A is obligatory	$=_{df}$	A′ is blameworthy.
A is indifferent	$=_{df}$	Both A and A′ are neither praiseworthy nor blameworthy.
A is excusable	$=_{df}$	A′ is praiseworthy.
A is forbidden	$=_{df}$	A is blameworthy.

Thus we have succeeded in defining the five new classes of actions in terms of two basic concepts, namely the phenomenological concepts of praiseworthiness and blameworthiness (i.e. strong positive and strong negative value-response).

However, while the three old classes (i.e. the three deontic concepts 'obligatory', 'permissible' and 'forbidden') are interdefinable, the two new basic concepts cannot be defined in terms of each other. This is the price that must be paid for the extension of the classification.[6]

References

[1] Boswell, James. *The Life of Dr Johnson*. Great Books of the Western World, vol. 44, Encyclopaedia Britannica Inc., Chicago 1952.

[2] Chisholm, R. M. "Supererogation and Offence". *Ratio*, vol. 5 (1963). Reprinted in R. M. Chisholm, *Brentano and Meinong Studies*, Rodopi, Amsterdam 1982.

[3] Hartmann, Nicolai. *Ethik*, 4th ed., Walter de Gruyter & Co., Berlin, 1962. *Ethics*, vol. 2, Allen and Unwin, London, 1932.

[4] Kant, I. *Grundlegung zur Metaphysik der Sitten*, 1785. Translated by T. K. Abbot as *Fundamental Principles of the Metaphysics of Morals* in vol. 42 of the Great Books of the Western World, Encyclopaedia Britannica Inc., Chicago 1952.

[5] Meinong, Alexius. *Psychologisch-ethische Untersuchungen zur Werththeorie*. Gesamtausgabe, vol. 3, Akademische Druck- u. Verlagsanstalt, Graz 1968.

[6] Meinong, Alexius. *Ethische Bausteine*. Gesamtausgabe, vol. 3. Akademische Druck- u. Verlagsanstalt, Graz 1968.

[7] Urmson, J. O. "Saints and Heroes". In *Essays in Moral Philosophy*, (ed.) A. I. Melden. University of Washington Press, Seattle, 1958.

[6] Lately I have come to entertain doubts about Hartmann's originality. A strikingly similar classification of moral actions is given by Alexius Meinong in ([5], § 29). Cf. also [6]. Meinong published the first-mentioned book 32 years before Hartmann's *Ethik*.

A system of natural deduction for GL

by

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The system GL_{ax} is a propositional modal logic, defined by the axioms:

(0) all tautologies are axioms;

(1) $\vdash_{[GL_{ax}]} \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(2) $\vdash_{[GL_{ax}]} \Box A \rightarrow \Box \Box A$

(3) $\vdash_{[GL_{ax}]} \Box (\Box A \rightarrow A) \rightarrow \Box A$

and the rules Modus Ponens and Necessitation

$$\text{NEC:} \quad \frac{\vdash_{[GL_{ax}]} A}{\vdash_{[GL_{ax}]} \Box A}$$

In the arithmetic interpretation, propositional letters are mapped into sentences of Peano Arithmetic and $\Box$ is taken to be the provability predicate *Bew* (Boolos 1979).

Our language contains a propositional constant $\perp$ for the absurd, denumerably many propositional letters, the connectives $\rightarrow$ and & ($\sim A$ being defined as $A \rightarrow \perp$, etc.) and the modal operator $\Box$. Letters X and Y will be used as abbreviations for sequences of sentences, with the convention that if X is $\{A_1, \ldots, A_m\}$ then $\Box X$ is $\{\Box A_1, \ldots, \Box A_m\}$. Uppercase Greek letters Π, Σ, Φ, Ξ, Δ, Λ will denote derivations.

The main result of this paper, the correction of Leivant's proof of the Cut Elimination theorem for GL, was announced in an abstract sent to the Abstracts AMS March 31, 1982. A first version of this paper was submitted for publication in *Theoria* April 30, 1982. The paper has been rewritten following suggestions of Dr. T. Franzén. I wish to thank Giovanni Sambin and Silvio Valentini for introducing me to the subject, Dag Prawitz, Solomon Feferman, Helmut Schwichtenberg, Wilfried Sieg and Albert Visser, for their help in these years and particularly Tony Ungar, whose competence in the field of relations between Cut Elimination and Normalization has been of extraordinary value.

Our system GL_{nat} contains the usual natural deduction rules for $\rightarrow$ and & (Prawitz 1965), $\perp_C$ (reductio ad absurdum) in the form

$$\begin{array}{c} [\sim A] \\ \Pi \\ \dfrac{\perp}{A} \end{array}$$

where A must be either atomic and different from $\perp$ or of the form $\Box B$, and the only modal rule GLR described below.

GLR is of the form

$$\dfrac{\begin{array}{cccccc} & & & & & \overset{(1)}{[B_1]} \ldots \overset{(n)}{[B_n]}\ \overset{(0)}{[\Box A]}\ \overset{(n+1)}{[\Box B_{n+1}]} \ldots \overset{(n+p)}{[\Box B_{n+p}]} \\ \Sigma_1 & \Sigma_n & \Sigma_{n+1} & & \Sigma_{n+p} & \Sigma \\ \Box B_1 \ldots & \Box B_n & \Box B_{n+1} & \ldots & \Box B_{n+p} & A \end{array}}{\begin{array}{cccccc} \uparrow(1) & \uparrow(n) & \uparrow(n+1) & & \uparrow(n+p) & \Box A \end{array}}\ (0)$$

where $[B_1], \ldots, [B_n], [\Box A], [\Box B_{n+1}], \ldots, [\Box B_{n+p}]$ stand for all the classes of open assumptions in the derivation Σ of A. $\Box B_1, \ldots, \Box B_{n+p}$ are called the *major premises* and A the *minor premise* of the rule application. All the open assumptions, which the minor premise depends on, must be discharged by the rule application and we say that the assumption class $[B_i]$ (or $[\Box B_{n+j}]$) is discharged *in virtue of* the major premise $\Box B_i$ (or $\Box B_{n+j}$), for $i = 1, \ldots, n$ and $j = 1, \ldots, p$. A fully detailed definition of deduction in GL_{nat} would include the specification of which assumption is discharged in virtue of which major premise, for every application of GLR. We shall sometimes write an arrow and a number under a major premise, to indicate the assumption class discharged in virtue of that premise.

An assumption of the form $\Box A$, discharged by a GLR application with minor premise A, is called *diagonal assumption* if it does not belong to any assumption class discharged in virtue of some major premise.

GLR may be thought of as a rule of *introduction*, since it introduces the symbol $\Box$ in the conclusion $\Box A$, as well as a rule of *elimination*, since it eliminates assumptions (actually, all assumptions) by using the "intended meaning" of $\Box$.

From this point of view, GLR obviously violates the Inversion Principle (Prawitz 1965, pp 32–33), if the assumption class (0) is not empty: the elimination of the diagonal assumptions is not justified as the inverse operation of GLR, used as a rule of introduction. Indeed it could not be accomplished by replacing the open assumptions in the derivation of the minor premise with proofs of the major premises.

We recall that an application $\mathcal{R}$ of $\perp_C$ is said to be *redundant* if there is an assumption $\sim A$ discharged by $\mathcal{R}$ such that
(i) $\sim A$ is a major premise of an $\rightarrow$ elimination $\mathcal{S}$, and
(ii) the conclusion A of $\mathcal{R}$ depends on all the assumptions, which the minor premise A of $\mathcal{S}$ depends on.

In a similar way, a major premise of a modal inference is said to be redundant if no assumption is discharged in virtue of it.

A redundant $\perp_C$ application is simplified as follows:

$$\begin{array}{cc} & [X] \\ & \Sigma_1 \quad (1) \\ \mathcal{S} & \dfrac{A \quad \sim A}{\perp} \\ & \Sigma_2 \\ \mathcal{R} & \dfrac{\perp}{A}(1) \end{array}$$

is transformed into

$$\begin{array}{c} [X] \\ \Sigma_1 \\ A \end{array}$$

To simplify a redundant major premise of a GLR is to erase this parasitic premise (together with its derivation). We decide that redundant $\perp_C$ applications, as well as redundant premises of modal inferences, are simplified without mention.

A *branch* in derivation is a sequence $C_1, \ldots, C_n$ of formula occurrences such that
(i) C_1 is either an assumption that is not discharged by a GLR or a diagonal assumption of a GLR;

(ii) if C_j is neither the minor premise of an $\rightarrow$ elimination nor a major premise of a GLR, then C_{j+1} is the formula occurrence immediately below C_j;
(iii) if C_j is a major premise of a GLR, then C_{j+1} is any assumption discharged in virtue of C_j;
(iv) C_n is either the minor premise of an $\rightarrow$ elimination or the endformula of the deduction.

A *segment* in a branch is a sequence $C_j, \ldots, C_{j+n}$ of occurrences of the same formula $\Box B$ such that
(i) C_{j-1} and C_{j+n+1} are not occurrences of $\Box B$;
(ii) $C_j, \ldots, C_{j+n-1}$ are major premises of GLR:
(iii) $C_{j+1}, \ldots, C_{j+n}$ are assumptions discherged in virtue of $C_j, \ldots, C_{j+n-1}$ respectively. Call a segment *proper* if its length m is greater than 1.

Given a derivation Π, the immediate subderivation(s) of Π is (are) the derivation(s) of the premise(s) of the last inference of Π.

The correctness of GL_{nat} with respect to GL_{ax} can be proved as follows (see Sambin and Valentini 1980). Let $\bigwedge X$ be the conjunction of all the formulas in X.

THEOREM 0 $\vdash_{[GL_{ax}]} \bigwedge X \rightarrow A$ *iff* $X \vdash_{[GL_{nat}]} A$

Proof. (Only if) Modal axioms and Necessitation Rule are easily derivable in GL_{nat}. It is easy to see that nothing is lost by assuming that reductio ad absurdum is applied with the above restrictions.

(If) Assume $X \vdash_{[GL_{nat}]} A$ and show $\vdash_{[GL_{ax}]} \bigwedge X \rightarrow A$, by induction on the length of the derivation. Let Π be a natural deduction ending with a GLR application

$$\dfrac{\begin{array}{ccc} [X_1] & [X_{n+p}] & \\ \Sigma_1 & \Sigma_{n+p} & \Sigma \\ \Box B_1 & \Box B_{n+p} & A \end{array}}{\Box A}$$

We have by induction hypothesis, for i=1, ..., n+p

$$\vdash_{[GL_{ax}]} \bigwedge X_i \rightarrow \Box B_i$$

and by induction hypothesis and propositional logic

$$\vdash_{[GL_{ax}]} (B_1 \rightarrow \ldots \rightarrow (B_n \rightarrow (\Box B_{n+1} \rightarrow \ldots \rightarrow (\Box B_{n+p} \rightarrow (\Box A \rightarrow A))\ldots))\ldots)$$

By Necessitation we obtain

$$\vdash_{[GL_{ax}]} (\Box B_1 \rightarrow \ldots \rightarrow (B_n \rightarrow (\Box B_{n+1} \rightarrow \ldots \rightarrow (\Box B_{n+p} \rightarrow (\Box A \rightarrow A))\ldots))\ldots)$$

Using several instances of Axioms (1), (2), (3) and propositional logic we get

$$\vdash_{[GL_{ax}]} (\Box B_1 \rightarrow \ldots \rightarrow (\Box B_n \rightarrow (\Box B_{n+1} \rightarrow \ldots \rightarrow (\Box B_{n+p} \rightarrow \Box A)\ldots))\ldots)$$

Therefore

$$\vdash_{[GL_{ax}]} \bigwedge X_1 \& \ldots \& \bigwedge X_{n+p} \rightarrow \Box A.$$

A GLR application is called *proper* if it discharges some diagonal assumptions; it is called *K4R application* otherwise. A K4R application is called *KR application* if $[B_1], \ldots, [B_n]$ are all the assumption classes, which the minor premise depends on. Thus $K4_{nat}$ (K_{nat}) is the natural deduction system in which only the restricted modal rule K4R (KR) is used. It is easy to see that $K4_{nat}$ corresponds to GL_{ax} without Löb axiom (axiom 3) and that K_{nat} corresponds to GL_{ax} without axioms 2 and 3.

A formula occurrence in a derivation is called *maximal* if it stands at the same time as the conclusion of an introduction or of a modal inference and as a major premise of an elimination or of a modal inference. A segment $C_j, \ldots, C_{j+m}$ is said to be maximal if C_j is the consequence of a modal inference.

Normal deductions

Normal deductions are usually defined as derivations in which there are no maximal formulas. The normal form in an interesting property of derivations, if a normal deduction enjoys the Subformula Property, i.e. if every formula occurring in a deduction of A from X is a subformula either of A or of some formula in X.

It is well known that the rules for negation, and particularly $\perp_C$ (reductio ad absurdum), create problems for the proof of Normalization Theorem, already in propositional logic (Prawitz 1965, 1971). In GL_{nat} a deduction which is normal according to this definition need not enjoy the Subformula Property, as it is shown by the following easy counterexample:

$$\mathcal{R}_2\ \dfrac{\mathcal{R}_1\ \dfrac{\perp}{\Box(B\&C)} \qquad \dfrac{B\&C}{B}}{\Box B}$$

(notice that here only the intuitionistic negation rule is applied).

Moreover no reduction of the kind given by Prawitz (1965) is available to reduce the logical complexity of a consequence $\Box A$ of an application $\mathcal{C}$ of $\perp_C$ (i.e. we cannot replace $\mathcal{C}$ with an application $\mathcal{C}'$ giving A, followed by a GLR application introducing $\Box A$).

Nevertheless we can modify the definition of normal form, by adding the requirement that no major premise of a modal rule be consequence of $\perp_C$ (the intuitionistic negation rule is considered as a case of the classical one with no assumption discharged). We define the appropriate reduction step by constructing a new derivation; here the formula that in the given deduction is the conclusion of the modal inference is inferred instead by the negation rule (in our example take $\Box B$ as consequence of $\mathcal{R}_1$; in general in the case of an intuitionistic negation rule we replace its conclusion and simply delete the modal inference). Of course, if some assumption is discharged by the negation rule, then we have to change those assumptions as well. We show that this is always possible, by inserting copies of the modal inference in the "upper part" of the derivation. (This reduction amounts to a permutation of inferences in the proof of cut elimination for the corresponding system of calculus of sequents for GL).

A formula occurrence $\Box B$ which is a consequence of an $\perp_C$ application and a major premise of a modal inference is called $\perp_C$/GLR maximal. Our official definition of normal deduction is:

A derivation in GL_{nat} *is said to be normal if it contains no maximal formula and no* $\perp_C$/GLR *maximal formula.*

It will be proved that a normal deduction enjoys the Subformula Property, with the exception of the assumptions discharged by $\perp_C$ and of the occurrences of $\perp$, occurring immediately below such assumptions.

Let us call a formula occurrence in a branch *minimal* if it is a consequence of an elimination and a premise of an introduction.

EXPANSION LEMMA. If A *is a minimal formula in a branch then the deduction can be expanded so that the branch contains an atomic minimal formula, which is a subformula of* A. *Moreover if* $\sim$A *is an assumption discharged by an application of* $\perp_C$ *then the deduction can be expanded so that the assumption discharged by the* $\perp_C$ *is a major premise of an* $\rightarrow$ *elimination.*

Proof. Trivial.

We shall use the second statement of the Expansion Lemma to make sure that every deduction has the appropriate form whenever we apply the $\perp_C$/GLR reduction below.

Reductions

Let Π be a derivation with a set X of open assumptions, let $\mathcal{R}$ be any modal inference in Π and let Y be the set of open assumptions which the minor premise of $\mathcal{R}$ depends on. Since all assumptions in Y are discharged by $\mathcal{R}$, X and Y are disjoint sets; therefore no substitution of a derivation for an open assumption in X may modify the correctness of $\mathcal{R}$. This remark is relevant to the $\rightarrow$ Reduction, the K4R Reduction and the Segment Reduction below.

($\rightarrow$ REDUCTION)

$$\mathcal{R}_2\ \dfrac{\begin{matrix}\Sigma_1\\ A\end{matrix} \qquad \mathcal{R}_1\ \dfrac{\begin{matrix}[A]\\ \Sigma_2\\ B\end{matrix}}{A\rightarrow B}}{B} \qquad \text{is transformed into} \qquad \begin{matrix}\Sigma_1\\ [A]\\ \Sigma_2\\ B\end{matrix}$$

(& REDUCTION)

$$\mathcal{R}_2\,\dfrac{\mathcal{R}_1\,\dfrac{\begin{matrix}\Sigma_1\\ A_1\end{matrix}\qquad\begin{matrix}\Sigma_2\\ A_2\end{matrix}}{A_1\& A_2}}{A_i}\qquad\text{is transformed into}\qquad\begin{matrix}\Sigma_i\\ A_i\end{matrix}$$

for i=1,2.

(K4R REDUCTION)

$$\mathcal{R}_2\,\dfrac{\mathcal{R}_1\,\dfrac{\begin{matrix}\Sigma_1\\ \Box X\end{matrix}\qquad\begin{matrix}\Sigma_2\\ A\end{matrix}}{\Box A}\qquad\begin{matrix}\Sigma_3\\ \Box Y\end{matrix}\qquad\begin{matrix}[A]\\ \Sigma_4\\ B\end{matrix}}{\Box B}$$

where $\mathcal{R}_1$ is a K4R application, is transformed into

$$\mathcal{R}_2^*\,\dfrac{\begin{matrix}\Sigma_1\\ \Box X\end{matrix}\qquad\begin{matrix}\Sigma_3\\ \Box Y\end{matrix}\qquad\begin{matrix}\Sigma_2\\ [A]\\ \Sigma_4\\ B\end{matrix}}{\Box B}$$

where every open assumption of Σ_2 discharged by $\mathcal{R}_1$ in virtue of some major premise in $\Box X$ is discharged by $\mathcal{R}_2^*$ in virtue of the same major premise in $\Box X$.

(SEGMENT REDUCTION STEP)

In the following reduction the notation

$$\mathcal{S}_i\,\dfrac{[\Box A]}{C_i}$$

for i = 1,...,h, means: $\mathcal{S}_i$ is an inference having among its premises a sequence $D_1,\ldots, D_h$ of occurrences of the formula $\Box A$.
Let the assumption class (1) be the union of the indicated classes of premises of $\mathcal{S}_i, \ldots, \mathcal{S}_k$: then

$$\mathcal{R}_2\,\dfrac{\mathcal{R}_1\,\dfrac{\begin{matrix}\Sigma_1\\ \Box X\end{matrix}\qquad\begin{matrix}\Sigma_2\\ A\end{matrix}}{\begin{matrix}\Box A\\ \uparrow (1)\end{matrix}}\qquad\begin{matrix}\Sigma_3\\ \Box Y\end{matrix}\qquad\begin{matrix}\mathcal{S}_1\dfrac{\overset{(1)}{[\Box A]}}{C_1}\ \ldots\ \mathcal{S}_k\dfrac{\overset{(1)}{[\Box A]}}{C_k}\\ \Sigma_4\\ B\end{matrix}}{\Box B}$$

is transformed into

$$\mathcal{R}_2^*\,\dfrac{\begin{matrix}\Sigma_1\\ \Box X\end{matrix}\qquad\begin{matrix}\Sigma_3\\ \Box Y\end{matrix}\qquad\begin{matrix}\Pi_1 & & \Pi_k\\ C_1 & \ldots & C_k\\ & \Sigma_4 & \\ & B & \end{matrix}}{\Box B}$$

where, for j = 1,...,k, Π_j is

$$\mathcal{S}_1\,\dfrac{\mathcal{R}_1\,\dfrac{\Box X\qquad\begin{matrix}\Sigma_2\\ A\end{matrix}}{[\Box A]}}{C_j}$$

and all the major premises of $\mathcal{R}_1$ belonging to $\Box X$ are open assumptions discharged in virtue of the corresponding major premise of $\mathcal{R}_2^*$.

($\bot_C$/GLR REDUCTION STEP)

Let Π be

$$\mathcal{R}_2\,\dfrac{\begin{matrix}\Lambda\\ \Box A\end{matrix}\qquad\begin{matrix}\Sigma_2\\ \Box X\end{matrix}\qquad\begin{matrix}\Sigma_3\\ B\end{matrix}}{\Box B}$$

where Λ is

$$\mathscr{S}_1\ \frac{\overset{\Sigma_0^1}{\Box A} \quad \overset{(1)}{\sim\Box A}}{\perp} \quad \cdots \quad \mathscr{S}_n\ \frac{\overset{\Sigma_0^n}{\Box A} \quad \overset{(1)}{\sim\Box A}}{\perp}$$

$$\mathscr{R}_1\ \frac{\overset{\Sigma_1}{\perp}}{\Box A}(1)$$

and the major premises of $\mathscr{S}_1, \ldots, \mathscr{S}_n$ are all the assumptions that are discharged by the inference $\mathscr{R}_1$. Then Π is transformed into

$$\mathscr{S}_1^*\ \frac{\overset{\Pi_1}{\Box B} \quad \overset{(1)}{\sim\Box B}}{\perp} \quad \cdots \quad \mathscr{S}_n^*\ \frac{\overset{\Pi_n}{\Box B} \quad \overset{(1)}{\sim\Box B}}{\perp}$$

$$\mathscr{R}_1^*\ \frac{\overset{\Sigma_1}{\perp}}{\Box B}(1)$$

where for $j = 1, \ldots, n$, Π_j is

$$\mathscr{R}_2\ \frac{\overset{\Sigma_0^j}{\Box A} \quad \overset{\Sigma_2}{\Box X} \quad \overset{\Sigma_3}{B}}{\Box B}$$

(SEGMENT REDUCTION)

We define recursively an operation **F** which applies to derivations and formula occurrences and gives derivations as result.

Let **F** (Π, D) be Π if D is not the first formula of a maximal segment of Π. Otherwise let Π have the form shown in the Segment Reduction Step. Let us assume first that for every $\mathscr{S}_i$ the assumption class $[\Box A]$ contains just one formula. Then if D is the indicated major premise $\Box A$ of $\mathscr{R}_2$

$$\mathbf{F}(\Pi, D) = \frac{\overset{\Sigma_1}{\Box X} \quad \overset{\Sigma_3}{\Box Y} \quad \overset{\overset{\mathbf{F}(\Pi_1, D_1)\ \mathbf{F}(\Pi_k, D_k)}{C_1 \cdots C_k}}{\overset{\Sigma_4}{B}}}{\Box B}$$

where $D_1, \ldots, D_k$ are the indicated premises of $\mathscr{S}_1, \ldots, \mathscr{S}_k$, respectively.

This definition can be easily generalized: if the assumption class $[\Box A]$ contains the premises $D_i^1, \ldots, D_i^h$ take

$$\underset{C_i}{\mathbf{F}\,(\ldots\ \mathbf{F}\,(\mathbf{F}\,(\Pi_1, D_i^1), D_i^2), \ldots, D_i^h)}$$

as derivation of the assumption C_i of Σ_4 (the complication of such generalization is mainly notational).

Clearly the set of segments starting from a given formula occurrence in a derivation is a finite tree and the operation **F** consists of a finite number of applications of the Segment Reduction Step. So **F** is well defined.

Let $C_j, \ldots, C_{j+m}$ be a segment starting with D in Π; it is clear how to identify in the derivation resulting from one step of Segment Reduction a new segment corresponding to the formulas $C_{j+1}, \ldots, C_{j+m}$. So it is clear in which sense a step of Segment Reduction reduces the length of a segment and the operation **F** eliminates proper segments.

F (Π, D) has the following *Property* (A):
Every proper segment starting with D *in* Π *is eliminated in* **F** (Π, D).

($\perp_C$/GLR REDUCTION)

The *degree* (or logical complexity) d(A) of a formula A is the number of logical connectives – different from $\perp$ – and of modal operators occurring in A.

We define recursively an operation $\mathbf{G}_\alpha$, transforming derivations into derivations as follows.

Let $\mathbf{G}_\alpha(\Pi)$ be Π if the last inference of Π does not contain any $\perp_C$/GLR maximal major premise of degree α.

Otherwise, let Π have the form shown in the $\perp_C$/GLR Reduction Step, and suppose that the indicated $\Box A$ is the leftmost $\perp_C$/GLR maximal major premise of degree α in $\mathcal{R}_2$. Then

$$\mathbf{G}_\alpha(\Pi) = \dfrac{\dfrac{\begin{matrix}\mathbf{G}_\alpha(\Pi_1) & (1)\\ \Box B & \sim\Box B\end{matrix}}{\perp} \quad \ldots \quad \dfrac{\begin{matrix}\mathbf{G}_\alpha(\Pi_n) & (1)\\ \Box B & \sim\Box B\end{matrix}}{\perp}}{\begin{matrix}\Sigma_1\\ \perp\end{matrix}} \qquad \dfrac{\perp}{\Box B}$$

We associate each formula of the form $\Box B$ which is a consequence of $\perp_C$ with a tree $\tau(\Box B)$ of formula occurrences, defined by recursion as follows:

i) $\langle A_1 \rangle$ belongs to $\tau(\Box B)$, where A_1 is the given $\Box B$;

ii) if $\langle A_1, \ldots, A_k \rangle$ is a branch of $\tau(\Box B)$, where A_k is $\Box B$ and is a consequence of an application $\mathcal{R}_k$ of $\perp_C$, let $C_{k,1}, \ldots, C_{k,h}$ be all the assumptions that are discharged by $\mathcal{R}_k$, and for $i = 1, \ldots, h$, let $A_{k+1,i}$ be the minor premise of the $\rightarrow$ elimination having $C_{k,i}$ as major premise; then, for $i = 1, \ldots, h$, $\langle A_1, \ldots, A_k, A_{k+1,i} \rangle$ belongs to $\tau(\Box B)$.

It is clear that the operation $\mathbf{G}_\alpha$ amounts to a finite number p of applications of the $\perp_C$/GLR reduction step (in fact p is the sum of the number of occurrences of formulas in every $\tau(\Box B_i)$, where $\Box B_i$ is a $\perp_C$/GLR maximal major premise of degree α occurring in $\mathcal{R}_2$). So $\mathbf{G}_\alpha$ is well defined.

$\mathbf{G}_\alpha(\Pi)$ has the following *Property (B)*: $\mathbf{G}_\alpha(\Pi)$ *contains copies* $\mathcal{R}_2^*$ *of the last inference* $\mathcal{R}_2$ *of* Π, *but no major premise in any* $\mathcal{R}_2^*$ *is a* $\perp_C$/GLR *maximal formula of degree* α.

Normalization theorem

THEOREM 1. (a) *Every derivation in* K_{nat} *can be transformed into a normal derivation of the same formula from the same set of assumptions.* (b) *The same for* $K4_{nat}$.

Proof. (b) We associate an index $I(\mathcal{R}) = \langle \alpha, \beta, \gamma \rangle$ with every *inference* $\mathcal{R}$ in a derivation as follows. Let $\mathcal{R}$ be an inference having $D_1, \ldots, D_n$ as major premises and D as conclusion. So let

α = $d(D_i)$, if $\mathcal{R}$ is $\rightarrow$ or & elimination and the (major) premise D_i is maximal;
= max $(d(E_1), \ldots, d(E_m))$, where $E_1, \ldots, E_m$ are those major premises among $D_1, \ldots, D_n$ that are maximal or $\perp_C$/GLR maximal, if $\mathcal{R}$ is a modal inference;
= 0, otherwise.

β = number of maximal or $\perp_C$/GLR maximal major premises of degree α that are first formulas of a segment of length greater than 1, if $\mathcal{R}$ is a modal inference;
= 0 otherwise.

γ = number of maximal or $\perp_C$/GLR maximal major premises of degree α if $\mathcal{R}$ is a modal inference;
= 0 otherwise.

Indices are ordered lexicographically.

The index $\mathcal{I}$ of a derivation is $\langle max, n \rangle$, where *max* is the maximum index of all its inferences and n the number of inferences having this maximum index.

To prove the theorem, given a derivation Γ of index $\mathcal{I}$, choose a subderivation Π such that its final inference $\mathcal{R}$ has index *max*. The immediate subderivations $\Pi_1, \ldots, \Pi_n$ have lower indices, so by induction hypothesis we may assume that $\Pi'_1, \ldots, \Pi'_m$ are normal derivations of the premises of $\mathcal{R}$ from the same sets of assumptions. Let Γ' be the result of replacing Π in Γ by

$$\Pi' = \quad \mathcal{R}' \dfrac{\Pi'_1 \qquad \Pi'_m}{D}$$

(notice that $m < n$ may occur, if some major premise of $\mathcal{R}$ becomes redundant because of some reduction in the derivation of the minor premise).

We first have to check that Γ' does not have index greater than $\mathcal{I}$. This is obvious, unless $\mathcal{R}'$ is modal. In this case, let α be the first member of *max*. First we show that there is no maximal major premise of $\mathcal{R}'$ of degree α that was not maximal as a premise of $\mathcal{R}$. A premise of $\mathcal{R}$ in Γ cannot become maximal because of a $\rightarrow$ or & reduction, since Γ did not contain maximal formulas of degree higher than α. If a major premise of $\mathcal{R}$ in Γ was an open assumption then it cannot become maximal by a segment reduction acting above $\mathcal{R}$. However there may very well be some $\perp_C$/GLR maximal major premise E of $\mathcal{R}'$ that was not such as a premise of $\mathcal{R}$ and is of degree α. We claim that such E was maximal, as a premise of $\mathcal{R}$ in Γ. E has the form $\Box$A and it was not an open assumption nor a consequence of $\perp_C$. If E was the conclusion of an elimination $\mathcal{S}$, then the major premise C of $\mathcal{S}$ was of degree higher than α. But $\mathcal{S}$ could have been replaced in Π' by an $\perp_C$ application only if C was maximal, against our assumption. Therefore E was the conclusion of a modal inference in Γ. This proves our claim.

Our next task is to show that the appropriate reductions applied to Π' actually produce a derivation of lower index. Let $I(\mathcal{R}') = \langle \alpha, \beta, \gamma \rangle$. If $\mathcal{R}'$ is not a modal inference, then the reduction for $\rightarrow$ or & gives a derivation Γ^* of index $\mathcal{I}^* = \langle I(\mathcal{Q}), m \rangle$, for some inference $\mathcal{Q}$ and some number m, where either $I(\mathcal{Q}) = I(\mathcal{R}')$ and $m < n$ or $I(\mathcal{Q}) < I(\mathcal{R}')$; indeed any new maximal formula resulting from such reduction is of lower degree.

So let $\mathcal{R}'$ be a modal inference. The proof splits in two parts.

Case a) $\mathcal{R}'$ does not have any $\perp_c$/GLR maximal major premise of degree α.

Consider the leftmost maximal major premise E of degree α; let E be $\Box$A. One of the following subcases applies:

Subcase (1) No segment of length greater than 1 starts at E.

By an application of the K4R reduction to Π' we obtain a new deduction Π^* whose maximal formulas can be only

(i) major premises of the final inference $\mathcal{R}^*$, or
(ii) formulas of the form A occurring as premises of some inference $\mathcal{T}$ in the derivation of the minor premise of $\mathcal{R}^*$.

The index $I(\mathcal{R}^*) = \langle \alpha^*, \beta^*, \gamma^* \rangle$ is less than $I(\mathcal{R}')$, since $\alpha^* \leq \alpha$, $\beta^* = \beta$ and $\gamma^* < \gamma$. On the other hand, since A is the only major premise of $\mathcal{T}$ and $d(A) = \alpha\text{-}1$, $I(\mathcal{T}) < I(\mathcal{R}')$.

Subcase (2) E is the first formula of a segment of length greater than 1.

By inspection of the inferences involved in a step of Segment Reduction we verify the following

Fact 1. Let Π have the form shown in the Segment Reduction Step, let the immediate subderivations of Π be normal and let $I(\mathcal{R}_2) = \langle \alpha, \beta, \gamma \rangle$. Then the derivation resulting from a step of Segment Reduction does not contain maximal formulas of degree greater than α.

Now consider $\mathbf{F}(\Pi', E)$: the only maximal formulas in it can be:
(i) major premises of the final inference $\mathcal{R}^*$ or
(ii) formulas of the form $\Box$A, corresponding to the last formula of a segment starting from E in Π'; let $\mathcal{T}_1, \ldots, \mathcal{T}_h$ be the inferences having any such $\Box$A as maximal major premise.

In the first case let $I(\mathcal{R}^*) = \langle \alpha^*, \beta^*, \gamma^* \rangle$. It follows from Fact 1 that $\alpha^* \leq \alpha$; moreover $\beta^* = \beta - 1$. So $I(\mathcal{R}^*) < I(\mathcal{R}')$.

In the second case let $I(\mathcal{T}_i) = \langle \alpha^+, \beta^+, \gamma^+ \rangle$. By induction on the definition of $\mathbf{F}$, using Fact 1, it follows that $\alpha^+ \leq \alpha$. It follows from Property A that $\beta^+ = 0$. So $I(\mathcal{T}_i) < I(\mathcal{R}')$.

Case b) $\mathcal{R}'$ has some $\perp_C$/GLR maximal major premise of degree α.

By inspection of the inferences involved in a step of $\perp_C$/GLR reduction we verify the following

Fact 2. Let Π have the form shown in the $\perp_C$/GLR Reduction Step and let the immediate subderivations of Π be normal. Then in the derivation resulting from the reduction step there may be maximal formulas only as major premises of some copy $\mathcal{R}'_2$ of $\mathcal{R}_2$.

Consider $\mathbf{G}_\alpha(\Pi')$: the only maximal formulas in it occur as major premises of some copy $\mathcal{R}^*$ of $\mathcal{R}'$ (this results from an induction on the definition of $\mathbf{G}_\alpha$ using Fact 2). $\mathcal{R}^*$ differs from $\mathcal{R}'$ only for having maximal major premises of degree α where $\mathcal{R}'$ had $\perp_C$/GLR maximal ones.

Conclude this case by applying the operations described in Case a) to every subderivation of $\mathbf{G}_\alpha(\Pi')$ ending with such inference $\mathcal{R}^*$.

Thus in every case we succeeded in obtaining a derivation Π^* where all inferences have index lower than $\mathcal{R}'$. So by arguing as in the case of propositional reductions we see that if Γ^* is the result of replacing Π' by Π^* in Γ' then Γ^* has index lower than Γ' and so than Γ. *Q.E.D.*

In order to extend this result to GL_{nat} we may still use all the previous reductions, but obviously we cannot use the K4R reduction if the maximal formula $\Box A$ is introduced by a proper GLR (any diagonal assumption would become an open assumption). If we try to transform such GLR into a K4R application, then our problem is: which derivation has to be given of the diagonal assumptions? It is not difficult to see that if we take the given subderivation of $\Box A$ again (Leivant 1981), then the normalization procedure never ends whenever a diagonal assumption is itself a major premise of a modal rule (the counterexample to Leivant's cut elimination procedure was found by Sambin and Valentini). Our proof shows that suitable parts of the given subderivation will do the job.

Let Π be a subderivation ending with $\Box A$. Let us call the final inference $\mathcal{R}$ of Π *primary*, if $\mathcal{R}$ is a modal rule. Let us call a modal inference of Π *secondary* if it has a diagonal assumption of the primary GLR as major premise. Π has the form

(1)
$$
\begin{array}{c}
\mathcal{S}_1\dfrac{\overset{(0)}{\Box A}\quad \overset{\Sigma_1}{C_1}}{\Box C_1}\ \cdots\ \mathcal{S}_k\dfrac{\overset{(0)}{\Box A}\quad \overset{\Sigma_k}{C_k}}{\Box C_k}\quad \overset{(0)}{[\Box A]}\\
\mathcal{R}\ \dfrac{\overset{\Sigma}{\Box X}\qquad\qquad\qquad \overset{\Sigma_0}{A}}{\Box A}\,(0)
\end{array}
$$

Now let Π_j be the result of erasing in Π the premises of the secondary inference $\mathcal{S}_j$ (of simplifying the resulting redundant premises) and of introducing the new major premise $\Box C_j$ in the final inference. Π_j is then

$$
\dfrac{\Box C_j \qquad \overset{\Sigma'}{\Box X'} \qquad\qquad \overset{\overset{(*)\qquad (0)}{[\Box C_j]\quad [\Box A]}}{\overset{\Lambda'}{A}}}{\uparrow (*)\qquad\qquad\qquad \Box A}\,(0)
$$

where Λ' is

$$
\begin{array}{c}
\mathcal{S}_1\dfrac{\overset{(0)}{\Box A}\quad \overset{\Sigma_1}{C_1}}{\Box C_1}\ \cdots(*)\cdots\ \Box C_j\ \cdots\ \mathcal{S}_k\dfrac{\overset{(0)}{\Box A}\quad \overset{\Sigma_k}{C_k}}{\Box C_k}\quad \overset{(0)}{[\Box A]}\\
\Sigma_0\\
A
\end{array}
$$

The assumption class (0) still contains diagonal assumptions but there are only k-1 secondary inferences. $\Box X'$ is a subset of $\Box X$. Σ' denotes those derivations in Σ that are derivations of formulas in $\Box X'$.

(GLR REDUCTION)

We define an operation $\mathbf{H}$ carrying derivations into derivations by recursion on the number of the secondary inferences of the given derivation as follows.

$\mathbf{H}(\Pi) = \Pi$ if the last inference $\mathcal{R}$ of Π is not a proper GLR.

Now suppose $\mathcal{R}$ is a proper GLR, so it has the form shown in (1), and let k be the number of secondary inferences in Π. If $k = 0$ then $\mathbf{H}(\Pi)$ is defined by steps (i)–(iii) below or, in other words, $\mathbf{H}(\Pi)$ is obtained by replacing every diagonal assumption $\Box A$ by Π. If $k > 0$, for $j = 1, \ldots, k$, let Πj be as in (2); then $\mathbf{H}(\Pi_j) = \Phi_j$ is defined and we may make the further assumption that Φ_j is of the form

(2)

$$\mathcal{T}\ \dfrac{\Box C_j \qquad \begin{matrix}\Sigma' \\ \Box X'\end{matrix} \qquad\qquad \begin{matrix}[\Box C_j] \\ \Xi_j \\ A\end{matrix}}{\Box A}$$

where $\mathcal{T}$ is a K4R application, $\Box X'$ is a subset of $\Box X$ and Σ' indicates as above the derivations of $\Box X'$ among those in Σ.

Then $\mathbf{H}(\Pi)$ is defined through the following steps:

(i) Introduce $k+1$ new major premises $D_0, \ldots, D_k$, all of the form $\Box A$, in $\mathcal{R}$; if $\Box A$ is a diagonal assumption occurring in the secondary inference $\mathcal{S}_j$, then discharge it in virtue of D_j; if $\Box A$ is a diagonal assumption that does not belong to any secondary inference, then discharge it in virtue of D_0.

(ii) For $i = 1, \ldots, k$, take Φ_j as derivation of D_j; take Π itself as derivation of D_0. The resulting derivation has the form

(3)

$$\mathcal{R}'\ \dfrac{\begin{matrix}\Pi \\ \Box A\end{matrix} \quad \begin{matrix}\Phi_1 \\ \Box A\end{matrix} \quad \ldots \quad \begin{matrix}\Phi_k \\ \Box A\end{matrix} \quad \begin{matrix}\Sigma \\ \Box X\end{matrix} \quad \begin{matrix}(0) \qquad (k) \\ [\Box A] \ldots [\Box A] \\ \Lambda \\ A\end{matrix}}{\uparrow(0) \quad \uparrow(1) \quad \ldots \quad \uparrow(k) \qquad\qquad \Box A}$$

where Λ is

$$\begin{matrix} \mathcal{S}_1\ \dfrac{\begin{matrix}(1) \\ [\Box A]\end{matrix} \quad \begin{matrix}\Sigma_1 \\ C_1\end{matrix}}{\Box C_1} \quad \ldots \quad \mathcal{S}_k\ \dfrac{\begin{matrix}(k) \\ [\Box A]\end{matrix} \quad \begin{matrix}\Sigma_k \\ C_k\end{matrix}}{\Box C_k} \quad \begin{matrix}(0) \\ [\Box A]\end{matrix} \\ \Sigma_0 \\ A \end{matrix}$$

(iii) Apply a step of Segment Reduction with respect to each new major premise $D_0, \ldots, D_k$. Let the result of this step be

(4)

$$\mathcal{R}''\ \dfrac{\begin{matrix}\Sigma^* \\ \Box X^*\end{matrix} \quad \Box C_1 \ldots \Box C_k \quad \begin{matrix}\Sigma \\ \Box X\end{matrix} \qquad \begin{matrix}[\Box C_1] \ldots [\Box C_k] \\ \Lambda_1 \\ A\end{matrix}}{\Box A}$$

(where Σ^* stands for the sequence $\Sigma', \Sigma'', \ldots, \Sigma'^{\cdots\prime}$ of subsets of Σ and) where Λ_1 has the form

$$\begin{matrix} \ldots \quad \mathcal{S}_j\ \dfrac{\mathcal{T}\ \dfrac{\Box C_j \quad \Box X'^{\cdots\prime} \quad \begin{matrix}(*) \\ [\Box C_j] \\ \Xi_j \\ A\end{matrix}}{\uparrow(*) \qquad\qquad \Box A} \qquad \begin{matrix}\Sigma_j \\ C_j\end{matrix}}{\Box C_j} \quad \ldots \\ \Sigma_0 \\ A \end{matrix}$$

(iv) In Λ_1, for $j = 1, \ldots, k$ consider each major premise $\Box A$ of $\mathcal{S}_j$. That became maximal as a consequence of step (iii): apply a K4R reduction to the subderivation ending with $\mathcal{S}_j$, if the formula discharged in virtue of such $\Box A$ is A (this is possible because $\mathcal{T}$ is a K4R application, by assumption); apply a step of segment reduction, if the formula discharged in virtue of it is $\Box A$.

(v) After step (iv) every Ξ_j occurs as a subderivation in the derivation of the minor premise of $\mathcal{S}_j$; discharge each $\Box C_j$ as diagonal assumption of $\mathcal{S}_j$. The result of the application of steps (iv) and (v) with respect to a major premise of $\mathcal{S}_j$ is thus either

(5)

$$\mathcal{R}^*\ \dfrac{\begin{matrix}\Sigma^* \\ \Box X^*\end{matrix} \quad \begin{matrix}\Sigma \\ \Box X\end{matrix} \quad \begin{matrix} \ldots\ \mathcal{S}_j\ \dfrac{\Box X' \qquad \begin{matrix}(*) \\ [\Box C_j] \\ \Xi_j \\ [A] \\ \Sigma_j \\ C_j\end{matrix}}{\Box C_j}(*) \\ \Sigma_0 \\ A \end{matrix}}{\Box A}$$

or

(6)

$$
\begin{array}{rccccr}
 & & & & (*) & \\
 & & & & [\Box C_j] & \\
 & & & & \Xi_j & \\
 & & & (+) & & \\
 & & & \Box C_j \; \Box X' & A & \\
 & & \mathcal{T} & \overline{} & \overline{} & \\
 & & & \uparrow (*) & [\Box A] & \\
 & & & & \Sigma_j & \\
 & & \Box X' & & C_j & \\
 & \ldots & \mathcal{S}_j & \overline{} & \overline{} & (+) \\
 & & & & \Box C_j & \\
 & \Sigma^* \quad \Sigma & & & \Sigma_0 & \\
 & \Box X^* \quad \Box X & & & A & \\
\mathcal{R}^* & \overline{} & \overline{} & \overline{} & \overline{} & \\
 & & & & \Box A &
\end{array}
$$

(Notice that after step (v) the new major premises $\Box C_j$ of have been cancelled as redundant). The construction of $\mathbf{H}(\Pi)$ is finished: $\mathbf{H}(\Pi)$ is a derivation of $\Box A$ from the same premises as Π, that ends with a K4R application.

THEOREM 2. Every derivation in GL_{nat} *can be transformed into a normal derivation of the same formula from the same set of assumptions.*

Proof. We modify the definition of index of an inference $\mathcal{R}$ by taking $I(\mathcal{R}) = \langle \alpha, \varkappa, \beta, \gamma \rangle$ where α, β, γ are as above and $\varkappa$ is the number of major premises of degree α in $\mathcal{R}$ that are consequence of a proper GLR application if $\mathcal{R}$ is a modal rule, 0 otherwise. Indices are still ordered lexicographically, and the index of a derivation is defined as above, using the modified notion of index of an inference.

Given a derivation Γ of index $\mathcal{J}$ and a subderivation Δ whose last inference $\mathcal{U}$ is of maximal index, let Δ' be the derivation obtained from Δ by normalizing the immediate subderivations. Following the above argument, we see that the result of replacing Δ with Δ' in Γ is not of index higher than $\mathcal{J}$ (in particular the number of major premises of $\mathcal{U}$ that are consequence of a proper GLR application, if any, is not affected). Let $\mathcal{U}'$ be the last inference of Δ' and let

$I(\mathcal{U}') = \langle \alpha, \varkappa, \beta, \gamma \rangle$. Now the argument proceeds as above, except that in Case a) we have

Subcase (0). E is a consequence of a proper GLR.

Let E be $\Box A$, let Π be its derivation and let Δ'' be the result of replacing Π with $\mathbf{H}(\Pi)$ in Δ'. We want to see that Δ'' has index lower than Δ'. Let $\mathcal{U}''$ be the last inference of Δ'': $I(\mathcal{U}'') = \langle \alpha, \varkappa^{-1}, \beta^+, \gamma^+ \rangle$ so $I(\mathcal{U}'')$ is lower than $I(\mathcal{U}')$. If we can show that all inferences in $H(\Pi)$ have indices lower than $I(\mathcal{U}')$ the proof will be finished.

This is done by induction on the definition of $\mathbf{H}$. Consider the derivation in (3) after step (ii): by induction hypothesis every inference in $\Phi_1, \ldots, \Phi_k$ has index lower than $I(\mathcal{U}')$. The inference $\mathcal{R}'$ contains $k+1$ maximal major premises $D_0, \ldots, D_k$ of degree α. Only D_0 (i.e. the premise in virtue of which the assumption class (0) is discharged) is a consequence of a proper GLR. In step (iii) by applying a segment reduction with respect to this premise we eliminate this maximal formula without generating any new maximal formula, since by construction the assumption class (0) contains only assumptions that do not belong to any secondary inference. (This takes care also of the base case).

Next consider the derivation in (4) after step (iii): for $j = 1 \ldots k$ $I(\mathcal{S}_j) = \langle \alpha, 0, \gamma', \delta' \rangle$ for some γ' and δ', so $I(\mathcal{S}_j)$ is certainly lower than $I(\mathcal{U}')$, since $\varkappa > 0$. We know from the induction hypothesis that every inference in Ξ_j has index lower than $I(\mathcal{U}')$. Moreover, no other inference in (4) contains maximal or $\perp_C$/GLR maximal formulas.

Step (iv) involves only K4R- or segment reduction; therefore, in order to conclude that every inference in the derivation after step (v) (see (5) and (6)) has index less than $I(\mathcal{U}')$, we have to see only that by discharging each $\Box C_j$ as diagonal assumption of $\mathcal{S}_j$ we do not increase the index of any inference. But this is obvious: the conclusion of any $\mathcal{S}_j$ cannot be maximal, since by assumption Σ_0 is normal. *Q.E.D.*

Example

It may be helpful to see the procedure of normalization in a concrete case. Let $\triangle$ be

$$\dfrac{\begin{matrix}\Box\sim\Box A\\ \Pi\\ \Box(\Box A\,\&\,\Box B)\end{matrix} \qquad \dfrac{\Box A\,\&\,\Box B}{\Box A}}{\Box\Box A}$$

where Π is

$$\mathcal{R}\,\dfrac{\Box\sim\Box A \qquad \dfrac{\begin{matrix}(5)\\ \Box(\Box A\,\&\,\Box B)\\ \Lambda_1\\ \Box A\end{matrix} \qquad \begin{matrix}(5)\\ \Box(\Box A\,\&\,\Box B)\\ \Lambda_2\\ \Box B\end{matrix}}{\Box A\,\&\,\Box B}}{\uparrow(6) \qquad \Box(\Box A\,\&\,\Box B)}\ (5)$$

and moreover Λ_1 is

$$\mathcal{S}_1\,\dfrac{\begin{matrix}(5)\\ \Box(\Box A\,\&\,\Box B)\end{matrix} \qquad \begin{matrix}(6)\\ \Box\sim\Box A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}(1)\\ \Box A\,\&\,\Box B\end{matrix}}{\Box A} \qquad \begin{matrix}(2)\\ \sim\Box A\end{matrix}}{\bot}}{A}}{\uparrow(1) \qquad \uparrow(2) \qquad \Box A}$$

and Λ_2 is

$$\mathcal{S}_2\,\dfrac{\begin{matrix}(5)\\ \Box(\Box A\,\&\,\Box B)\end{matrix} \qquad \begin{matrix}(6)\\ \Box\sim\Box A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}(3)\\ \Box A\,\&\,\Box B\end{matrix}}{\Box A} \qquad \begin{matrix}(4)\\ \sim\Box A\end{matrix}}{\bot}}{B}}{\uparrow(3) \qquad \uparrow(4) \qquad \Box B}$$

The reader should verify that the result of applying the normalization procedure to $\triangle$ is

$$\dfrac{\Box\sim\Box A \qquad \dfrac{\begin{matrix}(3)\\ \Box\sim\Box A\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}(1)\\ \Box A\end{matrix} \qquad \begin{matrix}(2)\\ \sim\Box A\end{matrix}}{\bot}}{A}}{\uparrow(2) \qquad \Box A}\ (1)}{\uparrow(3) \qquad \Box\Box A}$$

The first stage of the computation is the construction of $\mathbf{H}\,(\Pi)$. Let Π_1, (Π_2) be the result of erasing the subderivation Λ_1 (Λ_2) in Π. Similarly, let $\Pi_{1,2}$ be the result of erasing both Λ_1 and Λ_2 in Π. A part of the computation of $\mathbf{H}\,(\Pi)$ is the construction of a derivation $\mathbf{H}\,(\Pi_1) = \Phi_1$ of $\Box(\Box A\,\&\,\Box B)$ from $\Box\sim\Box A$ and $\Box A$ ending with a K4R application. This is done by using $\Pi_{1,2}$ to eliminate the diagonal assumption of $\mathcal{S}_1$ in Π_1. Similarly we construct a derivation $\mathbf{H}\,(\Pi_2) = \Phi_2$ of $\Box(\Box A\,\&\,\Box B)$ from $\Box\sim\Box A$ and $\Box B$ ending with a K4R application. Finally $\mathbf{H}\,(\Pi)$ is obtained by using Φ_1 and Φ_2 to eliminate the diagonal assumptions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The normalization of $\triangle$ is now obtained just by applying reductions different from the GLR reduction.

Subformula property

We can easily see that every branch $C_1,\ldots,C_n$ in a normal deduction consists of

i) an analytical part $C_1,\ldots,C_{m-1}$, in which every formula C_i is the major premise of an elimination or of a GLR and contains the immediately succeeding formula C_{i+1} as subformula;

ii) a minimum part C_m, (C_{m+1}) containing either a single formula or the premise of a $\bot_C$ application and its conclusion. In the first case such formula is subformula both of C_1 and of C_n and by the Expansion Lemma may be assumed atomic; in the second case the conclusion of the $\bot_C$ application is a subformula of C_n;

iii) a synthetical part (C_{m+1}), $C_{m+2},\dots,C_n$ in which every formula C_{j+1} is the conclusion of an introduction or of a GLR and contains the immediately preceeding formula C_j as a subformula.

COROLLARY 1. Every normal deduction of GL_{nat} has the subformula property (with the exception of assumptions discharged by $\bot_C$ and of occurrences of $\bot$ immediately following such assumptions).

Proof. The corollary follows from the above description of the branches in a normal deduction, by induction on the height of the branches (= number of branches occurring below the given one). The details are left to the reader. (Note that if $\Box A$ is a diagonal assumption, then it occurs also in the synthetical part of some branch of the same or lower height. Moreover if $\sim\Box A$ is an assumption discharged by $\bot_C$, and so an exception to the subformula property, then $\Box A$ belongs to the minimum part of some branch of the same or lower height.) Q. E. D.

It is immediate to see the following *facts* about normal deductions:

(*) *a branch starting with an assumption $\sim D$ discharged by $\bot_C$ contains only this assumption in its analytical part.*

Moreover it follows from (*) that

(**) *any major premise of a modal inference belongs to the elimination part of some branch and the first formula of such branch is not discharged by $\bot_C$.*

COROLLARY 2. Every closed normal deduction in GL_{nat} *ends with an introduction or with a* GLR *having no major premises.*

Proof. Let Π be a normal proof of $\Box A$. Π cannot end with an elimination, since then the first formula of the endbranch would be an open assumption. Suppose Π ends with an application $\mathcal{R}$ of $\bot_C$: we want to see that $\mathcal{R}$ is redundant.

Consider the tree $\tau(\Box A)$ associated with the final $\Box A$. We show by induction on $\tau(\Box A)$ that below every formula occurrence in the tree there are only applications of $\bot_C$ with $\Box A$ as conclusion. Suppose Π has the form

$$\dfrac{\mathcal{S}\,\dfrac{\Sigma}{B} \qquad \dfrac{\Sigma'}{\sim B}}{\mathcal{R}\,\dfrac{\bot}{\Box A}}$$

The endbranch has no synthetical part, so its first formula has the form $\sim\Box A$ and is discharged by $\bot_C$. Conclude by (*) that Σ' is just $\sim B$ and B is $\Box A$. A completely analogous argument gives the induction step. If $\tau(\Box A)$ contained more than 2 nodes we could easily make all the $\bot_C$ applications introducing its nodes redundant, except for one, let it be $\mathcal{R}$.

So $\mathcal{S}$ is not an $\bot_C$ application; it is not an elimination, since again the first formula of the branch, which B belongs to, would be an open assumption. So $\mathcal{S}$ is a GLR; we want to see that it is without major premises. If C was a major premise of $\mathcal{S}$, by (**) the first formula of its branch would be an open assumption. Therefore $\mathcal{S}$ has no major premises. But then $\mathcal{R}$ is redundant too. *Q. E. D.*

COROLLARY 3. If $\vdash_{GL_{nat}} \Box A$, *then* $\vdash_{GL_{nat}} A$.

Proof. This is an immediate consequence of the normalization theorem and of the preceeding corollary.

Since GL_{nat} is consistent (e.g. from Corollary 2), the unprovability of $\Box\bot$ follows from Corollary 3, as well as the unprovability of $\Box\sim\Box A$, for every A (since $\vdash_{GL_{nat}} \Box\sim\Box A \rightarrow \Box\bot$).

COROLLARY 4. If $\Box X \vdash_{GL_{nat}} \Box A_1 \vee \dots \vee \Box A_n$ *then* $\Box X' \vdash_{GL_{nat}} \Box A_i$, for some $i \le n$ and $X' \subseteq X$.

Proof. The hypothesis is equivalent to

$$\sim\Box A_1,\dots, \sim\Box A_{n-1}, \Box X \vdash_{GL_{nat}} \Box A_n$$

Take a normal deduction of $\Box A_n$ from $\sim\Box A_1,\dots,\sim\Box A_{n-1}, \Box X$ and consider the first occurrence B from bottom up which is neither a conclusion nor a premise of $\bot_C$ nor a major premise of an

$\rightarrow$ elimination. B is either the endformula $\Box A_n$ itself or the minor premise of an $\rightarrow$ elimination. In the latter case by an argument analogous to that of Corollary 2 using (*) we show that below B there are only $\perp_C$ applications whose conclusion is one of the formulas $\Box A_1 \ldots, \Box A_n$; so B must be one of these formulas too.

Let B be $\Box A_i$. If B belongs to $\Box X$ too, then the result is trivial. Otherwise B must be the conclusion of a GLR. But now by arguing as in Corollary 2 using (**) we show that the major premises of such GLR must belong to $\Box X$. Hence

$$\Box X' \vdash_{GL_{nat}} \Box A_i$$

Q. E. D.

References

A. Avron (1984) On Modal Systems Having Arithmetical Interpretations, J. S. L. 49-3, pp. 935–942.

G. Bellin (1982) Cut Elimination for GL, Abstract AMS 3-4, n. 82T-03-332, pp. 282–283.

G. Boolos (1979) The unprovability of consistency: an essay in modal logic, Cambridge, Cambridge University Press.

D. Leivant (1981) On the Proof Theory of the Modal Logic for Arithmetic Provability, J. S. L. 46-3, pp. 531–538.

D. Prawitz (1965) Natural Deduction, Stockholm, Almqvist & Wiksell.

D. Prawitz (1971) Ideas and results in Proof Theory, in Proceedings of the Second Scandinavian Logic Symposium, Oslo, Fenstad Editor, North Holland.

R. Solovay (1976) Provability interpretations of modal logic, Israel Journal of Mathematics 25, pp. 287–304.

G. Sambin and S. Valentini (1980) A modal sequent calculus for a fragment of arithmetic, Studia Logica 39, pp. 245–256.

Review

Saari, Heikki. ***Re-enactment: A study of R. G. Collingwood's philosophy of history.*** Acta Academiae Aboensis, Series A (Humaniora), vol. 63. Åbo, Finland: Åbo Akademi, 1984.

This monograph on Collingwood by Heikki Saari is Saari's dissertation at the Swedish-language university in Finland, Åbo Academy, in Åbo (the Finnish town name for which is Turku). The dissertation was written under the direction of Lars Hertzberg, a Wittgenstein scholar. I am pleased that Professor Hertzberg has sponsored such a study, for I think there are important contributions to analytic philosophy of history in general and to the interpretation of Collingwood's views in particular that can be made by students of Wittgenstein's thought. (And I have tried to make some such contributions myself in Martin, 1977, chap. 10, and in 1981.)

The most significant event in recent Collingwood scholarship is, of course, the availability of his unpublished writings at Oxford from 1978 on. Saari has not utilized this material directly, but he has made good use of the recent studies which draw on Collingwood's manuscripts and quote passages from the unpublished writings. Specifically, he draws on an article and a dissertation, the latter in Danish, by Margit Hurup Nielsen (1981 and 1980, respectively) and on an article and a path-breaking book by W. J. van der Dussen (1979 and 1981, respectively).

Saari is familiar with a good bit of the work in philosophy of history, especially that part which concerns Collingwood. And it is against this backdrop that I'll assess his contribution. Fittingly, his book begins with a brief but competent placing of Collingwood within the history of modern philosophy and philosophy of history.

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