concept used in a "spatial definition" there must exist at least one special "summoning" role. We will see in following section how the language $\mathcal{ALCRP}(\mathcal{D})$ overcomes this limit allowing the definition of complex roles based on concrete relations.

6.3.2 Spatial Reasoning with $\mathcal{ALCRP}(\mathcal{D})$ and $\mathcal{ALC}(\mathcal{C})$

Expressivity of $\mathcal{ALCRP}(\mathcal{D})$ and $\mathcal{ALC}(\mathcal{C})$

The language $\mathcal{ALC}(\mathcal{D})$ presented in the previous section is defined starting from the basic description logic $\mathcal{ALC}$. We saw that this language although augmented with a concrete domain remains rather counter intuitive. Many researches proposed different extensions to the basic $\mathcal{ALC}(\mathcal{D})$, among these we consider two languages:

- $\mathcal{ALCRP}(\mathcal{D})$ with concrete domain role constructors,
- $\mathcal{ALC}(\mathcal{C})$ with general TBoxes, proved to be decidable on special concrete domains called constraint systems.

We saw how the definition of spatial relations between objects requires special abstract roles to specify which is the concept one refers to. The language $\mathcal{ALCRP}(\mathcal{D})$ defined first by Haarslev et al. in [131] extends the basic $\mathcal{ALC}(\mathcal{D})$ by a complex role constructor (the formal definition is recalled in Definition 3.11). A *Concrete Domain Role* can be defined in $\mathcal{ALCRP}(\mathcal{D})$ to gain the availability of concrete relations as standard roles of the language. In order to give the idea of the expressive power and of how to use the syntax of the language we still consider the RCC-8 set of relations to define spatial examples. In order to use RCC-8 it is useful to define a concrete domain role for each of the spatial relations. In the following we keep as *concrete feature* the connection rep between an abstract object and its spatial representation.

$$
\begin{aligned}
\mathrm{eq} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{EQ} \\
\mathrm{po} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{PO} \\
\mathrm{dc} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{DC} \\
\mathrm{ec} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{EC} \\
\mathrm{ntpp} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{NTPP} \\
\mathrm{tpp} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{TPP} \\
\mathrm{ntppi} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{NTPPI} \\
\mathrm{tppi} &:= \exists \mathsf{rep}, \mathsf{rep}.\mathsf{TPPI}
\end{aligned}
$$

The concrete roles allow to overcome the limit of basic $\mathcal{ALC}(\mathcal{D})$ internalizing as standard roles the entire set of spatial relations. We can consider again the concept of "sea-town", which can now be easily defined as follows:

$$\mathsf{Sea\text{-}Town} := \mathsf{Town} \sqcap \exists \mathrm{ec}.\mathsf{Sea}$$

It is easy to see that with the introduction of concrete domain roles the notation is rather similar to the syntax of the undecidable $\mathcal{ALCI}_{\mathcal{RCC}8}$. The computational cost that we pay for this increase of expressivity is that $\mathcal{ALCRP}(\mathcal{D})$ turns out to be undecidable (this result by Haarslev et al. [131] is recalled in Theorem 3.12). In order to regain decidability Haarslev et al. have defined a syntactic restriction that limit the nesting of quantifications in order to avoid "problematic" concept definitions. This is a rather strong restriction since it does not allow complex definitions. We consider a variation of an example given by Wessel in [140] to understand the syntactic limit. Given the concepts of *country* and *German city* we consider the set of German cities crossed by a national river (in the sense of a river completely contained by Germany)

$$\mathsf{German\text{-}city} := \mathsf{city} \sqcap \exists \mathrm{pp}.\mathsf{Germany} \sqcap \forall \mathrm{po}.\neg \mathsf{country}$$

$$
\begin{aligned}
&\mathsf{German\text{-}city\text{-}with\text{-}national\text{-}river} := \\
&\qquad \mathsf{German\text{-}city} \ \sqcap \ \exists \mathrm{po}.(\mathsf{river} \ \sqcap \ \exists \mathrm{pp}.\mathsf{germany} \sqcap \\
&\qquad \forall \mathrm{pp}.\mathsf{country} \rightarrow \mathsf{germany})
\end{aligned}
$$

This last definition is not permitted with restricted $\mathcal{ALCRP}(\mathcal{D})$ because of the $\forall$ nesting into a $\exists$ quantification with complex roles.

The other extension of $\mathcal{ALC}(\mathcal{D})$ is given by *general TBoxes*, which allow one to define inclusions (corresponding to implications) between complex concepts. These kinds of implications are called general inclusion axioms (recalled formally in Definition 3.4) and are rather expressive. General TBoxes allow for general concept inclusions (GCIs) of the form $C \sqsubseteq D$, where C and D are (possibly) complex concepts, stating that C implies D. General TBoxes are highly desirable in knowledge representation, as they can capture complex constraints and dependencies in the application domain. However, they usually increase complexity of reasoning, and may introduce semantic and computational problems. For this reason Lutz and Miličić in [99] defined special concrete domains called *constraint systems*, that when fulfilling some particular properties, guarantee the decidability of $\mathcal{ALC}(\mathcal{D})$ even w.r.t. general TBoxes. This logic denoted by $\mathcal{ALC}(\mathcal{C})$ turns out to be decidable when used with ω-admissible constraint systems (see Definition 5.7 of ω-admissibility). Hereafter we recall the example presented by Lutz and Miličić in [99] considering as exemplification given by the case of RCC-8 relations, in order to understand the expressivity of $\mathcal{ALC}(\mathcal{C})$ with general TBoxes.

$$\begin{aligned}\mathsf{Hotel}\ &\sqsubseteq \forall \mathsf{hasRoom}.Room \sqcap \forall \mathsf{hasReception}.Reception\\ &\sqcap\ \forall \mathsf{hasCarPark}.CarPark\end{aligned}$$

$$\begin{aligned}\mathsf{Hotel}\ &\sqsubseteq \forall (\mathsf{hasRoom\ rep}), (\mathsf{rep}).\mathsf{tpp} \vee \mathsf{ntpp}\\ &\forall (\mathsf{hasRoom\ rep}), (\mathsf{hasRoom\ rep}).\mathsf{dc} \vee \mathsf{ec} \vee \mathsf{eq}\end{aligned}$$

$$\begin{aligned}\mathsf{CarFriendlyHotel}\ \doteq \mathsf{Hotel}\ &\sqcap \exists (\mathsf{hasReception\ rep}), (\mathsf{rep}).\mathsf{tpp}\\ &\sqcap \exists (\mathsf{hasCarPark\ rep}), (\mathsf{rep}).\mathsf{ec}\\ &\sqcap \exists (\mathsf{hasReception\ rep}), (\mathsf{hasReception\ rep}).\mathsf{ec}\end{aligned}$$

The example shows how general concepts inclusions allows to specify inclusions for the "Hotel" concept, which is non-primitive, stating first that a hotel is generally related with rooms, reception and car-park, then giving the proper spatial relations. It is clear that $\mathcal{ALC}(\mathcal{C})$ suffers as $\mathcal{ALC}(\mathcal{D})$ the lack of concrete roles and uses roles as hasReception and hasRoom to refer to the corresponding concepts ("Reception" and "Room" respectively) to define spatial relations. In the following section we present computational results for spatial hybridizations defined with $\mathcal{ALC}(\mathcal{C})$ and $\mathcal{ALCRP}(\mathcal{D})$.

Definable hybridizations and computational results

We have already pointed out that the *admissibility* of a concrete domain $\mathcal{D}$ is a sufficient condition to guarantee the decidability of the problem of concept satisfiability for the language $\mathcal{ALC}(\mathcal{D})$. Unlike for basic $\mathcal{ALC}(\mathcal{D})$, the admissibility property for a concrete domain does not imply decidability for $\mathcal{ALCRP}(\mathcal{D})$ and for $\mathcal{ALC}(\mathcal{C})$ with general TBoxes. Actually the problem of $\mathcal{ALCRP}(\mathcal{D})$-concept satisfiability has been proved to be undecidable. We summarized hereafter some relevant results about the computational properties of $\mathcal{ALCRP}(\mathcal{D})$ proved by Haarslev et al. in [131]:

- the concept satisfiability problem for $\mathcal{ALCRP}(\mathcal{D})$ is undecidable;
- if $\mathcal{D}$ is admissible and $\mathcal{D}$-satisfiability is in NP, then pure satisfiability of restricted $\mathcal{ALCRP}(\mathcal{D})$-concepts can be decided in NEXPTIME (see Theorem 3.14 page 50).

From these results follows the decidability and the computational properties of hybridizations with $\mathcal{ALCRP}(\mathcal{D})$ summarized in Figure 6.15. The spatial hybridizations that we investigate are based on the same set of admissible concrete domains used for the basic $\mathcal{ALC}(\mathcal{D})$ and listed in Figure 6.12 (page 126).

In order to extend $\mathcal{ALC}(\mathcal{D})$ with general TBox preserving decidability, Lutz and Miličić in [99]:

- restrict the set of considered concrete domains to special *constraint systems* only and
- require a more restricted condition of ω-admissibility.

For this reason we investigate hybridizations with $\mathcal{ALC}(\mathcal{C})$ with general TBoxes for QSRR formalisms with a corresponding constraint system. The QSRR formalisms hybridized are listed in Figure 6.16. For this set of hybrid spatial languages we have a decidability result which follows from the ω-admissibility of the constraint systems.

	QSRR Complexity	ALCRP(D) with syntactic restrictions	Decidability	Upper bound for pure concept-SAT
RCC8	NP-complete	ALCRP(S_2)		
BRCC8	NP-complete	ALCRP(Δ_{BRCC8}) *		
RCC5	NP-complete	ALCRP(S_{RCC5})		
BRCC5	NP-complete	ALCRP(Δ_{BRCC5}) *		
CDC	NP-complete	ALCRP(S_{CDC}) *	*Dec.*	*NExpTime*
RA	NP-complete	ALCRP(S_{RA}) *		
PDR	NP-complete	ALCRP(S_{PDR}) *		
PDR+	NP-complete	ALCRP(S_{PDR+}) *		
DIV9-RCC8	Polynomial	ALCRP($S_{DIV9-RCC8}$) *		

Fig. 6.15. Computational properties of $\mathcal{ALC}(\mathcal{D})$ hybridizations with QSRR formalisms.

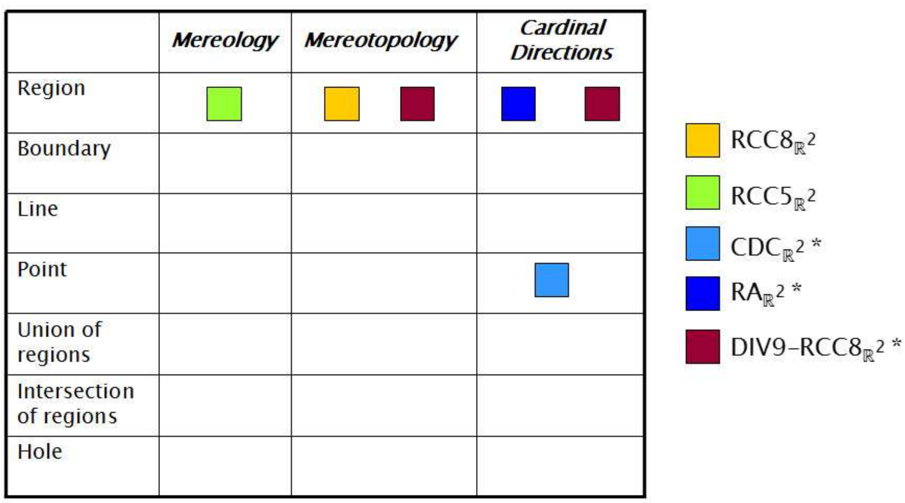

Fig. 6.16. ω-admissible constraint systems for $\mathcal{ALC}(\mathcal{C})$ hybridizations. We denote with * our contribution.

7

Conclusions and Further work

7.1 Qualitative Spatial Reasoning for $\mathcal{DL}$s

In Chapter 2 we presented the idea of qualitative spatial representation and reasoning and the reason why we decided to restrict our investigation to qualitative knowledge, preferring a commonsense approach to spatial knowledge respect to qualitative one. In particular we considered two important families of QSRR formalisms: those focused on topological and mereological relations and those on directional relations. For directional information we considered the Cardinal Direction Calculus (CDC) [89], the Rectangle Algebra (RA) [67], [108], [15], the Projection-Based Directional Relations (PDR) [62], [63], [124], [125], [126] for regions only or considering as primitives also points and lines (multi-dimensional PDR^{+}) and a combined approach to topological and directional relations (DIR9-RCC8) [88].

For topological relations we presented standard Region Connection Calculus (RCC) [40], [112] and [113], the variation of RCC represented by the Boolean RCC (BRCC) [143], [144], [84], [52], [53], and the Dimension Extended Method (DEM) [33], [32]. Since RCC and BRCC have been deeply investigated, there exists for these two formalisms well-known computational results. The DEM approach is one of the most expressive, nevertheless it lacks of formal results about computational complexity and even about decidability. Because of the lack of decidability results, no hybridization between a terminological language and DEM can be defined. In recent years Galton proposed in [54] a topological model for regions of different dimensions, but as pointed out by Galton himself, the work needs to be supplemented by further detailed investigations. We leave as an open problem if there is a correspondence between Galton's formal multi-dimensional framework and the expressive set of DEM relations, knowing that "the challenge to develop mathematical tools appropriate to the needs of the Knowledge Representation community is ongoing"(A. Galton).

7.2 $\mathcal{DL}$s for Qualitative Spatial Reasoning

A first attempt to introduce a "closure" operator into a description logic has been made by Cristani et al. in [39]. It was clear that this approach was not fully adequate for terminological spatial reasoning. For this reason I considered for the aim of this thesis other kinds of extensions of description logics. I identified some relevant terminological language well-suited for spatial reasoning: $\mathcal{ALCI}_{\mathcal{RCC}}$ and $\mathcal{ALC}(\mathcal{D})$. The first extends the basic logic with a composition-based Rbox, while the second considers an external concrete domain. In this thesis I close some points left as open by Wessel about the decidability $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, stating an undecidability result for both the languages, then provide a tableau technique to ensure the decidability in the cases of strong EQ semantics and finite maximal cardinality for admissible models. I investigate also the family of logics extended by concrete domain considering the basic $\mathcal{ALC}(\mathcal{D})$, its extension with concrete domain roles $\mathcal{ALCRP}(\mathcal{D})$ and the extension with general TBoxes $\mathcal{ALC}(\mathcal{C})$. My thesis provides a generalization of Wessel's idea of fixed role box for RCC relations and provides also a systematic investigation of expressivity and computational properties for all possible hybridizations between the considered QSRR formalisms and the most relevant families of $\mathcal{DL}$s for spatial reasoning. We leave as further work the investigation of a multidimensional language to manage mereo-topological relations between points and lines and of the ω-admissibility of constraint systems defined starting from the PDR approach. In particular it is worth noticing that a terminological language capable of the DEM expressivity would represent a milestone particularly for the GIS environment, since the DEM relations are used in the GeoUML language. Another open point is the definition w.r.t the language $\mathcal{ALCI}_{\mathsf{DIV9-RCC8}}$ of a proper combined RBox to manage the interference between the two QSRR formalisms not only to check concept-satisfiability but also at a reasoning level to guarantee implications as "if A is proper part with B, then A and B must be in the CC relation". I hope to work out these open points in the future.

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