Chapter 3

TWO PARADIGMS OF LOGICAL COMPUTATION IN AFFINE LOGIC?

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Abstract We propose a notion of *symmetric reduction* for a system of proof-nets for *Multiplicative Affine Logic with Mix* (**MAL +** Mix) (namely, multiplicative linear logic with the mix-rule the unrestricted weakening-rule). We prove that such a reduction has the strong normalization and Church–Rosser properties. A notion of irrelevance in a proof-net is defined and the *possibility* of cancelling the irrelevant parts of a proof-net without erasing the entire net is taken as one of the *correctness conditions*; therefore purely *local* cut-reductions are given, minimizing cancellation and suggesting a paradigm of *"computation without garbage collection"*. Reconsidering Ketonen and Weyhrauch's decision procedure for affine logic [15, 4], the use of the mix-rule is related to the non-determinism of classical proof-theory. The question arises, whether these features of classical cut-elimination are really irreducible to the familiar paradigm of cut-elimination for intuitionistic and linear logic.

Keywords: affine logic, proof-nets

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1. Introduction

1. *Classical Multiplicative Affine Logic* is classical multiplicative linear logic with the unrestricted rule of *weakening,* but without the rule of *contraction.* Classical affine logic is a much simpler system than classical logic, but it provides similar challenges for *logical computation,* both in the sense of *proof-search* and of *proof normalization* (or *cut-elimination*). For instance, the problem of *confluence* of cut-elimination (the *Church–Rosser property*) is already present in affine logic, but here we do not have the problem of *non-termination.* Affine logic is also simpler than linear logic from the point of view of proof-search: e.g., propositional linear logic is undecidable, yet becomes decidable when the unrestricted rule of weakening is added. Provability in *constant-only multiplicative linear logic* is NP-complete, yet it is decidable in linear time for *constant-only multiplicative affine logic,* as it is shown below.

The tool we will use here, proof-nets for affine logic, is older than the notion of a proof-net for linear logic. In a 1984 paper [15], J. Ketonen and R. Weyhrauch presented a decision procedure for first-order affine logic (called then *direct logic*) which essentially consists in building cut-free proof-nets, using the unification algorithm to determine the axioms. The 1984 paper is sketchy and it has been corrected (see [3, 4], where the relation between the decision procedure and proof-nets for $\mathbf{MLL}^-$ are discussed), but it contains the main ideas exploited in the present paper, namely, the construction of proof-nets *free from irrelevance* through *basic chains.* Yet neither the 1984 paper nor its 1992 revisitation contained a treatment of cut-elimination.[1]

2. The problem of non-confluence for classical affine logic is non-trivial: the following well-known example (given in Lafont's Appendix to [14]) reminds us that the Church–Rosser property is non-deterministic under the familiar *asymmetric* cut-reductions.

Example 1

$$\dfrac{\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A}\, w_1 \qquad \dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Delta, \neg A}\, w_2}{\vdash \Gamma, \Delta} \quad \text{reduces to} \quad \dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{\dfrac{weakenings}{\vdash \Gamma, \Delta}} \quad \text{or to} \quad \dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\dfrac{weakenings}{\vdash \Gamma, \Delta}}$$

Asymmetric reductions.

Indeed classical logic gives no justification for choosing between the two indicated reductions, the first commuting the cut-rule with the *left* application of the weakening-rule ("pushing d_2 up into d_1", thus erasing d_2), the second commuting the cut-rule with the *right* application of the weakening-rule ("pushing d_1 up into d_2", thus erasing d_1). Therefore the cut-elimination process in **MAL**, *a fortiori* in **LK**, is non-deterministic and non-confluent.

Compare this with normalization in intuitionistic logic. In the typed λ calculus a *cut / left weakening* pair corresponds to substitution of $t : A$ for a variable $x : A$ which does not occur in $u : B$; such a substitution is unambiguously defined as $u[t/x] = u$. Moreover in Prawitz's natural deduction **NJ** [19] the rule corresponding to *weakening-right* is the rule *"ex falso quodlibet"* and the normalization step for such a rule involves a form of η-**expansion**:

$$\frac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A \wedge B} \qquad \textit{reduces to} \qquad \frac{\dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A} \quad \dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{B}}{A \wedge B}$$

Such a reduction does *not* yield cancellation. Thus the cut-elimination procedure for the intuitionistic sequent calculus **LJ** inherits one sensible reduction strategy from natural deduction: *"push the left derivation up into the right one"*. In the case of a *weakening / cut* pair it is always the *left* dedution to be erased.

3. Here we are interested in exploring an obvious remark: for classical logic in addition to the *asymmetric* reductions of Example 1, there is a *symmetric* possibility, the *"Mix"* of d_1 and d_2.

Example 1 cont.

$$\frac{\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A}\, w_1 \qquad \dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Delta, \neg A}\, w_2}{\vdash \Gamma, \Delta} \qquad \text{reduces to} \qquad \frac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Gamma, \Delta}\, Mix$$

Symmetric reduction.

Instead of choosing a direction where to "push up" the cut-rule, we do both asymmetric reductions, using the mix-rule.

The idea is loosely related to a procedure well-known in the literature for the case when both cut-formulas result from a contraction-rule, with the name *cross-cut reduction.* Let d_1 and d_2 be derivations of the left and right premises of the cut-rule:

Example 2

$$d_1 = \dfrac{\dfrac{\begin{matrix} d_1' \\ \vdots \\ \vdash \Gamma, A, A \end{matrix}}{\vdash \Gamma, A} \quad \dfrac{\begin{matrix} d_2' \\ \vdots \\ \vdash \neg A, \neg A, \Delta \end{matrix}}{\vdash \neg A, \Delta}}{\vdash \Gamma, \Delta} = d_2$$

Let d_ℓ be obtained by commuting the cut-rule with the *left* application of the contraction-rule and symmetrically, let d_r be obtained by commuting the cut-rule with the *right* application of the contraction-rule. The *cross-cut* reduction is defined as follows:

$$d_\ell = \dfrac{\dfrac{\dfrac{\begin{matrix} d_1' \\ \vdots \\ \vdash \Gamma, A, A \end{matrix} \quad \dfrac{\begin{matrix} d_2' \\ \vdots \\ \vdash \neg A, \neg A, \Delta \end{matrix}}{\vdash \neg A, \Delta}}{\vdash \Gamma, \Delta, A}\, cut_1 \quad \dfrac{\dfrac{\begin{matrix} d_1' \\ \vdots \\ \vdash \Gamma, A, A \end{matrix}}{\vdash \Gamma, A} \quad \begin{matrix} d_2' \\ \vdots \\ \vdash \neg A, \neg A, \Delta \end{matrix}}{\vdash \neg A, \Gamma, \Delta}\, cut_2}{\dfrac{\vdash \Gamma, \Gamma, \Delta, \Delta}{\vdash \Gamma, \Delta}\, contractions}\, cut = d_r$$

Cross-cut reduction.

4. As it stands the *symmetric* reduction of Example 1 could not be taken very seriously as a confluent notion of cut-reduction. One issue is the fact that a *weakening* inference may be *permuted* with many other inferences in a sequent derivation, and such permutations may considerably modify the structure of the proof; it would therefore be useful to have some notion of a *normal form for weakening*. Two standard notions can be found in the literature: these amount to applying the weakening-rule either (i) as *high* as possible, i.e., at the level of axioms, or (ii) as *low* as possible. The first solution is not available for *multiplicative* connectives without the contraction-rule; the second solution still leaves room for many ambiguities.

But there is one case where no ambiguity is possible, that of a *weakening / multiplicative disjunction* pair, where the weakening-rule introduces a formula which is active in the disjunction-rule and the other active formula has ancestors in axioms (or, similarly, the case of a weakening-rule introducing a conclusion of the derivation). It turns out that every derivation can be transformed into one where all applications of the weakening-rule are of this form, through *weakening-reductions.* This property may adopted as a notion of *weakening normal form,* but there are two problems: first, if we apply the cut-rule to two derivations in weakening normal form, the resulting derivation may not be in weakening normal form and, second, if we define weakening-reductions like permutations of inferences in the sequent calculus, then they do not yield a unique normal form (Section 2).

A second issue is the nature of the mix-rule: this rule does not simply represent distinct possibilities of proof-transformation. On the contrary, it contributes to create proofs with very rich and complicated structure. However using the unrestricted weakening-rule if a sequent is derivable *with Mix* then it is derivable *without Mix,* i.e., the structural rule Mix is eliminable:

$$\dfrac{\begin{matrix} d_1 & d_2 \\ \vdots & \vdots \\ \vdash \Gamma & \Delta \end{matrix}}{\vdash \Gamma, \Delta}\ Mix \quad \text{may be transformed into} \quad \dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma \end{matrix}}{\text{weakenings}}}{\vdash \Gamma, \Delta} \quad \text{but also into} \quad \dfrac{\dfrac{\begin{matrix} d_2 \\ \vdots \\ \vdash \Delta \end{matrix}}{\text{weakenings}}}{\vdash \Gamma, \Delta}$$

The ambiguity may be resolved by taking *both* reducts. More generally, for all n we may introduce an n-ary rule *Additive Mix* building a derivation d of $\vdash \Gamma$ out of n derivations $d_1, \ldots, d_n$ of $\vdash \Gamma$:

$$\frac{d_1, \ldots, d_n}{\vdash \Gamma} \quad AM(n).$$

Then every application of Additive Mix may be permuted below other inferences. Thus the replacement of the *multiplicative* mix-rule with the *additive* mix-rule seems to capture the practice of disentangling simpler and more basic arguments from a more complicated one; conversely, the use of the multiplicative mix-rule may be explained as a compact notation unifying different ways of proving the same conclusions (Section 3). But the procedure **Sep** which eliminates the multiplicative mix-rule and permutes occurrences of Additive Mix below other inferences is computationally very expensive: therefore it would be desirable to find a more effective procedure to eliminate the multiplicative mix-rule.

5. If we look again at the *direct logic* decision procedure [15, 4], we see that two of our problems had already been solved there by the notion of a *chain.* Given a sequent Γ and a formula C in Γ, the procedure selects one (positive or negative) atomic subformula P in C and tries to find another subformula P of opposite polarity in some $D \in \Gamma$; in the terminology of proof-nets, it builds an axiom $\overline{P, P^{\perp}}$. Then the paths of subformulas from C to P and from $P^{\perp}$ to D are included in the chain; moreover, if a conjunct is in the chain, say A in $A \otimes B$, but the other conjunct B is not, then the procedure selects an atomic subformula Q of B and tries to find another axiom $\overline{Q, Q^{\perp}}$, and so on. The procedure will stop if all conjuncts have been matched by exactly one axiom. Now we apply this procedure *within a proof-net* $\mathcal{R}$ for *multiplicative affine logic with Mix* (**MAL** + Mix): it yields a path through the proof-net; if only one premise of a *par* link in in the path, then we introduce the other premise by a weakening-link; similarly for conclusions which are not reached by the path. The substructure $\mathcal{S}$ obtained in this way is a *proof-net* for **MAL**, *multiplicative affine logic without Mix*. If we repeat this procedure for all possible choices of axioms, we have a more efficient proof-net counterpart of the **Sep** procedure above. Moreover, the proof-net $\mathcal{S}$ corresponds to a sequent derivation in *weakening normal form*: we call it a proof-net *free from irrelevance*.

6. Given a proof-net for multiplicative affine logic with Mix, we may define a linear-time *pruning* algorithm, which yields a proof-net free from irrelevance, as follows ([4], Section 7.5):

(0) a weakening-formula is irrelevant;

(1) if the conclusion of a link is irrelevant, all its premises are irrelevant;

(2) if a formula in a logical axiom is irrelevant, so is the other;

(3) if one premise of a times-link or of a cut-link is irrelevant, so is the conclusion and the other premise;

(4) if both premises of a par-link are irrelevant, then the conclusion is irrelevant.

Clearly this resembles Girard's definition of the *empire* of a formula in $\mathbf{MLL}^-$ without Mix (cf. [11], Facts 2.9.4). What is important for us is that from the linear-time pruning algorithm we can draw two consequences, one relevant to problems of computational complexity, the other to the definition of confluent normalization procedures.

Given a proof-structure $\mathcal{S}$ without attachments for the weakening-links and satisfying the acyclicity property (of every Danos–Regnier graph), consider the problem of deciding whether attachments may be added to the weakening-links of $\mathcal{S}$ so that the resulting proof-structure $\mathcal{S}'$ is a proof-net. By a result of P. Lincoln and T. Winkler [16] this problem is NP-complete in the strong sense, thus it is as hard as the problem of finding a proof of Γ, given any $\mathbf{MLL}^-$ sequent Γ. But if we consider only proof-nets without irrelevance the problem of attaching the weakening-links is trivial, as they can be attached to a premise of a *par-link* or to a conclusion: the elimination of irrelevance algorithm yields a proof-structure free from irrelevance in linear time. It follows that provability in *constant-only affine logic* is decided in linear time by the pruning algorithm: since axioms are precisely the subformulas **1**, a sequent is provable if and only if the result of pruning the subtree of its formuals is non-empty.

Another remarkable feature of the pruning algorithm is that, regarded as a notion of reduction of proof-nets for **MAL** + Mix, it is *confluent*. It should be noticed that this algorithm *cannot be defined by permutations of inferences in* *the sequent calculus*: this follows from the study of the subnets of proof-nets [3], see examples in Section 4.3. It is this remark that allowed the research to take off, as it showed that proof-nets could be used to identify some invariants of proofs with respect to the behaviour of the *weakening-rule.*

7. The use of the proof-net representation of proofs and the notion of the *irrelevant part* $I(\mathcal{R})$ in a proof-net $\mathcal{R}$ are essential also to define a confluent notion of *cut-reduction.* Irrelevance is not stable under cut-elimination: if $\mathcal{R}_1$ reduces to $\mathcal{R}_2$ and $\mathcal{R}_2$ reduces to $\mathcal{R}_3$ then we may have $I(\mathcal{R}_1) \subset I(\mathcal{R}_2)$ but also $I(\mathcal{R}_3) \subset I(\mathcal{R}_2)$. Therefore we cannot actually eliminate irrelevance during cut-elimination: we may metaphorically say that we need *cut-elimination without garbage collection.* The goal is to eliminate cut through stricly *local* operations, e.g., by reducing the logical complexity of the cut-formula in a *weakening / cut* pair. This could be achieved if we could *expand weakening-links* following the model of natural deduction. Unfortunately such an expansion is *incorrect* in the sequent calculus: let P be atomic, then $\vdash \mathbf{1}, P \otimes P$ is derivable in **MAL** using one weakening-rule, but there is no way to expand the application of the weakening-rule into two weakening-rules with atomic conclusion P.

The solution proposed here is to define *proof-nets modulo irrelevance.* Given a proof-structure $\mathcal{S}$ without attachments for the weakening-links and satisfying the acyclicity property of every Danos–Regnier graph, we require $\mathcal{R} \setminus I\mathcal{R} \neq \emptyset$ as an additional *correctness condition*; namely, we require that *it should be possible to eliminate irrelevance without annihilating the proof-structure,* but we do not actually eliminate irrelevance as in [15, 4]. Then the “incorrect” expansions can be introduced, since they belong to the irrelevant part; the additional weakening-links introduced in this way will eventually be annihilated by the cut-elimination procedure. As a consequence, in the intermediate steps of the cut-elimination procedure only the *pruning* of $\mathcal{R}_i$, i.e., $\mathcal{R}_i \setminus I\mathcal{R}_i$ will be sequentializable. Moreover, during the cut-elimination procedure some weakening-links may be annihilated whose attachments guaranteed the connectedness condition for the original proof-net; therefore we may start with a proof-net with cut links in *multiplicative affine logic without Mix* and obtain a cut-free one in *multiplicative affine logic with Mix.*

8. Many researchers have studied the cut-elimination process for classical logic and defined well-behaved notions of reduction and reduction strategies, enjoying the confluence and strong normalization properties. A common feature of many works is that they recognize the non-determinism of classical cut-elimination as the root of both non-confluence and non-termination and then try to to eliminate non-deter- minism by *disambiguating* classical logic. A way to do so is through translations into intuitionistic and linear logic. Such a use of linear logic is exemplified in Girard [12] and it has been developed extensively by V. Danos, J-B. Joinet and H. Schellinx, see, e.g., [8, 9]. Through the modalities of linear logic the area of a proof-net which may be erased in a *weakening / cut* reduction is always determined *modulo* permutation of !-boxes. Interesting properties of classical cut-elimination are identified in this way, which can be related to known reduction strategies for the λ-calculus. However, it should be clear that the translations of classical logic into linear logic have the same function as the translations into intuitionistic logic, namely to extend the computational paradigm of intuitionistic logic to classical logic. In the case of affine logic, such translations yield *"computations with garbage collection"*.

Another approach is to look for dynamical properties of classical logic that are alternative and irreducible to those of intuitionistic and linear logic. It would be natural to expect that the classical cut-elimination should reflect the *symmetries* that arise from the fact that classical negation is an involution. In Girard [12] a mathematically precise notion of symmetry is given for the notion of cut-reduction in the proof-net representation. The set of substitutions $[p_i/p_i^\perp]$ where p_i is an atom in a proof-net may be regarded as the set of generators of a group acting on the graph. A notion of cut-reduction is *symmetric* if every proof-net which is invariant under the action of the group of substitutions is transformed by such a reduction into a proof-net which is also invariant. An example in [12] shows that a symmetric cut-reduction for classical logic cannot be defined without the use of the mix-rule. Our definition of cut-reduction for **MAL** with Mix would appear to be symmetric this technical sense. The cross-cut reduction of Example 2 is not symmetric, but a symmetric variant of it may perhaps be defined using the mix-rule (see Section 6 below).

Most recently, Bierman and Urban [7] have developed a term calculus for classical logic which represents a very large variety of strongly normalizing, but non-confluent reduction strategies for classical sequent calculus. Bierman and Urban also regard *non-determinism* as the distinguishing feature of classical dynamics. The treatment of Mix in the present paper, in particular the elimination of the multiplicative mix-rule through Additive Mix, suggests that our proof-net representation may provide a concise notation for classical non-determinism. Indeed the notion of *cut-elimination without garbage collegion* may also have a common-sense explanation in terms of non-determinism: if we cannot predict the future development of a process, we should be very conservative about what may be discarded or not.

Can we conclude with a positive answer to the question in the title? Do we really have a notion of cut-elimination for classical affine logic whose properties are alternative and irreducible to the familiar reduction strategies for intuitionistic and linear logic? It may be premature to give an answer. But the time may be ripe for a reconsideration of proof-nets for affine logic. Indeed an increasing number of researchers (e.g., [21]) seem to agree today that although the dynamics of classical logic may not be as elegant and pleasing as that of intuitionistic logic, a task of research is to develop tools to study it as it is, allowing the possibility that it may be very different from what we already know.

2. Sequent calculus of MAL + Mix

Definition 3 (i) The propositional language of Multiplicative Affine Logic **MAL** is built from the propositional constants **1** and $\bot$ and propositional variables, using the connectives $\otimes$ (*times*) and $\wp$ (*par*) of Multiplicative Linear Logic.

(ii) The language of *Constant-only* **MAL** is the fragment of **MAL** consisting of the formulas which do not contain propositional variables.

(iii) The sequent calculus for propositional **MAL** + Mix is given by the following axioms and rules:

identity rules

logical axiom: $\vdash A^{\perp}, A$ *1 axiom:* $\vdash \mathbf{1}$ *cut:* $$\frac{\vdash \Gamma, A^{\perp} \quad \vdash \Delta, A}{\vdash \Gamma, \Delta}$$

structural rules

exchange: $$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$ *weakening:* $$\frac{\vdash \Gamma}{\vdash \Gamma, A}$$ *mix:* $$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

logical rules

times: $$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B,}$$ *par:* $$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$$

The sequent calculus for classical **MAL** + Mix.

Given a derivation d in **MAL** + Mix, let us write $\#ax$, $\#\mathbf{1}$ for the number of occurrences of logical axioms and of **1** axioms, and $\#\otimes$, $\#cut$, $\#mix$, $\#w$ and $\#\wp$ for the number of applications of the *times-*, *cut-*, *mix-*, *weakening-* and *par-rules* and finally $\#conc$ for the number of conclusions in d.

Lemma 4 *Every sequent derivation in* **MAL** + *Mix satisfies the following equations:*

$$\#\text{ax} = \#\otimes + \#\text{cut} + \#\text{mix} + 1$$

$$\#\wp + \#\text{conc} + \#\mathbf{1} = \#\otimes + \#\text{w} + \#2\text{mix} + 2$$

PROOF. By induction on the length of the derivation. For instance, let the last inference of d be a *times-rule* with immediate subderivations d_1 and d_2; let

$\#ax_1$, $\#ax_2$ and $\#ax$ be the number of logical axioms in d_1, d_2 and in d, respectively and similarly for the other axioms and rules. Then we have

$$\begin{aligned}
\#ax &= \#ax_1 + \#ax_2 \\
&= (\#\otimes_1 + \#\otimes_2) + (\#cut_1 + \#cut_2) + (\#mix_1 + \#mix_2) + 2 \\
&= (\#\otimes_1 + \#\otimes_2 + 1) + \#cut + \#mix + 1 \\
&= \#\otimes + \#cut + \#mix + 1 \\
\wp + \#conc + \#\mathbf{1} &= (\#\wp_1 + \#\wp_2) + (\#conc_1 + \#conc_2 - 1) + (\#\mathbf{1}_1 + \#\mathbf{1}_2) \\
&= (\#\wp_1 + \#\wp_2) + (\#conc_1 + \#conc_2) + (\#\mathbf{1}_1 + \#\mathbf{1}_2) - 1 \\
&= (\#\otimes_1 + \#\otimes_2) + (\#w_1 + \#w_2) + 2(\#mix_1 + \#mix_2) + 4 - 1 \\
&= (\#\otimes_1 + \#\otimes_2 + 1) + \#w + 2\#mix + 2 \\
&= \#\otimes + \#w + 2\#mix + 2
\end{aligned}$$

■

2.1 Mix and weakening permutations

Definition 5 (i) Let m/r be a pair of consecutive inferences in a derivation d, where m is an instance of Mix and either r is not a *par-rule* or r is a par-rule but all the ancestors of its active formulas occur in the same branch above the mix-rule. Then the inferences m/r are permutable, i.e., we may obtain a derivation d' which is like d, except for having a consecutive pair of inferences r/w in place of w/r. Thus the only exceptions to the permutability of the mix-rule *below* other inferences are of the following form:

$$\dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma, A \end{matrix} \quad \begin{matrix} d_2 \\ \vdots \\ \vdash B, \Delta \end{matrix}}{\vdash \Gamma, A, B, \Delta}\ Mix}{\vdash \Gamma, A \wp B, \Delta}\ par$$

For every occurrence m of the mix-rule in d there is a set of applications of the par-rule $p_1, \ldots, p_n$ such that m cannot be permuted below $p_1, \ldots, p_n$.

(ii) Similarly, if r/m is a consecutive pair of inferences, where m is an instance of the mix-rule, then the inferences are permutable, but if r is a *times-rule, cut-rule* or *mix-rule* rule, then there are *two* ways of permuting, i.e., pushing the mix-rule up to the left premise of r or to the right one.

(iii) Let w/r be a pair of consecutive inferences in a derivation d, where w is an instance of the weakening-rule introducing a formula X and X is not active in r; then the inferences w/r are permutable, i.e., we may obtain a derivation d' which is like d, except for having a consecutive pair of inferences r/w in place of w/r.

(iv) Similarly, if r/w is a consecutive pair of inferences, where w is an instance of the weakening-rule, then the inferences are permutable, but if r is a *times-rule, cut-rule* or *mix-rule,* then there are *two* ways of permuting, i.e., pushing the weakening-rule up to the left premise of r or to the right one.

Remark 6 Weakening and Mix permutations are reversible.

2.2 Weakening reductions and expansions

Definition 7 (i) In the sequent calculus for **MAL** we have the following *weakening-reductions:*

weakening / par *reduction:*

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, A}\,\mathit{weak}}{\vdash \Gamma, A, B}\,\mathit{weak}}{\vdash \Gamma, A \wp B}\,\mathit{par} \qquad \text{reduces to} \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, A \wp B}\,\mathit{weak}$$

weakening / times *reduction to the right:*

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{\vdash \Delta}{\vdash \Delta, B}\,\mathit{weak}}{\vdash \Gamma, \Delta, A \otimes B}\,\mathit{times} \qquad \text{reduces to} \qquad \dfrac{\dfrac{\vdash \Delta}{\mathit{weakenings}}}{\vdash \Gamma, \Delta, A \otimes B}$$

weakening / times *reduction to the left:*

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, A}\,\mathit{weak} \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\mathit{times} \qquad \text{reduces to} \qquad \dfrac{\dfrac{\vdash \Gamma}{\mathit{weakenings}}}{\vdash \Gamma, \Delta, A \otimes B}$$

weakening / cut *reduction to the right:*

$$\dfrac{\vdash \Gamma, A \qquad \dfrac{\vdash \Delta}{\vdash \Delta, A^{\perp}}\,\mathit{weak}}{\vdash \Gamma, \Delta}\,\mathit{cut} \qquad \text{reduces to} \qquad \dfrac{\dfrac{\vdash \Delta}{\mathit{weakenings}}}{\vdash \Gamma, \Delta}$$

weakening / cut *reduction to the left:*

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, A}\,\mathit{weak} \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,\mathit{cut} \qquad \text{reduces to} \qquad \dfrac{\dfrac{\vdash \Gamma}{\mathit{weakenings}}}{\vdash \Gamma, \Delta}$$

(ii) The inference eliminated by any one of the above reduction will be called *irrelevant.* A sequent derivation which contains no irrelevant inference is said to be in *weakening normal form.*

Lemma 8 *In the sequent calculus for* **MAL** *every sequent derivation d can be transformed into a derivation* **d′** *in weakening normal form. The derivation* **d′** *is not unique.*

(ii) A derivation in weakening normal form has the following properties:

1 every formula introduced by an application of the weakening-rule becomes active only in the premise of an application of the par-rule *where the other active formula has an ancestor in an axiom;*

2 both premises of an application of the times-rule *have ancestors in (different) axioms.*

PROOF. (i) For every instance w of the weakening-rule, permute w below all inferences until either (a) the formula X introduced by w is active in the inference r immediately below, or (b) there is no logical inference below w. In the first case a weakening-reduction applies, unless r is a *par-rule* and the other active formula has not been introduced by a *weakening-rule.* The process is *non-deterministic*: in cases when *weakening / times* reductions to the right and to the left are both applicable, a random choice is required. (ii) By induction on the length of a derivation in weakening normal form. ∎

Remark 9 (i) *Weakening / par reductions* are always reversible, i.e., we have the following expansion rule:

weakening / par *expansion:*

$$\frac{\vdash \Gamma}{\vdash \Gamma, A \wp B}\ weak \qquad \text{expands to} \qquad \cfrac{\cfrac{\cfrac{\vdash \Gamma}{\vdash \Gamma, A}\ weak}{\vdash \Gamma, A, B}\ weak}{\vdash \Gamma, A \wp B}\ par$$

(ii) All other reductions are *irreversible* and produce a genuine loss of information.

(iii) In the sequent calculi for classical or intuitionistic logic the use of the weakening-rule can be standardized in two ways, by prescribing that all applications of the weakening-rule must occur either *as low as possible* or *as high as possible* (i.e., at the level of the *axioms*) in a derivation. In **MAL**, in absence of the *contraction-rule* only the first option is available for *multiplicative* connectives. In particular, in **MAL** + Mix we cannot assume that every formula introduced by Weakening is atomic: consider a cut-free derivation of $\vdash \mathbf{1}, P \otimes P$, where P is an atom different from **1.**

(iv) On the other hand, for *additive* connectives in absence of the *contraction-rule* only the second option is available, i.e., applying the weakening-rule as high as possible.

More precisely, we could define the propositional language of Additive Affine Logic **AAL** using the additive connectives & (*with*) and ⊕ (*plus*):

$$\textit{with:}\ \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}, \qquad \textit{plus}_i:\ \frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_0 \oplus A_1} \qquad i = 0, 1.$$

Proposition 10 *In the sequent calculus for* **AAL** *every sequent derivation* $\mathbf{d}$ *can be transformed into a unique derivation* $\mathbf{d}'$ *with the property that every formula introduced by the weakening-rule is atomic.*

PROOF. Repeatedly apply permissible permutations of inferences and the following *expansions:*

weakening / with *expansion:*

$$\dfrac{\begin{array}{c} d \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A\&B}\ weak \qquad \text{expands to} \qquad \dfrac{\dfrac{\begin{array}{c} d \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A}\ weak \qquad \dfrac{\begin{array}{c} d \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, B}\ weak}{\vdash \Gamma, A\&B}\ times$$

weakening / plus *expansion:*

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, A_0 \oplus A_1}\ weak \qquad \text{expands to} \qquad \dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, A_i}\ weak}{\vdash \Gamma, A_0 \oplus A_1}\ plus$$

∎

3. Additive mix

For all n we have an n-ary rule *Additive Mix* which builds a derivation d of $\vdash \Gamma$ out of n derivations $d_1, \ldots, d_n$ of $\vdash \Gamma$. We use the following notation:

$$\frac{d_1, \ldots, d_n}{\vdash \Gamma}\quad AM(n).$$

We have the following reduction:

Mix-elimination

$$\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ \Delta \end{array}}{\vdash \Gamma, \Delta}\ Mix \qquad \text{reduces to} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{weakenings}}{\vdash \Gamma, \Delta} \qquad \dfrac{\dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{weakenings}}{\vdash \Gamma, \Delta}}{\vdash \Gamma, \Delta}\ AM(2)$$

3.1 Properties of additive mix

The rule *Additive Mix* has the following properties:

1. *The order of the premises is immaterial,* i.e., we assume an *AM-exchange* rule of the form

$$\frac{d_1, \ldots, d_i, d_{i+1}, \ldots d_n}{\vdash \Gamma} \quad \sim \quad \frac{d_1, \ldots, d_{i+1}, d_i, \ldots d_n}{\vdash \Gamma}$$

2. *The derivations of the premises are paiwise distinct,* i.e., we assume an *AM-contraction* rule of the form

$$\frac{d_1, \ldots, d_{i-1}, d_i, d_i, d_{i+1}, \ldots d_n}{\vdash \Gamma} \quad \text{reduces to} \quad \frac{d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots d_n}{\vdash \Gamma}$$

3. *Consecutive applications of AM can be unified,* i.e., we assume an *AM-merging* rule of the form

$$\frac{\dfrac{d_1, \ldots, d_m}{\vdash \Gamma}\, AM(m) \qquad \dfrac{d_{n+1}, \ldots, d_{n+m}}{\vdash \Gamma}\, AM(n)}{\vdash \Gamma}\, AM(2) \quad \text{reduces to} \quad \frac{d_1, \ldots, d_{n+m}}{\vdash \Gamma}\, AM(n+m)$$

4. *The rule AM-contraction can be strengthened according to different notions of identity of proofs.* For instance, let us put $d \sim d'$ if and only if $(d)^-$ and $(d')^-$ are the same proof-net with attachments. Then the *AM-contraction* rule becomes:

$$\frac{d_1, \ldots, d_i, d_i', \ldots d_n}{\vdash \Gamma} \quad \text{where } d_i \sim d_i' \text{ reduces to} \quad \frac{d_1, \ldots, d_i, \ldots d_n}{\vdash \Gamma}$$

5. A sequent derivation in **MAL** + AM is in *AM normal form* if it has at most one application of Additive Mix as the last inference.

Convention. We assume that the *AM-contraction* in the strong form (4) is always applied without mention, and similarly for the *AM-merging* rules. We work with derivations *modulo* the AM-exchange rule.

3.2 Additive mix: reductions and permutations

AM / exchange *permutation:*

$$\cfrac{\cfrac{d_1, \ldots, d_n}{\vdash \Gamma, A, B}\, AM(n)}{\vdash \Gamma, B, A}\, exch \quad \text{reduces to} \quad \cfrac{\cfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, B, A}\, exch \quad \ldots \quad \cfrac{\begin{matrix} d_n \\ \vdots \\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, B, A}\, exch}{\vdash \Gamma, B, A}\, AM(n)$$

AM / weakening *permutation:*

$$\cfrac{\cfrac{d_1, \ldots, d_n}{\vdash \Gamma}\, AM(n)}{\vdash \Gamma, A}\, weak \quad \text{reduces to} \quad \cfrac{\cfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, A}\, weak \quad \ldots \quad \cfrac{\begin{matrix} d_n \\ \vdots \\ \vdash \Gamma\end{matrix}}{\vdash \Gamma, A}\, weak}{\vdash \Gamma, A}\, AM(n)$$

AM / par *permutation:*

$$\dfrac{\dfrac{d_1,\ldots,d_n}{\vdash \Gamma, A, B}\,AM(n)}{\vdash \Gamma, A \wp B}\,par \qquad \text{reduces to} \qquad \dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma, A, B \end{matrix}}{\vdash \Gamma, A\wp B}\,par \quad \ldots \quad \dfrac{\begin{matrix} d_n \\ \vdots \\ \vdash \Gamma, A, B \end{matrix}}{\vdash \Gamma, A\wp B}\,par}{\vdash \Gamma, A \wp B}\,AM(n)$$

AM / times *permutation:*

$$\dfrac{\dfrac{d_1,\ldots,d_m}{\vdash \Gamma, A}\,AM(m) \quad \dfrac{d'_1,\ldots,d'_n}{\vdash B, \Delta}\,AM(n)}{\vdash \Gamma, A \otimes B, \Delta}\,times \qquad \text{reduces to} \qquad \dfrac{\ldots \dfrac{\begin{matrix} d_i \\ \vdots \\ \vdash \Gamma, A \end{matrix} \quad \begin{matrix} d_j \\ \vdots \\ \vdash B, \Delta \end{matrix}}{\vdash \Gamma, A \otimes B, \Delta}\,cut \ldots}{\vdash \Gamma, A \otimes B, \Delta}\,AM(m\cdot n)$$

for $i \leq m, j \leq n$.

AM / cut *permutation:*

$$\dfrac{\dfrac{d_1,\ldots,d_m}{\vdash \Gamma, A}\,AM(m) \quad \dfrac{d'_1,\ldots,d'_n}{\vdash A^\perp, \Delta}\,AM(n)}{\vdash \Gamma, \Delta}\,cut \qquad \text{reduces to} \qquad \dfrac{\ldots \dfrac{\begin{matrix} d_i \\ \vdots \\ \vdash \Gamma, A \end{matrix} \quad \begin{matrix} d_j \\ \vdots \\ \vdash A^\perp, \Delta \end{matrix}}{\vdash \Gamma, \Delta}\,cut \ldots}{\vdash \Gamma, \Delta}\,AM(m\cdot n)$$

for $i \leq m, j \leq n$.

We define a procedure **Sep** of elimination of Mix, which takes a derivation d in **MAL** + Mix and returns a derivation $\mathbf{Sep}(d)$ in **MAL** + AM without Mix, in AM normal form:

1 Given a derivation d and a mix-rule m in d, permute m as low as possible, obtaining a derivation d_m where either m is the last inference or m is immediately followed by a par-rule p such that m cannot be permuted below p.

2 Apply the mix-elimination rule to the mix-rule m in d_m, yielding a derivation d'_m, which contains an application AM(2) of Additive Mix.

3 Permute the inference AM(2) below all inferences of d'_m, thus obtaining a derivation d^*_m ending with an application of Additive Mix.

4 Let $\mathbf{Sep}(d) = d^*_m$ **if** d^*_m contains no mix-rule; otherwise, select a new application of the mix-rule in d^*_m and start again with (1).

The procedure **Sep** has the desirable property that it does not create irrelevant inferences.

Theorem 11 *(i) Let* d *be a derivation of* Γ *in* **MAL** + *Mix. The result of applying the procedure* **Sep** *to* d

$$\mathbf{Sep}(d) = \frac{d_1, \ldots, d_n}{\Gamma}$$

is a derivation in **MAL** + *AM* without *Mix in AM normal form.*

(ii) If d *is in weakening normal form, then so is* $\mathbf{Sep}(d)$.

PROOF. (ii) If at the end of step 1 the mix-rule m is the last inference, then no irrelevant inference is created at steps (2) or (3), so d^*_m is still in weakening normal form. Otherwise at the end of step (1) the Mix m is followed by some *par-rule* $p_1, \ldots, p_k$ which are not permutable above m, therefore none of their active formulas is introduced by a weakening-rule; since d is in weakening normal form, it follows from Lemma 8.(ii) that all the formulas active in all p_i have ancestors in axioms. Therefore at the end of step (2) the Additive Mix AM(2) is immediately followed by an application of the par-rule which is not irrelevant and, moreover, after one application of the *AM / par* permutation both conclusions of the new par-rules have ancestors in some axiom. Finally, if a times-rule t occurs below m in d_m at the end of step (1), then a formula active in t is descendant of the conclusion of one of the p_i: otherwise, in step 1 t would be permutable above m. The same holds for cut-rules occurring in d_m below m. It follows that the Additive Mix permutations at step (3) do not introduce irrelevant inferences, i.e., d^*_m is still in weakening normal form. ∎

4. Proof-nets for MAL + Mix

4.1 Proof-structures and proof-nets

Definition 12 (i) A *proof-structure* for **MAL** + Mix is a directed graph whose edges are labelled with formulas and whose vertices (*links*) are of one of the forms in Figure 3.1.

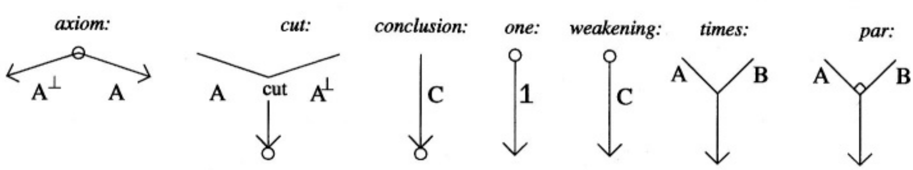

Figure 3.1. Links

In link the incoming edges are called the *premises* and the outgoing edges are the *conclusions* of the link. The transitive closure $\prec$ of the relation between links (equivalently, between edges) determined by the direction of the edges is a partial ordering (since the edges are typed with propositional formulas) and it is called the *structural orientation.* $A \prec B$ reads "*A* is a hereditary premise of *B*" or simply "*A* is above *B*". A *bottom* link is just a special case of a *weakening-link.*

(ii) Given a proof-structure $\mathcal{S}$ for **MAL** + Mix an *attachment* of a *weakening-link* w is an edge from the weakening-link to a new vertex which lies in another edge ℓ of $\mathcal{S}$. We write $\mathcal{S}^a$ for $\mathcal{S}$ toghether with a set of attachments for all the weakening-links of $\mathcal{S}$. The geometric properties of attachments are fully determined as follows:

- a *weakening-link* w with its attachment $a(w)$ has the same properties as a *logical axiom*;
- the new vertex where the attachment $a(w)$ and the edge ℓ meet has the same properties as a *times-link.*

(iii) A *switching* for a proof-structure $\mathcal{S}$ is a choice for every *par-link* in $\mathcal{S}$ of one of its premises.

(iv) Let $\mathcal{S}^a$ be a proof-structure *with attachments* for the weakening-links. Given a switching s for $\mathcal{S}^a$, the *Danos–Regnier graph* $s\mathcal{S}^a$ is the graph resulting from $\mathcal{S}^a$ by deleting from each *par-link* the edge which is *not* the premise chosen by s.

(v) A proof-structure $\mathcal{S}^a$ with attachments for the weakening-links is a *proof-net* if for every switching s the Danos–Regnier graph $s\mathcal{R}^a$ is acyclic (*correctness criterion* for **MAL** + Mix).

Theorem 13 *(i) There exists a map* $(\;)^-$ *from sequent derivations to proof-structures together with choices* $a_1, \ldots, a_r$ *of attachments for the weakening-links such that* $(d)^- = \mathcal{R}^{a_i}$ *is a proof-net, the attachments being given by any* a_i*, far* $i \leq r$.

(ii) (Sequentialization Theorem) Conversely, if $\mathcal{R}^a$ *is a proof-net, then there exists a sequent derivation* $\mathbf{d}$ *and a choice* $a_1, \ldots, a_n$ *of attachments for the weakening-links such that* $(\mathbf{d})^- = \mathcal{R}^a$*, and* $a = a_i$ *for some* $i \leq n$.

PROOF. (ii) For every *weakening-link* w in $\mathcal{R}^a$, let *A* be its conclusion and let ℓ be the edge, labelled with *B*, which $a(w)$ is attached to: replace w, $a(w)$ and ℓ with a logical axiom *ax* followed by a *times-link* t. The proof-structure $\mathcal{R}'$ thus obtained is labelled with different formulas than $\mathcal{R}^a$, but it has exactly the same geometric properties, so it is a proof-net in $\mathbf{MLL}^-$ + Mix and can be sequentialized as usual. In a sequent derivation d' such that $(d')^- = \mathcal{R}'$, let t' be the times-rule whose principal formula $A^\perp \otimes B$ corresponds to the new times-link t. Since the only active ancestor of $A^\perp$ is in the axiom *ax,* we may permute t' upwards to the left so that *ax* and t' are consecutive inferences and then replace them both with a *weakening-rule*:

$$\dfrac{\vdash A, A^{\perp} \quad \begin{array}{c}\vdots\\ \vdash B, \Gamma\end{array}}{\vdash A, A^{\perp} \otimes B, \Gamma} \quad \text{is replaced by} \quad \dfrac{\begin{array}{c}\vdots\\ \vdash B, \Gamma\end{array}}{\vdash A, B, \Gamma}$$

By repeating this procedure for all weakening-links of $\mathcal{R}^a$ we clearly obtain a derivation d such that $(d)^- = \mathcal{R}^a$. ∎

4.2 Computation and elimination of irrelevance

One of the uses of proof-nets for multiplicative logic *without Weakening* $\mathbf{MLL}^-$, is to *classify sequent derivations:* we prove that given sequent derivations d and d' in $\mathbf{MLL}^-$ we have $(d)^- = (d')^-$ if and only if there exists a sequence of derivations $d = d_1, d_2, \ldots, d_n = d'$ such that for all $i < n$, d_i and d_{i+1} differ only for a permutation of two consecutive inferences (cf. [6]). A similar result holds for $\mathbf{MLL}^-$ + Mix (cf. [3]).

In presence of the weakening-rule, thus already in **MLL** with Weakening restricted to a $\perp$-**rule**, we do not know how to obtain such a theorem. In [11] proof-nets are defined using *weakening-boxes,* thus the position of each weakening-rule in a given sequential proof d is fixed in the proof-net $(d)^-$.

The standard solution, which we have followed above, is the use of *attachments* for the weakening-links. But from the point of view of the classification of sequent derivations the notion of proof-nets with attachments is perhaps worse that that of proof-nets with weakening-boxes: given a box, there are several ways of making the attachment. In any event, this notion fails to identify sequential proofs *modulo permutations of weakening-rule.*

We turn now to the notion of a proof-net *modulo irrelevance,* which at least succeeds in minimizing the disturbances caused by weakening-links. As recalled in the Introduction, given a proof-structure $\mathcal{S}$ without attachments for the weakening-links and satisfying the acyclicity property (of every Danos–Regnier graph), the problem of deciding whether attachments may be added to the weakening-links of $\mathcal{S}$ so that the resulting proof-structure $\mathcal{S}'$ is a proof-net is NP-complete in the strong sense [16]. On the other hand in a proof-net free from irrelevance the weakening-links can only be attached to a premise of a par-link or to a conclusion. Moreover, the following algorithm identifies and possibly eliminates irrelevance in linear time.

Definition 14 (*computation of irrelevance algorithm,* cf. [11, 4]) Let $\mathcal{S}$ be a proof-structure *without attachments.*

(i) The *irrelevant part* $i(w)$ of $\mathcal{S}$ *determined by* a weakening-link w is the smallest subgraph closed under the following rules:

- the weakening-link w and its conclusion are in $i(w)$;
- if the premises A and B of a *par-link* v are in $i(w)$, then v and its conclusion $A \wp B$ are in $i(w)$;
- if either premise A or B of a *times-link* v is in $i(w)$, then v and its conclusion $A \otimes B$ are in $i(w)$; similarly, if A and B are premises of a *cut-link*;
- if $u \prec v$ and v is in $i(w)$, then u and its conclusions are in $i(w)$; in particular, if u is a *logical axiom* in $i(w)$, then both its conclusions are in $i(w)$.

(ii) The *doors* of $i(w)$ in $\mathcal{S}$ are the edges which are in $i(w)$ and are premises of links whose conclusions are not in $i(w)$. Clearly all the doors of $i(w)$ are either conclusions of $\mathcal{S}$ or premises of *par-links.*

(iii) Let w be a weakening-link in $\mathcal{S}$. Write $\mathcal{S}^w$ for $\mathcal{S} \setminus i(w)$, with the addition of a weakening-link w_D with conclusion D for every door D of $i(w)$. If $\mathcal{S}$ has attachments and w is attached to an edge ℓ through $a(w)$, then we may assume that $\ell \notin i(w)$, otherwise such an attachment would generate a cyclic D–R-graph and $\mathcal{S}$ could not be a *proof-net.* Therefore we may define a set of attachments a' for $\mathcal{S}^w$ as follows:

- $a'(w') = a(w')$, if w' occurs both in $\mathcal{S}$ and in $\mathcal{S}^w$;
- $a'(w_D)$ is attached to ℓ, for all doors D of $i(w)$, otherwise.

(iv) The *pruning map* given by $\mathcal{S} \mapsto \mathcal{S}^w$ for $w \in \mathcal{S}$ may be regarded as a notion of reduction, which has the Church–Rosser property.

Lemma 15 *Let w_1 and w_2 be weakening-links in $\mathcal{S}$. Then $\mathcal{S}^{w_1 w_2} = \mathcal{S}^{w_2 w_1}$.*

PROOF. Let D_1 be a door of $i(w_1)$, thus a premise of a par-link p. Suppose that the other premise D_2 of the same par-link is in $i(w_2)$ and apply the irrelevance computation algorithm and eliminate $i(w_2)$ in $\mathcal{S}^{w_1}$ and $i(w_1)$ in $\mathcal{S}^{w_2}$. In both cases the conclusion D of p is included in the irrelevant part $i(w_2)$ or $i(w_1)$ and the par-link p is removed both in $\mathcal{S}^{w_1 w_2}$ and in $\mathcal{S}^{w_2 w_1}$. ∎

Definition 16 (Definition 14 cont.) (v) Let $w_1, \ldots, w_n$ be all the weakening-links in $\mathcal{S}$. By Lemma 15 $\mathcal{S}^{w_1, \ldots, w_n} = \mathcal{S}^{w_{\sigma(1)}, \ldots, w_{\sigma(n)}}$ for any permutation σ of $1, \ldots, n$. The *irrelevant part* $I(\mathcal{S})$ of the proof-structure $\mathcal{S}$ is defined as

$\mathcal{S} \setminus \mathcal{S}^{w_1,\dots,w_n}$, where $w_1, \dots, w_n$ are all the weakening-links of $\mathcal{S}$. A proof-structure $\mathcal{R}$ is *irrelevance free* if $\mathcal{R} = \mathcal{R} \setminus I(\mathcal{R})$. We write $\mathbf{P}(\mathcal{R}) = \mathcal{R} \setminus I(\mathcal{R})$ for the *pruning operation* given by the computation of irrelevance algorithm.

(vi) A proof-structure $\mathcal{R}$ for **MAL** + Mix is a *proof-net modulo irrelevance* if $\mathcal{R} \setminus I(\mathcal{R})$ is non-empty and $\mathcal{R}$ (without attachments) satisfies the acyclicity condition for all D–R graphs.

Remark 17 Notice that if $\mathcal{S} = \mathbf{P}(\mathcal{R})$, then a *canonical attachment* a for $\mathcal{S}$ is given simply by a choice of a conclusion C to which we may attach all conclusions of $\mathcal{S}$ which are introduced by a weakening-link. Indeed all conclusions of a weakening-link which are premises of a par-link may be canonically attached to the other premise ℓ (which is not introduced by a weakening-link). Therefore if $\mathcal{S}$ has just one conclusion, there is only one canonical attachment a and we may omit mentioning it.

Theorem 18 *There exists a 'context-forgetting' map* $(\,)^{-}$ *from sequent derivations in* **MAL** + *Mix to proof-nets with the following properties:*

(i) Let $\mathbf{d}$ *be a sequent derivation, and let* $\mathcal{R}^a$ *be the proof-net such that* $(d)^{-} = \mathcal{R}^a$. *Then* $\mathbf{P}(\mathcal{R}^a)$ *is non-empty (in fact, it contains at least one axiom and at least one conclusion); if* $\mathbf{d}$ *is in weakening normal form, then* $\mathbf{P}(\mathcal{R}) = \mathcal{R}$

(ii) (Sequentialization) If $\mathcal{R}$ *is a proof-net modulo irrelevance, then there is a sequent calculus derivation d in weakening normal form such that* $\mathbf{P}(\mathcal{R}) = (d)^{-}$.

PROOF. (see [4] for the special case **MAL** without Mix.) (i) By induction on the length of d. (ii) In $\mathbf{P}(\mathcal{R})$ weakening-links can be given canonical attachments: as a consequence, the usual proof of sequentialization for $\mathbf{MLL}^{-}$ + Mix goes through without the *detour* used in the proof of Theorem 13 (ii). ∎

Corollary 19 *Provability in Constant-only* **MAL** + *Mix is in* **P.**

PROOF. In Constant-only **MAL** we may consider only proofs whose axioms are **1**-axioms (a logical axiom can be replaced by a **1** axiom together with a $\bot$-*weakening*). Thus to test whether $\vdash \Gamma$ is provable, we need only to check whether the proof-structure $\mathcal{R}$ which consists of the tree of subformulas of Γ is a proof-net *modulo irrelevance.* To decide this in linear time, apply the computation of irrelevance algorithm to $\mathcal{R}$ and check whether $\mathcal{R} \neq I(\mathcal{R})$. ∎

4.3 Examples and properties of irrelevance elimination

We need to justify our algorithm for the computation of irrelevance. For this purpose we recall some facts about the structure of subnets of proof-nets in $\mathbf{MLL}^-$ with Mix (cf. [3]).

Definition 20 (i) A *non-logical axiom* in a proof-structure is a link with no premise and n conclusions, for some n. Danos–Regnier graphs for proof-nets with non-logical axioms are defined as before and so is the notion of a *proof-net with non-logical axioms* for **MAL** + Mix (with attachment of *weakening-link*).

(ii) Let $\mathcal{R}^a$ be a proof-structure for **MAL** + Mix and let $\mathcal{S}$ be a substructure of $\mathcal{R}^a$ with conclusions $C_1, \ldots, C_n$. The *complementary substructure* $\overline{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{R}^a$ consists of all edges and links in $\mathcal{R}^a \setminus \mathcal{S}$ with the addition of a non-logical axiom with conclusions $C_1, \ldots C_n$.

(iii) Let $\mathcal{R}^a$ be a proof-net for **MAL** + Mix. A subnet $\mathcal{S}$ of $\mathcal{R}^a$ is a *normal subnet* if the complementary substructure $\overline{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{R}^a$ is a proof-net with a non-logical axiom. A normal subnet $\mathcal{S}$ of $\mathcal{R}^a$ has the property that there exists a sequent calculus derivation d and a subderivation d_0 of d such that $(d)^- = \mathcal{R}^a$ and $(d_o)^- = \mathcal{S}$.

(iv) Let $\mathcal{R}^a$ be a proof-net for **MAL** + Mix; let *A* be an edge and let v_0 be a link in $\mathcal{R}^a$. The *kingdom* kA [or the *empire* eA] of *A* in $\mathcal{R}^a$ is the smallest [the largest] *normal* subnet of $\mathcal{R}^a$ which has *A* as a conclusion. The *kingdom* kv_0 [or the *empire* ev_0] of v_0 in $\mathcal{R}^a$ is the smallest [the largest] *normal* subnet of $\mathcal{R}^a$ which has v_0 as a lowermost link (so that the conclusions of v_0 are all conclusions of $\mathcal{R}^a$).

(v) The kingdom of *A* in $\mathcal{R}^a$ is the smallest set closed under the inductive conditions:

1. $A \in kA$;

2. If v is an axiom with conclusions $X_1, \ldots, X_n$, then

   $kv = kX_1 = \ldots = kX_n;$

3. If v is a *times-link* with premises *X, Y,* then

   $kv = k(X \otimes Y) = kX \cup kY \cup \{X \otimes Y\};$

4. If v is *a par-link* with premises *X, Y,* then

   $kv = k(X \wp Y) = \bigcup_s \bigcup_{Z \in \mathbf{path}_s(X,Y)} kZ \cup \{X \wp Y\},$

   where s ranges over all switchings of $\mathcal{R}$.

(v) The empire of a link v in $\mathcal{R}^a$ is the smallest subnet of $\mathcal{R}^a$ containing the set of all links v' such that for no switching s there is a path $path_s(v, v')$ reaching v "from below" (in the structural orientation).

Remark 21 (i) The computation of irrelevance algorithm was presented first in Section 7.5 of [4]. It is similar to Girard's characterization of the *empire* of a formula in a proof-net for $\mathbf{MLL}^-$, cf. Facts 2.9.4 in [11]. More precisely, for any edge A in a proof-structure $\mathcal{R}$ write e^-A for the part of $\mathcal{R}$ satisfying Girard's characterization. Now let w be any weakening-link in a proof-structure $\mathcal{R}$ for **MAL** without attachments and let us introduce an attachment $a(w)$ connecting w to some other edge. Then $i(w) = e^-(a(w))$, i.e., $i(w)$ may be regarded as the result of applying Girard's algorithm for the empire of the edge $a(w)$. Notice that $e^-(A)$ has different properties in $\mathbf{MLL}^-$ and in $\mathbf{MLL}^-$ + Mix. In $\mathbf{MLL}^-$ + Mix e^-A is a subnet, but not necessarily a *normal* subnet (cf. [3] Section 2.3), i.e., it may not be possible to find a sequentialization d of $\mathcal{R}$ such that e^-A corresponds to a subderivation of d; this remains true in the case of **MAL** + Mix as it is shown by the following examples.

(ii) Consider the proof-net in Figure 3.2 with conclusions $(\mathbf{1}⅋B) \otimes (\mathbf{1}⅋C)$, $(C^{\perp}⅋B^{\perp}) \otimes A$, $\mathbf{1}$, where the irrelevant part (marked with a solid broken line) is a subnet which *is not normal*:

Example 22

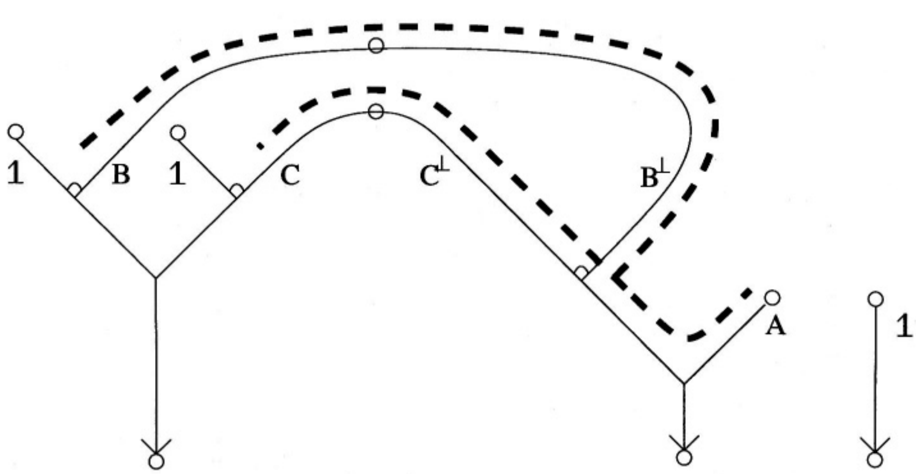

Figure 3.2. Prune

A sequentialization of such a proof-net is the following derivation $\boldsymbol{d}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathbf{1} \quad \vdash B, B^\perp}{\vdash \mathbf{1}, B, B^\perp}\,Mix}{\vdash \mathbf{1}\wp B, B^\perp} \quad \dfrac{\dfrac{\vdash \mathbf{1} \quad \vdash C, C^\perp}{\vdash \mathbf{1}, C, C^\perp}\,Mix}{\vdash \mathbf{1}\wp C, C^\perp}}{\vdash (\mathbf{1}\wp B)\otimes(\mathbf{1}\wp C), C^\perp, B^\perp}}{\vdash (\mathbf{1}\wp B)\otimes(\mathbf{1}\wp C), C^\perp\wp B^\perp} \quad \dfrac{\vdash \mathbf{1}}{\vdash \mathbf{1}, A}}{\vdash (\mathbf{1}\wp B)\otimes(\mathbf{1}\wp C), (C^\perp\wp B^\perp)\otimes A, \mathbf{1}}$$

The proof-net obtained by pruning the irrelevant part is sequentialized in the following derivation, which is not obtainable from $\boldsymbol{d}$ by permutation of inferences:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathbf{1}}{\vdash \mathbf{1}, B}}{\vdash \mathbf{1}\wp B} \quad \dfrac{\dfrac{\vdash \mathbf{1}}{\vdash \mathbf{1}, C}}{\vdash \mathbf{1}\wp C}}{\vdash (\mathbf{1}\wp B)\otimes(\mathbf{1}\wp C)} \quad \dfrac{\vdash \mathbf{1}}{\vdash (C^\perp\wp B^\perp)\otimes A, \mathbf{1}}}{\vdash (\mathbf{1}\wp B)\otimes(\mathbf{1}\wp C), (C^\perp\wp B^\perp)\otimes A, \mathbf{1}}\,Mix$$

(iii) Suppose *irrelevance computation algorithm* had been defined using the notion of *kingdom*: if the conclusion $A\wp B$ of a *par-link* is in $i(w)$, then we should let $k(A\wp B) \subset i(w)$. We show that this procedure, regarded as a notion of reduction, does not have the Church–Rosser property. Consider the following proof-net with conclusions $\mathbf{1}$, $X \otimes (A^\perp\wp B^\perp)$, $(B\wp C)\otimes(A\wp D^\perp)$, $(D\wp(G^\perp\wp G))\otimes((H\wp H^\perp)\wp E)$, $E^\perp \otimes F$, $(F\wp C^\perp)\otimes Y$, $\mathbf{1}$.

Example 23

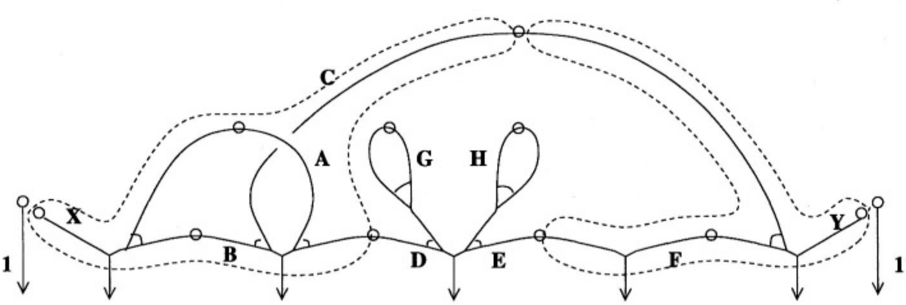

Figure 3.3. Big prune

(a) If we eliminate irrelevance starting from the weakening-link with conclusion *X*, then $k(A^\perp\wp B^\perp)$ is included in $i(X)$ and pruned first, so the link $(B\wp C)\otimes(A\wp D^\perp)$ is removed. Next we proceed to the weakening-link with conclusion *Y*, so $k(F^\perp\wp C^\perp)$ is pruned: but now the only links in it are the par-link itself, a weakening-link with conclusion $C^\perp$ and an axiom $\overline{F, F^\perp}$. Finally we prune $i(F)$ and we are left with a non-empty subnet with conclusions $(D\wp(G^\perp\wp G)) \otimes ((H\wp H^\perp)\wp E)$, in addition to two **1** links (and weakening-links).

(b) If we eliminate irrelevance starting with $i(Y)$ then $k(F^\perp\wp C^\perp)$ is pruned, and the times-links $(B\wp C)\otimes(A\wp D^\perp)$, $(D\wp(G^\perp\wp G))\otimes((H\wp H^\perp)\wp E)$ and $E^\perp \otimes F$ are removed as they belong to $k(F^\perp\wp C^\perp)$. Therefore after pruning $i(X)$ we are left with nothing else than two **1**-links (and weakening-links).

Notice that our official computation of irrelevance algorithm yields in the more conservative pruning which is also given by (a).

4.4 Mix-elimination

Definition 24 Let $\mathcal{R}^{a_0}$, $\mathcal{S}^{a_1}$ be proof-nets with attachments for *weakening-links.*

(i) A map $\varphi : \mathcal{R} \to \mathcal{S}$ *preserves links* (different from weakening-links) if it is a morphism of labelled directed graphs with respect to the links other than weakening-links. In other words, for every vertex v other than a weakening-link in $\mathcal{R}$, if v has incoming arrows labelled $A_1, \ldots, A_m$ and outgoing arrows labelled $B_1, \ldots, B_n$, then $\varphi(v)$ is a vertex of $\mathcal{R}'$ with incoming arrows also labelled $A_1, \ldots, A_m$ and outgoing arrows also labelled $B_1, \ldots, B_n$. However, φ may map a *weakening-link* to a link of another kind.

(ii) A one-to-one map of labelled graphs $\varphi : \mathcal{R}^a \to \mathcal{S}^b$ is an *embedding* if it preserves links and whenever a *weakening link* w of $\mathcal{R}^a$ has a non-canonical attachment $a(w) = \ell$, then $b(\varphi(w)) = \varphi(\ell)$.

(iii) A *covering* of $\mathcal{R}^a$ is a set of embeddings $\varphi_i : \mathcal{S}_i^{a_i} \to \mathcal{R}^a$, for $i \leq n$, such that every edge in $\mathcal{R}$ is in the image of some φ_i.

(iv) The same definitions obviously apply to proof-nets without attachments.

We are going to define a procedure **S** that given a proof-net $\mathcal{R}^a$ for **MAL** + Mix generates a covering $\{\varphi_i : \mathcal{S}_i^{a_i} \to \mathcal{R}^a\}_{i \leq n}$ (cf. [15, 4]).

(I) Let *C* be a conclusion of $\mathcal{R}^a$. We generate a data structure *chain*(*C*) as follows:

1 Select a link v with no premises such that $v \prec C$ and let $path_{C,v} \subset$ *chain*(*C*);

  i. if v is a logical axiom or a **1** axiom, then go to step (2);

  ii. if v is a weakening-link w, then consider the edge ℓ which $a(w)$ is attached to: if ℓ' is the lowermost edge such that $\ell \preceq \ell'$ and

$\ell' \notin chain(C)$ then let $\mathbf{path}_{\ell',\ell} \subset chain(C)$; finally, repeat step (1), choosing a link v' with no premises such that $v' \prec \ell$.

2 For every *times-link* or *cut-link* t in *chain*(*C*), select a link v with no premises such that $v \prec t$ and let $\mathbf{path}_{t,v} \subset chain(C)$; if v is a weakening-link, then go to step (1.ii.);

3 For every logical axiom *ax* in *chain*(*C*), if B is a conclusion of *ax* which is *not* in *chain*(*C*), then let ℓ be the lowermost edge such that $B \preceq \ell$ and $\ell \notin chain(C)$ and let $\mathbf{path}_{\ell,ax} \subset chain(C)$. Then return to step (2).

(II) We repeat this procedure making different choices at steps 1, 2, 3 and with different conclusions C', until no new chain is generated. Let $\{\mathbf{chain}_1^{a_1}, \ldots, \mathbf{chain}_n^{a_n}\}$ be the chains eventually obtained, where a_i is the restriction of a to $\mathbf{chain}_i$.

(II) For $i \leq n$, we transform $\mathbf{chain}_i$ into a proof-structure $\mathcal{S}_i^{a_i}$ by adding weakening-links with canonical attachments:

- let p be a vertex in $\mathbf{chain}_i$ corresponding to a par-link of $\mathcal{R}$ such that one premise ℓ is in $\mathbf{chain}_i$, but the other premises C is not: introduce a weakening-link w labelled C in $\mathbf{chain}_i$, together with an attachment $a_i(w)$ to the premise ℓ:
- let C' be a conclusion of $\mathcal{R}$ which does not belong to $\mathbf{chain}_i$: introduce a weakening-link w labelled C' in $\mathbf{chain}_i$, with an attachment $a_i(w)$ to a selected conclusion C.

(IV) Let $\{\mathcal{S}_1^{a_1}, \ldots, \mathcal{S}_n^{a_n}\}$ be the data thus obtained. We denote by $\mathbf{S}$ the operation such that $\mathbf{S}(\mathcal{R}) = \{\mathcal{S}_1^{a_1}, \ldots, \mathcal{S}_n^{a_n}\}$. Let $\mathcal{F} = \{\mathcal{S}_1^{a_1}, \ldots, \mathcal{S}_n^{a_n}\}$: we write $\mathbf{P}(\mathcal{F}) = \{\mathbf{P}(\mathcal{S}_1^{a_1}), \ldots, \mathbf{P}(\mathcal{S}_n^{a_n})\}$.

Theorem 25 *(i) Let $\mathcal{R}^a$ be a proof-net with conclusions Γ in* **MAL** + *Mix. The operation $\mathbf{S}(\mathcal{R}) = \{\mathcal{S}_1^{a_1}, \ldots, \mathcal{S}_n^{a_n}\}$ yields a covering $\{\varphi_i : \mathcal{S}_i^{a_i} \to \mathcal{R}^a\}_{i \leq n}$, where the $\mathcal{S}_i^{a_i}$ are proof-nets in* **MAL** without Mix.

(ii) $\mathbf{PS}(\mathcal{R}^a) = \mathbf{SP}(\mathcal{R}^a)$.

PROOF. (i) Let $\mathcal{R}^a$ be a proof-net with conclusions Γ. Each $\mathcal{S}_i$ is a proof-structure: the inclusion map $\iota : \mathcal{S}_i \to \mathcal{R}$ preserves *times-links* and *cut-links* by step (I.2.), *axiom* links by step (I.3.) and each $\mathcal{S}_i^{a_i}$ has conclusions Γ by step (III). Moreover $\iota : \mathcal{S}_i^{a_i} \to \mathcal{R}^a$ is an embedding because the attachments a_i contain the restriction of a to $\mathbf{chain}_i$ and all other weakening-links have canonical attachments. Also by construction it is clear that every edge v in $\mathcal{R}^a$ is in the image of some φ_i. Therefore $\mathbf{S}(\mathcal{R})$ yields a covering.

Each $\mathcal{S}_i^{a_i}$ is a proof-net: a cyclic Danos–Regnier graph in $\mathcal{S}_i^{a_i}$ would be one in $\mathcal{R}$. We need to show that $\mathcal{S}_i^{a_i}$ is a proof-net in **MAL** without Mix. The number of axioms, *times-links* and *cut-links* in $\mathcal{S}_i^{a_i}$ is determined by the procedure **S**, which after the first axiom selects an axiom for each *times-link* or *cut-link* encountered. Therefore by Lemma 4 any sequentialization of $\mathcal{S}_i^{a_i}$ is a derivation with no mix-rule.

(ii) Notice that $\mathbf{SP}(\mathcal{R}^a) \subset \mathbf{S}(\mathcal{R}^a)$. Indeed the chains obtained by the procedure **S** applied to $\mathbf{P}(\mathcal{R}^a)$ are also obtained by the procedure **S** applied to $\mathcal{R}^a$ starting with a conclusion C in $\mathbf{P}(\mathcal{R}^a)$ and remaining in $\mathbf{P}(\mathcal{R}^a)$ by selecting a logical axiom or a **1** axiom at steps (1) and (2) of the procedure; this is always possible since $C \in \mathbf{P}(\mathcal{R}^a)$. The chains $chain_1, \ldots, chain_k$ obtained in this way will be called *basic chains.* Let $\mathbf{SP}(\mathcal{R}) = \{\mathcal{S}_1^{a_1}, \ldots, \mathcal{S}_k^{a_k}\}$.

Furthermore notice that the procedure **P** applied to $\mathbf{S}(\mathcal{R}^a)$ transforms chains into basic chains. Indeed computing and eliminating irrelevance from a link w considered at a step (I.ii.) has the effect of removing the part of the chain visited up to that step: but this would also have been achieved by starting with the conclusion C' below ℓ' and by selecting $path_{C',\ell'}$ and so on. We conclude that **P** transforms $\mathbf{S}(\mathcal{R}^a)$ into $\mathbf{SP}(\mathcal{R}^a)$, and therefore $\mathbf{PS}(\mathcal{R}^a) = \mathbf{SP}(\mathcal{R}^a)$. ∎

Definition 26 Let $\mathcal{R}^a$ be a proof-structure for **MAL** + Mix such that $\mathcal{R} \neq I(\mathcal{R})$. Let *ax* be an axiom of $\mathcal{R}^a$ and let $\mathcal{R}^a_{ax}$ be the result of replacing *ax* in $\mathcal{R}^a$ with two weakening-links if *ax* is a logical axiom or with one weakening-link if *ax* is a **1** axiom. The set of axioms in $\mathcal{R}^a$ is *minimal* if for every *ax* in $\mathcal{R}^a$, $\mathcal{R}_{ax} = I(\mathcal{R}_{ax})$.

We give a direct proof that the connectedness condition of all D–R graph is equivalent to the minimality condition on axioms. We need the following fact:

Fact 27 *Given a proof-net with attachments $\mathcal{R}^a$ for* **MAL** + *Mix and any D–R switching s, any connected component of $s\mathcal{R}^a$ containing a weakening-link w contains also an axiom of $\mathcal{R}^a$, which is reached from w by crossing attachments and then always proceeding upwards in the structural orientation.*

PROOF. Let the attachment $a(w)$ be connected to an edge ℓ_1; let $\mathcal{S}_{\ell_1}$ be the smallest substructure ending with ℓ_1 and consider the uppermost links in $\mathcal{S}_{\ell_1}$ which are connected to ℓ_1 in $s\mathcal{R}^a$. If any such vertex is an axiom link, then we are done. Otherwise given a *weakening-link* w_1, let ℓ_2 be the edge which $a(w_1)$ is connected to, and consider the substructure ending with ℓ_2, and so on. In every case we proceed upwards in the structural orientation. Eventually we must reach an axiom. ∎

Proposition 28 ([4], Section 6) *Let $\mathcal{R}^a$ be a proof-net for* **MAL** + *Mix.* $\mathcal{R}^a$ *be a proof-net for* **MAL** *without Mix if and only if its set of axioms is minimal. Thus the minimality condition on axioms is equivalent to the condition of connectedness of all D–R graphs.*

PROOF. Since $\mathcal{R}^a$ is a proof-net, by Theorem 18 $I(\mathcal{R}) \neq \mathcal{R}$. Suppose $s\mathcal{R}^a$ is disconnected, for some switching s. If $\mathcal{R}^a$ consists of disconnected proof-structures, then the removal of an axiom in one substructure does not affect the computation of irrelevance in another disconnected substructure so $\mathcal{R}^a$ does not satisfy the minimality condition on axioms.
If $\mathcal{R}^a$ as a proof-structure is connected, then we may assume that there are par-links $p_1, \ldots p_k$ such that their premises belong to different connected components of $s\mathcal{R}^a$. Select a switching s_1 so that if the D–R graph determined by s_1 reaches a premise ℓ_1 of p_i then s_1 choses the other premise ℓ_2 and a switching s_2 which makes the opposite choices. Therefore the switchings s_1 and s_2 determine two connected components α and β of $s_1\mathcal{S}_i^{a_i}$ and $s_2\mathcal{S}_i^{a_i}$ respectively, where α and β contain different premises of the links p_i. It is easy to see that the removal of one axiom in α may at most make α irrelevant, but not β.

Conversely, if every D–R graph $s\mathcal{R}^a$ is connected, then $\mathcal{R}^a$, as a proof-net with attachments, has properties similar to those of proof-nets for $\mathbf{MLL}^-$, and it is easy to show by induction on the ordering of the kingdoms (cf. Lemma 3 in [6]) that the removal of one axiom in $\mathcal{R}^a$ induces $\mathcal{R}^a = I(\mathcal{R}^a)$. ∎

Example 29 (See Figure 3.4.)

Example 29 shows a proof-net $\mathcal{R}$ in $\mathbf{MLL}^-$ + Mix (top figure), representing a proof of

$$\vdash D \wp C^\perp, C \otimes (B^\perp \wp (B \wp A)), (A^\perp \wp D^\perp) \otimes E, E^\perp,$$

together with $\mathbf{S}(\mathcal{R})$, namely, $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$ (below, from left to right). Here:

- $\mathcal{S}_1$ is determined by the chain which starts either with $D \wp C^\perp$ or with $(A^\perp \wp D^\perp) \otimes E$ and reaches the axioms $\overline{D, D^\perp}$, and $\overline{E, E^\perp}$;
- $\mathcal{S}_2$ is determined by the chain which starts either with $D \wp C^\perp$ or with $C \otimes (B^\perp \wp (B \wp A))$, reaches the axioms $\overline{C^\perp, C}$ and chooses $\overline{B^\perp, B}$;
- $\mathcal{S}_2$ is determined either (i) by the chain which starts with $C \otimes (B^\perp \wp (B \wp A))$, reaches the axioms $\overline{C^\perp, C}$ and $\overline{E, E^\perp}$ after choosing $\overline{A, A^\perp}$ or (ii) by the chain which starts with $(A^\perp \wp D^\perp) \otimes E$ and reaches the axioms $\overline{E, E^\perp}$, $\overline{A, A^\perp}$ and $\overline{C^\perp, C}$.

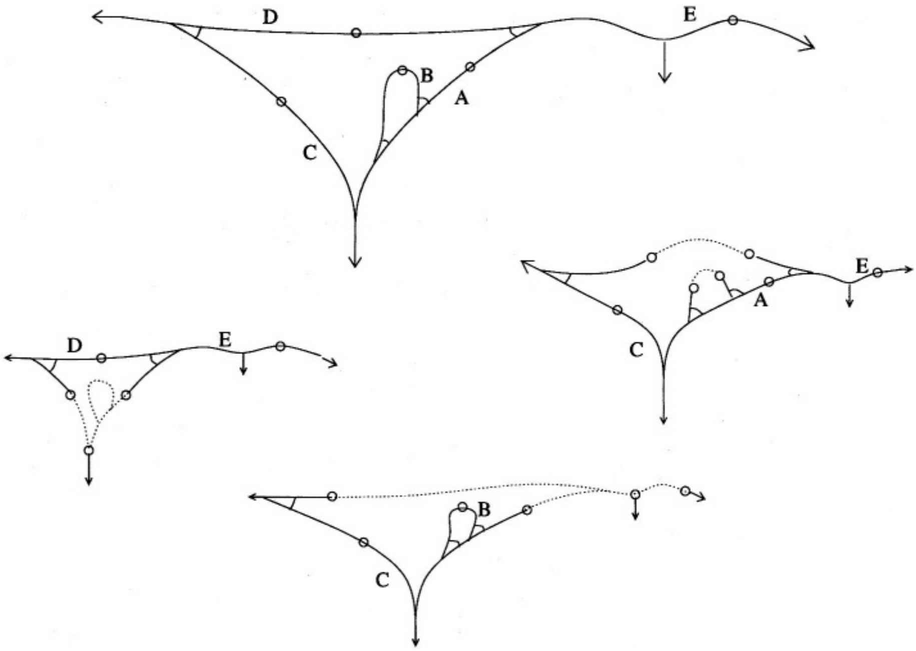

Figure 3.4. Cover

5. Cut-elimination modulo irrelevance

In this section we define a procedure of cut-elimination for proof-nets *modulo irrelevance*: this notion of reduction is based on a notion of *weakening-expansion* which applies to the irrelevant part of a proof-net, but is not admissible in general. Therefore this notion of reduction does not extend to proof-nets with attachments of the *weakening-links*.

It is essential to notice that the notion of the *irrelevant part* of a proof-net is highly unstable under our notion of reduction: if $\mathcal{R}_1$ reduces to $\mathcal{R}_2$ and $\mathcal{R}_2$ reduces to $\mathcal{R}_3$, it may very well be the case that $I(\mathcal{R}_1) \subset I(\mathcal{R}_2)$ but also $I(\mathcal{R}_2) \supset I(\mathcal{R}_3)$. Therefore in the cut-elimination process we will *mark* the irrelevant part of a proof-net, not *remove* it.

The cut-reduction for proof-nets *modulo irrelevance* for **MAL** + Mix (Figure 3.5) are the same as those of **MLL** without units. In addition, there is the *two weakenings / cut* reduction which annihilates a cut-link whose premises are both conclusions of weakening-links. Moreover there are the *weakening-expansions*:

- if a premise $A \otimes B$ [$A^{\perp} \wp B^{\perp}$] of a *cut-link* is conclusion of a *times-link* [*par-link*] and the other premise of the cut-link is conclusion of a *weakening-link* w, then w is replaced by a *par-link* [*times-link*] whose premises are conclusions of *weakening-links* w_1 and w_2.

Theorem 30 *(i) If $\mathcal{R}$ is qproof-net modulo irrelevance and $\mathcal{R}$ reduces to $\mathcal{R}'$, then $\mathcal{R}'$ is a proof-net modulo irrelevance.*

(ii) The cut-elimination process for proof-nets modulo irrelevance for **MAL** + *Mix has the strong normalization and Church–Rosser property.*

PROOF. (i) If $s\mathcal{R}$ is acyclic for every switching s, where there is no attachment for weakening links in $\mathcal{R}$, then the usual argument for $\mathbf{MLL}^-$ + Mix shows that $s\mathcal{R}'$ is acyclic for every switching s. If $\mathbf{P}(\mathcal{R})$ is non-empty, then $\mathbf{P}(\mathcal{R}')$ is non-empty by Lemma 31. (ii) Notice that if a cut-link occurs in $\mathcal{R}$, then exactly one among the five reduction or expansions applies to it. Moreover, such operations are strictly *local.* ∎

Lemma 31 *Let $\mathcal{R}$ be a proof-net modulo irrelevance for* **MAL** + *Mix.* If $\mathcal{R}$ *reduces to $\mathcal{R}'$, then $\mathbf{P}(\mathcal{R}') = \mathcal{R}' \setminus I(\mathcal{R}')$ is non-empty.*

PROOF. [of Lemma 31] First notice that *weakening-expansions* cannot make any relevant part of the proof-net irrelevant. Also in a *times /par* reduction if one premise of the times-link is in the irrelevant part, then the *cut-link* itself is irrelevant, and similarly if both premises of the par-link are irrelevant. Consider a *times /par* reduction where the par-link with conclusion $A^\perp \wp B^\perp$ has one and only one irrelevant premise, e.g., $B^\perp$ is conclusion of a weakening link w. Suppose both cut-links resulting from the reduction were irrelevant in $\mathcal{R}'$. Since $\{A, A^\perp\} \subset I(\mathcal{R}')$ but $A, A^\perp \notin I(\mathcal{R})$ it must be the case that by computing $I(w)$ in $\mathcal{R}'$ we reach either A or $A^\perp$. Now the algorithm for the computation of $I(w)$ starts with $B^\perp$, the conclusion of w, and continues as in the computation of $e^-(B)$: but since $\mathcal{R}'$ is like $\mathcal{R}$ except for the local rewriting of the *cut-link* in question, it follows that not only in $\mathcal{R}'$ but already in $\mathcal{R}$ we must have $A \in e^- B$ or $A^\perp \in e^- B$. It is easy to show that for some switching s there exists a path $path_s(A, B)$ or $path_s(A^\perp, B)$ in $s\mathcal{R}$. But this contradicts the acyclicity condition of the Danos–Regnier graphs for $\mathcal{R}$. ∎

Remark 32 Let $\mathcal{R}^a$ be a proof-net with attachments and suppose a *weakening-expansion* is applied to $\mathcal{R}$, yielding $\mathcal{S}$ with a new *times-link*: then there may be no system of attachments b such that $\mathcal{S}^b$ is a proof-net. This can be seen by Lemma 4: if $\mathcal{R}$ satisfies $\#ax = \#\otimes + \#cut + 1$ then in $\mathcal{S}$ the number of times-links is increased by one *coeteris paribus.* A stronger result would be to show that if $\mathcal{S}$ is the *cut-free* proof-structure resulting from cut-elimination, then there exists a system of attachments b such that $\mathcal{S}^b$ is a proof-net. We will not pursue the matter here.

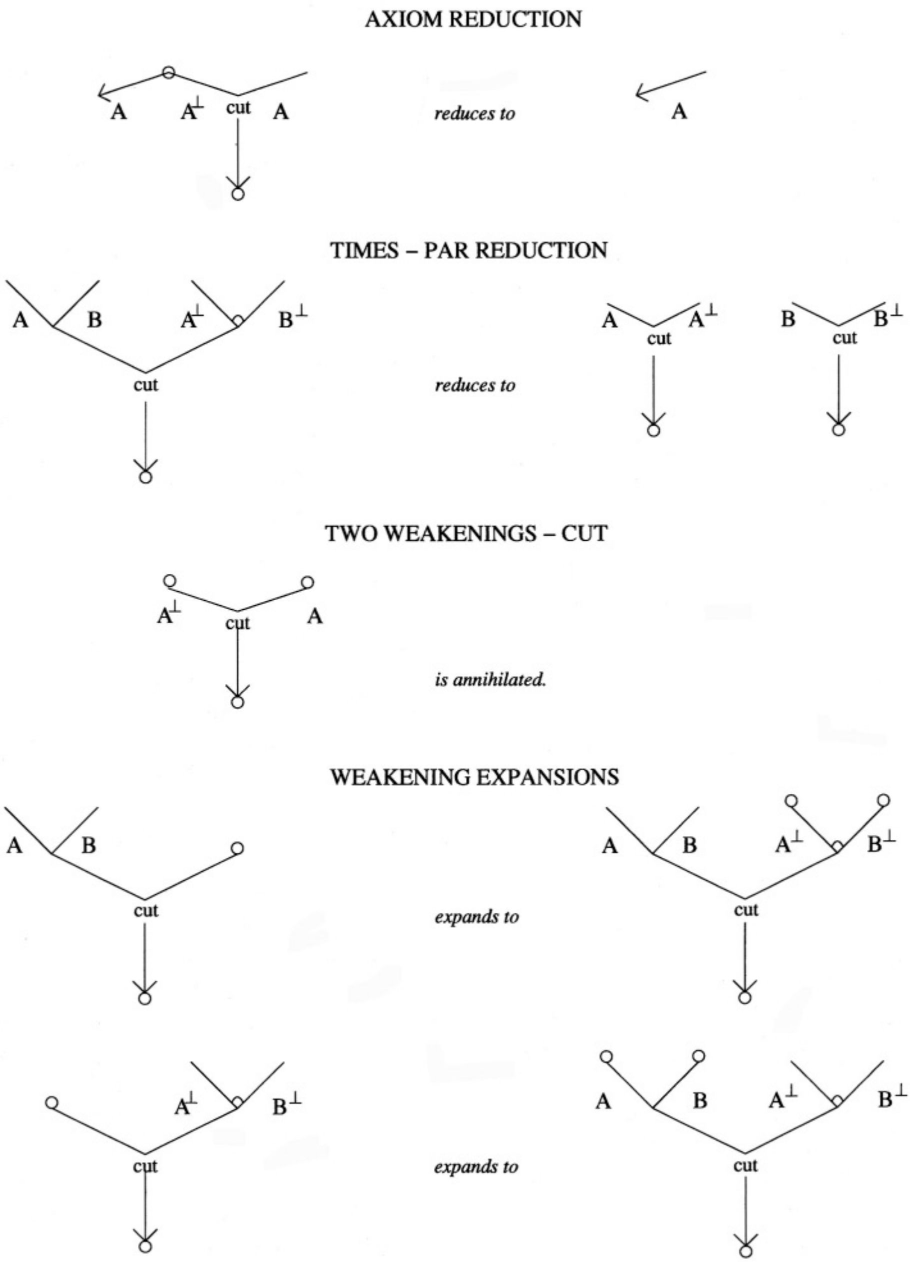

Figure 3.5. Cut

Example 33

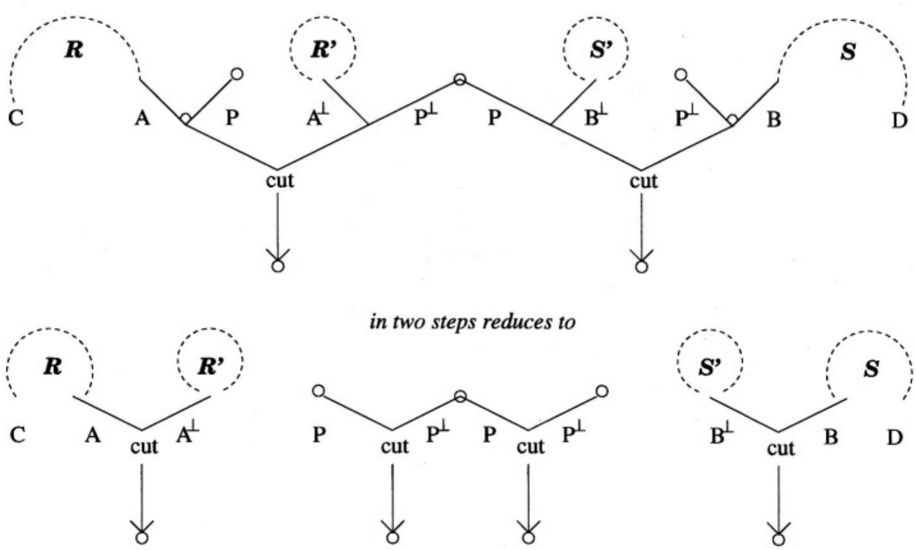

Figure 3.6. Example 33

Remark 34 Example 33 shows that the Church–Rosser property is lost if we *erase* the irrelevant part, instead of just *marking* it, during the cut-elimination process: e.g., after one reduction step, we would erase either $e(P)$ or $e^{-}(p^{\perp})$. Notice that after two more steps, (an *axiom reduction* and a *two weakenings / cut reduction*) we obtain two disconnected proof-nets: therefore our cut-elimination process essentially requires the use of the *mix-rule*.

6. Symmetric reductions require Mix

Can we define a symmetric reduction for *proof-nets* with contraction-links? An example in the Appendix B2 of Girard [12] shows that in order to do so we need the rule Mix. Let a and b atomic formulas and consider the proof-net π associated with a proof of $\vdash a \wedge b, a \wedge \neg b, \neg b \wedge a, \neg a \wedge \neg b$, and the proof-net π' to which π reduces as indicated in Figure 3.7.

Example 35

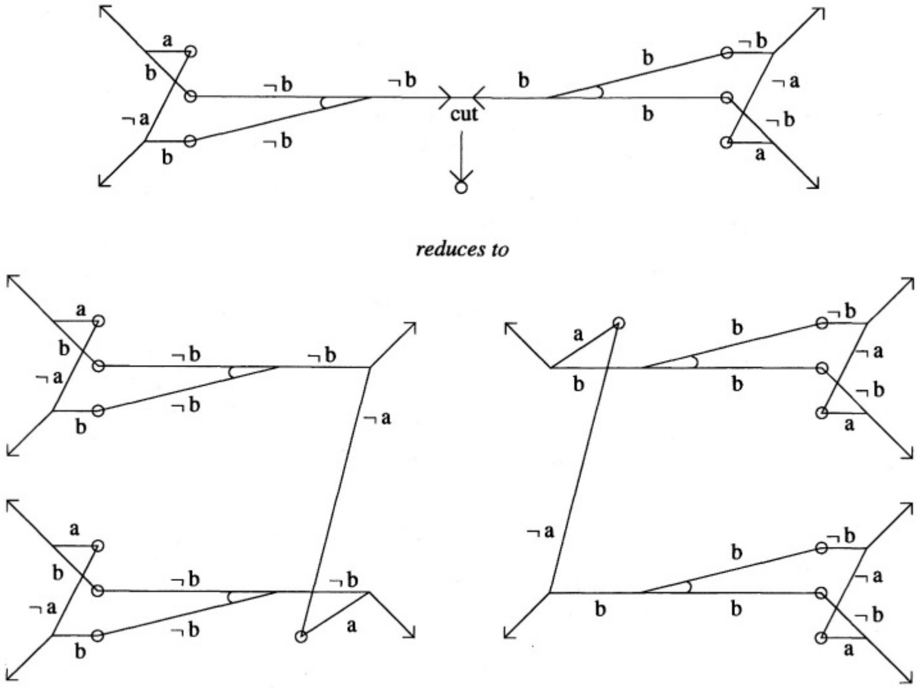

Figure 3.7. Mix

The reduction in Example 35 is the only one which has the following property of symmetry. Let $a\backslash\neg a$ the substitution operation which replaces every edge a with an edge $\neg a$ throughout the proof-net. Consider the group of substitutions **S** consisting of

$$\{\text{identity},\ a\backslash\neg a,\ b\backslash\neg b,\ a\backslash\neg a \text{ and } b\backslash\neg b\}$$

on the graphs π and π'. This group acts transitively on both π and π'; both π and π' are invariant under the substitutions of **S** – under the assumption that axioms, cut-links and contraction-links are symmetric, i.e., not ordered. It is easy to see that every *asymmetric* reduction, sending π to one of the connected components of π', does *not* preserve the property of invariance under substitution. Therefore Girard's example shows that there cannot be any symmetric cut-elimination for classical logic which does not use the rule Mix. It is also an easy exercise to see that (the proof-net translation of) the cross-cut reduction (as described in the intoduction) applied to the proof-net π yields a proof-net π'' which is *not* invariant under some substitution. The problem of finding a proof-net representation for classical logic is therefore entirely open.

Notes

1. Ketonen's notion of a chain was independently rediscovered by A. Asperti [1] in his characterization of proof-nets for **MLL**$^-$ + Mix through distributed processes. Therefore all the results in this paper can be interpreted in terms of Asperti's distributed processes.

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